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Consensus in the two-state Axelrod model

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Abstract

The Axelrod model is a spatial stochastic model for the dynamics of cultures which, similar to the voter model, includes social influence, but differs from the latter by also accounting for another social factor called homophily, the tendency to interact more frequently with individuals who are more similar. Each individual is characterized by its opinions about a finite number of cultural features, each of which can assume the same finite number of states. Pairs of adjacent individuals interact at a rate equal to the fraction of features they have in common, thus modeling homophily, which results in the interacting pair having one more cultural feature in common, thus modeling social influence. It has been conjectured based on numerical simulations that the one-dimensional Axelrod model clusters when the number of features exceeds the number of states per feature. In this article, we prove this conjecture for the two-state model with an arbitrary number of features.

MSC: 60K35

Keywords: Interacting particle systems; Axelrod model; Annihilating random walks

1. Introduction

There has been in the past decade a rapidly growing interest in agent-based models in an attempt to understand the long-term behavior of complex social systems. These models are characterized by heuristic rules that govern the outcome of an interaction between two agents, and a graphical structure, modeling either physical space or a social network, that encodes

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the pairs of agents that may interact due to, e.g., geographical proximity or friendship. The main objective of research in this field is to deduce the macroscopic behavior that emerges from the microscopic rules, which also depends on the structure of the network of interactions. The mathematical term for agent-based models is interacting particle systems though, as scientific fields, the former involves numerical simulations whereas the latter is based on rigorous mathematical analyses. While there is a common effort from sociologists, economists, psychologists, and statistical physicists to understand such models, interacting particle systems of interest in social sciences have been so far essentially ignored by mathematicians, with the notable exception of the voter model. This paper is motivated by this lack of analytical results and continues the study initiated in [5] for one of the most popular models of social dynamics: the Axelrod model [2]. The effort to collect analytical results is mainly justified by the fact that stochastic spatial simulations are generally difficult to interpret. This is especially true for the Axelrod model which, in contrast with the voter model, has a number of absorbing states that grows exponentially with the size of the network, and for which simulations of the finite system can freeze in atypical configurations, thus exhibiting behaviors which are not symptomatic of the long-term behavior of their infinite counterpart.

The Axelrod model [2] has been proposed by political scientist Robert Axelrod as a stochastic model for the dissemination of culture. The heuristic microscopic rules include two important social factors: homophily, which is the tendency of individuals to interact more frequently with individuals who are more similar, and social influence, which is the tendency of individuals to become more similar when they interact. Note that the voter model [3,4] accounts for the latter but not for the former: individuals are characterized by one of two competing opinions which they update at a constant rate by mimicking one of their neighbors chosen uniformly at random. In particular, any two individuals in the voter model either totally agree or totally disagree, which prevents homophily from being incorporated in the model. In contrast, individuals in the Axelrod model are characterized by a vector, also called a culture, that consists of F coordinates, called cultural features, each of which assumes one of q possible states. Homophily can thus be naturally modeled in terms of a certain cultural distance between two individuals: pairs of neighbors interact at a rate equal to the fraction of features they have in common. Social influence is then modeled as follows: each time two individuals interact, one of the cultural features for which the interacting pair disagrees (if any) is chosen uniformly at random, and the state of one of the two individuals is set equal to the state of the other individual for this cultural feature. More formally, the Axelrod model on the infinite one-dimensional lattice, which is the network of interactions considered in this paper, is the continuous-time Markov chain whose state space consists of all spatial configurations

$$\eta : \mathbb{Z} \longrightarrow \{1, 2, \ldots, q\}^F.$$

To describe the dynamics of the Axelrod model, we let

$$F(x, y) = \frac{1}{F} \sum_{i=1}^{F} \mathbf{1}\{\eta(x, i) = \eta(y, i)\},$$

where $\eta(x, i)$ refers to the ith coordinate of the vector $\eta(x)$, denote the fraction of cultural features vertex x and vertex y have in common. In addition, we introduce the operator $\sigma_{x,y,i}$ on the set of spatial configurations defined by

$$(\sigma_{x,y,i}\, \eta)(z, j) = \begin{cases} \eta(y, i) & \text{if } z = x \text{ and } j = i \\ \eta(z, j) & \text{otherwise} \end{cases} \qquad \text{for } x, y \in \mathbb{Z} \text{ and } i \in \{1, 2, \ldots, F\}.$$

In other words, the configuration $\sigma_{x,y,i}\,\eta$ is obtained from the configuration η by setting the ith feature of the individual at vertex x equal to the ith feature of the individual at vertex y and leaving the state of all the other features in the system unchanged. The dynamics of the Axelrod model is then described by the Markov generator L defined on the set of cylinder functions by

$$Lf(\eta) = \sum_{|x-y|=1} \sum_{i=1}^{F} \frac{1}{2F} \left[\frac{F(x,y)}{1-F(x,y)}\right] \mathbf{1}\{\eta(x,i) \neq \eta(y,i)\}\, [f(\sigma_{x,y,i}\,\eta) - f(\eta)].$$

Note that the expression of the Markov generator indicates that the conditional rate at which the ith feature of vertex x is set equal to the ith feature of vertex y given that these two vertices are nearest neighbors that disagree on their ith feature can be written as

$$\frac{1}{2F}\left[\frac{F(x,y)}{1-F(x,y)}\right] = F(x,y) \times \frac{1}{F\,(1-F(x,y))} \times \frac{1}{2}$$

which, as required, equals the fraction of features both vertices have in common, which is the rate at which the vertices interact, times the reciprocal of the number of features for which both vertices disagree, which is the probability that any of these features is the one chosen to be updated, times the probability one half that vertex x rather than vertex y is chosen to be updated.

The main question about the Axelrod model is whether or not the population converges to a consensus when starting from a random configuration. For simplicity, we assume that the initial cultures at different sites are independent and identically distributed, and that at a given site, each of the q^F possible initial cultures appears with the same probability. The term "consensus" is defined mathematically in terms of clustering of the infinite system: the model is said to cluster if

$$\lim_{t\to\infty} P\,(\eta_t(x,i) = \eta_t(y,i)) = 1 \quad \text{for all } x, y \in \mathbb{Z} \text{ and } i \in \{1, 2, \ldots, F\}$$

and is said to coexist otherwise. The dichotomy between clustering and coexistence for the finite model is unclear since, as mentioned above, the finite system can hit an absorbing state in which different cultures are present even though its infinite counterpart clusters. In order to characterize the transition between the two regimes, Vilone et al. [7] considered the random variable $s_{\max}$ which refers to the length of the largest interval in which all individuals share the same culture in the absorbing state hit by the finite system, and distinguished between the two regimes depending on whether the expected value of this random variable scales like the population size or is uniformly bounded. Denoting the population size by N, their spatial simulations suggest that

$$\lim_{N\to\infty} E(N^{-1} s_{\max}) > 0 \quad \text{for } F > q \quad \text{and} \quad \lim_{N\to\infty} E(s_{\max}) < \infty \quad \text{for } F < q$$

so we conjecture clustering when $F > q$ and coexistence when $F < q$ for the one-dimensional infinite system. Motivated by this dichotomy, we call the critical case the parameter region in which both parameters are equal: $F = q$. The analysis of the Axelrod model initiated in [5] strongly suggests the coexistence part of the conjecture for a certain subset of the parameter region. More precisely, letting n_c denote the number of cultural domains in the absorbing state hit by the finite system, it is proved based on duality-like techniques and a coupling with a simple urn problem that

$$\lim_{N\to\infty} E(N^{-1}\, n_c) > 0 \quad \text{for } F < c \times q \text{ where } c \approx 0.567 \text{ satisfies } c = e^{-c}.$$

