PH 207: Introduction to Special Relativity
Solution to 2021 Endsem

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"Relativity is simple, that does not make it easy."
I made that quote up.
Anyway, I hope this solution booklet helps you. If there is anything that you feel has been solved wrongly, please let me know.

Problem 1

a) If a particle maintained constant acceleration in the S frame, then it would eventually gain the speed of light, and cross it. We know that this is not possible because c is the limiting speed of all matter.

b) What you know here is that there are two frames: S, the original frame and S', the MCRF at that instant of time. The particle has a velocity v at this instant, so to be at rest in a certain frame, that frame (the MCRF) must be moving with the same velocity v with respect to S. Check using the velocity addition rule that this is indeed true.
Now, in frame S' you wait at a particular x', you stand there and watch your watch tick off a small time $d\tau$, what is the time elapsed between these two events (x',τ) and (x',$\tau + d\tau$) measured in the S frame?

Using the Lorentz transforms, we obtain:

$$dt/\gamma = d\tau \tag{1.1}$$

c) The particle's velocity in S' increases by an amount $du = ad\tau$. To find the corresponding increase dv in the S frame, apply the velocity addition formula, transforming the changed velocity in S' which is $0 + du$ to the changed velocity $v + dv$ in S'

$$v\ +\ dv = \frac{du + v}{1 + v du/c^2} \tag{1.2}$$

Simplifying...

$$v\ +\ dv = \frac{ad\tau + v}{1 + \frac{vad\tau}{c^2}} \tag{1.3}$$

Using the fact that $vad\tau \ll c^2$, we can make use of binomial approximation.

$$v + dv = (ad\tau + v)(1 - \frac{vad\tau}{c^2}) \tag{1.4}$$

$$= ad\tau(1 - \frac{v^2}{c^2}) + v - v\frac{(ad\tau)^2}{c^2} \tag{1.5}$$

Neglecting the second order differential terms and cancelling v on both sides, we obtain:

$$dv = a(1 - \frac{v^2}{c^2})d\tau \tag{1.6}$$

Yay!

d) Integration time.

$$\frac{dv}{1 - \frac{v^2}{c^2}} = ad\tau \tag{1.7}$$

Let $v/c = \tanh(\alpha)$. This gives:

$$dv = csech^2(\alpha)d\alpha \tag{1.8}$$

and therefore our original equation simplifies to:

$$\frac{csech^2(\alpha)d\alpha}{1 - tanh^2(\alpha)} = ad\tau \tag{1.9}$$

One small remark: I recommend that you brush up your hyperbolic trigonometry. It is quite handy throughout this course. Using one such hyperbolic trigonometric identity, the numerator and denominator of the LHS of Equation (1.9) cancel, and we are left with a simple integral.

$$d\alpha = \frac{ad\tau}{c} \tag{1.10}$$

Integrating and setting the constant of integration to zero because $v = 0$ (which implies $\alpha = 0$) at $\tau = 0$:

$$\alpha = \frac{a\tau}{c} \tag{1.11}$$

Taking tanh on both sides:

$$tanh(\alpha) = tanh(\frac{a\tau}{c}) \tag{1.12}$$

Replace α and you have $v(\tau)$!

$$v = ctanh(\frac{a\tau}{c}) \tag{1.13}$$

e) We have $v(\tau)$, and we know that it is equal to dx/dt at t=τ. To find x(τ) and t(τ), we are going to use the chain rule, and the fact that we already know t(τ). (Think why?)

$$v(\tau) = \frac{dx/d\tau}{dt/d\tau} \tag{1.14}$$

From equation (1.1):

$$\frac{dt}{d\tau} = \gamma \tag{1.15}$$

Plug in $\gamma(\tau)$ using Equation (1.13) to obtain the following:

$$dt = cosh(a\tau/c)d\tau \tag{1.16}$$

Integrate and you have t(τ)!

