



# A Hard-Core Predicate for all One-Way Functions

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## Abstract

A central tool in constructing pseudorandom generators, secure encryption functions, and in other areas are “hard-core” predicates  $b$  of functions (permutations)  $f$ , discovered in [Blum Micali 82]. Such  $b(x)$  cannot be efficiently guessed (substantially better than 50-50) given only  $f(x)$ . Both  $b, f$  are computable in polynomial time.

[Yao 82] transforms any one-way function  $f$  into a more complicated one,  $f^*$ , which has a hard-core predicate. The construction applies the original  $f$  to many small pieces of the input to  $f^*$  just to get one “hard-core” bit. The security of this bit may be smaller than any constant positive power of the secu-

rity of  $f$ . In fact, for inputs (to  $f^*$ ) of practical size, the pieces effected by  $f$  are so small that  $f$  can be inverted (and the “hard-core” bit computed) by exhaustive search.

In this paper we show that every one-way function, padded to the form  $f(p, x) = (p, g(x))$ ,  $\|p\| = \|x\|$ , has by itself a hard-core predicate of the same (within a polynomial) security. Namely, we prove a conjecture of [Levin 87, sec. 5.6.2] that the scalar product of boolean vectors  $p, x$  is a hard-core of every one-way function  $f(p, x) = (p, g(x))$ . The result extends to multiple (up to the logarithm of security) such bits and to any distribution on the  $x$ 's for which  $f$  is hard to invert.

## 1 Introduction

One-way functions are fundamental to many aspects of Theory of Computation. Loosely speaking, one-way are those functions which are easy to evaluate but hard to invert. However, many applications such as pseudorandom generators [Blum Micali 82, Yao 82] and secure probabilistic encryption [Goldwasser Micali 82] require that the function has a “hard-core” predicate  $b$ . This  $b(x)$  should be easy to evaluate on input  $x$ , but hard

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to guess (with a noticeable correlation) when given only the value of  $f(x)$ . Intuitively, the hard-core predicate “concentrates” the one-wayness of the function in a strong sense.

Clearly, permutations with hard-core predicates must be one-way. A natural question of practical and theoretical importance is *which* one-way functions have one. So far only partial answers have been given:

1. [Blum Micali 82] showed that the discrete exponentiation function, if it is one-way, has a hard-core predicate.<sup>1</sup> Analogous results for the RSA and Rabin functions (i.e. raising to a power modulo a composite integer) have been shown in [Alexi Chor Goldreich Schnorr 84].
2. [Yao 82] showed how any one-way permutation  $f$  can be used to construct another one-way permutation  $f^*$  which has a hard-core predicate. The function  $f^*$  partitions its input into many shorter inputs and applies  $f$  to each of them in parallel:  $f^*(x_1 \dots x_{k^3}) = f(x_1) \dots f(x_{k^3})$ ,  $\|x_i\| = k$ . (For a proof and more refined analysis see [Levin 87].)

The drawback of the first set of results is their dependence on a specific intractability assumption (e.g. the hardness of the discrete logarithm problem). The second result offers a function  $f^*$  with security smaller than any constant power of the security of  $f$ .

This paper resolves the above question by providing a hard-core predicate for every one-way function. More specifically, for any  $s$  (e.g.  $s(n) = n$ , or  $s(n) = 2^{\sqrt{n}}$ ), the following tasks are equivalent for probabilistic algorithms running in time  $s(\|x\|)^{O(1)}$ :

1. Given  $f(x)$  find  $x$  for at least a fraction  $s(\|x\|)^{-O(1)}$  of the  $x$ 's.
2. Given  $f(x)$  and  $p$ ,  $\|p\| = \|x\|$ , guess the Boolean inner-product  $B(x, p)$  of  $x$  and  $p$  with a correlation (i.e. the difference between the success and failure probabilities) of  $s(\|x\|)^{-O(1)}$ .