It is also proved that the infinite system clusters in the critical case $F = q = 2$. In this paper, we extend this result to all values of the number of features when the number of states per feature is again equal to two, which proves the clustering part of the conjecture stated above when $q = 2$. More precisely, we have the following theorem.

Theorem 1. *The F-feature 2-state Axelrod model on $\mathbb{Z}$ clusters, starting from the random initial configuration in which the cultures at different sites are independent and identically distributed and at a given site, each of the 2^F possible initial cultures appears with the same probability.*

The proof when $F = q = 2$ is carried out in [5] based on duality techniques for the voter model through the existence of a natural coupling between the two-feature two-state Axelrod model and the voter model obtained by identifying cultures with no feature in common. This coupling, however, fails for any other values of the parameters, so a different approach is needed to extend the result to a larger number of features. The first step is to construct a coupling between the two-state Axelrod model and a certain collection of non-independent systems of annihilating symmetric random walks that keep track of the disagreements between nearest neighbors. Clustering of the Axelrod model is equivalent to extinction of these systems of annihilating random walks. The proof of the latter is inspired by a symmetry argument introduced by Adelman [1] which is combined with certain parity properties of the collection of non-independent systems of random walks.

2. Systems of annihilating random walks

In this section, we represent the Axelrod model on $\mathbb{Z}$ by a particle system that keeps track of the interfaces between cultural domains, thus looking at the disagreements along the edges of the graph rather than the actual cultures on the vertices. This approach is motivated by the fact that consensus in the Axelrod model is equivalent to the extinction of its interfaces. In order to obtain a well-defined Markov process, it is necessary to keep track of the features for which neighbors disagree rather than simply the number of these features. Therefore, we think of each edge of the graph as having F levels, and place a particle on edge $e = (x, y)$ at level i if and only if vertex x and vertex y disagree on their ith feature. That is, we define

$$\xi_t(u, i) = \mathbf{1}\{\eta_t(u - 1/2,\ i) \neq \eta_t(u + 1/2,\ i)\} \quad \text{for all } u \in \mathbb{D} := \mathbb{Z} + 1/2$$

and place a particle at site $u \in \mathbb{D}$ at level i whenever $\xi_t(u, i) = 1$. These particles will jump as the process evolves and the interfaces between cultural domains move. To study the dynamics of this system and the rates at which particles jump, it will be useful to also keep track of the number of particles per site so we introduce

$$\zeta_t(u) = \sum_{i=1}^{F} \xi_t(u, i) \quad \text{for all } u \in \mathbb{D}$$

and call u a j-site whenever it has a total of j particles: $\zeta_t(u) = j$. To understand the evolution, we first observe that when a particle jumps, it moves right or left with equal probability, unless another particle already occupies the site on which the particle tries to jump in which case the particles annihilate each other. Thus, these processes induce a collection of F non-independent systems of annihilating symmetric random walks. The symmetry is due to the fact that, when two neighbors interact, each of them is equally likely to be the one chosen to be updated. Also, the reason why a collision between two particles results in an annihilation is that when two

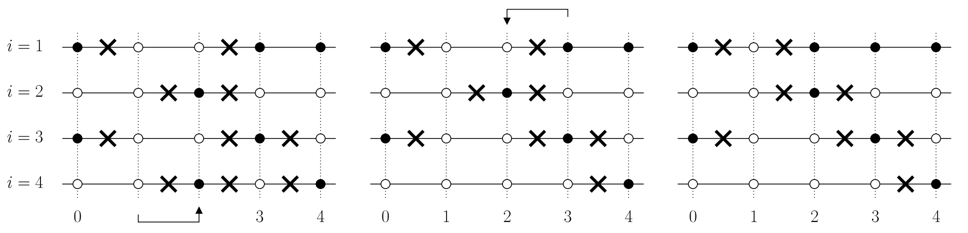

Fig. 1. Coupling between the Axelrod model and annihilating random walks.

individuals disagree with a third one on a given feature, these two individuals must agree on this feature, which happens when $q = 2$. (Note however that for larger values of q collisions between particles would result in either an annihilating event or a coalescing event depending on the configuration of the underlying Axelrod model.) Fig. 1 gives an illustration of these evolution rules through a dynamical picture of the two-state Axelrod model along with the coupled systems of annihilating random walks that keep track of the discrepancies. Black and white dots represent the two possible states for each individual's cultural feature while the crosses indicate the position of the random walks. The two imitation events for the Axelrod model represented in this example translate into two jumping events in the systems of random walks, the first one inducing in addition the annihilation of two particles. Even though the evolution of the particle system at a single level is somewhat reminiscent of the evolution of the interfaces of the one-dimensional voter model, it is in fact much more complicated due to the presence of strong dependencies among the different levels. These dependencies result from the inclusion of homophily in the model, which implies that particles jump at varying rates. More precisely, since two adjacent vertices that disagree on exactly j of their features interact at rate $1 - j/F$, which is the fraction of features they share, and the site between these vertices is a j-site, we see that if u is a j-site, then each particle at site u jumps at rate

$$r(j) = \left(1 - \frac{j}{F}\right) \frac{1}{j} = \frac{1}{j} - \frac{1}{F} \quad \text{for } j \neq 0. \tag{1}$$

This represents the rate at which both vertices interact times the probability that any of the j particles is the one selected to jump. Note that the rates given in the picture are equal to half of the rate $r(j)$ since they include one more piece of information: which of the two interacting vertices is the one selected to be updated, or looking at the systems of random walks, in which direction the particle selected to jump moves. The expression of the rate in (1) indicates that the motion of the particles is slowed down at sites that contain more particles, with the dynamics being frozen at sites with F particles. In the following sections, we will call particles frozen or active depending on whether these particles are located at an F-site or not, respectively. We refer the reader to Fig. 2 for simulation pictures of the systems of annihilating random walks when $F = 3$.

