অন্তরীকরণ

প্রশ্নমালা IX A

নিম্নের সীমাগুলির মান নির্ণয় কর ঃ

১। $\lim\limits_{x \to 2} \dfrac{x^2-4}{x^2-5x+6}$

সমাধান ঃ ধরি $x = 2 + h$. $\therefore$ $h \to 0$, যখন $x \to 2$

$$\lim_{x \to 2} \frac{x^2-4}{x^2-5x+6}$$

$$= \lim_{h \to 0} \frac{(2+h)^2-4}{(2+h)^2-5(2+h)+6}$$

$$= \lim_{h \to 0} \frac{4+4h+h^2-4}{4+4h+h^2-10-5h+6}$$

$$= \lim_{h \to 0} \frac{h(h+4)}{h(h-1)} = \lim_{h \to 0} \frac{h+4}{h-1}$$

$$= \frac{0+4}{0-1} = -4 \text{ (Ans.)}$$

বিকল্প পদ্ধতি ঃ $\lim\limits_{x \to 2} \dfrac{x^2-4}{x^2-5x+6}$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x \to 2} \frac{x+2}{x-3}$$

$$= \lim_{x \to 2} \frac{x+2}{x-3} = \frac{2+2}{2-3} = -4 \text{ (Ans.)}$$

1(b) $\lim\limits_{x \to 0} \dfrac{(x+4)^3-(x-8)^2}{x(x-3)}$

$$= \lim_{x \to 0} \frac{x^3+12x^2+48x+64-x^2+16x-64}{x(x-3)}$$

$$= \lim_{x \to 0} \frac{x^3+12x^2+48x+64-x^2+16x-64}{x(x-3)}$$

$$= \lim_{x \to 0} \frac{x^3+11x^2+64x}{x(x-3)}$$

$$= \lim_{x \to 0} \frac{x(x^2+11x+64)}{x(x-3)}$$

$$= \lim_{x \to 0} \frac{x^2+11x+64}{x-3} = \frac{0^2+11.0+64}{0-3}$$

$$= \frac{64}{-3} = -21\frac{1}{3} \text{ (Ans.)}$$

2(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+3x}-\sqrt{1-4x}}{x}$ [সি.'০৩]

$$= \lim_{x \to 0} \frac{(\sqrt{1+3x}-\sqrt{1-4x})(\sqrt{1+3x}+\sqrt{1-4x})}{x(\sqrt{1+3x}+\sqrt{1-4x})}$$

$$= \lim_{x \to 0} \frac{(\sqrt{1+3x})^2-(\sqrt{1-4x})^2}{x(\sqrt{1+3x}+\sqrt{1-4x})}$$

$$= \lim_{x \to 0} \frac{1+3x-1+4x}{x(\sqrt{1+3x}+\sqrt{1-4x})}$$

$$= \lim_{x \to 0} \frac{7x}{x(\sqrt{1+3x}+\sqrt{1-4x})}$$

$$= \lim_{x \to 0} \frac{7}{\sqrt{1+3x}+\sqrt{1-4x}}$$

$$= \frac{7}{\sqrt{1+3.0}+\sqrt{1-4.0}} = \frac{7}{1+1} = \frac{7}{2}$$

2(b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$ [ব. '০৯,'১৩]

$$= \lim_{x \to 0} \frac{(\sqrt{1+2x}-\sqrt{1-3x})(\sqrt{1+2x}+\sqrt{1-3x})}{x(\sqrt{1+2x}+\sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{(\sqrt{1+2x})^2-(\sqrt{1-3x})^2}{x(\sqrt{1+2x}+\sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{1+2x-1+3x}{x(\sqrt{1+2x}+\sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{5x}{x(\sqrt{1+2x}+\sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{5}{\sqrt{1+2x}+\sqrt{1-3x}}$$

$$= \frac{5}{\sqrt{1+2.0}+\sqrt{1-3.0}} = \frac{5}{1+1} = \frac{5}{2} \text{ (Ans.)}$$

2(c) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

$$= \lim_{x\to 0}\left\{\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\times\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}}\times\frac{\sqrt{1+x^3}+\sqrt{1+x}}{\sqrt{1+x^3}+\sqrt{1+x}}\right\}$$

$$= \lim_{x\to 0}\frac{(1+x^2-1-x)(\sqrt{1+x^3}+\sqrt{1+x})}{(1+x^3-1-x)(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \lim_{x\to 0}\frac{x(x-1)(\sqrt{1+x^3}+\sqrt{1+x})}{x(x^2-1)(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \lim_{x\to 0}\frac{(x-1)(\sqrt{1+x^3}+\sqrt{1+x})}{(x^2-1)(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \frac{(0-1)(\sqrt{1+0^3}+\sqrt{1+0})}{(0^2-1)(\sqrt{1+0^2}+\sqrt{1+0})} = \frac{2}{2}=1$$

3(a) $\displaystyle\lim_{x\to\infty}\frac{2x^4-3x^2+1}{6x^4+x^3-3x}$

$$= \lim_{x\to\infty}\frac{x^4(2-\frac{3}{x^2}+\frac{1}{x^4})}{x^4(6+\frac{1}{x}-\frac{3}{x^3})}$$

$$= \lim_{x\to\infty}\frac{2-\frac{3}{x^2}+\frac{1}{x^4}}{6+\frac{1}{x}-\frac{3}{x^3}} = \frac{2-0+0}{6+0-0} = \frac{2}{6}=\frac{1}{3}$$

3(b) $\displaystyle\lim_{x\to\infty}\frac{3^x-3^{-x}}{3^x+3^{-x}}$ **[চ.'০০]**

$$= \lim_{x\to\infty}\frac{3^x(1-\frac{1}{3^{2x}})}{3^x(1+\frac{1}{3^{2x}})} = \lim_{x\to\infty}\frac{1-\frac{1}{3^{2x}}}{1+\frac{1}{3^{2x}}}$$

$$= \frac{1-0}{1+0} = \frac{1-0}{1+0} = 1$$

3(c) $\displaystyle\lim_{x\to\infty}\{\ln(2x-1)-\ln(x+5)\}$ **[প্র.ভ.প.'০৪]**

$$= \lim_{x\to\infty}\ln\frac{2x-1}{x+5} = \lim_{x\to\infty}\ln\frac{x(2-\frac{1}{x})}{x(1+\frac{5}{x})}$$

$$= \lim_{x\to\infty}\ln\frac{2-\frac{1}{x}}{1+\frac{5}{x}} = \ln\frac{2-0}{1+0}$$

$= \ln 2$ (Ans.)