For any polynomial time computable  $f, b$ , there is always the smallest (within a polynomial) such  $s$  called the *security* of  $f$  and  $b$ , respectively. The security is a constructible function, and can be computed by trying all small guessing algorithms. It is assumed to grow very fast (at least  $n^{1/o(1)}$ ).

## 2 Conventions

Let  $S$  be the set of finite and  $\Omega$  of infinite strings in the alphabet  $\{0, 1\}$ ; let  $\|x\|$  be the length of  $x$ ,  $S_n \subset S$  be the set of strings of length  $n$ , and  $x \circ y$  be the concatenation of  $x$  and  $y$ . We identify  $S$  (in lexicographical order) with  $\mathbf{N} = \{0, 1, \dots\}$ .

For simplicity, we consider only length preserving functions  $f : S \rightarrow S$ ,  $\|f(x)\| = \|x\|$ . Let  $F$  be the set of such functions. The set of functions computable in polynomial time is denoted  $P$ . We almost always refer to predicates as having range  $\{\pm 1\}$  instead of  $\{0, 1\}$ . The Boolean value  $\sigma$  corresponds to  $(-1)^\sigma \in \{\pm 1\}$ , and the exclusive-or of Boolean values corresponds to multiplication.

When the probability distribution of  $x$  is clear from the context,  $E_x f(x)$  denotes the expected value of  $f$ . Thus, a Boolean predicate  $P$  holds with probability  $E_x P(x)$ .

Time bounds of the (adversarial) algorithms is assumed here to be huge comparative to input lengths. Thus we allow this

<sup>1</sup>Extended to all Abelian groups in [Kaliski 88].

preliminary draft to ignore factors of  $\|x\|^{O(1)}$  in the running time. So time bound of  $T(x)$  means that the algorithm may make  $\|x\|^{O(1)}T(x)$  steps. These factors are easy to figure out from the context. This convention makes the choice of model of computation not so important.

### 3 The Results

Below, the function  $d$  merely generates a probability distribution of instances (and the condition  $d \in F$  is not really essential). A simple case of interest is  $d(x) = x$ .

Let  $I(\omega, y)$  be a probabilistic algorithm which attempts to invert  $f \in F$ , i.e. to compute a list containing  $x$  from  $y = f(x)$ , using  $\omega \in \Omega$  as the source of internal coin flips. Its performance has two aspects: the running time  $T_I(\omega, y)$  and the probability of success:

$$R_{I,f,d}(n) \stackrel{\text{def}}{=} E_{r,\omega}(x \in I(\omega, f(x))),$$

where  $r \in S_n$ ,  $x = d(r)$ . We can combine these two measures (i.e. absorb the running time into  $R_{I,f,d}$ ) in the following way. Without loss of generality, we require the mean running time  $E_\omega T_I(\omega, y)$  of all *inverting algorithms* to be  $O(1)$ . Any algorithm can be modified to satisfy this requirement. For this purpose,  $I$  may use its power of flipping coins to set itself random time limits  $2^t$  with probability, say,  $2^{-t}/t^2$ . This will decrease the probability of success in proportion to  $I$ 's original running time. Then  $R_{I,f,d}$  accounts for running time as well as the probability of success and is called the *inverting rate* of  $I$  for  $f$  on  $d$ . It reflects the reciprocal of the time needed to notice by sampling the instances where  $I$  inverts  $f$ .

One-way functions are those invertible with a negligible rate only. Namely,

**Definition 1** (One-way Functions): A function  $f \in F$  is *one-way* on  $d \in F$  with security  $s: \mathbb{N} \rightarrow \mathbb{N}$  if  $R_{I,f,d}(n) = o(1)/s(n)^\epsilon$ , for some  $\epsilon > 0$  and all probabilistic inverting algorithms  $I$ . The security is *strict* if  $\epsilon > 1$ .