3. Showing that the process cannot become frozen

In this section, we prove that no site can remain an F-site forever, which is the key to proving consensus in the two-state Axelrod model. From the point of view of the particle system described in the previous section, this means that if some site u is completely filled with particles

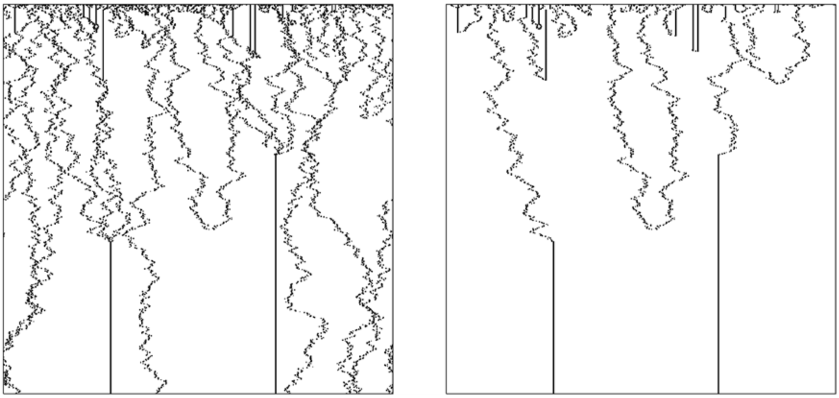

Fig. 2. Pictures of the system of coalescing random walks obtained from a realization of the two-state Axelrod model on the one-dimensional torus with 600 vertices with $F = 3$ features. The left picture gives a superposition of all three levels of random walks while only the first level is extracted in the right picture.

at some time, then eventually another particle will jump onto the site u, annihilating one of the frozen particles on the site and making the other $F - 1$ particles active.

Proposition 2. *Assume that site $u \in \mathbb{D}$ is an F-site at time t. Then,*

$$T := \inf\{s > t : \zeta_s(u) \neq F\} < \infty \quad a.s.$$

It will be useful in the proof of the proposition to construct graphically the particle system described in the previous section using the following collections of independent Poisson processes and random variables: for each pair of site and feature $(v, i) \in \mathbb{D} \times \{1, 2, \ldots, F\}$,

- we let $(N_{v,i}(t) : t \geq 0)$ be a rate one Poisson process,
- we denote by $T_{v,i}(n)$ its nth arrival time: $T_{v,i}(n) = \inf\{t : N_{v,i}(t) = n\}$,
- we let $(B_{v,i}(n) : n \geq 1)$ be a collection of independent Bernoulli variables with

$$P(B_{v,i}(n) = +1) = P(B_{v,i}(n) = -1) = 1/2,$$

- and we let $(U_{v,i}(n) : n \geq 1)$ be a collection of independent Uniform (0, 1) random variables.

The system of annihilating random walks is constructed as follows. At time $t = T_{v,i}(n)$, we draw an arrow labeled i from site v to site $v + B_{v,i}(n)$ to indicate that if

$$\xi_{t-}(v, i) = 1 \quad \text{and} \quad U_{v,i}(n) \leq r(\zeta_{t-}(v)) = \frac{1}{\zeta_{t-}(v)} - \frac{1}{F}$$

then the particle at site v at level i jumps to site $v + B_{v,i}(n)$. Note that this construction indeed produces the desired rates given by Eq. (1).

The above construction can be extended naturally to any subgraph G of the lattice by using the same collections of independent Poisson processes and random variables and killing all the particles that jump onto a site v which is not the center of an edge of the graph. Consider now the case in which the graph G is the one induced by the vertex set $\mathbb{N}$. Suppose the initial configuration is such that the left-most edge has F particles, one at every level, while every level of every other edge independently has a particle at time zero with probability $1/2$. Let

$$p := P\,(\text{the left-most edge has } F \text{ particles at all times}).$$

That is, p is the probability that no particle ever tries to jump onto one of the particles on the left-most edge. We will later see that $p = 0$.

Returning now to the setting of Proposition 2, fix a site $u \in \mathbb{D}$ and a time $t > 0$, and suppose that particles at that space–time point are frozen: $\zeta_t(u) = F$. We will consider only the sites to the right of u and show that eventually some particle from the right must jump onto u, unless the site u has already been hit from the left. Lemma 5 will allow us to break the process into stages, and give a lower bound for the probability that a particle at u is annihilated at each stage. We start by proving Lemma 4 which is a key preliminary result.

Definition 3. The interval $\{u, u+1, \ldots, v\} \subset \mathbb{D}$ is said to be active at time t if the numbers of particles it contains at two different levels differ in their parity, i.e.,

$$\sum_{w=u}^{v} \xi_t(w, i) \neq \sum_{w=u}^{v} \xi_t(w, j) \bmod 2 \quad \text{for some } i \neq j.$$

Lemma 4. *Assume that $\{u, \ldots, v\}$ is active at time t and $\zeta_t(u) = \zeta_t(v) = F$. Then,*

$$T := \inf\{s > t : \zeta_s(u) + \zeta_s(v) \neq 2F\} < \infty \quad a.s.$$

Proof. Seeking a contradiction, we assume that $\zeta_s(u) = \zeta_s(v) = F$ for all $s > t$. Under this assumption, the parity of the number of particles at each level between site u and site v is preserved since particles annihilate in pairs at each level. In particular, $\{u, \ldots, v\}$ is active at every later time, which implies that it contains at least one site which is neither a 0-site nor an F-site since intervals with only 0-sites or F-sites have the same number of particles at each level, and thus cannot be active. This further implies that the interval contains at least one active particle at every later time. Since this particle jumps at a positive rate, it must hit one of the boundaries u or v in an almost surely finite time, which leads to a contradiction. □

Lemma 5. *There exists a sequence of random times $t = t_0 < t_1 < t_2 < \cdots < \infty$ such that, if A_k denotes the event that at some time $s \in (t_{k-1}, t_k]$, a particle at site u is annihilated, then*

$$P\left(A_k \,\middle|\, \bigcap_{j=1}^{k-1} A_j^c\right) \geq \frac{p}{2} \quad \text{for all } k > 0. \tag{2}$$

Proof. The proof relies in part on delicate symmetry arguments and is modeled after the construction of Adelman [1]. We refer the reader to Fig. 3 for a picture of this construction in our context. We must analyze in detail the process at each stage.

For all $w \in \mathbb{D}$ with $w \geq u$ and all $s \geq 0$, let

$$\begin{aligned}\mathcal{F}_w(s) = \sigma\,(&\xi_0(v,i), (N_{v,i}(r) : 0 \leq r \leq s), (B_{v,i}(n) : 1 \leq n \leq N_{v,i}(s)),\\ &(U_{v,i}(n) : 1 \leq n \leq N_{v,i}(s)) : u \leq v \leq w,\ 1 \leq i \leq q)\end{aligned}$$

be the σ-field generated by the graphical representation of the system of random walks over the spatial interval $\{u, u+1, \ldots, w\}$ through time s. Call a pair of random variables (V, T) a stopping pair whenever the following three conditions are satisfied:

- the random variable V is $\{u, u+1, \ldots\}$-valued,
- the random variable T is $[0, \infty)$-valued, and
- $\{V \leq w\} \cap \{T \leq s\} \in \mathcal{F}_w(s)$ for all $w \in \mathbb{D}$ with $w \geq u$ and all $s \geq 0$.