3.(d) $\displaystyle\lim_{x\to\infty}2^x\sin\frac{b}{2^x}$ **[সি.'০৫]**

ধরি, $\frac{b}{2^x}=\theta$. এখানে $x\to\infty$ বলে $2^x\to\infty$

$\theta=\frac{b}{2^x}\to 0$

$$\lim_{x\to\infty}2^x\sin\frac{b}{2^x} = \lim_{\theta\to 0}\frac{b}{\theta}\sin\theta$$

$$= b\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = b\,.\,1 = b$$

4.(a) $\displaystyle\lim_{x\to a}\frac{x^{7/2}-a^{7/2}}{\sqrt{x}-\sqrt{a}}$ **[ঢা.'০৩]**

$$= \frac{\lim_{x\to a}(x^{7/2}-a^{7/2})}{\lim_{x\to a}(x^{1/2}-a^{1/2})} = \frac{\lim_{x\to a}\frac{x^{7/2}-a^{7/2}}{x-a}}{\lim_{x\to a}\frac{x^{1/2}-a^{1/2}}{x-a}}$$

$$= \frac{\frac{7}{2}a^{\frac{7}{2}-1}}{\frac{1}{2}a^{\frac{1}{2}-1}} \quad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$= \left(\frac{7}{2}\times\frac{2}{1}\right)a^{\frac{7}{2}-1-\frac{1}{2}+1} = 7a^{\frac{7}{2}-\frac{1}{2}} = 7a^3 \text{ (Ans.)}$$

4(b) $\displaystyle\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{x^{3/5}-a^{3/5}}$

$$= \frac{\lim_{x\to a}(x^{5/2}-a^{5/2})}{\lim_{x\to a}(x^{3/5}-a^{3/5})} = \frac{\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{x-a}}{\lim_{x\to a}\frac{x^{3/5}-a^{3/5}}{x-a}}$$

$$= \frac{\frac{5}{2}a^{\frac{5}{2}-1}}{\frac{3}{5}a^{\frac{3}{5}-1}} \quad [\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}]$$

$$= (\frac{5}{2}\times\frac{5}{3})a^{\frac{5}{2}-1-\frac{3}{5}+1} = \frac{25}{6}a^{\frac{5}{2}-\frac{3}{5}}$$

$$= \frac{25}{6}a^{\frac{25-6}{10}} = \frac{25}{6}a^{\frac{19}{10}} \text{ (Ans.)}$$

5(a) $\lim\limits_{x\to 0}\frac{1-\cos 3x}{3x^2}$ [প্র.ভ.প.'৮৫]

$$= \lim_{x\to 0}\frac{2\sin^2\frac{3x}{2}}{3x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{3x}{2}}{\frac{9x^2}{4}.\frac{4}{3}}$$

$$= \frac{2.3}{4}\lim_{x\to 0}\left\{\frac{\sin(3x/2)}{3x/2}\right\}^2 = \frac{3}{2}.1 = \frac{3}{2}$$

5.(b) $\lim\limits_{x\to 0}\frac{1-\cos 7x}{3x^2}$ [সি.'০৮,'১২;কু.'১১; রা.'০৭,'১০; চ.'০৬; য.'০৮,'১২; ব.'০৮; ঢা.'১০; দি.'১১]

$$= \lim_{x\to 0}\frac{2\sin^2\frac{7x}{2}}{3.\frac{49x^2}{4}.\frac{4}{49}}$$

$$= (\frac{2}{3}\times\frac{49}{4})\lim_{x\to 0}\left\{\frac{\sin(7x/2)}{7x/2}\right\}^2$$

$$= \frac{49}{6}.1 = \frac{49}{6} \text{ (Ans.)}$$

6. (a) $\lim\limits_{x\to 0}\frac{\cos 2x-\cos 3x}{x^2}$

[ব.'০১; মা.'০৫ সি.'০৪]

$$= \lim_{x\to 0}\frac{2\sin\frac{1}{2}(2x+3x)\sin\frac{1}{2}(3x-2x)}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{5x}{2}\sin\frac{x}{2}}{x^2}$$

$$= 2\lim_{x\to 0}\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\times\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\times\frac{5}{2}\times\frac{1}{2}$$

$$= 2\times 1\times\frac{5}{4} = \frac{5}{2}\text{(Ans.)}$$

6(b) $\lim\limits_{x\to 0}\frac{\cos 2x-\cos 4x}{x^2}$ [কু.'০৩]

$$= \lim_{x\to 0}\frac{2\sin\frac{1}{2}(2x+4x)\sin\frac{1}{2}(4x-2x)}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin 3x\sin x}{x^2}$$

$$= 2.\lim_{x\to 0}\frac{\sin 3x}{3x}\times\lim_{x\to 0}\frac{\sin x}{x}\times 3$$

$$= 2\times 1\times 1\times 3 = 6 \text{ (Ans.)}$$

6. (c) $\lim\limits_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$ [ব'১২; য.'১৩]

$$= \lim_{x\to 0}\frac{2\sin\frac{1}{2}(ax+bx)\sin\frac{1}{2}(bx-ax)}{x^2}$$

$$= 2\lim_{x\to 0}\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}\times\frac{a+b}{2}\times \lim_{x\to 0}\frac{\sin\frac{(b-a)x}{2}}{\frac{(b-a)x}{2}}\times\frac{b-a}{2}$$

$$= 2\times 1\times\frac{a+b}{2}\times 1\times\frac{b-a}{2} = \frac{1}{2}(b^2-a^2)$$

6(d) $\lim\limits_{x\to 0}\frac{1-2\cos x+\cos 2x}{x^2}$

[য.'০৫; কু.'১৪]

$$= \lim_{x\to 0}\frac{1-2\cos x+2\cos^2 x-1}{x^2}$$

$$= \lim_{x\to 0}\frac{2\cos x(\cos x-1)}{x^2}$$

$$= \lim_{x \to 0} \frac{2\cos x(-2\sin^2 \frac{x}{2})}{x^2}$$

$$= -4 \lim_{x \to 0} \left\{\frac{\sin(x/2)}{x/2}\right\} \times \frac{1}{4} \times \lim_{x \to 0} \cos x$$

$$= -4 \times 1 \times \frac{1}{4} \times \cos 0 = -1 \times 1 = -1$$

6(e) $\lim\limits_{x \to 0} \dfrac{x(\cos x + \cos 2x)}{\sin x}$ [য.'০৯; রা.'১১; চ,'১৩]

