$= \frac{1}{2}\{3 + \cos(2A - 240°) + \cos(2A + 240°) + \cos 2A\}$

$= \frac{1}{2}\{3 + 2\cos 2A \cos 240° + \cos 2A\}$

$= \frac{1}{2}\{3 + 2\cos 2A \cos(180° + 60°) + \cos 2A\}$

$= \frac{1}{2}\{3 + 2\cos 2A(-\cos 60°) + \cos 2A\}$

$= \frac{1}{2}\{3 + 2.\cos 2A(-\frac{1}{2}) + \cos 2A\}$

$= \frac{1}{2}(3 - \cos 2A + \cos 2A) = \frac{3}{2} =$ R.H.S.

2(e) $\cos^2\frac{A}{2} + \cos^2(\frac{\pi}{3} + \frac{A}{2}) + \cos^2(\frac{A}{2} - \frac{\pi}{3}) = \frac{3}{2}$

L.H.S. $= \cos^2\frac{A}{2} + \cos^2(\frac{\pi}{3} + \frac{A}{2}) + \cos^2(\frac{A}{2} - \frac{\pi}{3})$

$= \frac{1}{2}\{1 + \cos 2.\frac{A}{2} + 1 + \cos 2(\frac{\pi}{3} + \frac{A}{2}) + 1 + \cos 2(\frac{\pi}{3} - \frac{A}{2})\}$

$= \frac{1}{2}\{3 + \cos A + \cos(\frac{2\pi}{3} + A) + \cos(\frac{2\pi}{3} - A)\}$

$= \frac{1}{2}\{3 + \cos A + 2\cos\frac{2\pi}{3}\cos A\}$

$= \frac{1}{2}\{3 + \cos A + 2(-\frac{1}{2})\cos A\}$

$= \frac{1}{2}\{3 + \cos A - \cos A\} = \frac{3}{2} =$ R.H.S.

f) $\tan(\alpha + \frac{\pi}{3}) + \tan(\alpha - \frac{\pi}{3}) = \frac{4\sin 2\alpha}{1 - 4\sin^2\alpha}$

L.H.S. $= \tan(\alpha + \frac{\pi}{3}) + \tan(\alpha - \frac{\pi}{3})$

$= \frac{\sin(\alpha + \frac{\pi}{3})}{\cos(\alpha + \frac{\pi}{3})} + \frac{\sin(\alpha - \frac{\pi}{3})}{\cos(\alpha - \frac{\pi}{3})}$

$= \frac{\sin(\alpha + \frac{\pi}{3})\cos(\alpha - \frac{\pi}{3}) + \cos(\alpha + \frac{\pi}{3})\sin(\alpha - \frac{\pi}{3})}{\cos(\alpha + \frac{\pi}{3})\cos(\alpha - \frac{\pi}{3})}$

$= \frac{\sin(\alpha + \frac{\pi}{3} + \alpha - \frac{\pi}{3})}{\frac{1}{2}(\cos 2\alpha + \cos 2\frac{\pi}{3})} = \frac{2\sin 2\alpha}{\cos 2\alpha + (-\frac{1}{2})}$

$= \frac{4\sin 2\alpha}{2\cos 2\alpha - 1} = \frac{4\sin 2\alpha}{2(1 - 2\sin^2\alpha) - 1}$

$= \frac{4\sin 2\alpha}{1 - 4\sin^2\alpha} =$ R.H.S. (Proved)

3.(a) $\cos^3 x + \cos^3(60° - x) + \cos^3(60° + x) = \frac{1}{4}(6\cos x - \cos 3x)$

L.H.S. $= \cos^3 x + \cos^3(60° - x) + \cos^3(60° + x)$

$= \frac{1}{4}\{3\cos x + \cos 3x + 3\cos(60° - x) + \cos 3(60° - x) + 3\cos(60° + x) + \cos 3(60° + x)\}$

$= \frac{1}{4}[3\{\cos x + \cos(60° + x) + \cos(60° - x)\} + \cos 3x + \cos(180° + 3x) + \cos(180° - 3x)]$

$= \frac{1}{4}[3(\cos x + 2\cos 60° \cos x) + \cos 3x - \cos 3x - \cos 3x]$

$= \frac{1}{4}[3(\cos x + 2.\frac{1}{2}\cos x) - \cos 3x]$

$= \frac{1}{4}(3.2\cos x - \cos 3x)$

$= \frac{1}{4}(6\cos x - \cos 3x) =$ R.H.S. (Proved)

