# Approximation speed of quantized vs. unquantized ReLU neural networks and beyond

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# ReLU neural networks

parameters

$$(\underbrace{A_1, \ldots, A_L}_{\text{matrices}}, \underbrace{b_1, \ldots, b_L}_{\text{vectors}})$$

function represented

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$$\underbrace{(A_1, \ldots, A_L, b_1, \ldots, b_L)}_{\theta}$$

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| parameters | | function represented |
| --- | --- | --- |
| $\underbrace{(A_1, \ldots, A_L, b_1, \ldots, b_L)}_{\theta}$ | $x \mapsto$ | $A_1 x + b_1$ |

# ReLU neural networks

$$\underbrace{(A_1, \ldots, A_L, b_1, \ldots, b_L)}_{\theta}^{\text{parameters}} \quad x \mapsto \quad \overset{\text{function represented}}{\rho(A_1 x + b_1)}$$

with $\rho(x) = \max(0, x) = x_+$ the ReLU function

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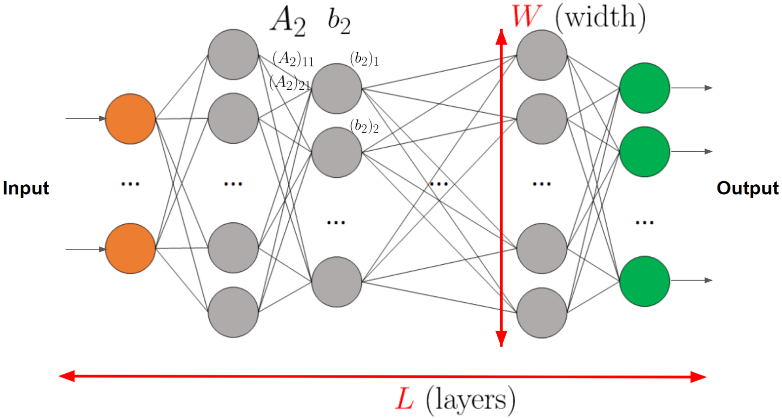
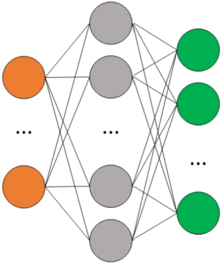


# Context: approximation with increasing complexity

**Consider:** $\Sigma_M$ set of networks with complexity increasing with $M$



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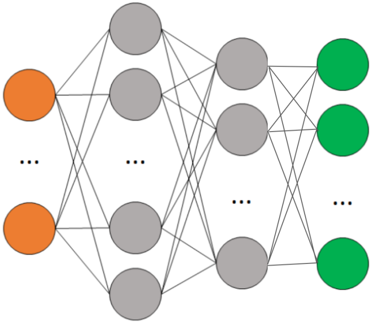

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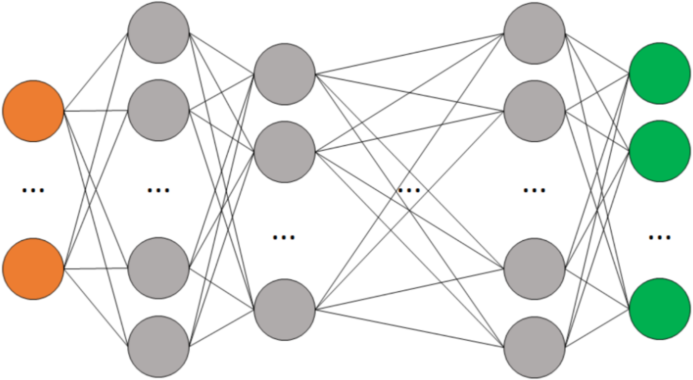