$$= \lim_{x\to 0} \frac{x}{\sin x} \times \lim_{x\to 0} (\cos x + \cos 2x)$$

$$= 1 \times (\cos 0 + \cos 0)$$

$$= 1 + 1 = 1 \text{ (Ans.)}$$

7.(a) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$ [রা.'০৯; ব.'১১, '১৪; কু.'১০; সি.'০৯; মা.'১৩]

$$= \lim_{x\to 0} \frac{\tan x(1-\cos x)}{x^3} = \lim_{x\to 0} \frac{\tan x . 2\sin^2 \frac{x}{2}}{x^3}$$

$$= 2 \lim_{x\to 0} \frac{\tan x}{x} \times \lim_{x\to 0} \left\{\frac{\sin(x/2)}{x/2}\right\}^2 \times \frac{1}{4}$$

$$= 2 \times 1 \times 1 \times \frac{1}{4} = \frac{1}{2} \text{ (Ans.)}$$

7(b) $\lim\limits_{x \to 0} \dfrac{\tan 2x - \sin 2x}{x^3}$ [মা.'০৪,'০৭]

$$= \lim_{x\to 0} \frac{\tan 2x(1-\cos 2x)}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan 2x . 2\sin^2 x}{x^3}$$

$$= 2 \lim_{x\to 0} \frac{\tan 2x}{2x} \times 2 \times \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2$$

$$= 2 \times 1 \times 2 \times 1 = 4 \text{ (Ans.)}$$

7(c) $\lim\limits_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$ [ঢা.'০৯]

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x\to 0} \frac{1-\cos x}{x \sin x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{x . 2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x\to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2} \text{ (Ans.)}$$

7(d) $\lim\limits_{x \to y} \dfrac{\sin x - \sin y}{x - y}$ [কু.'০৫]

$$= \lim_{x\to y} \frac{2\sin\frac{x-y}{2}\cos\frac{x+y}{2}}{x-y}$$

$$= 2. \lim_{x\to y} \frac{\sin\frac{x-y}{2}}{\frac{x-y}{2}} \times \frac{1}{2} \times \lim_{x\to y} \cos\frac{x+y}{2}$$

$$= 2 \times 1 \times \frac{1}{2} \cos\frac{y+y}{2} = \cos y \text{ (Ans.)}$$

7(e) $\lim\limits_{x\to\alpha} \dfrac{\tan x - \tan\alpha}{x-\alpha} = \lim\limits_{x\to\alpha} \dfrac{\frac{\sin x}{\cos x} - \frac{\sin\alpha}{\cos\alpha}}{x-\alpha}$

$$= \lim_{x\to\alpha} \frac{\sin x\cos\alpha - \cos x\sin\alpha}{(x-\alpha)\cos x\cos\alpha}$$

$$= \lim_{x\to\alpha} \frac{\sin(x-\alpha)}{(x-\alpha)\cos x\cos\alpha}$$

$$= \frac{1}{\cos\alpha} \lim_{(x-\alpha)\to 0} \frac{\sin(x-\alpha)}{x-\alpha} \times \lim_{x\to\alpha} \frac{1}{\cos x}$$

$$= \frac{1}{\cos\alpha} \times 1 \times \frac{1}{\cos\alpha} = \sec^2\alpha \text{ (Ans.)}$$

8.(a) $\lim\limits_{x \to 0} \dfrac{\tan ax}{\sin bx}$ [ঢা.'০৬]

$$= \lim_{x\to 0} \frac{\tan ax}{\sin bx} = \frac{\lim\limits_{ax\to 0} \frac{\tan ax}{ax} \times a}{\lim\limits_{bx\to 0} \frac{\sin bx}{bx} \times b}$$

$$= \frac{1 \times a}{1 \times b} = \frac{a}{b} \text{ (Ans.)}$$

8(b) $\lim_{x\to 0} \dfrac{1-\cos ax}{1-\cos bx}$ [চ.'০১]

$= \lim_{x\to 0} \dfrac{2\sin^2 \frac{ax}{2}}{2\sin^2 \frac{bx}{2}}$

$= \dfrac{\lim_{x\to 0}\left\{\dfrac{\sin(ax/2)}{ax/2}\right\}^2 \times \dfrac{a^2}{4}}{\lim_{x\to 0}\left\{\dfrac{\sin(bx/2)}{bx/2}\right\}^2 \times \dfrac{b^2}{4}} = \dfrac{1\times\dfrac{a^2}{4}}{1\times\dfrac{b^2}{4}} = \dfrac{a^2}{b^2}$

8(c) $\lim_{x\to 0} \dfrac{\cos 7x - \cos 9x}{\cos 3x - \cos 5x}$ [ঢা.'০৫; কু.'০৭]

$= \lim_{x\to 0} \dfrac{2\sin\frac{1}{2}(7x+9x)\sin\frac{1}{2}(9x-7x)}{2\sin\frac{1}{2}(3x+5x)\sin\frac{1}{2}(5x-3x)}$

$= \lim_{x\to 0} \dfrac{\sin 8x \sin x}{\sin 4x \sin x} = \lim_{x\to 0} \dfrac{2\sin 4x \cos 4x}{\sin 4x}$

$= 2\lim_{x\to 0} \cos 4x = 2\cos 0 = 2 . 1 = 2$

8(d) $\lim_{x\to 0} \dfrac{\sin 7x - \sin x}{\sin 6x}$ [চ.,মা.'০৩; দি.'১২]

$= \lim_{x\to 0} \dfrac{2\sin\frac{1}{2}(7x-x)\cos\frac{1}{2}(7x+x)}{\sin 6x}$

$= \lim_{x\to 0} \dfrac{2\sin 3x \cos 4x}{2\sin 3x \cos 3x} = \lim_{x\to 0} \dfrac{\cos 4x}{\cos 3x}$

$= \dfrac{\cos 0}{\cos 0} = \dfrac{1}{1} = 1$ (Ans.)

8(e) $\lim_{x\to \frac{\pi}{2}} \{\sec x(\sec x - \tan x)\}$ [ঢা.'০৭]

$= \lim_{x\to \frac{\pi}{2}} \dfrac{1}{\cos x}\left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right)$

$= \lim_{x\to \frac{\pi}{2}} \dfrac{1-\sin x}{\cos^2 x} = \lim_{x\to \frac{\pi}{2}} \dfrac{1-\sin x}{1-\sin^2 x}$

$= \lim_{x\to \frac{\pi}{2}} \dfrac{1-\sin x}{(1-\sin x)(1+\sin x)}$

$= \lim_{x\to \frac{\pi}{2}} \dfrac{1}{1+\sin x} = \dfrac{1}{1+\sin\frac{\pi}{2}} = \dfrac{1}{1+1} = \dfrac{1}{2}$

8. (f) $\lim_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right)$

[চ.'০৯; ব.'১০; সি.'১৪; প্র.ভ.প.'০৪]

$= \lim_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim_{x\to 0} \dfrac{1-\cos x}{\sin x}$

$= \lim_{x\to 0} \dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x\to 0} \tan\dfrac{x}{2}$

$= \tan\dfrac{0}{2} = \tan 0 = 0$ (Ans.)

