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*Note:* This document is a part of the lectures given during the Jan-May 2019 Semester.

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Box-Muller Method:

Perhaps the simplest method to implement (though not the fastest or necessarily the most convenient) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal distribution, each component of which is thus a univariate standard normal. The algorithm is based on the following two properties of the bivariate normal distribution:

If $Z \sim \mathcal{N}(0, I_2)$ then,

1. $R = Z_1^2 + Z_2^2$ is exponentially distributed with mean 2, *i.e.,*, $P(R \leq x)1 = 1 - e^{-x/2}$.
2. Given $R$, the point $(Z_1, Z_2)$ is uniformly distributed on the circle of radius $\sqrt{R}$ centered at the origin.

Thus, to generate $(Z_1, Z_2)$, we may first generate $R$ and then choose a point uniformly from the circle of radius $\sqrt{R}$. To sample from the exponential distribution we may set $R = -2\log(U_1)$, where $U_1 \sim \mathcal{U}[0, 1]$. To generate a random point on a circle, we may generate a random angle uniformly between 0 and $2\pi$ and then map the angle to a point on the circle. The random angle may be generated as $V = 2\pi U_2$, where $U_2 \sim \mathcal{U}[0, 1]$. The corresponding point on the circle has coordinates $\left(\sqrt{R}\cos(V), \sqrt{R}\sin(V)\right)$. The complete algorithm is:

1. Generate $U_1, U_2$ (independent) from $\mathcal{U}[0, 1]$.
2. $R = -2\log(U_1)$.
3. $V = 2\pi U_2$.
4. $Z_1 = \sqrt{R}\cos(V)$ and $Z_2 = \sqrt{R}\sin(V)$.
5. Return $Z_1$ and $Z_2$.

Marsaglia and Bray Method:

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by avoiding evaluation of the "cos" and "sin" functions. The Marsaglia and Bray method instead uses acceptance rejection method to sample points uniformly in the unit disc and then transforms these points to normal variables. The algorithm is as follows:

1. (While $X > 1$)
   Generate $U_1, U_2 \sim \mathcal{U}[0, 1]$
   $U_1 = 2U_1 - 1$
   $U_2 = 2U_2 - 1$
   $X = U_1^2 + U_2^2$.
2. $Y = \sqrt{-2\log(X)/X}$.
3. $Z_1 = U_1 Y$ and $Z_2 = U_2 Y$.
4. Return $Z_1$ and $Z_2$.

The transformation $U_i \to 2U_i - 1\ ,\ i = 1, 2$ makes $(U_1, U_2)$ uniformly distributed on the square $[-1, 1] \times [-1, 1]$. Accepting only those pairs for which $X = U_1^2 + U_2^2$ is less than or equal to 1 produces points uniformly distributed over the disc of radius 1 centered at the origin. Conditional on acceptance, $X$ is uniformly distributed between 0 and 1, so that $\log(X)$ has the same affect as $\log(U_1)$ for Box-Muller.