Note: This document is a part of the lectures given during the Jan-May 2019 Semester.

Beta Distribution:
The beta density on $[0,1]$ with parameters $\alpha_1, \alpha_2 > 0$ is given by,

$$f(x) = \frac{1}{B(\alpha_1, \alpha_2)} x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}, \ 0 \leq x \leq 1,$$

with

$$B(\alpha_1, \alpha_2) = \int_0^1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1} dx = \frac{\Gamma(\alpha_1)\ \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

where Γ is the gamma function. Varying the parameters α_1 and α_2 results in a variety of shapes, making this a versatile family of distribution. For example, the case, $\alpha_1 = \alpha_2 = 1/2$ is the arcsine distribution. If $\alpha_1, \alpha_2 \geq 1$ and at least one of the parameter exceeds 1, the beta density is unimodal and achieves its maximum at $\dfrac{\alpha_1 - 1}{\alpha_1 + \alpha_2 - 2}$. Let c be the value of the density f at this point. Then $f(x) \leq c \ \forall \ x$. For the purpose of acceptance rejection method, we may choose g to be the uniform density $\{g(x) = 1, 0 \leq x \leq 1\}$, which is in fact the beta density with parameters $\alpha_1 = \alpha_2 = 1$.
Algorithm:

1. Generate $U_1, U_2 \in \mathcal{U}[0,1]$ until $cU_2 \leq f(U_1)$.
2. Return U_1.

Normal from Double Exponential:
Fisher illustrated the use of acceptance-rejection method by generating half-normal samples from an exponential distribution. (A half normal random variable has the distribution of the absolute value of a normal random variable). This is important since the method can be used to generate normal random variables that is so critical in financial applications.

(i) The double exponential density on $(-\infty, \infty)$ is $g(x) = \exp(-|x|)/2$.

(ii) The normal density is $g(x) = \dfrac{\exp(-x^2/2)}{\sqrt{2\pi}}$.

(iii) The ratio is:

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2 + |x|} \leq \sqrt{\frac{2e}{\pi}} \approx 1.3155 = c.$$

Thus, the normal distribution is dominated by the scaled double exponential density $g(x)$. A sample from the double exponential can be generated (using the formula $X = -\theta \log(U)$, as already done) to draw a standard exponential random variables and then randomizing the sign. The rejection test $u > f(x)/cg(x)$ can be implemented as:

$$\begin{aligned} u &> \frac{e^{-x^2/2}}{\sqrt{2\pi}} \times \frac{1}{c \cdot \frac{e^{-|x|}}{2}} \\ &= \frac{e^{-x^2/2 + |x|}}{\sqrt{2\pi}} \times 2\sqrt{\frac{\pi}{2e}} \\ &= e^{-x^2/2 + |x|} \cdot e^{-1/2} \\ &= e^{-x^2/2 + |x| - 1/2} = e^{-\frac{1}{2}(|x|-1)^2}. \end{aligned}$$

In light of symmetry of f and g it suffices to generate a positive sample and determine the sign only if the sample is accepted. Absolute value is unnecessary in the rejection test.

Normal Random Variables and Vectors:

The standard univariate normal distribution has density:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

and cumulative distribution function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du.$$

The word "standard" indicates mean 0 and variance 1. More generally, the normal distribution with mean μ and variance σ^2 , $\sigma > 0$ has density:

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

and cumulative distribution:

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The notation $X \sim \mathcal{N}(\mu, \sigma^2)$ abbreviates the statement that the random variable X is normally distributed with mean μ and variance σ^2. If $Z \sim \mathcal{N}(0,1)$ (*i.e.*, Z has standard normal distribution), then $\mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$. Thus, given a method for generating samples $Z_1, Z_2, \ldots$ from the standard normal distribution, we can generate samples $X_1, X_2, \ldots$ from $\mathcal{N}(\mu, \sigma^2)$ by setting $X_i = \mu + \sigma Z_i$. It therefore suffices to consider methods for sampling from $\mathcal{N}(0,1)$.

Generating Univariate Normals:

We now discuss algorithms for generating univariate normal distribution. As noted, it suffices to consider sampling from $\mathcal{N}(0,1)$. We assume the availability of a sequence $U_1, U_2, \ldots$ of independent random variables uniformly distributed on the unit interval $[0,1]$ and consider methods for transforming these uniform random variables to normally distributed random variables. Before we proceed on to these methods we state a couple of Theorems.

1. Theorem 1: Suppose X is a random variable with density $f(x)$ and distribution $F(x)$. Further assume $h : S \to B$, with $S, B \in \mathbb{R}$, where S is the support of $f(x)$ (that is f is zero outside S) and let h be strictly monotonous.

   (a) Then $Y := h(X)$ is a random variable and its distribution F_Y in case $h' > 0$ is $F_Y(y) = F(h^{-1}(y))$.

   (b) If h^{-1} is absolutely continuous then for almost all y the density of $h(X)$ is:

   $$f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right|.$$

2. Theorem 2: Suppose X is a random variable in $\mathbb{R}^n$, with density $f(x) > 0$ on the support S. The transformation $h : S \to B$ $(S, B \subseteq \mathbb{R}^n)$ is assumed to be invertible and the inverse be continuously differentiable on B. $Y := h(X)$ is the transformed variable. Then Y has the density:

   $$f^{-1}(h(y)) \left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \right|, y \in B$$

   where $x = h^{-1}(y)$ and $\left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \right|$ is the determinant of the Jacobian matrix of all first order derivatives of $h^{-1}(y)$.