CS70 Discussion 1b Extra Problems Solutions

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Visual Proof

1. Find the sum of the first n odd numbers in terms of a closed-form expression. Start with one dot, and try to build a square by adding more and more dots (add 3 dots, then 5, ...).

   Solution. You should notice a pattern: $1, 1+3=4, 1+3+5=9\dots$ We keep completing a square. Thus, the sum should be n^2.

2. Now that you know what expression if for finding the sum of the first n odd numbers, find a closed-form expression for the first n even numbers.

   Solution. We know that the sum of the first n odd integers is n^2, and the sum of the first n even numbers is just the odd numbers shifted up by 1. Thus, for every odd number, we add 1 to the sum of our even numbers. This means that for the sum of the first n positive integers, we have $n^2 + n = n(n+1)$.

Too Many Socks

I love buying socks so let us say that I have 10 pairs of blue socks, 14 pairs of green socks, 3 pairs of yellow socks, and 2 pairs of orange socks. How many socks do I have to pull out of the drawer randomly to guarantee that I get a matching pair for the day?

Solution. An application of the pigeonhole principle. If I pull out one more than the total number of colors of socks, then I have to at least pull out two of the same color. Thus, we have to pull out 5 to guarantee that we will have a pair.

Ants on a Checkerboard

There is a game where we have a checkerboard that is 8 by 8 (it is also 8 by 8 inches), and there are a total of 66 ants crawling on it. I have a glass that has a circumference of $\frac{3\pi}{2}$ inches. My friend tells me that I win if I can place this glass on board and capture any two of these ants. Prove that no matter what the positions of the ants are, I can always capture at least 2 ants.

Solution. This is one of my favorite questions, especially since it may seem a little counter intuitive if one does not have a strategy to tackle the problem (you may just think to spread the ants out as much as possible). First, let us think of the checkerboard as a board of 64 1 by 1 inch squares. Since we have a total of 66 ants, at least 2 of the squares must have at least 2 ants by the pigeonhole principle. The circumference of our glass is $\frac{3\pi}{2}$, meaning that the diameter is 1.5 inches, which is greater than the diagonal of a an individual checker square, which is $\sqrt{2} \approx 1.4 < 1.5$. Since the glass can fully capture any one of the squares, just place the glass on top of one of the two square on the board that has at least 2 ants.