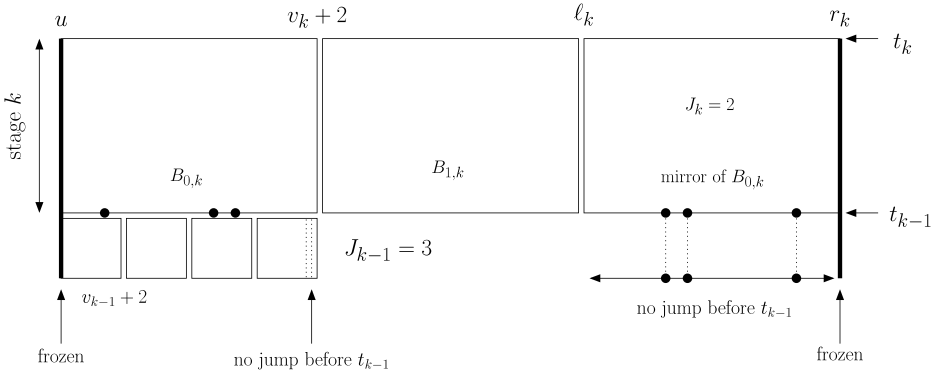

Fig. 3. Picture related to the proof of Lemma 5.

If (V, T) is a stopping pair, then define the σ-field $\mathcal{F}_V(T)$ to consist of all events A such that

$$A \cap \{V \leq w\} \cap \{T \leq s\} \in \mathcal{F}_w(s) \quad \text{for all } w \in \mathbb{D} \text{ with } w \geq u \text{ and all } s \geq 0.$$

We think of $\mathcal{F}_V(T)$ as being the σ-field representing the information from the graphical representation of the system of random walks over the interval $\{u, \ldots, V\}$ through time T.

We prove (2) by induction on k. As part of the induction hypothesis, we assume that at the beginning of the kth stage of the construction, we have a random site v_k and a random time t_{k-1} such that the following two conditions hold:

H1. $N_{v_k,i}(t_{k-1}) = N_{v_k+1,i}(t_{k-1}) = N_{v_k+2,i}(t_{k-1}) = 0$ for $i = 1, 2, \ldots, F$,
H2. $(v_k + 2, t_{k-1})$ is a stopping pair and $A_1, A_2, \ldots, A_{k-1} \in \mathcal{F}_{v_k+2}(t_{k-1})$.

Note that condition H2 implies that the Poisson processes at sites to the right of $v_k + 2$ are independent of v_k and t_{k-1}. For $k = 1$, condition H1 can be satisfied by choosing $t_0 = t$ and

$$v_1 = \min\{v > u : N_{v,i}(t) = N_{v+1,i}(t) = N_{v+2,i}(t) = 0 \text{ for } i = 1, 2, \ldots, F\}.$$

Then $(v_1 + 2, t_0)$ is a stopping pair, and so condition H2 is satisfied.

To prove inequality (2) by induction, we will show that

$$P\left(A_k | \mathcal{F}_{v_k+2}(t_{k-1})\right) \geq p/2 \tag{3}$$

and that t_k and v_{k+1} can be chosen to satisfy H1 and H2. The result (3) implies that

$$P\left(A_k \,\middle|\, \bigcap_{j=1}^{k-1} A_j^c\right) \geq \frac{p}{2} \quad \text{since } \bigcap_{j=1}^{k-1} A_j^c \in \mathcal{F}_{v_k+2}(t_{k-1})$$

according to condition H2.

Condition H1 above implies that, in the graphical representation, there is no arrow starting at either site v_k or site $v_k + 1$ by time t_{k-1}, from which it follows that

- particles starting to the right of v_k do not reach v_k by time t_{k-1}, and
- particles starting in $\{u, u + 1, \ldots, v_k\}$ do not reach $v_k + 1$ until after time t_{k-1}.

The assumption $N_{v_k+2,i}(t_{k-1}) = 0$ is not necessary at this stage of the proof but it will be useful later to obtain a lower bound of the probability that a certain interval is active. We partition the half-space starting at site u into intervals of the same length as $\{u, \ldots, v_k + 2\}$, thus each containing a total of $v_k - u + 3$ sites. Specifically, for each integer $j \geq 0$, we let

$$B_{j,k,1} = B_{j,k} = \{j v_k - (j-1)u + 3j, \ldots, (j+1)v_k - ju + 3j + 2\}.$$

Note that $B_{0,k} = \{u, u+1, \ldots, v_k + 2\}$ and that for each $j \in \mathbb{N}$ the right-most site in $B_{j,k}$ is the left neighbor of the left-most site in $B_{j+1,k}$, as depicted in Fig. 3. Now, let $J_{k,1} = J_k$ be the smallest positive integer j such that the following two conditions hold:

J1. We have $N_{w,i}(t_{k-1}) = 0$ for all $w \in B_{j,k}$ and $i = 1, 2, \ldots, F$, and
J2. We have

$$\text{translation}: \xi_0(j v_k - (j-1)u + 3j + m, i) = \xi_{t_{k-1}}(u + m, i)$$
$$\text{or reflection}: \xi_0((j+1)v_k - ju + 3j + 2 - m, i) = \xi_{t_{k-1}}(u + m, i)$$

for $m = 0, 1, \ldots, v_k - u + 1$ and $i = 1, 2, \ldots, F$.

The second requirement ensures that the initial configuration in $B_{J_k,k}$ excluding its right-most site is the same as the configuration in $B_{0,k}$ excluding its right-most site at time t_{k-1}, or else the initial configuration in $B_{J_k,k}$ excluding its left-most site is the mirror image of the configuration in $B_{0,k}$ excluding its right-most site at time t_{k-1}. In the former case, we say that translation occurs; in the latter case, we say that reflection occurs.

Probability of a reflection—We claim that the probability of a reflection is equal to at least one half. To see this, note that whether or not translation or reflection occurs depends only on the initial configuration at the sites in the interval $B_{J_k,k}$. Recall that, by assumption, we begin from a configuration in which the cultures at different sites are independent and the possible cultures appear at each site with equal probability. In particular, the two possible cultural features at each level of each site in $\mathbb{Z}$ appear independently with equal probability one half. As a result, when we consider the particle system that keeps track of interfaces, at time zero each level of each site in $\mathbb{D}$ independently has a particle with probability one half. Because the Poisson processes are independent of the initial configuration, knowing that J1 holds for a particular j provides no information about the initial configuration in $B_{j,k}$. Furthermore, by condition H2, the initial configuration to the right of $v_k + 2$ is independent of the site v_k and the time t_{k-1}. Consequently, conditional on the values of v_k and t_{k-1} and on the event that condition J1 holds for a particular j, all possible values for $\xi_0(w, i)$ with $w \in B_{j,k}$ and $i = 1, 2, \ldots, F$ are equally likely. In particular, this configuration is exactly as likely to be the same as the configuration in the interval $B_{0,k}$ excluding its right-most site at time t_{k-1} as it is to be the same as the mirror image of the configuration in the interval $B_{0,k}$ excluding its right-most site at time t_{k-1}. Thus, the probabilities of translation and reflection are the same. Because either translation or reflection must occur, and both will occur if the configuration between u and $v_k + 1$ at time t_{k-1} is a mirror image of itself, it follows that the probability of reflection is larger than one half.

