PH 207: Introduction to Special Relativity
Solution to 2022 Endsem

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Problem 1

a) To see if these transformations are different, we simply need to compare the transformation matrix in the two cases. In the first case it is the X boost first, followed by the Y boost. This would be represented by this matrix:

$$\begin{pmatrix} \gamma_y & 0 & -\gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_x & -\gamma_x\beta_x & 0 & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After some simple matrix multiplication, it becomes:

$$\begin{pmatrix} \gamma_y\gamma_x & -\gamma_y\gamma_x\beta_x & -\gamma_y\beta_y & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ -\gamma_y\gamma_x\beta_y & \gamma_y\gamma_x\beta_x\beta_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the second case, the matrices are multiplied in the reverse order.

$$\begin{pmatrix} \gamma_x & -\gamma_x\beta_x & 0 & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_y & 0 & -\gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and that is:

$$\begin{pmatrix} \gamma_y\gamma_x & -\gamma_x\beta_x & -\gamma_x\gamma_y\beta_y & 0 \\ -\gamma_x\beta_x\gamma_y & \gamma_x & \gamma_y\gamma_x\beta_x\beta_y & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly, the overall transformations are **not** the same.

b) I hope that you have derived in class that $E^2 - c^2B^2$ is a Lorentz invariant. The quantity in question $E^2 + c^2B^2$ can simply be rewritten as $E^2 - c^2B^2 + 2c^2B^2$. The first term of this sum is a Lorentz invariant, but the second is not. It's super easy to see why. Just find B in one frame and B' in another and compare magnitudes. It will not be an invariant.

Therefore, since it is a sum of an invariant and a non-invariant, the whole thing is non-invariant and we conclude that $E^2 + c^2B^2$ is **not** a Lorentz invariant.

c) You've learnt the formula for relativistic force in class, it has an extra term compared to usual.

$$\vec{F} = \frac{\vec{F}.\vec{u}}{c^2}\vec{u} + m\vec{a}$$

Can $\vec{F}$ be perpendicular to $\vec{a}$? If it were possible, then $\vec{F}.\vec{a}$ would be zero right? So let's write that out and see if we run into any problems. If we don't that means this funny unintuitive motion is indeed possible!

$$\vec{F}.\vec{a} = \frac{\vec{F}.\vec{u}}{c^2}\vec{u}.\vec{a} + m|\vec{a}|^2$$

Is it possible for the RHS of this equation to be zero? Why not? Just set up the body with appropriate magnitudes and directions of $\vec{F}$, $\vec{u}$ and $\vec{a}$ such that the RHS is zero. This statement is therefore, **true**.

d) This statement is **false**. Events that are causally related in one frame, are causally related in **all** frames. The temporal order of two unrelated events can be flipped on switching from one frame to another, but if one is the cause for the other then the cause always happens first, in **all frames of reference**.

e) This statement is also false. Like I mentioned earlier, $E^2 - c^2B^2$ is a relativistic invariant. If this term is positive, then it will be possible to find a frame where B is zero, but no frame will exist where E is zero. Think why. Similarly, if this is negative, then it will be possible to find a frame where E is zero, but no frame will exist where B is zero. Therefore, it is **not** always possible to find a frame where either of the field vanishes.

Problem 2

Start with a familiar equation:

$$\frac{d\vec{p}}{dt} = q\vec{E} \tag{2.1}$$

At $t = 0$, $\vec{p} = \vec{p_0}$,

$$\vec{p} = q\vec{E}t + \vec{p_0} \tag{2.2}$$

Notice how the question does not specify a coordinate frame that we are solving this question in. It is up to us to choose a system that makes calculations convenient for us. In this case, that convenient choice is to take the unit vector along and perpendicular to $\vec{E}$ as our basis. So that we can determine motion along and perpendicular to $\vec{E}$. Classically, the perpendicular part would be absent but relativistically, it is there. Therefore,

$$\vec{p} = (qEt + p_{0\parallel})\hat{E}_\parallel + (p_{0\perp})\hat{E}_\perp \tag{2.3}$$

Remember what you're trying to find. You want to arrive at $\vec{r}(t)$, but right now you have $\vec{p}(t)$, so the natural next step is to first find $\vec{u}(t)$ and then integrate. So let's do that!