(b) $\cos^3 x \cos 3x + \sin^3 x \sin 3x = \cos^3 2x$ **[য.'০৩]**

L.H.S. $= \cos^3 x \cos 3x + \sin^3 x \sin 3x$

$= \frac{1}{4}(\cos 3x + 3\cos x)\cos 3x +$

$\frac{1}{4}(3\sin x - \sin 3x)\sin 3x$

$= \frac{1}{4}(\cos^2 3x + 3\cos x\cos 3x +$
$3\sin x\sin 3x - \sin^2 3x)$

$= \frac{1}{4}\{\cos 2.3x + 3\cos(3x - x)\}$

$= \frac{1}{4}\{\cos 3.2x + 3\cos 2x\} = \cos^3 2x = \text{R.H.S.}$

3. (c) $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

L.H.S. $= \cos^4 x = (\cos^2 x)^2$

$= \{\frac{1}{2}(1+\cos 2x)\}^2$

$= \frac{1}{4}\{1 + 2\cos 2x + \cos^2 2x\}$

$= \frac{1}{4}\{1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\}$

$= \frac{1}{4}\{1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x)\}$

$= \frac{1}{4}\{\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)\}$

$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x = \text{R.H.S.}$

3(d) $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$

L.H.S.$= \sin^4 x + \cos^4 x$
$= (\sin^2 x)^2 + (\cos^2 x)^2$
$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$
$= 1^2 - \frac{1}{2}(2\sin x\cos x)^2 = 1 - \frac{1}{2}(\sin 2x)^2$
$= 1 - \frac{1}{2}\sin^2 2x = \text{R.H.S. (Proved)}$

4.(a) $\sec\theta = \frac{2}{\sqrt{2+\sqrt{2+2\cos 4\theta}}}$ [দি.'০৯; ঢা.'১৪]

L.H.S. $= \sec\theta = \frac{1}{\cos\theta} = \frac{2}{2\cos\theta}$

$= \frac{2}{\sqrt{4\cos^2\theta}} = \frac{2}{\sqrt{2(1+\cos 2\theta)}}$

$= \frac{2}{\sqrt{2+2\cos 2\theta}} = \frac{2}{\sqrt{2+\sqrt{4\cos^2 2\theta}}}$

$= \frac{2}{\sqrt{2+\sqrt{2(1+\cos 4\theta)}}} = \frac{2}{\sqrt{2+\sqrt{2+2\cos 4\theta}}}$

$= \text{R.H.S.}$

4.(b) $\frac{1}{\sin 10^0} - \frac{\sqrt{3}}{\cos 10^0} = 4$ [কু.'০৬; রা.'০৭; ঢা.'০৭; চ., ব.'০৮; দি.'১১; সি.'১২; য.'১৩]

L.H.S. $= \frac{1}{\sin 10^0} - \frac{\sqrt{3}}{\cos 10^0}$

$= \frac{\cos 10^0 - \sqrt{3}\sin 10^0}{\sin 10^0\cos 10^0}$

$= \frac{\frac{1}{2}\cos 10^0 - \frac{\sqrt{3}}{2}\sin 10^0}{\frac{1}{2}\sin 10^0\cos 10^0}$

$= \frac{\cos 60^0\cos 10^0 - \sin 60^0\sin 10^0}{\frac{1}{4}\sin 20^0}$

$= \frac{4\cos(60^0+10^0)}{\sin(90^0-70^0)} = \frac{4\cos 70^0}{\cos 70^0} = 4 = \text{R.H.S.}$

4(c) $\frac{\sqrt{3}}{\sin 20^0} - \frac{1}{\cos 20^0} = 4$ [ঢা.'১০;চ.'১৪]

L.H.S. $= \frac{\sqrt{3}}{\sin 20^0} - \frac{1}{\cos 20^0}$

$= \frac{\sqrt{3}\cos 20^0 - \sin 20^0}{\sin 20^0\cos 20^0}$

$= \frac{\frac{\sqrt{3}}{2}\cos 20^0 - \frac{1}{2}\sin 20^0}{\frac{1}{2}\sin 20^0\cos 20^0}$

$= \frac{\cos 30^0\cos 20^0 - \sin 30^0\sin 10^0}{\frac{1}{4}\sin 40^0}$

$= \frac{4\cos(30^0+20^0)}{\sin(90^0-50^0)} = \frac{4\cos 50^0}{\cos 50^0} = 4 = \text{R.H.S.}$

5. (a) $\tan\theta = \frac{1}{7}$ এবং $\tan\varphi = \frac{1}{3}$ হলে দেখাও যে, $\cos 2\theta = \sin 4\varphi$.

প্রমাণ : দেওয়া আছে , $\tan\theta = \frac{1}{7}$, $\tan\varphi = \frac{1}{3}$.