Let  $G(\omega, y) \in \{\pm 1, 0\}$  be a probabilistic algorithm which, given  $y = f(x)$ , attempts to guess a predicate  $b(x) \in \{\pm 1\}$ ,  $b \in P$ , using  $\omega$  as the source of internal coin flips (0 means a refusal to guess). As with inverting algorithms above, without loss of generality, we restrict its average running time:  $E_\omega T_G(\omega, y) = O(1)$ . We define the *guessing rate*  $R$  of  $G$  for  $b$  from  $f$  to reflect the reciprocal of the time needed to notice by sampling the correlation between  $b$  and  $G$ , i.e. the number of trials needed to evaluate the expectation of their product. Due to the  $O(1)$  restriction on expected running time, this number accounts for both the running-time and the correlation of  $G$  with  $b$ . The number of trials is determined by the reciprocal of

$$R_{G,f,b,d}(n) \stackrel{\text{def}}{=} \frac{(E_{\omega,r} b(x) G(\omega, y))^2}{E_{\omega,r} G(\omega, x)^2},$$

where  $r \in S_n$ ,  $x = d(r)$ ,  $y = f(x)$ ,  $\omega \in \Omega$ .

Hard-core predicates of a function are those that can be guessed from its output only with a negligible rate. Namely,

**Definition 2** (Hard-Core Predicates):

A predicate  $b$  is called a *hard-core* with security  $s$  for a function  $f \in F$  on  $d \in F$  if  $R_{G,f,b,d}(n) = o(1)/s(n)^\epsilon$ , for some  $\epsilon > 0$  and all probabilistic guessing algorithms  $G$ .

The security is *strict* if  $\epsilon > 1$ .

We call *padded* a function  $f \in F$  of the form  $f(x \circ p) = f'(x) \circ p$  for  $\|p\| = \|x\|$ . *Double-padded*  $f$  has  $\|p\| = 2\|x\|$ . Let  $B(x, p) = \pm 1$  depending on the inner product mod 2 of the Boolean vectors  $x$  and  $p$ ,  $\|p\| = \|x\|$ .

**Theorem 1** *Let  $f$  and  $d$  be arbitrary padded functions and  $f$  be one-way on  $d$  with security  $s$ . Then  $B$  is a hard-core predicate for  $f$  on  $d$  (with the same security  $s$ ).*

The theorem follows from the following Lemma, which efficiently reduces the task of retrieving  $x$  from  $f(x, p)$  to the task of approximating  $B(x, p)$  given  $f(x, p)$ . The rate  $\epsilon$  in the Lemma is chosen at random with distribution assuring the  $O(1)$  average running time. Let  $\tau(p, y) = \mathbb{E}_\omega |G(\omega, y, p)|$ ,  $\tau(y) = \mathbb{E}_p \tau(p, y)$  and  $\bar{\tau}(y) = \mathbb{E}_p \tau^2(p, y)$ . Note that the guessing rate of  $G$  does not exceed the average over  $x$  of its local (i.e. taken over fixed  $x, y = f'(x)$ ) guessing rates  $R(x) = (\mathbb{E}_{\omega, p} B(x, p) G(\omega, y, p))^2 / \tau(y)$ .

**Lemma 1 (Main):** *There exists an algorithm  $I$  that given a subroutine  $G$ , the coin-flip source  $\omega \in \Omega$ , and inputs  $y \in S, \epsilon \in (0, 1]$ , outputs a list of  $\bar{\tau}/\tau\epsilon$  strings including all  $x \in S_{||y||}$  with  $R(x) \geq \epsilon$ .  $I(\omega, y, \epsilon)$  may fail for  $\epsilon$  fraction of  $\omega$ . The first of the two stages of  $I$  takes  $1/\epsilon$  steps of  $I$  and  $G$ . The second stage does not call  $G$  and takes  $(\bar{\tau}/\tau\epsilon)^2 < \epsilon^{-2}$  steps.*

The proof is in Section 4. It seems likely that the second stage can also be sped up to  $1/\epsilon$  steps. The computation can be parallelized using  $\epsilon^{-O(1)}$  processors in  $-\log \epsilon$  time.