Probability of an active interval—We now consider the process conditioned on the event that $J_k = j$ as well as the event that a reflection occurs. We also, for now, truncate the process at the right edge of $B_{j,k}$, so the right-most site is

$$r_{k,1} = r_k := (j+1)\, v_k - ju + 3j + 2$$

and just consider the evolution of the process between the sites u and r_k. We refer to this as the truncated process. Let

$$\ell_{k,1} = \ell_k := j\, v_k - (j-1)\, u + 3j$$

be the left-most site in $B_{j,k}$ as indicated in Fig. 3. Because the particles at $v_k + 1$ and $\ell_k + 1$ have no opportunity to interact with other particles before time t_{k-1}, the configuration of particles at these two sites is the same at time t_{k-1} and at time zero. In particular, as explained above, each of the levels of these two sites is independently occupied by a particle with probability one half, independently of the configuration at other sites in $B_{j,k}$. Thus, the probability that the numbers of particles at two given levels of these two sites have the same parity is equal to $1/2$. It follows that the interval $\Gamma_k = \Gamma_{k,1} = \{u, u+1, \ldots, r_k\}$ is active in the sense of Definition 3 with probability at least $1/2$, in which case there will eventually be a change either to site u or to site r_k according to Lemma 4.

Since the probability of a reflection as well as the conditional probability that $\Gamma_{k,1}$ is active given that a reflection occurs are both larger than one half, the probability that both a reflection occurs and the interval $\Gamma_{k,1}$ is active is at least $1/4$. In case a reflection does not occur or the interval is not active, we partition the half-space starting at u into intervals of $(J_{k,1}+1)(v_k-u+3)$ sites, starting with $B_{0,k,2} = \Gamma_{k,1}$ instead of $B_{0,k,1}$. Also, we define $J_{k,2}$ as the smallest integer that satisfies conditions J1 and J2 but using this new partition. Our approach to estimating the probability of a reflection and the probability of the interval $\Gamma_{k,1}$ being active again applies, from which it follows that, after a finite number of steps, say n, we find a partition such that simultaneously reflection occurs and the interval $\Gamma_{k,n-1} = \{u, u+1, \ldots, r_{k,n-1}\}$ is active. More precisely, our estimates above show that n is stochastically smaller than a geometric random variable with parameter $1/4$. Since the rest of the proof is not sensitive to the value of the random variable n, to avoid cumbersome notation, we drop the subscript that refers to the value of n.

Law of the truncated process and its mirror image—The next step is to prove that the law of the truncated process is the same as the law of its mirror image. First, we observe that, because no particles in the interval $B_{j,k}$ can jump before time t_{k-1}, the configuration in this interval is the same at time zero and at time t_{k-1}. Therefore, the condition for a reflection implies that the left-most and the right-most $v_k - u + 2$ sites are mirror images of each other at time t_{k-1}, i.e.,

$$\xi_{t_{k-1}}(u+m, i) = \xi_{t_{k-1}}(r_k - m, i) \quad \text{for } m = 0, 1, \ldots, v_k - u + 1 \text{ and all } i. \tag{4}$$

Now, since there is no arrow starting or pointing at either site $v_k + 1$ or site $\ell_k + 1$ by time t_{k-1}, until this time, the particles located in $I_k = \{v_k+1, \ldots, \ell_k+1\}$ at time zero evolve independently of the particles located outside I_k at time zero. Since in addition the graphical representation is independent of the initial configuration, the law of the graphical representation in I_k is equal to the law of its mirror image up to time t_{k-1}. Moreover, conditional on $J_k = j$, all the possible configurations in the intervals $B_{i,k}$ for $i = 1, 2, \ldots, j-1$, are equally likely at time zero with the exception of the two configurations described in J2 which appear with probability zero. In particular, each possible initial configuration in the intervals $B_{i,k}$ is as likely as its mirror image, which also implies that the law of the initial configuration in the interval I_k is equal to the law of its mirror image. Because the configuration of the process in I_k up to time t_{k-1} only depends on the graphical representation and the initial configuration in this interval, it follows that the law of the configuration in I_k at all times until time t_{k-1} is equal to the law of its mirror image. This result combined with (4) implies that the law of the configuration in Γ_k at time t_{k-1} is the same as the law of its mirror image. Since in addition the law of the graphical representation after time t_{k-1} is equal to the law of its mirror image, the law of the truncated process after time t_{k-1} is equal to the law of its mirror image. In particular, the probability that there is a change to site u before there is a change to site r_k is by symmetry equal to $1/2$. That is, if D_k denotes the event that eventually there is a change to site u before there is a change to site r_k, i.e., D_k is the analog of A_k for the truncated process restricted to Γ_k, then

$$P\left(D_k | \mathcal{F}_{v_k+2}(t_{k-1})\right) = 1/2. \tag{5}$$

To complete the construction, we define time t_k to be the first time at which there is a change either to site u or to site r_k in this system.

We now return to the case in which only the sites to the left of u are discarded, which means there is a possibility that some particle could jump onto site r_k from the right. In this case, the particles at site r_k are frozen until at least time t_k, so thinking of site r_k as the left-most edge in the definition of p, we see that there is a probability larger than p that no particle will jump from site $r_k + 1$ to site r_k before time t_k. Using in addition (5), we deduce that there is a probability larger than $p/2$ that site u will change at time t_k, which gives (3). Finally, we let

$$v_{k+1} = \min\{v > r_k : N_{v,i}(t_k) = N_{v+1,i}(t_k) = N_{v+2,i}(t_k) = 0 \text{ for } i = 1, 2, \ldots, F\}.$$

To conclude the inductive step, we observe that, since no particle to the right of v_{k+1} can reach v_{k+1} by time t_k, the event $\{v_{k+1} + 2 \le w\} \cap \{t_k \le s\}$ depends only on the initial configuration and the graphical representation through time s at the sites $u, u+1, \ldots, w$. Therefore,

$$\{v_{k+1} + 2 \le w\} \cap \{t_k \le s\} \in \mathcal{F}_w(s)$$

and so $(v_{k+1} + 2, t_k)$ is a stopping pair. Also, again using that no particle to the right of v_{k+1} can reach v_{k+1} by time t_k, we have $A_k \in \mathcal{F}_{v_{k+1}+2}(t_k)$, which completes the proof. □

Proof of Proposition 2. Consider the system in which sites to the left of u are removed. We proceed by contradiction by assuming that there is a positive probability that the site u never changes after time t. We will show that this implies that $p > 0$. Thus, (2) will imply that

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$$

which means that with probability one, eventually some particle at site u will be annihilated. This contradiction will imply the result.