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(1/7)^2}{1+(1/7)^2}$$

$$= \frac{1-1/49}{1+1/49} = \frac{49-1}{49+1} = \frac{48}{50} = \frac{24}{25}$$

$$\sin 4\varphi = 2\sin 2\varphi\cos 2\varphi$$

$$= 2\frac{2\tan\varphi}{1+\tan^2\varphi}\,\frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

$$= \frac{4.\frac{1}{3}(1-\frac{1}{9})}{(1+\frac{1}{9})^2} = \frac{4.\frac{1}{3}.\frac{8}{9}}{(\frac{10}{9})^2} = \frac{32}{27}\times\frac{81}{100} = \frac{24}{25}$$

$\cos 2\theta = \sin 4\varphi$ (Showed)

5.(b) $2\tan\alpha = 3\tan\beta$ হলে প্রমাণ কর যে, $\tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$

প্রমাণ : দেওয়া আছে , $2\tan\alpha = 3\tan\beta$

$$\Rightarrow \tan\alpha = \frac{3}{2}\tan\beta$$

$$\text{L.H.S.} = \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{(\frac{3}{2}-1)\tan\beta}{1+\frac{3}{2}\tan^2\beta} = \frac{\tan\beta}{2+3\tan^2\beta}$$

$$= \frac{\frac{\sin\beta}{\cos\beta}}{2+3\frac{\sin^2\beta}{\cos^2\beta}} = \frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}$$

$$= \frac{2\sin\beta\cos\beta}{2.2\cos^2\beta+3.2\sin^2\beta}$$

$$= \frac{\sin 2\beta}{2(1+\cos 2\beta)+3(1-\cos 2\beta)}$$

$$= \frac{\sin 2\beta}{2+2\cos 2\beta+3-3\cos 2\beta} = \frac{\sin 2\beta}{5-\cos 2\beta}$$

= R.H.S. (Proved)

6.(a) $x = \sin\frac{\pi}{18}$ হলে দেখাও যে,

$$8x^4 + 4x^3 - 6x^2 - 2x + \frac{1}{2} = 0$$

প্রমাণ : আমরা জানি, $4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\therefore\ 4\sin^3\frac{\pi}{18} = 3\sin\frac{\pi}{18} - \sin 3\frac{\pi}{18}$$

$$\Rightarrow 4x^3 = 3x - \sin\frac{\pi}{6} \qquad [\ x = \sin\frac{\pi}{18}]$$

$$\Rightarrow 4x^3 - 3x + \frac{1}{2} = 0$$

এখন, $8x^4 + 4x^3 - 6x^2 - 2x + \frac{1}{2}$

$$= 2x(4x^3 - 3x + \frac{1}{2}) + 1(4x^3 - 3x + \frac{1}{2})$$

$= 2x \times 0 + 1 \times 0 = 0$ (Showed)

6(b)প্রমাণ কর : $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ [রা.'১১]

প্রমাণ : $\cos 5\theta = \cos(3\theta + 2\theta)$

$= \cos 3\theta\cos 2\theta - \sin 3\theta\sin 2\theta$

$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta).\ 2\sin\theta\cos\theta$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 2\cos\theta(3\sin^2\theta - 4\sin^4\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2\cos\theta\{3(1-\cos^2\theta) - 4(1-\cos^2\theta)^2\}$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2\cos\theta\{3 - 3\cos^2\theta - 4(1 - 2\cos^2\theta + \cos^4\theta)\}$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - (6\cos\theta - 6\cos^3\theta - 8\cos\theta + 16\cos^3\theta - 8\cos^5\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

7.(a) $\tan\alpha\tan\beta = \sqrt{\frac{a-b}{a+b}}$ হলে প্রমাণ কর যে , $(a - b\cos 2\alpha)(a - b\cos 2\beta) = a^2 - b^2$

প্রমাণ ঃ দেওয়া আছে , $\tan\alpha\tan\beta = \sqrt{\dfrac{a-b}{a+b}}$

$\Rightarrow \tan^2\alpha\tan^2\beta = \dfrac{a-b}{a+b}$

$\Rightarrow (a-b) = (a+b)\tan^2\alpha\tan^2\beta \cdots\cdots(1)$

L.H.S $= (a - b\cos 2\alpha)(a - b\cos 2\beta)$

$$= \left\{a - b\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right\}\left\{a - b\frac{1-\tan^2\beta}{1+\tan^2\beta}\right\}$$

$$= \frac{a + a\tan^2\alpha - b + b\tan^2\alpha}{1+\tan^2\alpha} \times \frac{a + a\tan^2\beta - b + b\tan^2\beta}{1+\tan^2\beta}$$

$$= \frac{(a-b)+(a+b)\tan^2\alpha}{1+\tan^2\alpha} \times \frac{(a-b)+(a+b)\tan^2\beta}{1+\tan^2\beta}$$

$$= \frac{(a+b)\tan^2\alpha\tan^2\beta+(a+b)\tan^2\alpha}{1+\tan^2\alpha} \times \frac{(a+b)\tan^2\alpha\tan^2\beta+(a+b)\tan^2\beta}{1+\tan^2\beta}$$

$$= \frac{(a+b)\tan^2\alpha(\tan^2\beta+1)}{1+\tan^2\alpha} \times \frac{(a+b)\tan^2\alpha(\tan^2\beta+1)}{1+\tan^2\beta}$$