8(g) $\lim_{\theta\to 0} \dfrac{1}{\theta}\left(\dfrac{1}{\sin\theta} - \dfrac{1}{\tan\theta}\right)$ [ঢা.'০১; রা.'১৩]

$= \lim_{\theta\to 0} \dfrac{1}{\theta}\left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right) = \lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta\sin\theta}$

$= \lim_{\theta\to 0} \dfrac{2\sin^2\frac{\theta}{2}}{\theta . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \lim_{x\to 0} \dfrac{\tan\frac{\theta}{2}}{\theta}$

$= \lim_{\theta\to 0} \dfrac{\tan\frac{\theta}{2}}{\frac{\theta}{2}} \times \dfrac{1}{2} = 1\times\dfrac{1}{2} = \dfrac{1}{2}$ (Ans.)

8(h) $\lim_{x\to 0} \dfrac{1+\sin x}{\cos x}$ [রা.'০৪]

$= \dfrac{1+\sin 0}{\cos 0} = \dfrac{1+0}{1} = 1$ (Ans.)

9(a) $\lim_{x\to 0} \dfrac{\sin 2x}{2x^2+x}$ [চ.'০২]

$= \lim_{x\to 0} \dfrac{\sin 2x}{x(2x+1)} = \dfrac{\lim_{x\to 0}\dfrac{\sin 2x}{2x}\times 2}{\lim_{x\to 0}(2x+1)}$

$= \frac{1\times 2}{2\times 0+1} = 2$ (Ans.)

9(b) $\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x^2\to 0} \frac{\sin x^2}{x^2}\times \lim_{x\to 0} x$

$= 1\times 0 = 0$ (Ans.)

10.(a) $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$

[য. '০৪; ব. '০৬; ঢা.'১৩ রা.'১৪]

ধরি, $x = \frac{\pi}{2} + h$. $x\to \frac{\pi}{2}$ $h\to 0$

$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{h\to 0} \frac{1-\sin(\frac{\pi}{2}+h)}{\cos(\frac{\pi}{2}+h)}$

$= \lim_{h\to 0} \frac{1-\cos h}{-\sin h} = \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{-2\sin\frac{h}{2}\cos\frac{h}{2}}$

$= -\lim_{h\to 0} \tan\frac{h}{2} = -\tan\frac{0}{2} = -\tan 0 = 0$

10(b) $\lim_{x\to \frac{\pi}{2}} (\frac{\pi}{2}-x)\tan x$ [চ.'১০]

ধরি, $\frac{\pi}{2} - x = h$. $x\to \frac{\pi}{2}$ $h\to 0$

$\lim_{x\to \frac{\pi}{2}} (\frac{\pi}{2}-x)\tan x$

$= \lim_{x\to \frac{\pi}{2}} h\tan(\frac{\pi}{2}-h) = \lim_{h\to 0} h\cot h$

$= \lim_{h\to 0} \frac{h}{\tan h} = 1$

10(c) $\lim_{x\to \frac{\pi}{2}} \frac{\sec x - \tan x}{\frac{\pi}{2}-x}$ [ব.'০২]

ধরি, $\frac{\pi}{2} - x = h$. $x\to \frac{\pi}{2}$ $h\to 0$

$\lim_{x\to \frac{\pi}{2}} \frac{\sec x - \tan x}{\frac{\pi}{2}-x}$

$= \lim_{h\to 0} \frac{\sec(\frac{\pi}{2}-h) - \tan(\frac{\pi}{2}-h)}{h}$

$= \lim_{h\to 0} \frac{\operatorname{cosec} h - \cot h}{h} = \lim_{h\to 0} \frac{\frac{1}{\sin h} - \frac{\cos h}{\sin h}}{h}$

$= \lim_{h\to 0} \frac{1-\cos h}{h\sin h} = \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{h.2\sin\frac{h}{2}\cos\frac{h}{2}}$

$= \lim_{h\to 0} \frac{\tan\frac{h}{2}}{\frac{h}{2}}\times\frac{1}{2} = 1\times\frac{1}{2} = \frac{1}{2}$ (Ans.)

10(d) $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{(\frac{\pi}{2}-x)^2}$ [য.'০৬,'১০; কু.'০৮]

ধরি, $\frac{\pi}{2} - x = h$. $x\to \frac{\pi}{2}$ $h\to 0$

$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{(\frac{\pi}{2}-x)^2} = \lim_{h\to 0} \frac{1-\sin(\frac{\pi}{2}-h)}{h^2}$

$= \lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{2\sin^2(h/2)}{(h/2)^2\times 4}$

$= \frac{1}{2}\lim_{h\to 0} \left\{\frac{\sin(h/2)}{h/2}\right\}^2 = \frac{1}{2}\times 1 = \frac{1}{2}$ (Ans.)

11.(a) $\lim_{x\to 0} \frac{\sin^{-1}x}{x}$

ধরি, $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$

$x\to 0$ $\theta\to 0$

$\lim_{x\to 0} \frac{\sin^{-1}x}{x} = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$

11(b) $\lim\limits_{x\to 0}\dfrac{\sin^{-1}(3x)}{4x}$

ধরি, $\sin^{-1}(3x)=\Theta \Rightarrow \sin\Theta = 3x$

$x\to 0 \qquad \theta\to 0$

$$\lim_{x\to 0}\frac{\sin^{-1}(3x)}{4x}=\lim_{\theta\to 0}\frac{\theta}{\frac{4}{3}\sin\theta}$$

$$=\frac{3}{4}\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=\frac{3}{4}\times 1=\frac{3}{4} \text{ (Ans.)}$$