In the case $t = 0$, the probability that the site u never changes after time t is exactly p, by definition. We still need to show that if site u becomes an F-site at $t > 0$, then having a positive probability that the site never changes after time t would still imply that $p > 0$. The strategy will be to argue that with positive probability, up to time t the process behaves within a finite region in such a way that makes it possible to reduce to the $t = 0$ case.

Because, with probability one, there are infinitely many sites $v > u$ such that

$$N_{v,i}(t) = N_{v+1,i}(t) = 0 \quad \text{for } i = 1, 2, \ldots, F,$$

there exists a site $v > u$ such that

$$P\left(\zeta_s(u) = F \text{ for all } s > t \text{ and } N_{v,i}(t) = N_{v+1,i}(t) = 0 \text{ for } i = 1, 2, \ldots, F\right) > 0.$$

The event $N_{v,i}(t) = N_{v+1,i}(t) = 0$ for $i = 1, 2, \ldots, F$ implies that there is no arrow connecting site v and site $v + 1$ by time t. Therefore conditional on this event, the evolution of the process on the interval $\{u, u+1, \ldots, v\}$ is independent of its evolution on $\{v+1, v+2, \ldots\}$ up to time t.

Because there are only finitely many possible configurations for the sites $u+1, \ldots, v-1$, there must exist numbers $c_{w,i} \in \{0, 1\}$ for $u < w \le v$ and $i = 1, 2, \ldots, F$ such that

$$P\big(\zeta_s(u) = F \text{ for all } s > t,\ \xi_t(w, i) = c_{w,i} \text{ for } u < w \le v \text{ and } i = 1, 2, \ldots, F,$$
$$\text{and } N_{v,i}(t) = N_{v+1,i}(t) = 0 \text{ for } i = 1, 2, \ldots, F\big) > 0.$$

Let $c_{u,i} = 1$ for $i = 1, 2, \ldots, F$. Clearly, there is a positive probability that

$$\xi_0(w,i) = c_{w,i} \quad \text{and} \quad N_{w,i}(t) = 0 \quad \text{for } w = u, u+1, \ldots, v \text{ and } i = 1, 2, \ldots, F$$

so it follows from the conditional independence noted above that

$$P\,(\zeta_s(u) = F \text{ for all } s > t,\ \xi_0(w,i) = c_{w,i} \text{ for } u \le w \le v \text{ and } i = 1, 2, \ldots, F, \\ \text{and } N_{w,i}(t) = 0 \text{ for } w = u, u+1, \ldots, v+1) > 0.$$

However, the probability on the left-hand side is at most

$$P\,(\zeta_s(u) = F \text{ for all } s > 0 \text{ and } \xi_0(u,i) = 1 \text{ for } i = 1, 2, \ldots, F),$$

which in turn is at most p. It follows that $p > 0$. □

4. Extinction of the particles

In this section, we prove almost sure extinction of the systems of annihilating random walks, which is equivalent to the fact that each site is eventually a 0-site:

$$\lim_{t\to\infty} P\,(\zeta_t(u) = 0) = 1 \quad \text{for all } u \in \mathbb{D}. \tag{6}$$

We use different strategies to deal with active particles and frozen particles. We start by proving extinction of the active particles since it is one of the keys to showing extinction of the frozen particles, but the main ingredient to prove the latter is the result of Proposition 2. To see that (6) indeed implies Theorem 1, we observe that for all $x, y \in \mathbb{Z}$ with $x < y$, we have

$$\begin{aligned} &\lim_{t\to\infty} P\,(\eta_t(x) \neq \eta_t(y)) \\ &\quad \le \lim_{t\to\infty} P\,(\eta_t(x+j-1) \neq \eta_t(x+j) \text{ for some } j = 1, 2, \ldots, y-x) \\ &\quad \le \lim_{t\to\infty} P\,(\zeta_t(x+j-1/2) \neq 0 \text{ for some } j = 1, 2, \ldots, y-x) \\ &\quad \le \lim_{t\to\infty} \sum_{j=1}^{y-x} P\,(\zeta_t(x+j-1/2) \neq 0) = 0. \end{aligned}$$

It remains to prove almost sure extinction of the systems of annihilating random walks, which is done in Lemma 7 for the active particles and in Lemma 8 for the frozen particles. The intuitive idea to prove extinction is that, as long as there are active particles in the system, annihilating events occur that decrease the number of particles. Since at equilibrium the number of particles cannot strictly decrease, the number of active particles must converge to zero. The argument is similar to the one invoked in page 385 of Liggett [6]. The somewhat less obvious extinction of the frozen particles is obtained by applying in addition Proposition 2. We now make these arguments rigorous in Lemmas 6–8, and point out that Lemmas 7 and 8 immediately imply (6).

Lemma 6. *The limits*

$$\lim_{t\to\infty} E(\zeta_t(u)\,\mathbf{1}\{\zeta_t(u) \neq F\}) \quad \textit{and} \quad \lim_{t\to\infty} E(\zeta_t(u)\,\mathbf{1}\{\zeta_t(u) = F\})$$

exist and do not depend on the choice of $u \in \mathbb{D}$.

Proof. First of all, since the initial configuration as well as the evolution rules of the process are translation invariant in space, the probability that u has i particles at time t does not depend on the choice of a specific site $u \in \mathbb{D}$, i.e.,

$$P(\zeta_t(u) = i) = P(\zeta_t(v) = i) \quad \text{for all } u, v \in \mathbb{D}.$$

This implies that the limit superior and limit inferior of the expected values in the statement do not depend on u. Now, if $s < t$, then $\zeta_t(u) - \zeta_s(u)$ is the number of particles that jump onto the site u between times s and t minus the number of particles that jump away from the site u between times s and t minus the number of particles annihilated at the site u between times s and t. Since each active particle jumps to the left and to the right with equal probabilities, the expected number of particles that jump onto the site u between times s and t is equal to the expected number of particles that jump away from site u between times s and t. Therefore, $E(\zeta_t(u)) - E(\zeta_s(u))$ is the negative of the expected number of particles annihilated at site u between times s and t, which is nonpositive. It follows that the expected number of particles at site u is a nonincreasing function of time. In particular, it has a limit as $t \to \infty$ and we write

$$\limsup_{t\to\infty} E(\zeta_t(u)) = \liminf_{t\to\infty} E(\zeta_t(u)) = \lim_{t\to\infty} E(\zeta_t(u)) = L. \tag{7}$$