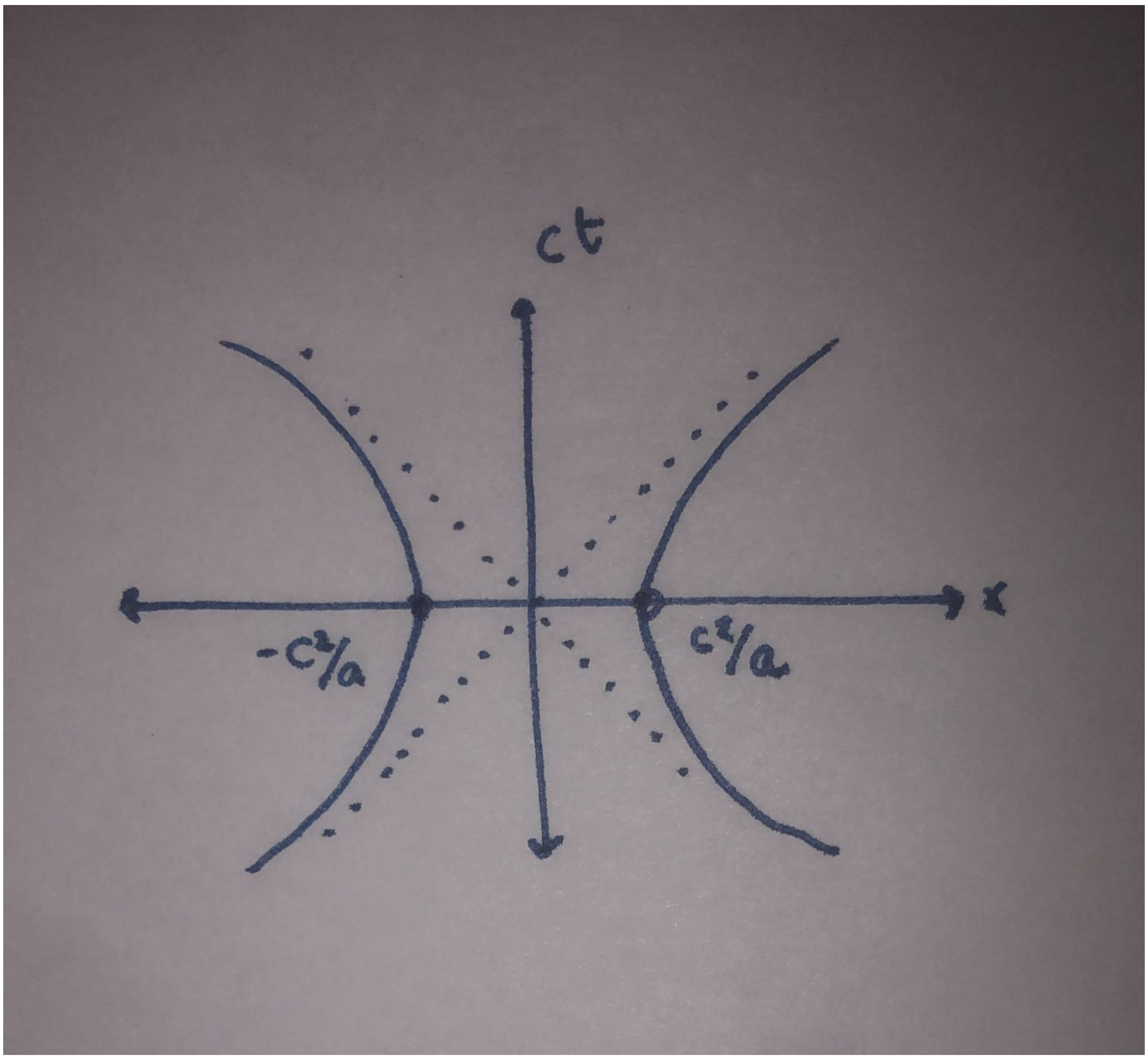
$$t(\tau) = \frac{c}{a}sinh(a\tau/c) \tag{1.17}$$

Now, I'm going to skip a few steps so that you do some hard work. I'll give you hints though. Use Equation (1.13) and (1.14) and arrive at:

$$x(\tau) = \frac{c^2}{a}cosh(a\tau/c) \tag{1.18}$$

f) Eliminate the trigonometric terms to obtain a relation between x and t.

$$x^2 - (ct)^2 = c^4/a^2 \tag{1.19}$$



Your sketch should look like this. The two asymptotes are $x = ct$ and $x = -ct$

g) This question might scare you. It definitely did scare me when I first read it, but let's take it step by step. Notice that the question says A and B have a common MCRF. This implies that v(τ) must be the same function for both.(Why?)