12. (a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-(1+x)^7}{\ln(1+x)}$

$$=\lim_{x\to 0}\frac{\{1+2x+\frac{(2x)^2}{2!}\cdots\}-(1+7x+21x^2+\cdots)}{x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots}$$

$$=\lim_{x\to 0}\frac{(2-7)x+(2-21)x^2+\cdots}{x(1-\frac{x}{2}+\frac{x^2}{3}-\cdots)}$$

$$=\lim_{x\to 0}\frac{-5-19x+\cdots}{1-\frac{x}{2}+\frac{x^2}{3}-\cdots}$$

$$=\frac{-5-19\times 0+0+\cdots}{1-\frac{0}{2}+\frac{0^2}{3}-0+\cdots}$$

$$=\frac{-5}{1}=-5 \text{ (Ans.)}$$

12(b) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$

$$=\lim_{x\to 0}\frac{\{1+x\ln a+\frac{(x\ln a)^2}{2!}+\cdots\}-1}{x}$$

$$=\lim_{x\to 0}\frac{x\{\ln a+\frac{x(\ln a)^2}{2!}+\frac{x^2(\ln a)^3}{3!}+\cdots\}}{x}$$

$$=\lim_{x\to 0}\{\ln a+\frac{x(\ln a)^2}{2!}+\frac{x^2(\ln a)^3}{3!}+\cdots\}$$

$$=\ln a+\frac{0\times(\ln a)^2}{2!}+\frac{0^2(\ln a)^3}{3!}+\cdots\cdots\}$$

$$=\ln a$$

12(c) $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\sin x}$ [কু.'০১; মা.বো.'০৯; রা.'১২]

$$=\lim_{x\to 0}\frac{\{1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\cdots)-1}{\sin x}$$

$$=\lim_{x\to 0}\frac{\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\cdots}{\sin x}$$

$$=\lim_{x\to 0}(1+\frac{\sin x}{2!}+\frac{\sin^2 x}{3!}+\cdots)$$

$$=1+\frac{\sin 0}{2!}+\frac{\sin^2 0}{2!}+\cdots=1+0+0\cdots$$

$$=1$$

12(d) $\lim\limits_{x\to 0}\dfrac{a^x-a^{-x}}{x}$ [প্র.ভ.প.'০৬]

$$=\lim_{x\to 0}\frac{1}{x}[\{1+x\ln a+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{3!}+\cdots\}$$
$$-\{1-x\ln a+\frac{(x\ln a)^2}{2!}-\frac{(x\ln a)^3}{3!}+\cdots\}]$$

$$=\lim_{x\to 0}\frac{1}{x}\{2x\ln a+2\frac{(x\ln a)^3}{3!}+\cdots\cdots\}$$

$$=2\lim_{x\to 0}\{\ln a+\frac{x^2(\ln a)^3}{3!}+\frac{x^4(\ln a)^5}{5!}+\cdots\}$$

$$=2\lim_{x\to 0}\{\ln a+\frac{0^2(\ln a)^3}{3!}+\frac{0^4(\ln a)^5}{5!}+\cdots\}$$

$$=2\ln a \text{ (Ans.)}$$

12(e) $\lim\limits_{x\to\infty}(1+\dfrac{b}{x})^{\frac{x}{a}}, a>0, b>0$

$$=\lim_{x\to\infty}(1+\frac{b}{x})^{\frac{x}{a}}$$

$$=\lim_{x\to\infty}\{1+\frac{\frac{x}{a}}{1!}.\frac{b}{x}+\frac{\frac{x}{a}(\frac{x}{a}-1)}{2!}(\frac{b}{x})^2+$$

$$+\frac{\frac{x}{a}(\frac{x}{a}-1)(\frac{x}{a}-2)}{3!}(\frac{x}{a})^3+\cdots\cdots\}$$

$$=\lim_{x\to\infty}\{1+\frac{b}{a}+\frac{\frac{x^2}{a^2}(1-\frac{a}{x})}{2!}\frac{b^2}{x^2}+\frac{\frac{x^3}{a^3}(1-\frac{a}{x})(1-\frac{2a}{x})}{3!}\frac{b^3}{x^3}+\cdots\}$$

$$=\lim_{x\to\infty}\{1+\frac{b}{a}+\frac{1-\frac{a}{x}}{2!}\frac{b^2}{a^2}+\frac{(1-\frac{a}{x})(1-\frac{2a}{x})}{3!}\frac{b^3}{a^3}+\cdots\}$$

$$=1+\frac{b}{a}+\frac{1-0}{2!}\frac{b^2}{a^2}+\frac{(1-0)(1-0)}{3!}\frac{b^3}{a^3}+\cdots$$

$$=1+\frac{b}{a}+\frac{1}{2!}(\frac{b}{a})^2+\frac{1}{3!}(\frac{b}{a})^3+\cdots\cdots=e^{\frac{b}{a}}$$

12(i) $f(x)=\sin x$ হলে, $\lim\limits_{h\to 0}\frac{f(x+nh)-f(x)}{h}$ এর মান নির্ণয় কর। [প্র.ভ.প.'০০]

$$\lim_{h\to 0}\frac{f(x+nh)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+nh)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\frac{nh}{2}\cos\frac{1}{2}(2x+nh)}{h}$$

$$=2\lim_{h\to 0}\frac{\sin\frac{nh}{2}}{\frac{nh}{2}}\times\frac{n}{2}\lim_{h\to 0}\cos\frac{1}{2}(2x+nh)$$

$$=2\times 1\times\frac{n}{2}\times\cos\frac{1}{2}(2x+n\times 0)$$

$$=n\cos x \text{ (Ans.)}$$

13. (a) $\lim\limits_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}$

$$=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$=\lim_{n\to\infty}\frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6n^3}$$

$$=\lim_{n\to\infty}\frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6}=\frac{(1+0)(2+0)}{6}$$

$$=\frac{2}{6}=\frac{1}{3}\text{ (Ans.)}$$

13(b) $\lim\limits_{n\to\infty}\frac{1}{n^4}\sum\limits_{r=1}^{n}r^3$

$$=\lim_{n\to\infty}\frac{1}{n^4}(1^3+2^3+3^3+\cdots+n^3)$$

$$=\lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4}=\lim_{n\to\infty}\frac{n^4(1+\frac{1}{n})^2}{4n^4}$$

$$=\lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{4}=\frac{(1+0)^2}{4}=\frac{1}{4}\text{(Ans.)}$$

13(c) $\lim\limits_{n\to\infty}\frac{1.3+2.4+\cdots\cdots+n(n+2)}{n^3}$

সমাধান : মনে করি, $1.3+2.4+\cdots+n(n+2)$ ধারার nতম পদ u_n.

