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Mean Time To Failure of a Coherent System Equipped with a Single Cold Standby Component having Dependent Components

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Abstract

This project emphasizes on a special case of a coherent system, characterized by the fact that, when the system fails, either all components are inactive, or the remaining functional components are arranged in a parallel configuration. The components within the coherent system are considered to be exchangeable. The standby component turns operational if the coherent system fails. The active spare unit reactivates the system together with remaining original components. A copula function models the dependency between simultaneous operating components. First, we attain the common distribution function of the still active units of the coherent system at the time of the system's failure. Next, we calculate the system under consideration's mean time to failure. We also present some numerical results to analyze theoretical results.

Keywords

Coherent System, Copula, Exchangeable Component, Residual Life Function, Standby Redundancy

Introduction

The system is coherent. if its structural function is monotone and does not consist of any irrelevant component. The study of coherent systems holds significant importance within the field of reliability theory. Over the years, various statisticians have delved into the realm of coherent systems exploring their reliability characteristics. Using cold standby sparing is an efficient method for improving system reliability. Here, the standby component remains inactive and does not experience any failure rate while it is in the standby state. A cold standby unit is frequently chosen over a hot and warm standby when substituting times are brief since the component stays inactive until the system experiences a failure. A typical example of a cold standby unit is the backup battery in electronic devices. If the primary battery fails, the backup can be immediately used to power the device, allowing it to continue functioning. In the aerospace industry, critical spare parts for spacecraft are kept on cold standby, ready to be deployed for repairs, thereby minimizing mission downtime. Likewise, reserve communication systems in critical infrastructure are kept in a cold standby state, ready to be activated immediately if the primary communication network fails, ensuring continuous connectivity. In financial institutions, backup transaction processing units are maintained in cold standby mode, ready to take over in case of primary system failures, thus guaranteeing uninterrupted financial operations.

Most studies focus on cases where units are considered to be iid. Regardless of that, in real-world scenarios, the lifetimes of components are often interconnected, making it reasonable to assume a degree of dependence among them. There is considerable interest in evaluating the reliability of systems composed of dependent components. The lifetimes of components within the same operating environment can be modeled using exchangeable dependence.

Related work

[1-4], [11], [14], [15] and other statisticians have explored the application of cold standby redundancy. Various works have demonstrated that cold standby units can be utilized in multiple means to enhance the reliability of a system. An approach involves using a single cold standby unit to restart the considered system, alongwith the reinforcement of the remaining active original components.

A working “k-out-of-n system”, consisting of iid units, is functional iff at least k (where $k \leq n$) units are working. [12] explored the mean residual life functions of a k-out-of-n system equipped with a cold standby unit, referred to as a k-out-of-n cold standby system, under the condition of identical and independent units. [4] obtained an expression for the “random variable” (T_1), representing “lifetime of the k-out-of-n cold standby system”, defined by

$$T = X_{n-k+1:n} + \min(X_{n-k+2:n} - X_{n-k+1:n}, Z), \tag{1}$$

for $k = 2, \ldots, n$, and $T = X_{n:n} + Z$ for $k = 1$, where $X_{1:n} < X_{n:n}$ are ordered lifetimes of active original units $X_1, \ldots, X_n$.

They also derived “$P(T > t), t > 0$." Additionally, three MRL functions were calculated :

$$E[T - t \mid T > t],$$

$$E[T - t \mid Z_{n-k+1:n} > t],$$

$$E[T - t \mid X_{1:n} > t]$$

for $t >= 0$.

(2)

[4]extended the work by determining $E[T_1 - s|X_{j:n} > t]$, for j = 1, 2,...,n−k+1.