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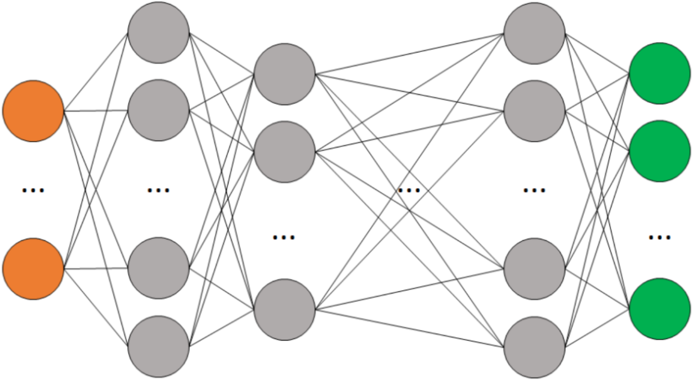

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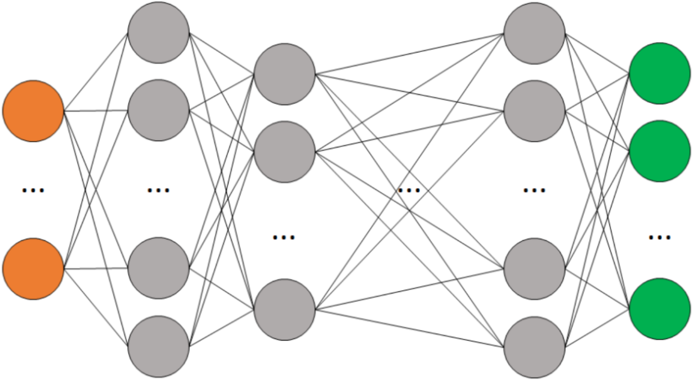


$$\textbf{complexity} \nearrow \Longrightarrow \textbf{approximation error} \searrow$$

**Typical known result:**

$$f \text{ "}\gamma\text{-smooth"} \Longrightarrow d(f, \Sigma_M) \lesssim M^{-\gamma}$$

# Challenges

**Definition:**

$$\gamma^{*\text{approx}}(\mathcal{C}|\Sigma) := \text{largest } \gamma > 0 \text{ s.t. } \sup_{f\in\mathcal{C}} d(f, \Sigma_M) \underset{M\to\infty}{=} \mathcal{O}(M^{-\gamma})$$

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**Question: Unified framework?**

# Part 1: quantization vs. approximation

# A key tool: bound on the Lipschitz constant of the parameterization

$$\|f - R_{Q(\theta)}\| \leqslant \|f - R_\theta\| + \underbrace{\|R_\theta - R_{Q(\theta)}\|}_{\textit{quantization error}}$$

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*Explicit bounds on $K_\Theta$?*

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**To understand the tradeoff: new explicit bounds in terms of the depth, the width and a bound on the weights of the network**

[1] Neural network approximation. R. DeVore et al. 2021.

# Contribution 1: explicit bounds on the Lipschitz parameterization of ReLU networks and its consequences

Under mild assumptions, there exists $c > 0$ s.t.

$$\boxed{\frac{1}{c} L B^{L-1} \leqslant K_{\Theta_{L,W}(B)} \leqslant cWL \times LB^{L-1}}$$

**Definition of $\Theta_{L,W}(B)$:**

- depth $= L \in \mathbb{N}^*$
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**Ingredient provided by our work:** bounds on $K_{\Theta_{L,W}(B)}$

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# Consequence 1: $\varepsilon$-accuracy with naive uniform quantization

**Fixed:** depth $L$, width $W$, bound $B$ on the parameters, desired quantization error $\varepsilon > 0$

**Look for:** smallest number of bits/weight to provide $\varepsilon$-error with $Q_\eta(x) = \lfloor x/\eta \rfloor \eta$ applied coordinatewise:

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Improvement of the multiplicative constant compared to a known result[2]

[2] Deep Neural Network Approximation Theory. D. Elbrächter et al. 2021.

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**Applying this recipe:** **Recovery of a known result**[3] in Sobolev spaces using an external ingredient [4]

[3] On the Universal Approximability and Complexity Bounds of Quantized ReLU Neural Networks. Y. Ding et al. 2019.