The information provided to us is that, at the same time in the common MCRF, B is ahead of A by ξ, so let's turn that into math:

$$t'_B = t'_A \tag{1.20}$$

$$x'_B = x'_A + \xi \tag{1.21}$$

Transforming x'_B back to the S frame:

$$x_B = \gamma(x'_B + vt'_B) \tag{1.22}$$

The '+' sign in between is because S is moving a relative velocity -v with respect to S'. Substitute x'_B and t'_B using (1.20) and (1.21).

$$x_B = \gamma(x'_A + \xi + vt'_A) \tag{1.23}$$

$$x_B = \gamma(x'_A + vt'_A) + \gamma\xi \tag{1.24}$$

$$x_B = x_A + \gamma\xi \tag{1.23}$$

$$x_B = \frac{c^2}{a}cosh(a\tau/c) + \xi cosh(a\tau/c) \tag{1.24}$$

Therefore,

$$x_B(\tau) = (\frac{c^2}{a} + \xi) cosh(a\tau/c) \tag{1.25}$$

You know $v_B(\tau)$ so use that and (1.25) to obtain $t_B(\tau)$. You get:

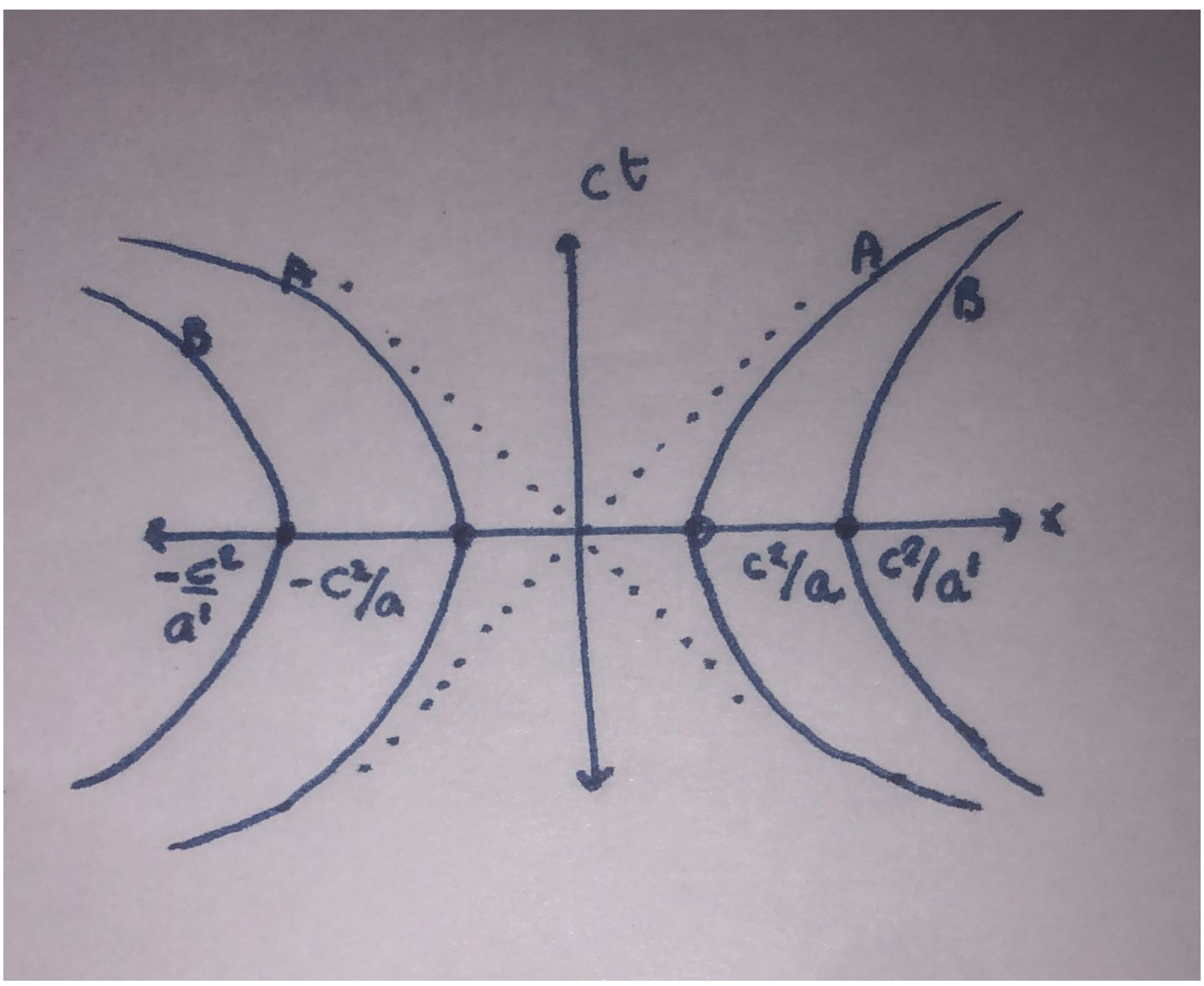
$$ct_B(\tau) = (\frac{c^2}{a} + \xi) sinh(a\tau/c) \tag{1.26}$$

h) Notice that $x_A = \frac{c^2}{a} cosh(a\tau/c)$ where a is the acceleration of A in the MCRF. Similarly we can write $x_B = \frac{c^2}{a'} cosh(a\tau/c)$ where a' would be the acceleration of B in the MCRF. The good thing is that we have already achieved this form of $x_B(\tau)$ in (1.25). Use this to find *a'*.