Note that the first limit in the statement of the lemma is the limit of the expected number of active particles at the site u, while the second limit is the limit of the expected number of frozen particles at the site u. In view of (7), to prove that both limits exist, it suffices to show convergence of the expected number of active particles. We proceed by contradiction. If the limit does not exist, then we can find L_- and L_+ such that

$$L_- = \liminf_{t\to\infty} E(\zeta_t(u)\, \mathbf{1}\{\zeta_t(u) \neq F\}) < \limsup_{t\to\infty} E(\zeta_t(u)\, \mathbf{1}\{\zeta_t(u) \neq F\}) = L_+ \tag{8}$$

and let $\epsilon > 0$ such that $5\epsilon = L_+ - L_-$. In view of (7) there exists $t_0 < \infty$ such that

$$L - \epsilon < E(\zeta_t(u)) < L + \epsilon \quad \text{for all } t > t_0. \tag{9}$$

Also, in view of (8) there exists an infinite sequence of times

$$t_0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_j < t_j < \cdots < \infty$$

such that for all integers $j > 0$, we have

$$E(\zeta_{s_j}(u)\, \mathbf{1}\{\zeta_{s_j}(u) \neq F\}) < L_- + \epsilon \quad \text{and} \quad E(\zeta_{t_j}(u)\, \mathbf{1}\{\zeta_{t_j}(u) \neq F\}) > L_+ - \epsilon. \tag{10}$$

It directly follows from (9) and (10) that

$$\begin{aligned}
&E(\zeta_{t_j}(u)\, \mathbf{1}\{\zeta_{t_j}(u) = F\}) - E(\zeta_{s_j}(u)\, \mathbf{1}\{\zeta_{s_j}(u) = F\}) \\
&\quad = E(\zeta_{t_j}(u)) - E(\zeta_{s_j}(u)) + E(\zeta_{s_j}(u)\, \mathbf{1}\{\zeta_{t_j}(u) \neq F\}) - E(\zeta_{t_j}(u)\, \mathbf{1}\{\zeta_{s_j}(u) \neq F\}) \\
&\quad < (L + \epsilon) - (L - \epsilon) + (L_- + \epsilon) - (L_+ - \epsilon) = L_- - L_+ + 4\epsilon = -\epsilon
\end{aligned}$$

for all $j > 0$ from which we deduce that

$$\begin{aligned}
&P(\zeta_{t_j}(u) = F) - P(\zeta_{s_j}(u) = F) \\
&\quad = F^{-1}\, E(\zeta_{t_j}(u)\, \mathbf{1}\{\zeta_{t_j}(u) = F\}) - F^{-1}\, E(\zeta_{s_j}(u)\, \mathbf{1}\{\zeta_{s_j}(u) = F\}) < -\epsilon\, F^{-1}.
\end{aligned} \tag{11}$$

Now observe that, in view of the evolution rules, active particles can only be created when an F-site becomes an $(F-1)$-site, which will involve the annihilation of one active particle and one frozen particle. Therefore, over any finite interval, the number of particles annihilated between times s and t is at least twice the number of times an F-site becomes an $(F-1)$-site between times s and t, which in turn is at least twice the number of F-sites at time s minus twice the number of F-sites at time t. Taking the expected value per site in the previous statement, and using that $E(\zeta_t(u)) - E(\zeta_s(u))$ is the negative of the expected number of particles annihilated at site u between times s and t, we get

$$E(\zeta_t(u)) - E(\zeta_s(u)) \leq 2 \times P(\zeta_t(u) = F) - 2 \times P(\zeta_s(u) = F) \quad \text{for all } s < t. \tag{12}$$

Letting $s = s_j$ and $t = t_j$ and combining (11) and (12) gives

$$E(\zeta_{t_j}(u)) - E(\zeta_{s_j}(u)) < -2\epsilon F^{-1}. \tag{13}$$

In particular, using once more that the expected number of particles at the site u is a nonincreasing function of time, and applying F times inequality (13), we obtain

$$\begin{aligned} E(\zeta_{t_F}(u)) - E(\zeta_{s_1}(u)) &= \sum_{j=1}^{F} \big(E(\zeta_{t_j}(u)) - E(\zeta_{s_j}(u))\big) \\ &\quad + \sum_{j=1}^{F-1} \big(E(\zeta_{s_{j+1}}(u)) - E(\zeta_{t_j}(u))\big) \\ &\leq \sum_{j=1}^{F} \big(E(\zeta_{t_j}(u)) - E(\zeta_{s_j}(u))\big) < -2\epsilon, \end{aligned}$$

which, since $t_F > s_1 > t_0$, contradicts (9). In particular, (8) does not hold therefore the expected number of active particles at the site u has a limit as time goes to infinity. To prove that the second limit in the statement of the lemma also exists, we simply observe that

$$E(\zeta_t(u)\, \mathbf{1}\{\zeta_t(u) = F\}) = E(\zeta_t(u)) - E(\zeta_t(u)\, \mathbf{1}\{0 < \zeta_t(u) < F\})$$

and invoke the fact that both expected values on the right-hand side have a limit as time goes to infinity. This completes the proof. □

Lemma 7. *There is extinction of the active particles, i.e.,*

$$\lim_{t \to \infty} P(0 < \zeta_t(u) < F) = 0 \quad \textit{for all } u \in \mathbb{D}.$$

Proof. The key to proving the result is to observe that, as long as the expected number of active particles per site is positive, annihilating events occur that reduce the expected number of particles. In particular, if the limiting expected number of active particles is strictly positive then the expected number of particles drops below zero, a contradiction that we obtain by counting the number of a certain type of events that may occur at each site. To make our intuition rigorous, we note first that, by Lemma 6, the expected number of active particles at a site does not depend on the choice of a specific site and has a limit as time goes to infinity. Seeking a contradiction, we assume that this limit is positive: $L_+ > 0$. In particular,

$$\begin{aligned} E(\zeta_t(u)\, \mathbf{1}\{0 < \zeta_t(u) < F\}) &= \sum_{i=1}^{F-1} i \times P(\zeta_t(u) = i) \\ &\geq \sum_{i=1}^{F-1} P(\zeta_t(u) = i) = P(0 < \zeta_t(u) < F) \\ &\geq L_+/2 > 0 \end{aligned} \tag{14}$$

for all t large enough. Now, we say that an event occurs at site u at time t if one of the following two conditions at space–time point (u, t) is satisfied.

1. Annihilating event: $\zeta_t(u) + \zeta_t(u+1) = \zeta_{t-}(u) + \zeta_{t-}(u+1) - 2$.
2. Freezing event: $\zeta_{t-}(u) = F - 1$ and $\zeta_t(u) = F$.