[4] Error bounds for approximations with deep ReLU networks. D. Yarotsky. 2017

# Consequence 3: approximation speed of quantized ReLU networks

**Consider:** Increasingly complex $\Sigma = (\Sigma_M)_{M \in \mathbb{N}}$ with weights in $\mathbb{R}$

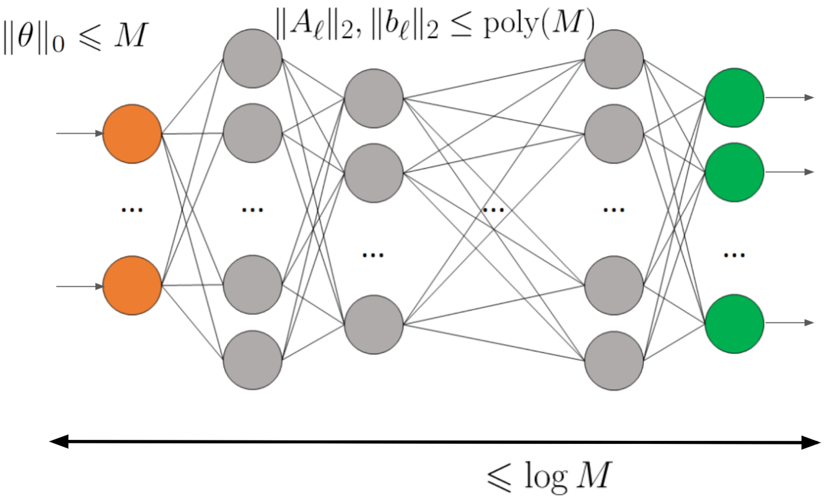
**Can we guarantee:**

$$\gamma^{*\text{approx}}(\mathcal{C}|\boldsymbol{Q(\Sigma)}) = \gamma^{*\text{approx}}(\mathcal{C}|\Sigma)\textbf{?}$$

Recall the definition:
$\gamma^{*\text{approx}}(\mathcal{C}|\Sigma) :=$ largest $\gamma > 0$ s.t. $\sup_{f \in \mathcal{C}} d(f, \Sigma_M) \underset{M \to \infty}{=} \mathcal{O}(M^{-\gamma})$

# Consequence 3: same approximation speed with quantized networks

$(\log M)^2$ **bits per weight is enough** for $\Sigma_M :=$ functions represented by a ReLU network:



## Part 2: on a relation between approximation- and information-theoretic quantities

# Challenge: unified framework for a known inequality

**Known:** in **certain situations** (arbitrary $C$, sequence $\Sigma$: dictionaries[5], ReLU neural networks[6])

$$\underbrace{\gamma^{*\mathsf{approx}}(\mathcal{C}|\Sigma)}_{\text{approximation theory}} \leqslant \underbrace{\gamma^{*\mathsf{encod}}(\mathcal{C})}_{\text{information theory}}$$

$\gamma^{*\mathsf{encod}}(\mathcal{C})$ measures how well $\mathcal{C}$ can be encoded into bit sequences

**Challenge:** **Unified framework?**

[5] Optimally sparse data representations. P. Grohs. 2015

[6] Deep Neural Network Approximation Theory. D. Elbrächter et al. 2021

# We need to control the increasing complexity

**Can we:** identify a property of $\Sigma$ to guarantee $\gamma^{*\text{approx}}(\mathcal{C}|\Sigma) \leqslant \gamma^{*\text{encod}}(\mathcal{C})$

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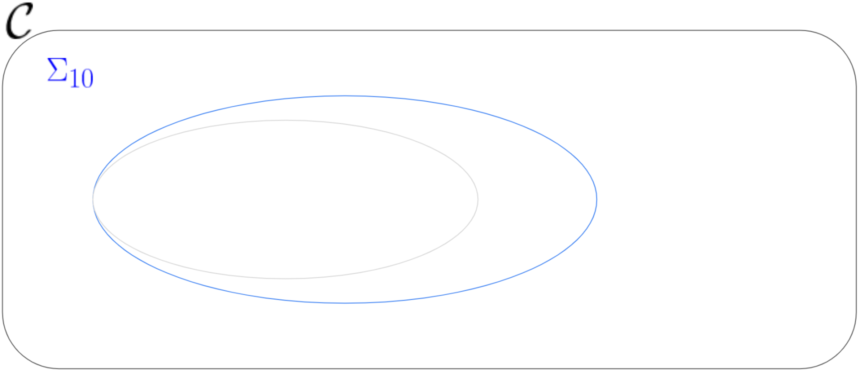