$$\frac{c^2}{a'} = \frac{c^2}{a} + \xi \tag{1.27}$$

Simplify and you have your answer.

$$a' = \frac{a}{1 + \frac{a\xi}{c^2}} \tag{1.28}$$



And this is what your sketch should look like.

i) If the expression $\frac{c^2}{a} + \xi$ were negative, it would imply that a' is negative. (Why?) This would mean B is moving on the left branch on the hyperbola while A is moving on the right branch. The distance between them is clearly increasing in the S frame, and is definitely not a constant in the MCRF as we would have wanted. Therefore, $\frac{c^2}{a} + \xi$ must be positive so that B maintains a constant separation from A in their common MCRF. And tada!

$$\xi > -\frac{c^2}{a} \tag{1.29}$$

Problem 2

Conserve energy and momentum. Keep in mind that:

$$\Delta E = mc^2 - m'c^2 \tag{2.1}$$

and the internal energy of atom **decreases** (the question has a typo). How else would the photon have gained energy? It is because the atom lost some mass, and that mass provided the energy for the photon and atom's KE. So, we have using conservation of energy:

$$mc^2 = h\nu + E' \tag{2.2}$$

E' is the **total** energy of the resultant atom, **not** just the rest mass energy or the kinetic energy. This is important, otherwise you might get confused later on. Now, conserve momentum.

$$0 = \frac{h}{\lambda} + p' \tag{2.3}$$

λ is the wavelength of the emitted photon and p' is the momentum of the resultant atom. Now we have to do some grindwork, unfortunately. Start with (2.2).

$$mc^2 - h\nu = E' \tag{2.4}$$

Square both sides.

$$m^2c^4 + (h\nu)^2 - 2mh\nu c^2 = p'^2c^2 + m'^2c^4 \tag{2.5}$$

We know $(h\nu)^2 = p'^2c^2$ from (2.3).

$$m^2c^4 - 2mh\nu c^2 = m'^2c^4 \tag{2.6}$$

Keep this aside for now. Rewrite (2.1) as:

$$\Delta E - mc^2 = m'c^2 \tag{2.7}$$

Square both sides. Replace the RHS with the LHS of (2.6)

$$\Delta E^2 + m^2c^4 + 2m\Delta Ec^2 = m^2c^4 - 2mh\nu c^2 \tag{2.8}$$

Bring $h\nu$ to one side and you have what you want!

$$h\nu = \Delta E(1 - \Delta E/2mc^2) \tag{2.9}$$

Not so hard, but definitely some tricky simplifications that you had to do here.

Problem 3

Hope your brain cells haven't turned to mush by now, but just one more question and we're done. So, buckle up.

a) Start by writing a familiar equation.

$$\frac{d\vec{p}}{dt} = q\vec{E_0} \tag{3.1}$$

This also implies that,

$$\frac{d|\vec{p}|}{dt} = q|\vec{E_0}| \tag{3.2}$$

Substitute $|\vec{p}|$ as γmu and apply the product rule of differentiation.

$$m\gamma\frac{du}{dt} + mu\frac{d\gamma}{dt} = qE_0 \tag{3.3}$$

This should leave you with:

$$\frac{ma}{(1-u^2/c^2)^{1/2}} + \frac{mau^2}{c^2(1-u^2/c^2)^{3/2}} = qE_0 \tag{3.4}$$

Simplify and you will get:

$$\frac{ma}{(1-u^2/c^2)^{3/2}} = qE_0 \tag{3.5}$$

And we're done!

$$a = \frac{qE_0}{m}(1-\frac{u^2}{c^2})^{3/2} \tag{3.6}$$

b) Integrate (3.1) and set the constant of integration to 0 because $p = 0$ at t = 0.

$$\gamma mu = qE_0 t \tag{3.7}$$

Rewriting...

