CS70 Discussion 8a (Exam Review Session)

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1 Can You Guess Their Age?

There are three siblings in a family and it is known that sibling A is the youngest, sibling B is the middle child, and sibling C is the oldest (we only know their ages in years). Moreover, we know that the siblings are all at least of age 10 and at most of age 20. Agnibho, Srishti, and Khalil take some guesses at the number of ways that we could assign ages to each of the siblings.

1. Agnibho is the most stupid, and forgets that the siblings have different ages and who the youngest, middle, and oldest are. How many combinations of ages could Agnibho guess from?

   Solution. Any of the siblings could have any age from 10 to 20, leaving them with 11 choices each for their age. Agnibho can guess from 11^3

2. Srishti is a little less forgetful, but did not listen carefully. She does not know that none of the siblings have the same age as another, but understands that $C \geq B \geq A$. How many combinations of ages could Srishti guess from?

   Solution. We can model this situation as stars and bars, with each bin representing an age, and the stars as each sibling. We can then create a one-to-one relationship between the number of valid combinations and arrangements of the stars and bars by placing 3 stars in 11 bins (10 bars) since there is exactly one ordering for each arrangement that gives a valid combination. $\binom{13}{3}$.

3. Khalil is the most attentive of them all, and knows that the sibling ages must be different and that $C > B > A$. How many combinations of ages could Khalil guess from?

   Solution. This question reduces down to choosing 3 ages from a set of 15 since each age has to be distinct, and there is only one ordering from each set of 3 ages chosen that gives $C > B > A$. $\binom{11}{3}$.

2 Increasing and Decreasing, Nearly

Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order. (Source: 2020 AIME I)

Solution. We can deal with one of the cases first, say the ascending case, and multiply by 2 since we know that for each ascending case there is a unique corresponding decreasing case. All valid sequences can be constructed by taking all numbers but one, order them in an increasing sequence, and then insert the final number, say some number k, at any location. This is valid because k can be deleted and the sequence is still ascending. Since there are 6 possible values of k, and 6 locations that they can be inserted into, we have a total of $\binom{6}{5} * 6$ choices.

However, we have overcounted here. Let us take the cases of inserting a 1 and inserting a 2 and look at all the sequences:

123456 213456

213456 123456

231456 132456

234156 134256

234516 134526

234561 134562

The first two sequences of each list are the same, and this is because the sequence 123456 is being counted in every single list, and we are over counting the case of the inserting a number k and $k+1$. Since we only want 5 of the 6 sequences in the first case and there are 5 additional of over counting from consecutive digits, the answer should be $(\binom{6}{5} * 6 - 5 - 5) * 2$

3 Gardening

1. There is a landscaper who wants to plant flowers in a row, which consist of s sunflowers, b bluebells, and r roses, and wants the roses to be spread out. How many arrangements exists where no two roses are next to each other?

   We want the roses to be the most spread out as possible, and we can do this by pre-placing a non-rose in between every single rose and perform stars and bars on the rest. lets consider x to represent a flower that is a bluebell or sunflower. We end up with a sequence like this:

   $$r_1 \ x_1 \ r_2 \ldots r_{r-1} \ x_{r-1} \ r_r$$

   This means that we have a total of $s+b-(r-1)$ remaining flowers to allocate amongst $r+1$ bins (r bars). This yields $\binom{s+b-(r-1)+r}{r} = \binom{s+b+1}{r}$. We have under counted here, however, because not all our stars are indistinguishable, thus, we need to multiply by the number of ways that we need to order the sunflower and bluebells, which yields:

   $$\binom{s+b+1}{r} * \binom{s+b}{b}$$

2. How many arrangements exist where at least two other flowers are separating each rose?

   This is similar to the last question, except now we need to allocate 2 flowers between each rose, leading to a stars and bars with $s+b-2(r-1)$ flowers to allocate now. This would yield:

   $$\binom{s+b-2(r-1)+r}{r} * \binom{s+b}{b} = \binom{s+b-r+2}{r} * \binom{s+b}{b}$$

4 Fixed Permutations

For each permutation σ of 1 through n, let $\sigma(i)$ denote the value at position i. For example, if the permutation is $2,4,1,3$ we have $\sigma(1) = 2$ and $\sigma(2) = 4$ (Source: CS 70 SP19 MT2)

1. For a fixed $1 \leq k \leq n$, how many permutations σ of 1 through n are there where for all $i < k, \sigma(i) < \sigma(k)$? Express your answers in terms of n and k.

   $\frac{n!}{k}$. $\frac{1}{k}$ of the permutations have $\sigma(k)$ being the largest of the first k elements in $\sigma(\cdot)$

2. How many permutations of 1 through n are there such that for each i, $\sigma(\sigma(i)) = i$ and $\sigma(i) \neq i$? (For example, the permutation $3,4,1,2$ is such a permutation, since for example $\sigma(\sigma(1)) = \sigma(3) = 1$. You may assume n is even.

   We essentially want all sequences where two indices are swapped, which is equivalent to pairing off two indices at a time to be switched with each other. Thus, we can write the total combinations as:

   $$\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2} \cdots = \frac{n!}{2^{n/2}}$$

   We have over-counted here, however, since there is no labeling to each of the "pairs" of indices, thus, we need to divide by the number of ways to order the pairs, which is $(\frac{n}{2})!$, leading to a total of:

   $$\frac{n!}{2^{n/2}(\frac{n}{2})!}$$