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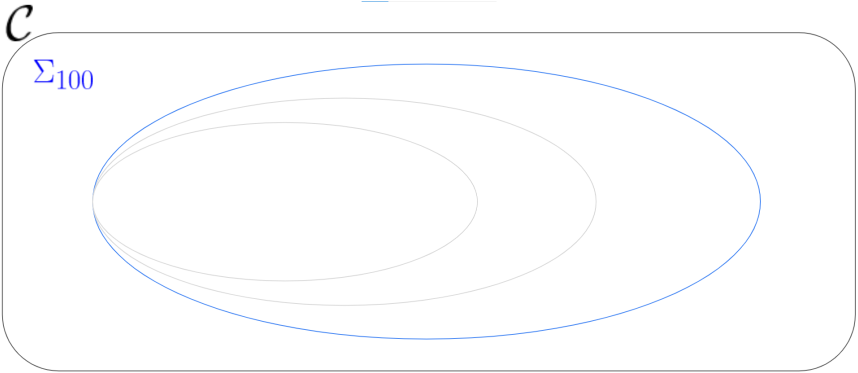


**Observation:** intuitively, if $\Sigma_M$ "grows" too fast then $\gamma^{*\text{approx}}(\mathcal{C}|\Sigma)$ may be unreasonably large

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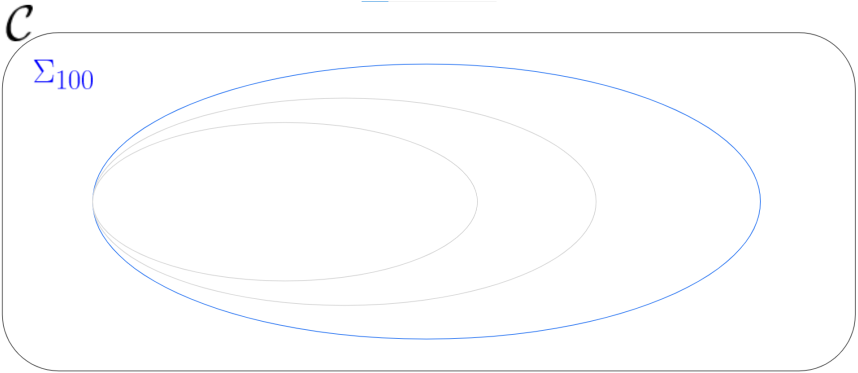


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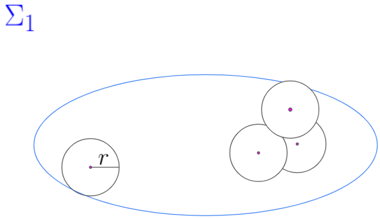
Extreme case : $\Sigma_1 = \cdots = \Sigma_M = \cdots = \mathcal{C}$ where $\gamma^{*\text{approx}}(\mathcal{C}|\Sigma) = \infty$

# Contribution 2: encodability property that allows unification

**A way to measure the "size" of $\Sigma_M$:** covering numbers



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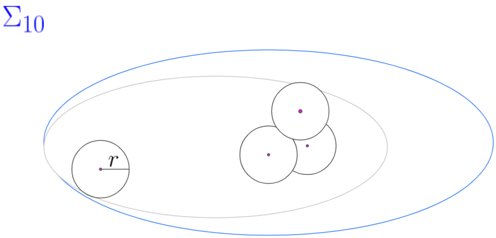

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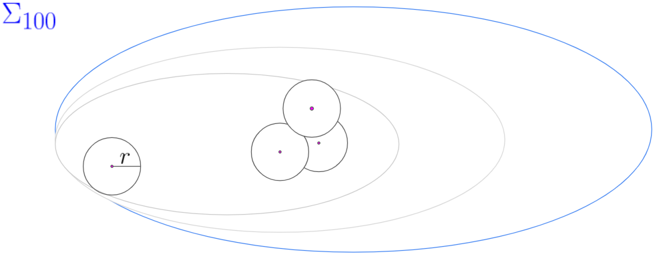