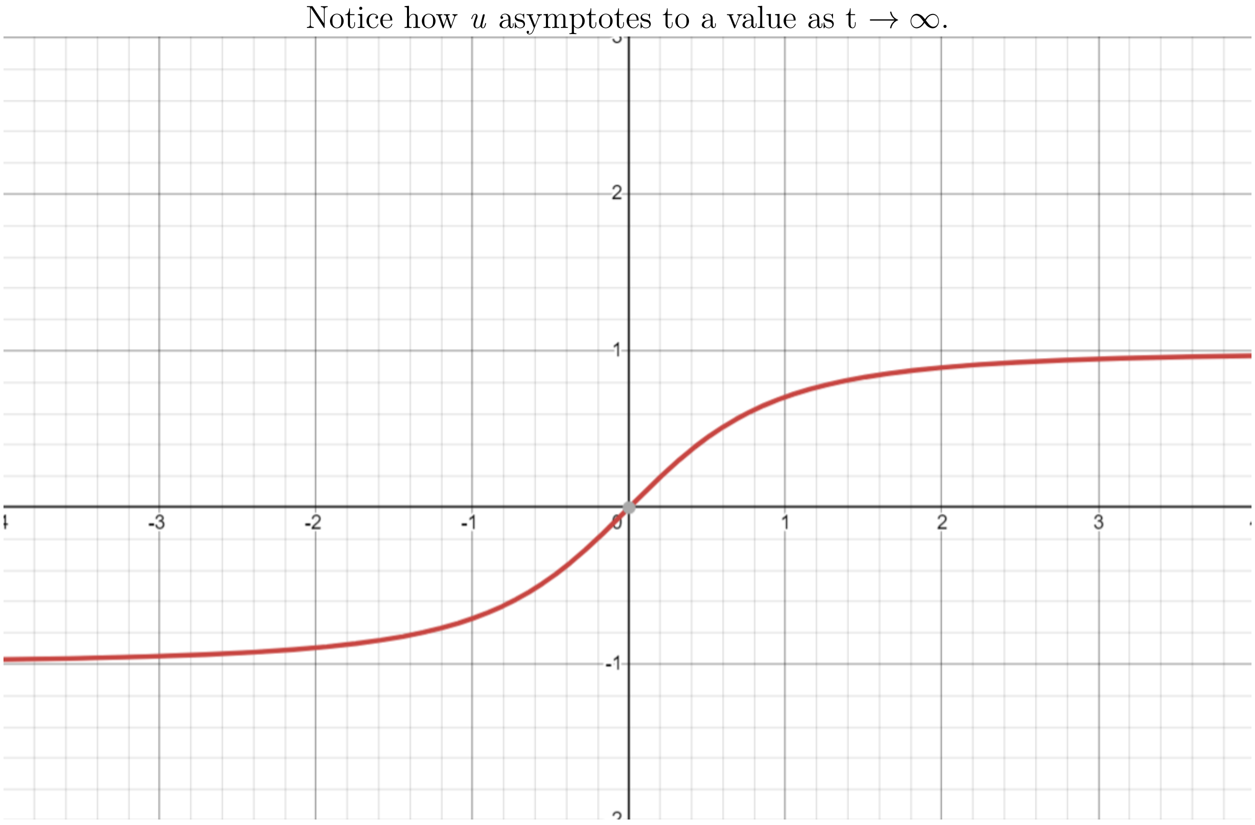
$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{qE_0 t}{m} \tag{3.8}$$

Now all you have to do is find u. You'll end up with:

$$u(t) = \frac{qE_0 t/m}{\sqrt{1+(qE_0 t/mc)^2}} \tag{3.9}$$

c)

Notice how u asymptotes to a value as t $\to \infty$.



That value can be obtained from (3.9) to be c.

d) This is a scenario you have definitely come across during JEE. A body with some constant mass experiences a constant force to due to magnetic field (which is always perpendicular to its velocity). What motion does it undergo? Uniform circular motion. The force due to magnetic field provides the centripetal acceleration for this motion. Note that in this reference frame, the particle is at rest and it's mass $m' = \gamma m$.

$$F = \frac{m'u^2}{R} \tag{3.10}$$

$$quB_0 = \frac{m'u^2}{R} \tag{3.11}$$

And we have,

$$R = \frac{m'u}{qB_0} \tag{3.12}$$

$$\omega = u/R \tag{3.13}$$

$$= \frac{qB_0}{m'} \tag{3.14}$$

$$= \frac{qB_0}{m}\sqrt{1 - \frac{u^2}{c^2}} \tag{3.15}$$

And we're done. Thank you for reading all the way till here. Any feedback is always welcome.