The theorem extends to  $\log s(n)$  secure bits. For  $c \in \{0, 1\}$ ,  $a_0, a_1 \in S$ , define  $\text{Pr}_c(a_0, a_1) = a_c$ .

**Definition 3 (Hard-core function):** A function  $h: S \rightarrow S$  in  $\mathbf{P}$  is called a *hard-core* (with security  $s$ ) of  $f \in \mathbf{F}$  on  $d \in \mathbf{F}$  if  $b(x, r, c) = (-1)^c$  is a hard-core predicate (with security  $s$ ) on  $d$  for the function  $f_h(x, r, c) = f(x) \circ \text{Pr}_c(r, h(x)) \circ 0$ ,  $\|r\| = \|h(x)\|$ .

So, given  $f(x)$ , the output of a hard-core  $h(x)$  should be indistinguishable in feasible time from a randomly chosen string  $r$ .

A *Toeplitz* matrix is a matrix  $M$  such that for all  $i, j$ ,  $M_{i,j} = M_{i+1,j+1}$ . Let  $k: \mathbb{N} \rightarrow \mathbb{N}$  and  $M_p$  be  $k(n) \times n$  Boolean Toeplitz matrix with the first row and column determined by the corresponding bits of  $p$ ,  $\|p\| \geq k(n) + n - 1$ . Let Boolean vector  $H_k(x, p)$  be the matrix product  $M_p$  times  $x$ , for  $\|p\| = 2\|x\|$ .

**Corollary 1** *Let  $f, d$  be arbitrary double-padded functions and  $f$  is one-way on  $d$  with security  $s \in \mathbf{P}$ . Then, for some  $\epsilon > 0$ ,  $k(n) = \epsilon \log s(n)$ ,  $H_k$  is a hard-core function for  $f$  on  $d$  with the same security  $s$ .*

The proof is in Section 5. This number of pseudorandom bits cannot be improved without additional assumptions or a major breakthrough in complexity theory. The reason is that a one-way function with security  $s$  may act only on  $\log s(\|x\|)$  of the bits of  $x$  and leave the rest unchanged.<sup>2</sup> A restriction of the “optimality” claim to bits extractable through linear transformations (as in the Corollary) can be easily proven.

## 4 Proof of Main Lemma

Our inverting algorithm  $I(\omega, y, \epsilon)$  lists all strings  $x$  for which  $G$  guesses  $B$  with a local rate  $R(x) = (\mathbb{E}_{p, \omega} B(x, p) G(\omega, y, p))^2 / \tau(y) \geq \epsilon$ .  $I$  constructs the list  $L$  containing these strings and all their prefixes bit by bit, in  $n = \|y\|$  rounds. The  $k$ -th round of  $I$  generates  $L_k = L \cap S_k$ . During the next round  $I$

<sup>2</sup>To exclude this possibility one must rule out the existence of one-way functions  $f$  with security  $2^{\epsilon n}$ . Otherwise  $f_*(x'x'') = f(x')x''$ ,  $\epsilon\|x'\| = \log s(\|x'x''\|)$  has security  $s$ .

examines all one-bit extensions of strings in  $L_k$ , discards some of them and keeps the rest.

Throughout the proof  $y, k, \varepsilon$  are fixed and used implicitly. So, let  $g$  be the matrix with components  $g_{r,s} = E_\omega G(\omega, y, rs)$ , and  $b$  be the matrix with components  $b_{z,r} = B(z, r)$  for  $r, z \in S_k$ .

**Note 1** Matrix  $2^{-k/2}b$  is symmetric and orthonormal, i.e.  $b = b^T = 2^k b^{-1}$ .

**Proof:** Indeed,  $\sum_r (B(z, r))^2 = 2^k$ . When  $z_1, z_2$  differ by  $i$ -th digit, let  $z = z_1 \oplus z_2$  and  $r'$  be  $r$  with  $i$ -th digit changed.