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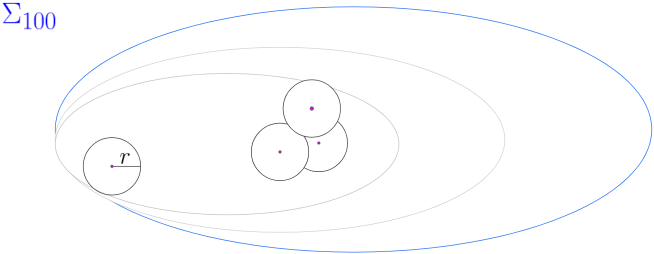


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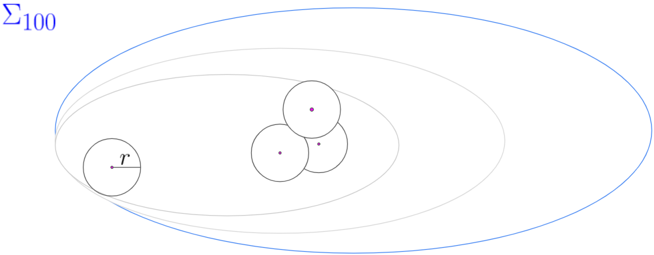


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**Unification with this encodability property**

# Contribution 2: recovery and generalization of $\gamma^{*\mathrm{approx}}(\mathcal{C}|\Sigma) \leqslant \gamma^{*\mathrm{encod}}(\mathcal{C})$

1. encodability property $\Longrightarrow \gamma^{*\mathrm{approx}}(\mathcal{C}|\Sigma) \leqslant \gamma^{*\mathrm{encod}}(\mathcal{C})$ for every $\mathcal{C}$

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2. satisfied for $\Sigma$: dictionaries, ReLU neural networks

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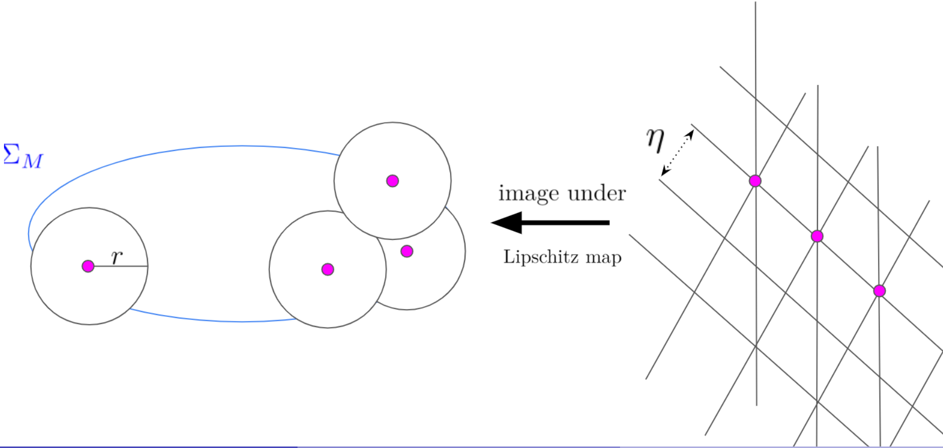
**Proof of the encodability property for ReLU networks:**



# Summary and perspectives

# Conclusion

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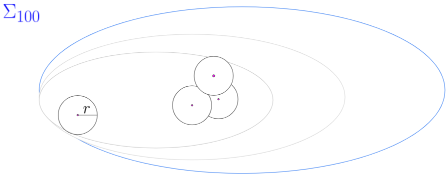

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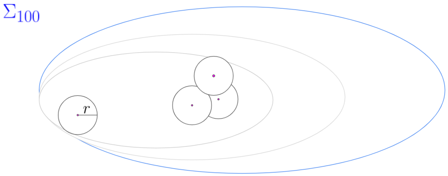