Similarly, [1] studied a "k-out-of-n system" with two cold standby units when units are independent and identical. They examined the potential to enhance the reliableness of the "k-out-of-n system" by using two cold standby units. In their study, the cold standby units are activated sequentially, not simultaneously. Specifically, when one standby unit begins to operate, the other remains in standby mode and only starts operating when the system stops operating again after the first "standby component" has been used. [3] examined a special case of a coherent system where, once the standby unit is turned on, the system fails subsequent to the failure of the upcoming unit. Such a situation occurs if the remaining operational units, along with the "standby component," are connected serially. [10] analyzed the impact of a "cold standby component" on the "mean residual life functions of a coherent system." They elucidated the lifespan of T with a standby unit Z, as follows:

$$T_c = \begin{cases} Z_{i:n} + \min(Z_{i+1:n} - Z_{i:n}, Z) & \text{if } T = Z_{i:n},\ i = 1, 2, ..., n-1, \\ Z_{n:n} + Z & \text{if } T = Z_{n:n}, \end{cases} \tag{3}$$

Let $Z_{1:n} \leq \ldots \leq Z_{n:n}$ represent the "order statistics" related to iid random variables $Z_1, \ldots, Z_n$. [3] obtained a straightforward expression for the "reliability function" "$P(T_c > s)$" based on the "system signature." Not only this they also, computed two "mean residual functions":

$$M(t) = E(T_c - t | T > t)$$

$$H(t) = E(T_c - t | Z_{1:n} > t) \tag{4}$$

for $t \geq 0$.

[2] considered a special case of a coherent system where a system failure results in either the absence of functional units or the remaining functional units being placed parallelly.

Based on these systems they computed the "lifetime of the coherent system" as defined by

$$T_c^* = \begin{cases} Z_{i:n} + \min(Z_{n:n} - Z_{i:n}, Z) & \text{if } T^* = Z_{i:n},\ i = 1, 2, ..., n-1, \\ Z_{n:n} + Z & \text{if } T^* = Z_{n:n}, \end{cases} \tag{5}$$

Here, T_c^* symbolizes life span T, which includes a standby component Z. However, T_c^* cannot be regarded as a "coherent system" because the "cold standby component Z" does not impact the system's efficacy while in "the standby state." The investigation of these coherent systems is a recent area of interest in cold standby sparing. Three different "mean residual life functions" that they examined are expressed as follows -

$$E(T_c^* - t | T_c^* > t), \tag{6}$$

$$E(T_c^* - t | T^* > t), \tag{7}$$

$$E(T_c^* - t | Z_{1:n} > t), \tag{8}$$

for t >= 0.

Another method of utilizing "cold standby components" is by renewing the system by changing the "failed components" in the place of "standby ones." [14] obtained the "reliability function" for a "coherent system" equipped with a "cold standby component", given the condition that the system fails upon the failure of its first unit. [15] expanded on [14]'s work by examining a type of "coherent system" that breaks down upon the failure of its s^{th} unit. However, [15] noted that their "formula" becomes challenging to apply to complex structures, even for systems with a few components particularly when the units have "exponential lifetime distributions." [1] investigated the restoration process of a coherent system that could break down, either upon the failure of its first unit or its second unit. They developed the reliability characteristics for this system, which is backed by two cold spare units.

In nature, the units that constitute a system might be dependent, often because of sharing the same load, like heat and tasks. [18] explored a k-out-of-n system where the lifetimes of the units are connected via an Archimedean (survival) copula, without any dependence on the lifetime of a cold standby unit.

[11] examined the reliability characteristics of a “two-component series system” that includes a cold standby, where the lifetimes of the concurrently operating components share an exchangeable joint distribution, established using copulas.

$$P(Z_1 \leq z_1, Z_2 \leq z_2) = C(F(z_1), F(z_2)), \text{ and } P(Z_1^* \leq z_1, Z \leq z) = C(F^*(z_1), F(z)). \quad (9)$$