You know this:

$$\frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \vec{p} \tag{2.4}$$

So, given $\vec{p}(t)$, you can find $\vec{u}(t)$. That should yield:

$$\vec{u} = \frac{\vec{p}c}{\sqrt{m^2c^2 + |\vec{p}|^2}} \tag{2.5}$$

Substitute $\vec{p}$ and get a super scary equation.

$$\vec{u} = c\frac{(qEt + p_{0\parallel})\hat{E}_\parallel + (p_{0\perp})\hat{E}_\perp}{\sqrt{m^2c^2 + (qEt + p_{0\parallel})^2 + p_{0\perp}^2}} \tag{2.6}$$

Now you can integrate this to find $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{u}(t)dt \tag{2.7}$$

$$\vec{r}(t) = c\int \frac{(qEt + p_{0\parallel})\hat{E}_\parallel + (p_{0\perp})\hat{E}_\perp}{\sqrt{m^2c^2 + (qEt + p_{0\parallel})^2 + p_{0\perp}^2}}dt \tag{2.8}$$

This big horrifying integral has two terms. Let the first term be I_1 and the second be I_2. We'll evaluate them separately.

$$I_1 = c \int \frac{(qEt + p_{0\|})\hat{E_\|}}{\sqrt{m^2c^2 + (qEt + p_{0\|})^2 + p_{0\perp}^2}} dt \tag{2.9}$$

Let $m^2c^2 + (qEt + p_{0\|})^2 = \alpha$. Then $d\alpha = 2(qEt + p_{0\|})qEdt$.

$$I_1 = \frac{c}{2qE} \int \frac{d\alpha}{\sqrt{\alpha^2 + p_{0\perp}^2}} \hat{E_\|} \tag{2.10}$$

$$I_1 = \frac{c}{qE} \sqrt{\alpha^2 + p_{0\perp}^2} \hat{E_\|} \tag{2.11}$$

I'll account for the constant of integration in the end. Moving on to I_2 now.

$$I_2 = c \int \frac{(p_{0\perp})\hat{E_\perp}}{\sqrt{m^2c^2 + (qEt + p_{0\|})^2 + p_{0\perp}^2}} dt \tag{2.12}$$

Let $\beta^2 = (qEt + p_{0\|})^2$. Then $d\beta = qEdt$.

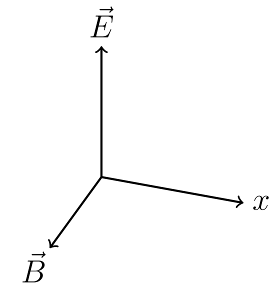
$$I_2 = \frac{c}{qE} \int \frac{(p_{0\perp})\hat{E_\perp}}{\sqrt{\beta^2 + (m^2c^2 + p_{0\perp}^2)}} d\beta \tag{2.13}$$

$$I_2 = \frac{cp_{0\perp}}{qE} ln|\beta^2 + \sqrt{\beta^2 + (m^2c^2 + p_{0\perp}^2)}|\hat{E_\perp} \tag{2.14}$$

Now, substitute β back in and put I_1 and I_2 together to get your final answer. Don't forget to add a term which accounts for $\vec{r}$ at $t = 0$ to be equal to $\overrightarrow{r_0}$.

Problem 3

The question does not specify a coordinate system, so we'll have to choose one that makes life simple for us. We'll choose $\vec{E}$ to be along y and $\vec{B}$ to be along z.



Don't you think it's going to be a gruesome task to solve for the motion of a particle under the influence of both an electric and magnetic field? There's a neat trick to overcome this. We can switch frames and go to a different one where either of the fields vanish, solve for trajectory there and transform back! Question. Which field will vanish in a different frame?

The electric field! Why? The question mentions $E < cB$. That means the Lorentz invariant $E^2 - c^2B^2$ is negative in this frame and therefore, negative in **all** frames. So, we can find another frame where E becomes zero and work there. Not possible to find a frame where B becomes zero though.

Our desired frame moves along the positive x axis with velocity $v = B/E$. We get this value simply from making $E'_y = 0$ in the Lorentz transform for electric and magnetic fields. The other components of electric field are zero in this frame as well, which should be evident again from the transformation equations.

Time to find the new magnetic field.

$$B'_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Substitute v.

$$B'_z = \gamma(B - \frac{E^2}{c^2B})$$

$$B'_z = \frac{\gamma}{c^2B}(c^2B^2 - E^2)$$

Let $c^2B^2 - E^2 =$ k, as it is a Lorentz invariant. The x and y components of $\vec{B}$ are zero, from the transformation equations. We end up with:

$$\vec{B'} = \frac{\gamma k}{c^2B}\hat{z}$$

What motion does a particle with fixed mass and charge with an initial velocity undergo in the presence of a constant magnetic field? Uniform circular motion! Note that the particle was at rest in the original frame, but in our shifted frame it is moving in the $-x$ direction with speed v. It experiences a force $\vec{F} = q(\vec{v} \times \vec{B})$ which is along the $+y$ direction.