Denote by $\mathcal{A}_t(u)$ and $\mathcal{F}_t(u)$ the number of annihilating and freezing events that occur at site u by time t, and observe that the joint distribution of these two random variables does not depend on the specific choice of the site u. This follows by again invoking the translation invariance in space of the initial configuration and the evolution rules of the process. Since active particles evolve according to symmetric random walks run at a positive rate, and one-dimensional symmetric random walks are recurrent, given any two active particles at the same level at time s, at some almost surely finite random time $t > s$, either one of these two particles is killed due to a collision with a third particle, or the particle becomes frozen, or both particles annihilate each other. In other words, each active particle is doomed to either annihilation or freezing. Therefore, if the expected number of active particles at a site is bounded away from zero at all large times, as stated in (14), then

$$\lim_{t\to\infty} E(\mathcal{A}_t(u)) + \lim_{t\to\infty} E(\mathcal{F}_t(u)) = \infty. \tag{15}$$

To obtain a contradiction, we now prove that (15) is not true. To do this, we first observe that each particle can get annihilated only once and that particles annihilate in pairs. In particular, looking at the process in a fixed interval between two fixed times, the number of particles that are destroyed in this space–time window is twice the number of annihilating events that occur in this window. Using that each site has initially at most F particles and taking the expected value per site in the previous statement, we obtain

$$E(\mathcal{A}_t(u)) \le F/2 < \infty \quad \text{for all } t \ge 0. \tag{16}$$

In addition, if site u experiences two freezing events, one at time s and one at time $t > s$, then it must have changed from an F-site to an $(F-1)$-site between time s and time t, which also indicates the presence of an annihilating event at either site u or site $u-1$ at that time: two consecutive freezing events at site u are therefore separated in time by an annihilating event that occurs either at site u or at site $u-1$. In particular, the number of freezing events at site u is at most equal to one plus the number of annihilating events at sites u and $u-1$. Taking the expected value in the previous statement and using (16), we obtain the upper bound

$$E(\mathcal{F}_t(u)) \le 1 + E(\mathcal{A}_t(u)) + E(\mathcal{A}_t(u-1)) = 1 + 2\,E(\mathcal{A}_t(u)) \le 1 + F < \infty \tag{17}$$

for all $t \ge 0$. Combining (16) and (17) gives

$$\lim_{t\to\infty} E(\mathcal{A}_t(u)) + \lim_{t\to\infty} E(\mathcal{F}_t(u)) \le 3\,F/2 + 1 < \infty$$

which contradicts (15). In particular, (14) does not hold and $L_+ = 0$. □

Lemma 8. *There is extinction of the frozen particles, i.e.,*

$$\lim_{t\to\infty} P(\zeta_t(u) = F) = 0 \quad \textit{for all } u \in \mathbb{D}.$$

Proof. By Lemma 6, the limit to be estimated exists and does not depend on the choice of the site u. Let $\epsilon > 0$ and observe that

$$\begin{aligned} E\left(\zeta_t(u)\,\mathbf{1}\{0<\zeta_t(u)<F\}\right) &= \sum_{i=1}^{F-1} i \times P\left(\zeta_t(u)=i\right) \\ &\le (F-1) \sum_{i=1}^{F-1} P\left(\zeta_t(u)=i\right) \\ &= (F-1) \times P\left(0<\zeta_t(u)<F\right). \end{aligned}$$

Therefore, Lemma 7 implies the existence of a time $t_0 < \infty$ such that

$$E\left(\zeta_t(u)\,\mathbf{1}\{0<\zeta_t(u)<F\}\right) < \epsilon/2 \quad \text{for all } t \ge t_0. \tag{18}$$

To prove extinction of the frozen particles, we apply successively Proposition 2 to obtain the existence of an increasing sequence of times $\{t_j : j \ge 0\}$ tending to infinity such that

$$P\left(\zeta_s(u)=F \text{ for all } s \in (t_j, t_{j+1}) | \zeta_{t_j}(u)=F\right) < 1/2 \quad \text{for all } j \ge 0. \tag{19}$$

In particular, since each time an F-site becomes an $(F-1)$-site, there are two particles that annihilate each other, the previous inequality implies that

$$E\left(\zeta_{t_j}(u)\right) - E\left(\zeta_{t_{j+1}}(u)\right) > 2 \times \left(1-\frac{1}{2}\right) P\left(\zeta_{t_j}(u)=F\right) = P\left(\zeta_{t_j}(u)=F\right) \tag{20}$$

for all integers $j \ge 0$. To see this, note that if a site is frozen at time t_j, then with probability at least one half according to (19), this site will be visited by an active particle before t_{j+1}, resulting in the annihilation of two particles. Therefore, the expected number of particles killed per site between times t_j and t_{j+1} is larger than the probability of a site being an F-site at time t_j, which gives (20). Now, using (20) and then (18), we deduce that

$$\begin{aligned} F \times P\left(\zeta_{t_{j+1}}(u)=F\right) &= E\left(\zeta_{t_{j+1}}(u)\,\mathbf{1}\{\zeta_{t_{j+1}}(u)=F\}\right) \le E\left(\zeta_{t_{j+1}}(u)\right) \\ &\le E\left(\zeta_{t_j}(u)\right) - P\left(\zeta_{t_j}(u)=F\right) \\ &= E\left(\zeta_{t_j}(u)\,\mathbf{1}\{\zeta_{t_j}(u)=F\}\right) + E\left(\zeta_{t_j}(u)\,\mathbf{1}\{0<\zeta_{t_j}(u)<F\}\right) \\ &\quad - P\left(\zeta_{t_j}(u)=F\right) \\ &\le F\, P\left(\zeta_{t_j}(u)=F\right) + \epsilon/2 - P\left(\zeta_{t_j}(u)=F\right) \\ &= (F-1)\, P\left(\zeta_{t_j}(u)=F\right) + \epsilon/2. \end{aligned}$$

In particular, a simple induction gives

$$\begin{aligned} P\left(\zeta_{t_n}(u)=F\right) &\le \left(1-\frac{1}{F}\right) P\left(\zeta_{t_{n-1}}(u)=F\right) + \frac{\epsilon}{2F} \\ &\le \left(1-\frac{1}{F}\right)^n P\left(\zeta_{t_0}(u)=F\right) + \frac{\epsilon}{2F} \sum_{j=0}^{n-1} \left(1-\frac{1}{F}\right)^j \\ &\le \left(1-\frac{1}{F}\right)^n + \frac{\epsilon}{2F}\,\frac{1}{1-(1-1/F)} = \left(1-\frac{1}{F}\right)^n + \frac{\epsilon}{2} \le \epsilon \end{aligned}$$

for sufficiently large n. In view of Lemma 6, the previous inequality implies that

$$\lim_{t\to\infty} P\left(\zeta_t(u)=F\right) = \lim_{n\to\infty} P\left(\zeta_{t_n}(u)=F\right) \le \epsilon \quad \text{for all } u \in \mathbb{D}.$$

Since this holds for all $\epsilon > 0$ arbitrarily small, the lemma follows. $\square$

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