[16] extended this study by examining a multi-unit parallel system. [2] explored a particular type of coherent system where, upon system failure, either no active units remain, or the surviving active units are arranged in parallel. The units within this system are assumed to be interchangeable. The units that operate simultaneously are considered dependent, and this dependency is represented using a copula function. They defined the lifetime of the considered system as

$$T = Z_{n-k+1:n} + \min\{Z_1^*, \ldots, Z_{k-1}^*, Z\}, \ k = 2, \ldots, n; \quad (10)$$

$$T = Z_{n:n} + Z; \ k = 1. \quad (11)$$

[2] assumed that the dependency among the original units of the “k-out-of-n system” is represented by a symmetric copula, C.

$$P(Z_1 \leq z_1, \ldots, Z_n \leq z_n) = C(F(z_1), \ldots, F(z_n)) \quad (12)$$

While the relation of the remaining lifespans of the surviving units and the cold spare unit Z is assumed to be represented by the copula C^*

$$P(Z_1^* \le z_1, \ldots, Z_{k-1}^* \le z_{k-1}, Z \le z) = C^*(F^*(z_1), \ldots, F^*(z_{k-1}), G(z)) \quad (13)$$

Here, Z_1^*, ..., Z^*_{k-1} represents the residual lifetimes of the (k−1) components still functioning at the time of the k-out-of-n system's failure.

Proposed Methodology and Methods

After conducting thorough literature review, it is observed that no work has been done in the field of coherent systems other than k-out-of-n systems.

That's why we consider the coherent system：

$$T^* = \varphi(X_1, \ldots, X_n) \quad (14)$$

which is outlined by: when the system breaks down, either no active units remain, or the remaining functional units are connected in parallel configuration. There are two systems of this kind with an order of 2, four with an order of 3, and seven with an order of 4 {see **Table 1**}

Table 1. Considered coherent systems with two, three and four components.

$T^* = \phi(Z_1, Z_2)$	**p**
$\min(Z_1, Z_2)$ (series system)	$(1, 0)$
$\max(Z_1, Z_2)$ (parallel system)	$(0, 1)$
$T^* = \phi(Z_1, Z_2, Z_3)$	**p**
$\min(Z_1, \max(Z_2, Z_3))$	$(\frac{1}{3}, \frac{2}{3}, 0)$
$Z_{2:3}$ (2-out-3 system)	$(0, 1, 0)$
$\max(Z_1, \min(Z_2, Z_3))$	$(0, \frac{2}{3}, \frac{1}{3})$
$Z_{3:3} = \max(Z_1, Z_2, Z_3)$ (parallel system)	$(0, 0, 1)$
$T^* = \phi(Z_1, Z_2, Z_3, Z_4)$	**p**
$\min(\max(Z_1, Z_2), \max(Z_3, Z_4))$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$
$Z_{3:4}$ (2-out-4 system)	$(0, 0, 1, 0)$
$Z_{4:4} = \max(Z_1, Z_2, Z_3, Z_4)$ (parallel system)	$(0, 0, 0, 1)$
$\max(Z_1, \min(Z_4, Z_3), \min(Z_2, Z_4))$	$(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$
$\max(Z_1, Z_2, \min(Z_3, Z_4))$	$(0, 0, \frac{1}{2}, \frac{1}{2})$
$\max(Z_1, \max(\min(Z_2, Z_3), \min(Z_2, Z_4), \min(Z_3, Z_4)))$	$(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$
$\min(Z_1, \max(Z_2, Z_3, Z_4))$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$

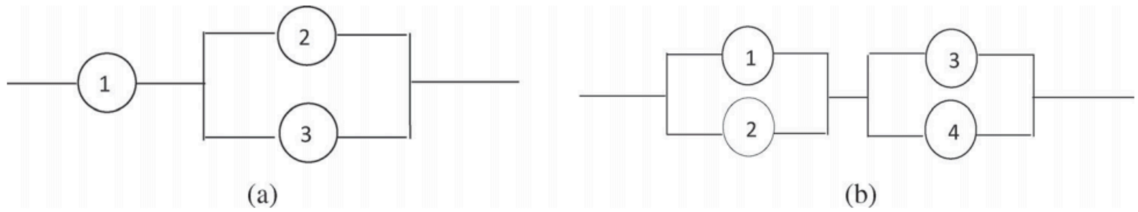

Figure 1. (a) $\min(Z_1, \max(Z_2, Z_3))$ (b) $\min(\max(Z_1, Z_2), \max(Z_3, Z_4))$.