$= (a+b)^2\tan^2\alpha\tan^2\beta = (a+b)^2 \dfrac{a-b}{a+b}$

$= a^2 - b^2 =$ R.H.S. (Proved)

7. (b) যদি α ও β কোণদ্বয় ধনাত্মক ও সূক্ষ্ম এবং $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$ হয়, তবে দেখাও যে, $\tan\alpha = \pm\sqrt{2}\tan\beta$

প্রমাণ ঃ দেওয়া আছে, $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$

$\Rightarrow \dfrac{1}{\cos 2\alpha} = \dfrac{3-\cos 2\beta}{3\cos 2\beta - 1}$

$\Rightarrow \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha} = \dfrac{3-\cos 2\beta - 3\cos 2\beta + 1}{3 - \cos 2\beta + 3\cos 2\beta - 1}$

$\Rightarrow \dfrac{2\sin^2\alpha}{2\cos^2\alpha} = \dfrac{4(1-\cos 2\beta)}{2(1+\cos 2\beta)}$

$\Rightarrow \tan^2\alpha = \dfrac{2.2\sin^2\beta}{2\cos^2\beta} = 2\tan^2\beta$

$\therefore \tan\alpha = \pm\sqrt{2}\tan\beta$ (Showed)

7(c) $\cos A \sin\left(A - \dfrac{\pi}{6}\right)$ এর মান বৃহত্তম হলে A এর মান নির্ণয় কর।

সমাধান : $\cos A \sin\left(A - \dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}.2\cos A\cos\left(A - \dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}\left\{\sin\left(A + A - \dfrac{\pi}{6}\right) - \sin\left(A - A + \dfrac{\pi}{6}\right)\right\}$

$= \dfrac{1}{2}\left\{\sin\left(2A - \dfrac{\pi}{6}\right) - \sin\dfrac{\pi}{6}\right\}$

$= \dfrac{1}{2}\left\{\sin\left(2A - \dfrac{\pi}{6}\right) - \dfrac{1}{2}\right\}$

ইহা বৃহত্তম হলে , $\sin\left(2A - \dfrac{\pi}{6}\right) = 1$

$\Rightarrow \sin\left(2A - \dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{2}$

$\therefore 2A - \dfrac{\pi}{6} = \dfrac{\pi}{2} \Rightarrow 2A = \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{3\pi+\pi}{6}$

$\Rightarrow 2A = \dfrac{4\pi}{6} \therefore A = \dfrac{\pi}{3}$ (Ans.)

অতিরিক্ত প্রশ্ন (সমাধানসহ)

প্রমাণ কর যে,

1(a) $\tan\theta(1 + \sec 2\theta) = \tan 2\theta$

L.H.S.$= \tan\theta(1 + \sec 2\theta)$

$= \tan\theta\left(1 + \dfrac{1}{\cos 2\theta}\right)$

$= \tan\theta\left(1 + \dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= \tan\theta\left(\dfrac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= \dfrac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta =$ R.H.S. (proved)

1.(b) $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}=\tan A$

L.H.S.= $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}$

$=\dfrac{\sin A+2\sin A\cos A}{1+\cos A+2\cos^2 A-1}$

$=\dfrac{\sin A(1+2\cos A)}{\cos A(1+2\cos A)}=\tan A=$ R.H.S.

1(c) $\dfrac{\cos^3 x+\sin^3 x}{\cos x+\sin x}=1-\dfrac{1}{2}\sin 2x$

L.H.S.= $\dfrac{\cos^3 x+\sin^3 x}{\cos x+\sin x}$

$=\dfrac{(\cos x+\sin x)(\cos^2 x+\sin^2 x-\cos x\sin x)}{\cos x+\sin x}$

$=1-\cos x\sin x=1-\dfrac{1}{2}\sin 2x=$ R.H.S.