It moves on a circle in the XY plane, centred at (0,R,0) where R is the radius of the circular motion $= \frac{mv}{qB'}$. The mass m here is not the rest mass, it is γ times the rest mass, m_0. The equations of motion in the particle are therefore simple sine and cosine functions. The angular velocity $\omega = \frac{v}{R}$

$$x'(t) = -Rsin(\omega t)$$

$$y'(t) = R - Rcos(\omega t)$$

$$z'(t) = 0$$

You can see what it looks like here: `https://www.desmos.com/calculator/bfhcwaxgl8`

Congratulations! You have obtained the trajectory of the particle in the S' frame. Transform back to obtain it in the S frame. Keep in mind that S moves with a velocity $-v$ with respect to the S' frame.

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma(t' + \frac{v}{c^2}x')$$

On substituting, you will obtain $x(t')$, $y(t')$, $z(t')$. That is **not** what you want. You want $x(t)$, $y(t)$, $z(t)$. It would not be of much use to express the trajectory in a frame with the time of another frame would it? So substitute all the t' terms with its Lorentz transform in terms of the unprimed coordinates. You end up with implicit expressions of trajectory of the particle in the S frame, and job done :).

Problem 4

Here, we are going to make use of some clever manipulation using momentum four vectors. First we will conserve four momentum in the lab frame. I have put $c = 1$ everywhere.

$$\begin{pmatrix} E_a \\ \vec{p_a} \end{pmatrix} + \begin{pmatrix} m_b \\ 0 \end{pmatrix} = \begin{pmatrix} \sum E_i \\ \sum \vec{p_i} \end{pmatrix}$$

The first component of the four momentum of particle b is simply m_b because it is at rest and only possesses rest mass energy. The RHS is the total four momentum of all the formed particles.

We are yet to impose the condition that the incoming energy is the *threshold* energy i.e the minimum energy required to make this collision possible. How will we do that?

Let's go into the center of momentum frame. In this frame, the net momentum of all the particles is zero and we can find the relative velocity of this frame using a simple Lorentz transform between momentum four vectors from our original frame and this one, but that won't be needed.

For the incoming energy to be minimum, all the products should be at rest in the COM frame. Why? If it was not the minimum energy, then there would be some "extra" energy that would cause one or more particles to be in motion in the COM frame, but we don't want that. To ensure that we have taken the least possible incoming energy, we make the final energy as less as possible too, by letting all the products be at rest. This is the situation where incoming energy is the threshold energy. I hope you understood this, if not think about it a bit more.

Therefore, we can conserve the four momentum in the COM frame as well!

$$\begin{pmatrix} E'_a \\ \vec{p'_a} \end{pmatrix} + \begin{pmatrix} E'_b \\ \vec{p'_b} \end{pmatrix} = \begin{pmatrix} \sum m_i \\ \sum 0 \end{pmatrix}$$

Understand the above equation completely. There is a lot going on.

Firstly, the energy of particle a is now E'_a not E_a like earlier because we shifted frames. Same for it's momentum. Particle b is not at rest anymore either, it has momentum. Also, this is the COM frame, which means net momentum is **zero**. So $p'_a + p'_b = 0$. On the RHS, all products are at rest, so they only possess rest mass energy. Now comes the clever step.

You know that the norm of a four-momentum is a Lorentz invariant. It has the same value in all frames. We are going to make use of that. Keep in mind that you cannot equate $p_{initial}$ in frame 1 to p_{final} in frame 2. What you can do is equate their norms. This might be confusing, but read it over a few times and this should settle into your head.

$p_i = p_f$ and $p'_i = p'_f$, all of them have the same norm(primed four momentum vectors are the ones in the COM frame). We are going to equate the norms of p_i and p'_f.

$$(p_i).(p_i) = p_a^2 - (E_a + m_b)^2$$

$$(p_i).(p_i) = p_a^2 - E_a^2 - m_b^2 - 2E_a m_b$$

You know that:

$$E_a^2 = p_a^2 + m_a^2$$

Substitute this back into the earlier equation:

$$(p_i).(p_i) = -m_a^2 - m_b^2 - 2E_a m_b$$

Now find the norm of p'_f.

$$(p'_f).(p'_f) = -(\Sigma m_i)^2$$

And equate the two as we planned earlier.

$$-m_a^2 - m_b^2 - 2E_a m_b = -(\Sigma m_i)^2$$

Rearrange.

$$E_{a|threshold} = \frac{(\sum m_i)^2 - m_a^2 - m_b^2}{2m_b}$$

The end! Hope these solutions helped you.