The system $\min(Z_1, \max(Z_2, Z_3))$ depicted in **Figure 1**(a). It is observed that if unit number 2 fails first and unit number 1 fails second, then it will fail. Here, the spare should be located at the position of unit 1 to form a "series system," instead of the position of unit 2, where it would create a parallel configuration with the rest of the active unit at position 3. The system $\min(\max(Z_1, Z_2), \max(Z_3, Z_4))$ shown in **Figure 1**(b) is considered. It is observed that if the unit at position 2 fails first, followed by the failure of the unit at position 3, and then at position 4, the coherent system will fail. Here, the "standby unit" must be placed at unit number 3 or 4 to establish a "series system", instead of unit number 2, where it would result in a " parallel system" with the active unit number 3. In this work our main target is to study the important reliability characteristics of the considered coherent system.

Result and Analysis

It is clear that

$$T^* \in \{X_{1:n}, \ldots, X_{n:n}\} \tag{15}$$

whose probability is 1. We assume that "$X_1, .., X_n$" are exchangeable and dependent random variables and this dependency is modeled by a copula C,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(F(x_1), \ldots, F(x_n)) \tag{16}$$

If $T^* = X_{i:n}$; i=1,2,...,n-1, then the system breaks down at the time of i^{th} unit failure which leads to (n-i) working units inside the system. At that moment, $X_{i:n,}$ the standby unit Z (with CDF G(x)) is active at random time $X_{i:n}$ and resumes it with the remaining working units. Now, if

$X_1^*,\ldots, X_{n-i}^*$ signify the residual lifespan of the (n−i) still functioning units at the moment of breakdown of the coherent system,

Then $X_j^{(i)} = X_j - X_{i:n} \mid X_j > X_{i:n}$, $T^* = X_{i:n}$; j= 1,2,...n-i. It is obvious that $X_j^{(i)}$ is identical. We assume that $F^{(i)}t = P(X_j^{(i)} \le t)$, i=1,...,n. It is easy to observe that the joint distribution of $X_1^*,\ldots, X_{n-i}^*$, and X are no longer exchangeable. We model this dependence between $X_1^*,\ldots, X_{n-i}^*$ and X by the (n-i+1)-dimensional copula function C^*. Thus, by using Sklar's theorem:

$$P(X_1^* \le x_1,\ldots, X_{n-i}^* \le x_{n-1,} X \le x) = C^*(F^{(i)}(x_1),\ldots, F^{(i)}(x_{n-i}),G(x)) \tag{17}$$

Therefore the lifespan of the system is defined by

$$T_z^* = \left\{ \begin{array}{ccc} X_{i:n} + \min(X_1^{(i)}, \ldots, X_{n-i}^{(i)}, Z) & \text{if} & T^* = X_{i:n}, i = 1, \ldots, n-1, \\ X_{n:n} + Z & \text{if} & T^* = X_{n:n} \end{array} \right\} \tag{18}$$

Till now we have defined the lifetime of the system. Our main target is to find out the $E(T_z^*)$.

$$E(T_z^*) = \sum_{i=1}^{n} p_i E(T_z^*)|T^* = Z_{i:n}) = \sum_{i=1}^{n-1} p_i E(T_z^*)|T^* = Z_{i:n}) + p_n E(T_z^*)|T^* = Z_{n:n}) \tag{19}$$

Conclusion and Future Scope

In the last two months, we have majorly worked on analyzing, understanding, and carrying out a literature review to come up with a problem statement "Mean Time To Failure of a Coherent System Equipped with a Single Cold Standby Component having Dependent Components" about which we have discussed above. We have found out the lifetime of the system. In the future, we want to find out the expectations of the system in explicit form and with numerical examples.

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