2. $\dfrac{\tan^2(\theta+\frac{\pi}{4})-1}{\tan^2(\theta+\frac{\pi}{4})+1}=\sin 2\theta$

L.H.S.= $\dfrac{\tan^2(\theta+\frac{\pi}{4})-1}{\tan^2(\theta+\frac{\pi}{4})+1}$

$=-\dfrac{1-\tan^2(\theta+\frac{\pi}{4})}{1+\tan^2(\theta+\frac{\pi}{4})}=-\cos 2(\theta+\frac{\pi}{4})$

$=-\cos(\frac{\pi}{2}+2\theta)=-(-\sin 2\theta)$

$=\sin 2\theta=$ R.H.S (Proved)

3 $4\cos^3 x\sin 3x+4\sin^3 x\cos 3x=3\sin 4x$

L.H.S. $=4\cos^3\sin 3x+4\sin^3 x\cos 3x$

$=(\cos 3x+3\cos x)\sin 3x+(3\sin x-\sin 3x)\cos 3x$

$=\cos 3x\sin 3x-\sin 3x\cos 3x+3(\sin 3x\cos x+\sin x\cos 3x)$

$=3\sin(3x+x)$

$=3\sin 4x=$ R.H. S **(Proved)**

4. $\tan^2\theta=1+2\tan^2\varphi$ হলে দেখাও যে, $\cos 2\varphi=1+2\cos 2\theta$

প্রমাণ ঃ দেওয়া আছে , $\tan^2\theta=1+2\tan^2\varphi$

এখন , $1+2\cos 2\theta=1+2\dfrac{1-\tan^2\theta}{1-\tan^2\theta}$

$=\dfrac{1+\tan^2\theta+2-2\tan^2\theta}{1+\tan^2\theta}=\dfrac{3-\tan^2\theta}{1+\tan^2\theta}$

$=\dfrac{3-1-2\tan^2\varphi}{1+1+2\tan^2\varphi}=\dfrac{2(1-\tan^2\varphi)}{2(1+\tan^2\varphi)}$

$=\dfrac{1-\tan^2\varphi}{1+\tan^2\varphi}=\cos 2\varphi$

$\cos 2\varphi=1+\cos 2\theta$ (Showed)

বিকল্প পদ্ধতি: দেওয়া আছে , $\tan^2\theta=1+2\tan^2\varphi$

$\Rightarrow\tan^2\theta-1=2\tan^2\varphi$

$\Rightarrow\dfrac{1}{\tan^2\varphi}=\dfrac{2}{\tan^2\theta-1}$

$\Rightarrow\dfrac{1-\tan^2\varphi}{1+\tan^2\varphi}=\dfrac{2-\tan^2\theta+1}{2+\tan^2\theta-1}$

[যোজন-বিয়োজন করে]

$\Rightarrow\cos 2\varphi=\dfrac{3-\tan^2\theta}{1+\tan^2\theta}$

$=\dfrac{1+\tan^2\theta+2(1-\tan^2\theta)}{1+\tan^2\theta}$

$=\dfrac{1+\tan^2\theta}{1+\tan^2\theta}+2\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\therefore\cos 2\varphi=1+2\cos 2\theta$

5. $\cos\alpha=\dfrac{1}{2}(x+\dfrac{1}{x})$ হলে প্রমাণ কর যে , $\cos 2\alpha=\dfrac{1}{2}(x^2+\dfrac{1}{x^2})$, $\cos 3\alpha=\dfrac{1}{2}(x^3+\dfrac{1}{x^3})$, $\cos 4\alpha=\dfrac{1}{2}(x^4+\dfrac{1}{x^4})$

প্রমাণ ঃ দেওয়া আছে , $\cos\alpha=\dfrac{1}{2}(x+\dfrac{1}{x})$

$\cos 2\alpha=2\cos^2\alpha-1$

$$= 2.\left(\frac{1}{2}(x+\frac{1}{x})\right)^2 - 1$$

$$= 2.\frac{1}{4}(x^2 + 2.x.\ \frac{1}{x} + \frac{1}{x^2}) - 1$$

$$= \frac{1}{2}(x^2 + 2 + \frac{1}{x^2} - 2) = \frac{1}{2}(x^2 + \frac{1}{x^2})$$

$$\cos 2\alpha = \frac{1}{2}(x^2 + \frac{1}{x^2})$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$= 4\left(\frac{1}{2}(x+\frac{1}{x})\right)^3 - 3.\ \frac{1}{2}(x+\frac{1}{x})$$

$$= 4\ .\ \frac{1}{8}(x^3 + 3x^2.\ \frac{1}{x} + 3x\frac{1}{x^2} + \frac{1}{x^3})$$

$$- 3.\ \frac{1}{2}(x+\frac{1}{x})$$

$$= \frac{1}{2}(x^3 + 3x + 3.\frac{1}{x} + \frac{1}{x^3} - 3x - 3.\ \frac{1}{x})$$