Then  $\sum_r B(z_1, r)B(r, z_2) = \sum_r B(z, r) = 0$ , since  $B(z, r) = -B(z, r')$ . ■

We discard prefixes  $z$  from  $L$  using an upper bound  $c(z)$  for  $R(x)\tau(y)$ ,  $x = zz'$ . It is  $c(z) \stackrel{\text{def}}{=} E_s(E_{r,\omega} B(zz', rs)G(\omega, y, rs))^2 = E_s(E_r B(z', s)B(z, r)g_{r,s})^2$ , which then simplifies to  $E_s(E_r B(z, r)g_{r,s})^2 = E_s(2^{-k}bg_{z,s}^2)$ , since  $B(z', s) = \pm 1$ .

The number of  $z \in S_k$  with  $c(z) > \varepsilon\tau(y)$  remains limited:

**Lemma 2**  $\sum_z c(z) = E_{r,s}g_{r,s}^2 \leq 1$ .

**Proof:** Any matrix multiplied by orthonormal matrix  $2^{-k/2}b$  preserves its mean square of the elements. So,  $\sum_z c(z) = 2^k E_z c(z) = E_{z,s}(2^{-k/2}bg_{z,s})^2 = E_{r,s}g_{r,s}^2$ . ■

All the above does not yield an efficient algorithm, since the straightforward computation of  $c(z)$  takes exponential time. However, the exact value is not needed, a good approximation suffices:

Let  $\tau_p = \tau(p, y)$  for  $\|p\| = n$ ,  $\tau_s = E_r \tau_{r,s}$  for  $\|rs\| = n$ ;  $\bar{\tau}_s = E_r \tau_{r,s}^2$ .

**Lemma 3** *There is a probabilistic algorithm  $A(\omega, z, y, \delta)$  (using  $\omega$  for its internal coin flips) that outputs an approximation  $\tilde{c}(z)$  to*

$c(z)$  with accuracy  $O(\delta)$  and probability of failure  $\delta^3$ . *A halts within  $\tau/\delta$  steps, including the steps of the subroutine  $G$ . Only its last  $\bar{\tau}/\delta$  steps depend on  $z$ .*

**Proof Sketch:** The approximation,  $\tilde{c}(z)$ , is computed as  $\sum_i 4^{-i} E_s(c_s(z) > 2^{-i})$ . If we get estimates with standard deviation  $\alpha$ , repeating them  $l = O(-\log \delta)$  times and taking the median gives an approximation deviating from  $c_s$  by  $2\alpha$  with exponentially small, in  $l$ , probability. For every  $i$  we will need (within log factors)  $4^{-i}/\delta$  samples of  $s$  and  $4^i \tau_s$  samples of  $(r, \omega')$ , of which  $4^i \tau_s^2$  will produce a  $\pm 1$  guess. These “productive” samples may be reused for each  $z$ . ■

A speed up may be achieved by computing  $\tilde{c}(z)$  simultaneously for many  $z$ .

Clearly all steps of passing from  $L_k$  to  $L_{k+1}$  can be made in parallel. Still each  $k \leq n$  takes a sequential iteration. A parallel speed-up can be obtained by keeping the lists of candidates for all substrings of a particular length. Let  $L_{i,l}$  be the list of candidates for the  $l$ -bit long substring of  $x$  starting at location  $i$ . Then  $L_{i,2l}$  is formed by all strings  $z$  in the concatenation of  $L_{i,l}, L_{i+l,l}$  having  $c_i(z) \stackrel{\text{def}}{=} E_{s',s''}(E_r B(z, r)g_{s'r,s''})^2 > \varepsilon\tau$ .