$$u_n=n(n+2)=n^2+2n$$

$$1.3+2.4+\cdots+n(n+2)=\sum_{n=1}^{n}n^2+2\sum_{n=1}^{n}n$$

$$=\frac{n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}$$

$$=n(n+1)(\frac{2n+1}{6}+1)$$

$$=n(n+1)\frac{2n+1+6}{6}=\frac{n(n+1)(n+7)}{3}$$

$$\lim_{n\to\infty}\frac{1.3+2.4+\cdots\cdots+n(n+2)}{n^3}$$

$$=\lim_{n\to\infty}\frac{n(n+1)(n+7)}{6n^3}$$

$$= \lim_{n\to\infty} \frac{n^3(1+\frac{1}{n})(1+\frac{6}{n})}{6n^3}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{2}{n})}{6} = \frac{(1+0)(1+0)}{6}$$

$$= \frac{1}{6} \text{ (Ans.)}$$

14. যদি $f(x) = \dfrac{2x}{1-x}$ হয়, তবে (i) $\lim\limits_{x\to 1+} f(x)$ এবং $\lim\limits_{x\to 1-} f(x)$ এর মান নির্ণয় কর।

সমাধান ঃ ধরি $x = 1 + h$

$$\therefore \lim_{x\to 1+} f(x) = \lim_{h\to 0+} \frac{2(1+h)}{1-(1+h)} = \lim_{h\to 0+} \frac{2+2h}{1-1-h}$$

$$= \lim_{h\to 0+} \frac{2+2h}{-h} = \lim_{h\to 0+} (-\frac{2}{h}-2)$$

$$= -\infty - 2 = -\infty \text{ (Ans.)}$$

$$\lim_{x\to 1-} f(x) = \lim_{h\to 0-} \frac{2(1+h)}{1-(1+h)} = \lim_{h\to 0-} \frac{2+2h}{1-1-h}$$

$$= \lim_{h\to 0-} \frac{2+2h}{-h} = \lim_{h\to 0-} (-\frac{2}{h}-2)$$

$$= +\infty - 2 = +\infty \text{ (Ans.)}$$

(ii) $\lim\limits_{x\to\infty} f(x)$ এবং $\lim\limits_{x\to -\infty} f(x)$ এর মান নির্ণয় কর।

সমাধান ঃ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2x}{1-x}$

$$= \lim_{x\to\infty} \frac{2x}{x(\frac{1}{x}-1)} = \lim_{x\to\infty} \frac{2}{\frac{1}{x}-1}$$

$$= \frac{2}{0-1} = -2 \text{ (Ans.)}$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{2x}{1-x}$$

$$= \lim_{x\to -\infty} \frac{2x}{x(\frac{1}{x}-1)} = \lim_{x\to -\infty} \frac{2}{\frac{1}{x}-1}$$

$$= \frac{2}{-0-1} = -2 \text{ (Ans.)}$$

15. স্যান্ডউইচ উপপাদ্যের সাহায্যে মান নির্ণয় কর:

(a) $\lim\limits_{x\to 0} x^2 \sin(\frac{1}{x})$

সমাধান: আমরা পাই, $-1 \le \sin(\frac{1}{x}) \le 1, x \ne 0$

এবং $x^2 \ge 0$

$$-x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$$

এখন, $\lim\limits_{x\to 0} (-x^2) = -0^2 = 0$ অদ্রূপ, $\lim\limits_{x\to 0} x^2 = 0$

স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই,

$$\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$$

(b) $\lim\limits_{x\to 0} x \sin(\frac{1}{x})$

$x \ne 0$ এর জন্য আমরা পাই, $-1 \le \sin(\frac{1}{x}) \le 1$

$x>0$ এর জন্য, $-x \le x \sin(\frac{1}{x}) \le x$ এবং

$x < 0$ এর জন্য, $-x \ge x \sin(\frac{1}{x}) \ge x$

$$\Rightarrow x \le x \sin(\frac{1}{x}) \le -x$$

যেহেতু, $\lim\limits_{x\to 0} (-x) = 0 = \lim\limits_{x\to 0} x$, সুতরাং স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই, $\lim\limits_{x\to 0} x\sin(\frac{1}{x}) = 0$

(c) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

সমাধান ঃ আমরা পাই, $-1 \le \sin x \le 1$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}, [\because x \to \infty, \therefore x>0]$$

এখন, $\lim_{x\to\infty}(-\frac{1}{x})=0$ এবং $\lim_{x\to\infty}(\frac{1}{x})=0$

স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই,

$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

15. (d) $\lim_{x\to\infty}\frac{2-\cos x}{x+3}$

সমাধান : আমরা পাই, $-1\le\cos x\le+1$

$\Rightarrow +1\ge-\cos x\ge-1$, [উভয় পক্ষকে (-1) দ্বারা গুণ করে।]

$\Rightarrow -1\le-\cos x\le+1$

$\Rightarrow 2-1\le2-\cos x\le2+1$

$\Rightarrow \frac{1}{x+3}\le\frac{2-\cos x}{x+3}\le\frac{3}{x+3}$

$[\because x\to\infty, \therefore x+3>0]$

যেহেতু $\lim_{x\to\infty}\frac{1}{x+3}=0=\lim_{x\to\infty}\frac{3}{x+3}$, স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই, $\lim_{x\to\infty}\frac{2-\cos x}{x+3}=0$

15. (e) $\lim_{x\to\infty}\frac{\cos^2(2x)}{3-2x}$

সমাধান : আমরা পাই, $-1\le\cos(2x)\le+1$

$\Rightarrow 0\le\cos^2(2x)\le1$

$\Rightarrow \frac{0}{3-2x}\ge\frac{\cos^2(2x)}{3-2x}\ge\frac{1}{3-2x}$

$[\because x\to\infty, \therefore 3-2x>0]$

$\Rightarrow \frac{1}{3-2x}\le\frac{\cos^2(2x)}{3-2x}\le\frac{0}{3-2x}$

যেহেতু $\lim_{x\to\infty}\frac{1}{3-2x}=0=\lim_{x\to\infty}0$, স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই, $\lim_{x\to\infty}\frac{\cos^2(2x)}{3-2x}=0$

15. (f) $\lim_{x\to0^-}x^3\cos(\frac{2}{x})$

সমাধান : আমরা পাই, $-1\le\cos(\frac{2}{x})\le+1$

$\Rightarrow -x^3\ge x^3\cos(\frac{2}{x})\ge+x^3$

$[\because x\to0^-, \therefore x^3<0]$

$\Rightarrow x^3\le x^3\cos(\frac{2}{x})\le-x^3$

যেহেতু $\lim_{x\to0^-}x^3=0=\lim_{x\to0^-}(-x^3)$, স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই, $\lim_{x\to0^-}x^3\cos(\frac{2}{x})=0$