**New consequence: Unifies** situations where $\gamma^{*\text{approx}}(\mathcal{C}|\Sigma) \leqslant \text{complexity}(\mathcal{C})$

# Possible future directions on quantization

**Current theory = number of bits/weight must be linear in the depth:** if $Q_\eta(x) = \lfloor x/\eta \rfloor \eta$ gives $\varepsilon$-quantization error

$$\max_{\theta \in \Theta_{L,W}(B)} \max_{x \in [0,1]^d} \|R_\theta(x) - R_{Q_\eta(\theta)}(x)\|_p \leqslant \varepsilon$$

**Practice = 1 bit/weight is enough[7]:** quantization-aware training, same performance on MNIST for a 3 hidden-layers with 1024 neurons per layer

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**Thank you!**

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# Definition of the encoding speed

$$L(\varepsilon, \mathcal{C}) := \inf \Big\{ \ell \in \mathbb{N}, \exists E : \mathcal{C} \mapsto \{0,1\}^\ell, \\ \exists D : \{0,1\}^\ell \mapsto \mathcal{F}, \sup_{f \in \mathcal{C}} d\left(f, D(E(f))\right) \leqslant \varepsilon \Big\}$$

$$\gamma^{*\mathrm{encod}}(\mathcal{C}) := \sup \left\{ \gamma > 0, L(\varepsilon, \mathcal{C}) = \mathcal{O}_{\varepsilon \to 0}\left(\varepsilon^{-1/\gamma}\right) \right\}.$$

# Known approximation speeds[8]

| $\mathcal{C}$ := unit ball of | | $\Sigma$ | $\gamma^{*\text{approx}}(\mathcal{C}\|\Sigma) = \gamma^{*\text{encod}}(\mathcal{C})$ |
|---|---|---|---|
| $\alpha$-Hölder | $C^\alpha([0,1])$ | Wavelet basis | $\alpha$ |
| $L^p$-Sobolev[a] | $W_p^m([0,1]^d)$ | Wavelet frame | $\frac{m}{d}$ |
| Besov[b] | $B_{p,q}^m([0,1]^d)$ | Wavelet frame | $\frac{m}{d}$ |

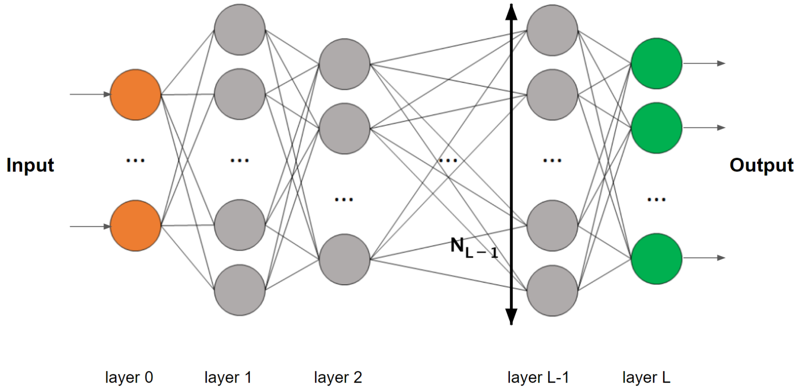
[a] where $p \in [1,\infty]$, $m > d\max(1/p - 1/2, 0)$
[b] where $p, q \in (0,\infty]$, $m > d\max(1/p - 1/2, 0)$

[8] Deep Neural Network Approximation Theory. D. Elbrächter et al. 2021.

# Notations : ReLU neural networks

**Architecture:** $(L, \mathbf{N})$ where

- $L \in \mathbb{N}$ is the number of layers : the depth
- $\mathbf{N} = (N_0, \ldots, N_L) \in \mathbb{N}^{L+1}$ with $N_\ell$ the width (number of neurons) of layer $\ell$



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**Realization of the network:** $\boxed{R_\theta : \mathbb{R}^{N_0} \to \mathbb{R}^{N_L} \text{ the realization of } \theta\text{:}}$

$$R_\theta(x) = A_L\rho(\cdots(A_2\rho(A_1x + b_1) + b_2)\cdots) + b_L \text{ avec } \rho(x) = \max(0, x).$$