## 5 Proof of the Corollary

Corollary 1 is a special case of the following Lemma 4, as its conditions are obviously satisfied by the family of Toeplitz matrices. Another simple case is a family of all Boolean matrices. Let  $a, k : \mathbf{N} \rightarrow \mathbf{N}$  be in  $\mathbf{P}$ . Let  $\{M_p : p \in S_{a(n)}\}$  be a family of  $k(n) \times n$  Boolean matrices,  $P(u, q, \omega)$  be an algorithm selecting (on inputs  $u, q$  and coin tosses  $\omega$ ) an index  $p$  such that  $uM_p = q$ , and  $M_{P(u,q,\omega)}$  be computable in polynomial time.

**Lemma 4** Suppose  $M, P$  are as above and, for each  $u \neq 0$ ,  $E_{q,z}(p = P(u, q, z)) = 2^{-\|p\|}$ . Then the function  $H(x, p) = M_p x$  is hard-core for  $f$  on  $d$  with strict security  $s$ , if the predicate  $B(x, q)$  is hard-core with strict security  $s(n)4^{k(n)}$ .

The proof of Lemma 4 is based on the idea of [Vazirani 87]. It also incorporates the “XOR condition” of [Vazirani Vazirani 84], proving that a function is hard-core iff the exclusive-or of any non-empty subset of its bits is.

**Proof:** The orthonormal functions  $B_u: r \mapsto B(r, u)$  form a linear basis in the Euclidean space of all real functions on  $S_k$ . So, any function can be expressed as  $g(r) = \sum_u c_u B_u(r)$ , where  $c_u = E_v B(v, u)g(v)$ . A general form for the case  $c_0 = E_r g(r) = 0$  is  $g(r) = NE_{u \neq 0, v} B(v, u)g(v)B(r, u)$ , with  $N = 2^{\|r\|} - 1$ .

Let an algorithm  $G_{\omega, y, p}(r_c)$  guess  $(-1)^c$  with correlation  $\varepsilon$  given  $(y, p) = f(x, p) = (f'(x), p)$ , the source  $\omega$  of internal coin flips and  $r_c = \text{Pr}_c(M_p x, r)$ . Without loss of generality we may assume  $E_{\omega, r} G_{\omega, y, p}(r) = 0$ , which may be achieved by modifying  $G$  so that with probability  $1/2$  it is applied to a random string instead of its argument  $r$  and the sign of the output changed. Then  $\overline{G}_{y, p}(r) \stackrel{\text{def}}{=} E_{\omega} G_{\omega, y, p}(r) = NE_{u \neq 0, v} B(v, u) \overline{G}_{y, p}(v) B(r, u)$ .

$G$ 's correlation is:  $E(-1)^c \overline{G}_{f'(x), p}(r_c) =$

$$NE(-1)^c E_{u \neq 0, v} B(v, u) \overline{G}_{f'(x), p}(v) B(r_c, u),$$

where  $r_c = \text{Pr}_c(M_p x, r)$  and  $E$  averages over  $x, p, r, c$ . For  $c=1$ , it is

$$-NE_{u \neq 0, v, x, p, r} B(v, u) \overline{G}_{f'(x), p}(v) B(r, u) = 0,$$

since  $E_r B(r, u) = 0$ , for each  $u \neq 0$ .

For the term  $c = 0$ , we use  $B(Mx, u) = B(x, uM)$  and express  $(1/N)$  of it as

$$\begin{aligned} & E_{u \neq 0, v, x, p} B(v, u) \overline{G}_{f'(x), p}(v) B(x, uM_p) \\ = & E_{\omega', x, q} g(\omega', f'(x), q) B(x, q), \end{aligned}$$

where  $g(\omega', y, q) \stackrel{\text{def}}{=} B(v, u)G_{\omega, y, P(u, q, w)}(v)$ , with  $u \neq 0, v, w, \omega$  generated with uniform distribution from  $\omega'$ . So,  $g$  guesses  $B(x, q)$  with correlation  $\varepsilon/N$ . ■

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