$$= \frac{1}{2}(x^3 + \frac{1}{x^3})$$

$$\therefore \cos 3\alpha = \frac{1}{2}(x^3 + \frac{1}{x^3})$$

$$\cos 4\alpha = \cos 2.2\alpha = 2\cos^2 2\alpha - 1$$

$$= 2.\ \{\frac{1}{2}(x^2 + \frac{1}{x^2})\}^2 - 1$$

$$= \frac{1}{2}(x^4 + 2.x^2\ \ \frac{1}{x^2} + \frac{1}{x^4}) - 1$$

$$= \frac{1}{2}(x^4 + 2 + \frac{1}{x^4} - 2)$$

$$\cos 4\alpha = (x^4 + \frac{1}{x^4})$$

6 $\tan\theta = \dfrac{\tan x + \tan y}{1 + \tan x \tan y}$ **হলে দেখাও যে,**

$$\sin 2\theta = \frac{\sin 2x + \sin 2y}{1 + \sin 2x . \sin 2y}$$

প্রমাণঃ দেওয়া আছে, $\tan\theta = \dfrac{\tan x + \tan y}{1 + \tan x \tan y}$

$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y + \sin x \sin y}$$

$$\therefore \quad \tan\theta = \frac{\sin(x+y)}{\cos(x-y)}$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\dfrac{\sin(x+y)}{\cos(x-y)}}{1 + \{\dfrac{\sin(x+y)}{\cos(x-y)}\}^2}$$

$$= \frac{2\sin(x+y)}{\cos(x-y)} \times \frac{\cos^2(x-y)}{\cos^2(x-y) + \sin^2(x+y)}$$

$$= \frac{2\sin(x+y)\cos(x-y)}{\frac{1}{2}\{1+\cos 2(x-y)\} + \frac{1}{2}\{1-\cos 2(x+y)\}}$$

$$= \frac{\sin(x+y+x-y) + \sin(x+y-x+y)}{\frac{1}{2}\{2 + \cos 2(x-y) - \cos 2(x+y)\}}$$

$$= \frac{\sin 2x + \sin 2y}{1 + \frac{1}{2}.2\sin\frac{2(x-y)+2(x+y)}{2}\sin\frac{2(x+y)-2(x-y)}{2}}$$

$$\therefore \quad \sin 2\theta = \frac{\sin 2x + \sin 2y}{1 + \sin 2x + \sin 2y} \quad \text{(Showed)}$$

7. $\tan\theta = \dfrac{y}{x}$ **হলে দেখাও যে,**

$$x\cos 2\theta + y\sin 2\theta = x.$$

প্রমাণ ঃ দেওয়া আছে , $\tan\theta = \dfrac{y}{x}$

$$x\cos 2\theta + y\sin 2\theta$$

$$= x\ \frac{1-\tan^2\theta}{1+\tan^2\theta} + y\ \frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \quad x\frac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}} + y\frac{2\dfrac{y}{x}}{1+\dfrac{y^2}{x^2}}$$

$$= \quad x\frac{x^2-y^2}{x^2+y^2} + y(\frac{2y}{x} \times \frac{x^2}{x^2+y^2})$$

$$= \quad \frac{x^3 - xy^2}{x^2+y^2} + \frac{2xy^2}{x^2+y^2}$$

$$\frac{x^3 - xy^2 + 2xy^2}{x^2 + y^2} = \frac{x(x^2 + y\ \ )}{x^2 + y^2}$$

$x \cos 2\theta + y \sin 2\theta = x$ (Showed)

৮. $\sqrt{2}$ cosA=cosB+ $\cos^3$B এবং $\sqrt{2}$ sinA = sinB−$\sin^3$B হলে দেখাও যে, $\sin(A-B) = \pm\frac{1}{3}$.

প্রমাণ : দেওয়া আছে, $\sqrt{2}\cos A = \cos B + \cos^3 B$

$\sqrt{2}\sin A = \sin B - \sin^3 B$

$\sin(A - B) = \sin A \cos B - \sin B \cos A$

$= \frac{1}{\sqrt{2}}(\sin B - \sin^3 B)\cos B - \frac{1}{\sqrt{2}}\sin B(\cos B + \cos^3 B)$

$\Rightarrow \sqrt{2}\sin(A-B) = \sin B\cos B - \sin^3 B\cos B - \sin B\cos B - \sin B\cos^3 B$