15. (g) $\lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100}$

সমাধান : আমরা পাই, $-1\le\sin x\le+1$

$\Rightarrow 0\le\sin^2 x\le1\Rightarrow 2\le2+\sin^2 x\le3$

$\Rightarrow 2x^2\le x^2(2+\sin^2 x)\le3x^2$

$\Rightarrow \frac{2x^2}{x+100}\le\frac{x^2(2+\sin^2 x)}{x+100}\le\frac{3x^2}{x+100}$

$[\because x\to\infty, \therefore x+100>0]$

এখন, $\lim_{x\to\infty}\frac{2x^2}{x+100}=\lim_{x\to\infty}\frac{2x^2}{x(1+\frac{100}{x})}$

$=\lim_{x\to\infty}\frac{2x}{1+\frac{100}{x}}=\frac{2\times\infty}{1+0}=\infty$

তদ্রূপ, $\lim_{x\to\infty}\frac{3x^2}{x+100}=\infty$

স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই,

$\lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100}=\infty$ (বিদ্যমান নাই)

15. (h) $\lim\limits_{x\to-\infty} \dfrac{5x^2-\sin(3x)}{x^2+10}$

সমাধান ঃ আমরা পাই, $-1 \le \sin(3x) \le +1$

$\Rightarrow +1 \ge -\sin(3x) \ge -1$

$\Rightarrow -1 \le -\sin(3x) \le +1$

$\Rightarrow 5x^2-1 \le 5x^2-\sin(3x) \le 5x^2+1$

$\Rightarrow \dfrac{5x^2-1}{x^2+10} \ge \dfrac{5x^2-\sin(3x)}{x^2+10} \ge \dfrac{5x^2+1}{x^2+10}$

$[\because x\to-\infty, x^2+10<0]$

$\Rightarrow \dfrac{5x^2+1}{x^2+10} \le \dfrac{5x^2-\sin(3x)}{x^2+10} \le \dfrac{5x^2-1}{x^2+10}$.

এখন, $\lim\limits_{x\to\infty} \dfrac{5x^2+1}{x^2+100} = \lim\limits_{x\to\infty} \dfrac{x^2(5+1/x^2)}{x^2(1+100/x^2)}$

$= \lim\limits_{x\to\infty} \dfrac{5+1/x^2}{1+100/x^2} = \dfrac{5+0}{1+0} = 5$

তদ্রূপ, $\lim\limits_{x\to\infty} \dfrac{5x^2-1}{x^2+100} = 5$

স্যান্ডউইচ এর উপপাদ্য অনুসারে পাই,

$\lim\limits_{x\to-\infty} \dfrac{5x^2-\sin(3x)}{x^2+10} = 5$

অতিরিক্ত প্রশ্ন (সমাধানসহ)

1. $\lim\limits_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right)$

$= \lim\limits_{x\to 1} \dfrac{3x+5-2x-6}{(x-1)(x+3)(3x+5)}$

$= \lim\limits_{x\to 1} \dfrac{x-1}{(x-1)(x+3)(3x+5)}$

$= \lim\limits_{x\to 1} \dfrac{1}{(x+3)(3x+5)} = \dfrac{1}{(1+3)(3.1+5)}$

$= \dfrac{1}{4.8} = \dfrac{1}{32}$ (Ans.)

2.(a) $\lim\limits_{x\to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}}$

$= \lim\limits_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})}$

$= \lim\limits_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{3^2-(x^2+5)}$

$= \lim\limits_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{9-x^2-5}$

$= \lim\limits_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2}$

$= \lim\limits_{x\to 2}(3+\sqrt{x^2+5}) = 3+\sqrt{2^2+5}$

$= 3+3 = 6$ (Ans.)

2(b) $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}$ [প্র.ভ.প.'৮৩]

$= \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x^2-1}-\sqrt{x-1})}{(\sqrt{x^2-1}+\sqrt{x-1})(\sqrt{x^2-1}-\sqrt{x-1})}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x^2-1}-\sqrt{x-1})}{(x^2-1)-(x-1)}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x^2-1}-\sqrt{x-1})}{x^2-1-x+1}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(\sqrt{x^2-1}-\sqrt{x-1})}{x(x-1)}$

$= \lim\limits_{x\to 1} \dfrac{\sqrt{x^2-1}-\sqrt{x-1}}{x}$

$= \dfrac{\sqrt{1^2-1}-\sqrt{1-1}}{1} = \dfrac{0}{1} = 0$ (Ans.)

2(c) $\lim\limits_{h\to 0} \dfrac{(x+h)^{1/2}-x^{1/2}}{h}$ [সি.'০১]

$= \lim\limits_{h\to 0} \dfrac{\{(x+h)^{1/2}-x^{1/2}\}\{(x+h)^{1/2}+x^{1/2}\}}{h\{(x+h)^{1/2}+x^{1/2}\}}$

$= \lim\limits_{h\to 0} \dfrac{\{(x+h)^{1/2}\}^2-\{x^{1/2}\}^2}{h\{(x+h)^{1/2}+x^{1/2}\}}$

$$= \lim_{h\to 0} \frac{x+h-x}{h\{(x+h)^{1/2}+x^{1/2}\}}$$

$$= \lim_{h\to 0} \frac{h}{h\{(x+h)^{1/2}+x^{1/2}\}}$$

$$= \lim_{h\to 0} \frac{1}{(x+h)^{1/2}+x^{1/2}}$$

$$= \frac{1}{(x+0)^{1/2}+x^{1/2}} = \frac{1}{x^{1/2}+x^{1/2}} = \frac{1}{2\sqrt{x}}$$

2.(d) $\displaystyle\lim_{x\to 0} \frac{a-\sqrt{a^2-x^2}}{x^2}$

$$= \lim_{x\to 0} \frac{a-\sqrt{a^2-x^2}}{x^2}$$

$$= \lim_{x\to 0} \frac{(a-\sqrt{a^2-x^2})(a+\sqrt{a^2+x^2})}{x^2(a+\sqrt{a^2-x^2})}$$

$$= \lim_{x\to 0} \frac{a^2-(\sqrt{a^2-x^2})^2}{x^2(a+\sqrt{a^2-x^2})}$$

$$= \lim_{x\to 0} \frac{a^2-a^2+x^2}{x^2(a+\sqrt{a^2-x^2})}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(a+\sqrt{a^2-x^2})}$$

$$= \lim_{x\to 0} \frac{1}{a+\sqrt{a^2-x^2}}$$

$$= \lim_{x\to 0} \frac{1}{a+\sqrt{a^2-0^2}} = \frac{1}{a+a} = \frac{1}{2a}$$

3. $\displaystyle\lim_{x\to\infty} \frac{2x^2+1}{6+x-3x^2}$

$$= \lim_{x\to\infty} \frac{x^2(2+\frac{1}{x^2})}{x^2(\frac{6}{x^2}+\frac{1}{x}-3)}$$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{\frac{6}{x^2}+\frac{1}{x}-3} = \frac{2+0}{0+0-3} = -\frac{2}{3}$$

