CS70 Discussion 1c Extra Problems Solutions

Agnibho Roy

Summer 2020

Cubes and Divisors

1. Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 $\forall n \geq 0$.

   Base Case: for $n = 0$, $5^1 + 2^1 = 7$, which is divisible by 7.

   Inductive Hypothesis: For some $k < n$, assume that $5^{2n+1} + 2^{2n+1}$

   Inductive Step: We prove the inductive hypothesis for $k+1$. We want to prove that the expression evaluated at $k+1$ in $\mod 7$ is 0:

   $$5^{2(k+1)+1} + 2^{2(k+1)+1} = 25(5^{2k+1}) + 4(2^{2k+1}) = 4(5^{2k+1}) + 4(2^{2k+1}) = 4(5^{2k+1} + 2^{2k+1}) \mod 7$$

   Since the second part of this expression is divisible by 7 by the inductive hypothesis, we know that this expression is also divisible by 7.

2. Prove using induction that the sum of the first n cubes $(1^3 + 2^3 \ldots n^3)$ is $(\frac{n(n+1)}{2})^2$

   Base Case: for $n = 1$, $(\frac{1(2)}{2})^2 = 1$, which is 1^3

   Inductive Hypothesis: For some $k < n$, assume that $(\frac{k(k+1)}{2})^2$ is the sum of the first k cubes.

   Inductive Step: We prove the inductive hypothesis for $k+1$ given k:

   $$\begin{aligned} 1^3 + 2^3 \ldots (k+1)^3 &= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} \\ &= (\frac{(k+1)(k+2)}{4})^2 \end{aligned}$$

Fibonacci Closed-Form

The nth fibonacci number is represented by the sum of its preceding two values in the sequence: $F_n = F_{n-1} + F_{n-2}$. Prove that we can write the nth Fibonacci number as:

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

For all $n \geq 2$. (*hint*: do we need strong induction here?)
Base Case: We know that the sequence goes $0, 1, 1, \ldots$ so $F_2 = 1$ and $F_3 = 2$. Using the formula:

$$\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2) = \frac{1}{\sqrt{5}}(\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}) = \frac{1}{\sqrt{5}}(\frac{4\sqrt{5}}{4}) = 1$$

$$\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^3 - (\frac{1-\sqrt{5}}{2})^3) = (\text{algebra}) = 2$$

We need to prove two base cases in this case because in our inductive step, we need to use the two previous iterations of the inductive hypothesis to be able to calculate our next value in the sequence.

Inductive Hypothesis: We assume that for $k < n$, $F_k = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k)$ holds

Inductive Step: We want to prove that the inductive hypothesis holds for $k+2$ given that we know it holds for k and $k+1$. We can represent F_{k+2} as:

$$\begin{aligned}
&= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k) + \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}) \\
&= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k + (\frac{1+\sqrt{5}}{2})^{k+1}) - \frac{1}{\sqrt{5}}((\frac{1-\sqrt{5}}{2})^k + (\frac{1-\sqrt{5}}{2})^{k+1}) \\
&= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k + (\frac{1+\sqrt{5}}{2})^{k+1} - ((\frac{1-\sqrt{5}}{2})^k + (\frac{1-\sqrt{5}}{2})^{k+1})) \\
&= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{k+2} - (\frac{1-\sqrt{5}}{2})^{k+2})
\end{aligned}$$

You can convince yourself with a little bit of algebra that:

$$(\frac{1+\sqrt{5}}{2})^k - (\frac{1+\sqrt{5}}{2})^{k+1} = (\frac{1+\sqrt{5}}{2})^{k+2} - (\frac{1-\sqrt{5}}{2})^{k+2}$$