$\Rightarrow \sqrt{2}\sin(A-B = -\sin B\cos B(\sin^2 B + \cos^2 B)$

$\Rightarrow \sqrt{2}\sin(A-B) = -\frac{1}{2}\sin 2B$

$\Rightarrow 2\sqrt{2}\sin(A-B) = -\sin 2B \cdots\cdots (1)$

$\sqrt{2}\cos(A-B) = \sqrt{2}\cos A\cos B - \sqrt{2}\sin A\sin B$

$= (\cos B + \cos^3 B)\cos B - \sin B(\sin B - \sin^3 B)$

$= \cos^2 B + \sin^2 B + \cos^4 B - \sin^4 B$

$= 1 + (\cos^2 B + \sin^2 B)(\cos^2 B - \sin^2 B)$

$\sqrt{2}\cos(A-B) = 1 + \cos 2B$

$\Rightarrow \sqrt{2}\cos(A-B) - 1 = \cos 2B \cdots\cdots\cdots\cdots (2)$

(1) ও (2) বর্গ করে যোগ করলে আমরা পাই ,

$(2\sqrt{2})^2\sin^2(A-B) + (\sqrt{2})^2\cos^2(A-B) + 1 - 2\sqrt{2}\cos(A-B) = \sin^2 2B + \cos^2 2B$

$\Rightarrow 8\{1 - \cos^2(A-B)\} + 2\cos^2(A-B) + 1 - 2\sqrt{2}\cos(A-B) = 1$

$\Rightarrow 8 - 8\cos^2(A-B) + 2\cos^2(A-B) - 2\sqrt{2}\cos(A-B) = 0$

$\Rightarrow 6\cos^2(A-B) - 2\sqrt{2}\cos(A-B) - 8 = 0$

$\Rightarrow 3\cos^2(A-B) - \sqrt{2}\cos(A-B) - 4 = 0$

$\Rightarrow 3\cos^2(A-B) - 3\sqrt{2}\cos(A-B) + 2\sqrt{2}\cos(A-B) - 4 = 0$

$\Rightarrow 3\cos(A-B)\{\cos(A-B) - \sqrt{2}\} + 2\sqrt{2}\{\cos(A-B) - \sqrt{2}\} = 0$

$\Rightarrow \{\cos(A-B) - \sqrt{2}\}\{3\cos(A-B) + 2\sqrt{2}\} = 0$

$\therefore \cos(A-B) = \sqrt{2}$ অথবা, $\cos(A-B) = -\frac{2\sqrt{2}}{3}$

কিন্তু $-1 \le \cos\theta \le 1$ বলে $\cos(A-B) \ne \sqrt{2}$

$\therefore \cos(A-B) = -\frac{2\sqrt{2}}{3}$

$\therefore \sin(A-B) = \pm\sqrt{1 - \sin^2(A-B)}$

$= \pm\sqrt{1 - \left(-\frac{2\sqrt{2}}{3}\right)^2} = \pm\sqrt{1 - \frac{8}{9}}$

$\therefore \sin(A-B) = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$

9 . দেখাও যে, $\frac{\tan 2^n\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots\cdots(1 + \sec 2^n\theta)$

প্রমাণ : $\tan\theta(1 + \sec 2\theta) = \tan\theta\left(1 + \frac{1 + \tan^2\theta}{1 - \tan^2\theta}\right) = \tan\theta\left(\frac{1 - \tan^2\theta + 1 + \tan^2\theta}{1 - \tan^2\theta}\right)$

$= \tan\theta\frac{2}{1 - \tan^2\theta} = \frac{2\tan\theta}{1 - \tan^2\theta} = \tan 2\theta$

$\therefore \frac{\tan 2\theta}{\tan\theta} = 1 + \sec 2\theta$

অনুরূপভাবে আমরা পাই, $\frac{\tan 2^2\theta}{\tan 2\theta} = 1 + \sec 2^2\theta, \frac{\tan 2^3\theta}{\tan 2^2\theta} = 1 + \sec 2^3\theta, \cdots, \frac{\tan 2^n\theta}{\tan 2^{n-1}\theta} = 1 + \sec 2^n\theta$

$\therefore$

$\frac{\tan 2\theta}{\tan\theta}\cdot\frac{\tan 2^2\theta}{\tan 2\theta}\cdot\frac{\tan 2^3\theta}{\tan 2^2\theta}\cdots\cdots\frac{\tan 2^n\theta}{\tan 2^{n-1}\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)$

$(1 + \sec 2^3\theta) \cdots\cdots (1 + \sec 2^n\theta)$

$\Rightarrow \dfrac{\tan 2^n\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)($
$1 + \sec 2^3\theta) \cdots\cdots\cdots (1 + \sec 2^n\theta)$

10.(a) দেখাও যে, $\dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)$ $(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\cdots(2\cos 2^{n-1} - 1)$