4.(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin^2\frac{x}{2}}{2.\frac{x^2}{4}} = \frac{1}{2}\lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}.1 = \frac{1}{2} \text{ (Ans.)}$$

4(b) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x}$

$$= \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \times \lim_{x\to 0} \sin\frac{x}{2} = 1.\ \sin\frac{0}{2}$$

$$= 1.0 = 0 \text{ (Ans.)}$$

5. $\displaystyle\lim_{x\to 0} \frac{3\sin\pi x - \sin 3\pi x}{x^3}$

$$= \lim_{x\to 0} \frac{4\sin^3\pi x}{x^3} = 4\lim_{x\to 0} (\frac{\sin\pi x}{\pi x})^3.\pi^3$$

$$= 4\times 1\times\pi^3 = 4\pi^3$$

6.(a) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \frac{\lim_{x\to 0}\frac{\sin 5x}{5x}\times 5}{\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3}$

$$= \frac{1\times 5}{1\times 3} = \frac{5}{3} \text{ (Ans.)}$$

6(b) $\displaystyle\lim_{x\to 0} \frac{6x-\sin 2x}{2x+3\sin 4x}$

$$= \lim_{x\to 0} \frac{x(6-\frac{\sin 2x}{x})}{x(2+3\frac{\sin 4x}{x})} = \frac{6-\lim_{x\to 0}\frac{\sin 2x}{2x}\times 2}{2+3\lim_{x\to 0}\frac{\sin 4x}{4x}\times 4}$$

$$= \frac{6-1\times2}{2+3\times1\times4} = \frac{6-2}{2+12} = \frac{4}{14} = \frac{2}{7} \text{ (Ans.)}$$

7(a) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\sin 2x}{\cos 2x}$ [প্র.ভ.প. ৮৬]

ধরি, $x = \dfrac{\pi}{4} + h$. $\quad x \to \dfrac{\pi}{4} \quad h \to 0$

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\sin 2x}{\cos 2x} = \lim_{h\to0} \frac{1-\sin 2(\frac{\pi}{4}+h)}{\cos 2(\frac{\pi}{4}+h)}$$

$$= \lim_{h\to0} \frac{1-\sin(\frac{\pi}{2}+2h)}{\cos(\frac{\pi}{2}+2h)} = \lim_{h\to0} \frac{1-\cos 2h}{-\sin 2h}$$

$$= \lim_{h\to0} \frac{2\sin^2 h}{-2\sin h\cos h} = -\lim_{h\to0} \tan h$$

$$= -\lim_{h\to0} \frac{\tan h}{h} \times h = -1\times0 = 0 \quad \text{(Ans.)}$$

7(b) $\lim\limits_{x\to\pi} \dfrac{\sin x}{\pi - x}$

ধরি, $\pi - x = h$. $\quad x \to \pi \quad h \to 0$

$$\lim_{x\to\pi} \frac{\sin x}{\pi - x} = \lim_{h\to0} \frac{\sin(\pi-h)}{h}$$

$$= \lim_{h\to0} \frac{\sin h}{h} = 1 \text{ (Ans.)}$$

8(a) $\lim\limits_{x\to0} \dfrac{x-\ln(1+x)}{1+x-e^x}$

$$= \lim_{x\to0} \frac{x-(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots)}{1+x-(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)}$$

$$= \lim_{x\to0} \frac{x-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots}{1+x-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots}$$

$$= \lim_{x\to0} \frac{x^2(\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}-\cdots\cdots)}{x^2(-\frac{1}{2!}-\frac{x}{3!}-\cdots\cdots)}$$

$$= \lim_{x\to0} \frac{\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}-\cdots\cdots}{-\frac{1}{2!}-\frac{x}{3!}-\cdots\cdots}$$

$$= \frac{\frac{1}{2}-\frac{0}{3}+\frac{0^2}{4}-\cdots}{-\frac{1}{2!}-\frac{0}{3!}-\cdots} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$$

8(b) $\lim\limits_{x\to0} \dfrac{\ln(1+5x)}{\ln(1-5x)}$

$$= \lim_{x\to0} \frac{5x-\frac{(5x)^2}{2}+\frac{(5x)^3}{3}-\frac{(5x)^4}{4}+\cdots}{-5x-\frac{(5x)^2}{2}-\frac{(5x)^3}{3}-\frac{(5x)^4}{4}-\cdots}$$

$$= \lim_{x\to0} \frac{5-\frac{5^2x}{2}+\frac{5^3x^2}{3}-\frac{5^4x^3}{4}+\cdots}{-5-\frac{5^2x}{2}-\frac{5^3x^2}{3}-\frac{5^4x^3}{4}-\cdots}$$

$$= \frac{5-\frac{5^2.0}{2}+\frac{5^3 0^2}{3}-\frac{5^4 0^3}{4}+\cdots}{-5-\frac{5^2.0}{2}-\frac{5^3 0^2}{3}-\frac{5^4 0^3}{4}-\cdots}$$

$$= \frac{5}{-5} = -1 \text{ (Ans.)}$$

8(c) $\lim\limits_{x\to0}(1+2x)^{(2x+5)/x} = \lim\limits_{x\to0}(1+2x)^{2+5/x}$

$$= \lim_{x\to0}(1+2x)^2 \lim_{x\to0}(1+2x)^{\frac{5}{x}}$$

$$= (1+2.0)^2 \times \{\lim_{x\to0}(1+2x)^{\frac{1}{2x}}\}^{10}$$

$$= e^{10} \quad \text{(Ans.)}$$

9. (a) $f(x) = \begin{cases} e^{-|x|/2}, & \text{যখন } -1 < x < 0 \\ x^2, & \text{যখন } 0 < x < 2 \end{cases}$ হলে

$\lim\limits_{x\to0} f(x)$ এর মান কি বিদ্যমান আছে?

সমাধান : $x = 0$ বিন্দুতে

ডানদিকবর্তী লিমিট $= \lim\limits_{x\to0_+} f(x) = \lim\limits_{x\to0_+} x^2 = 0^2 = 0$