প্রমাণ : আমরা পাই ,

$(2\cos\theta + 1)(2\cos\theta - 1) = 4\cos^2\theta - 1$

$= 4 \cdot \dfrac{1}{2}(1 + \cos 2\theta) - 1 = 2 + 2\cos 2\theta - 1$

$2\cos\theta - 1 = \dfrac{2\cos 2\theta + 1}{2\cos\theta + 1}$

অনুরূপভাবে,

$2\cos 2\theta - 1 = \dfrac{2\cos 2^2\theta + 1}{2\cos 2\theta + 1}$

$2\cos 2^2\theta - 1 = \dfrac{2\cos 2^3\theta + 1}{2\cos 2^2\theta + 1}$

$2\cos 2^{n-1}\theta - 1 = \dfrac{2\cos 2^n\theta + 1}{2\cos 2^{n-1}\theta + 1}$

গুণ করে আমরা পাই ,

$(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)$
$\cdots\cdots (2\cos 2^{n-1}\theta - 1)$

$\dfrac{2\cos 2\theta + 1}{2\cos\theta + 1} \cdot \dfrac{2\cos 2^2\theta + 1}{2\cos 2\theta + 1} \cdot \dfrac{2\cos 2^3\theta + 1}{2\cos 2^2\theta + 1}$

$\cdots\cdots\cdots \dfrac{2\cos 2^n\theta + 1}{2\cos 2^{n-1}\theta + 1} = \dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$

$\dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)((2\cos 2\theta - 1)$
$(2\cos 2^2\theta - 1)\cdots\cdots (2\cos 2^{n-1}\theta - 1)$

10.(b) $13\theta = \pi$ হলে দেখাও যে, $\cos\theta. \cos 2\theta.$ $\cos 3\theta. \cos 4\theta. \cos 5\theta. \cos 6\theta = \dfrac{1}{2^6}$

প্রমাণ : $\cos\theta \cos 2\theta \cos 3\theta \cos 4\theta \cos 5\theta \cos 6\theta$

আমরা জানি, $2\sin\theta\cos\theta = \sin 2\theta$

$\Rightarrow \sin\theta\cos\theta = \dfrac{1}{2}\sin 2\theta$

$\therefore \sin\theta\cos\theta\cos 2\theta = \dfrac{1}{2}\sin 2\theta\cos 2\theta$

$= \dfrac{1}{2^2}\sin 4\theta$

অনুরূপভাবে, $\sin\theta\cos\theta\cos 2\theta.\cos 4\theta = \dfrac{1}{2^3}\sin 8\theta$

$\sin\theta\cos\theta\cos 2\theta.\cos 4\theta\cos 8\theta = \dfrac{1}{2^3}\sin 16\theta$

$\sin\theta\cos\theta\cos 2\theta.\cos 4\theta\cos 8\theta\cos 16\theta$
$\cos 32\theta = \dfrac{1}{2^6}\sin 64\theta$

$\Rightarrow \sin\theta\cos\theta\cos 2\theta.\cos 4\theta\cos(13\theta - 5\theta)$
$\cos(13\theta + 3\theta)\cos(26\theta + 6\theta)$
$= \dfrac{1}{2^6}\sin(65\theta - \theta)$

$\Rightarrow \sin\theta\cos\theta\cos 2\theta.\cos 4\theta\cos(\pi - 5\theta)$
$\cos(\pi + 3\theta)\cos(2\pi + 6\theta)$
$= \dfrac{1}{2^6}\sin(5\pi - \theta)$

$\Rightarrow \sin\theta\cos\theta\cos 2\theta.\cos 4\theta(-\cos 5\theta)$
$(-\cos 3\theta).\cos 6\theta = \dfrac{1}{2^6}(\sin\theta)$

$\therefore \cos\theta\cos 2\theta\cos 3\theta\cos 4\theta$
$\cos 5\theta\cos 6\theta = \dfrac{1}{2^6}$ (Showed)

10.(c) $\theta = \dfrac{\pi}{2^n + 1}$ হলে প্রমাণ কর যে , $2^n\cos\theta$ $\cos 2\theta\cos 2^2\theta \cdots\cdots\cdots \cos 2^{n-1}\theta = 1$.

প্রমাণ : দেওয়া আছে, $\theta = \dfrac{\pi}{2^n + 1} \Rightarrow 2^n\theta + \theta = \pi$

$\Rightarrow 2^n\theta = \pi - \theta \Rightarrow \sin 2^n\theta = \sin(\pi - \theta)$

$\Rightarrow 2\sin 2^{n-1}\theta\cos 2^{n-1}\theta = \sin\theta$

$\Rightarrow 2\cos 2^{n-1}\theta(2\sin 2^{n-2}\theta\cos 2^{n-2}\theta) = \sin\theta$

$\Rightarrow 2^2\cos 2^{n-1}\theta\cos 2^{n-2}\theta\sin 2^{n-2}\theta = \sin\theta$