2022-2023 James E Davis Trimester 1 Algebra 2 Week 2 Class Notes

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Previously...

We went over the additive property of matrices, so if we have two matrices

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 7 & -2 \\ 3 & 5 \end{bmatrix}$$

then if the matrices have the same dimensions (i.e., the same rows and columns), then the matrix addition is done by adding each component up individually, so we for instance have

$$A + B = \begin{bmatrix} 8 & 1 \\ 6 & 9 \end{bmatrix}$$

Example 1. Let's say that

$$C = \begin{bmatrix} 1 & 8 \\ 6 & -9 \end{bmatrix}, \ D = \begin{bmatrix} -11 \\ 5 \end{bmatrix}$$

Then $C + D$ is not well-defined since the number of columns are different.2.

Warm up

Let

$$A = \begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 7 & 9 \\ -5 & 6 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 10 & 25 & 66 \\ -27 & 5 & -2 \end{bmatrix}, \ D = \begin{bmatrix} 7 & -1 \\ 67 & -20 \end{bmatrix}, \ E = \begin{bmatrix} \pi & a \\ 5 + x & 72 \end{bmatrix}$$

Determine if the following is well-defined, and if so, determine the sum

1. $A + B$

2. $C + D$

3. $B + C$

4. $A + D + E$

5. $B + D + C$

What are Matrices? (Cont.)

We'll next learn about matrix multiplication, which is a bit more sophisticated than matrix addition. For instance, when we multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

How we got that is through multiplying the rows of the first matrix, by the columns of the second.

Multiplying Matrices

There's two types matrix multiplication

Type 1. "scalar multiplication", i.e., multiplication of a scalar value c with an $n \times m$ matrix A, which we write this as

$c \cdot A$ or cA (usually we write the constant first, then the number, sometimes don't use the middle $\cdot$ when the context is clear)

This gives us an $n \times m$ matrix (the same dimensions as before) achieved by just pairwise multiplying each of the entries of the matrix by c

For example, take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and some arbitrary scalar h. Then

$$h \cdot A = \begin{bmatrix} h \cdot a & h \cdot b \\ h \cdot c & h \cdot d \end{bmatrix}$$

Usually pretty straightforward

Type 2. "matrix multiplication". We take an $n \times m$ matrix A and an $m \times p$ matrix B and multiply the matrices together to get $A \cdot B$ or AB (don't use the middle dot if the context is obvious)

This gives us an $n \times p$ matrix, and as we'll discuss more in detail, in order for the matrix multiplication to make sense, then the columns of the first matrix MUST agree with the number of rows of the second matrix (i.e., there is m columns for$A$ and m for B . Otherwise, you can't multiply the matrix in the way described.

As I'm sure you can imagine, this process is much more involved than the previous multiplication of scalar multiplication. We can summarize the process of multipliying a matrix in three steps. They are as follows:

Step 1. Check if the columns of the first matrix match the second. Take

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix} \times \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}$$

$$2 \times 3 \quad 3 \times 2$$

If this match does not occur, then then the we are unable to do take the arithmatic steps necessary to solution this problem, and there's no solution.

If the matrix multiplication is well-defined, then the product matrix will inherit the number of rows of the first matrix and the number of columns of the second matrix, i.e. in the example above, we have the following rows for

Step 2. Identify each row of the first matrix; each column of the second, and for every i, j entry of the matrix , for every a number i that is between 1 and the number of rows matrix, for every number j that is between 1 and the number of columns in the second matrix. For the given matrices, for example, we have

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix}, \text{ row 1} = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}, \text{ row } 2 = \begin{bmatrix} 9 & -7 & 2 \end{bmatrix}$$
$$B = \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}, \text{ Column 1} = \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix}, \text{ column } 2 = \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix}$$

For each entry of AB, which as a result of matrix multiplication, we have a set of four product entries, which are made up of arithmetic (to be step three) of the correspond row and column.

$$AB = \begin{bmatrix} \text{row } 1 \times \text{ col. 1} & \text{row } 1 \times \text{col. 2} \\ \text{row } 2 \times \text{col } 1 & \text{row } 2 \times \text{col.2} \end{bmatrix}$$

i.e., we have

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} & \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} \\ \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} & \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} \end{bmatrix}$$

Step 3. Didn't get to this; will cover tomorrow

Preview for Tomorrow

N/A

9/27

Previously...

We gave the three steps for multiplying two matrices

Step 1. Check if the columns of the first matrix match the second. Take

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix} \times \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}$$

$$2 \times 3 \qquad 3 \times 2$$

If this match does not occur, then then the we are unable to do take the arithmatic steps necessary to solution this problem, and there's no solution.

If the matrix multiplication is well-defined, then the product matrix will inherit the number of rows of the first matrix and the number of columns of the second matrix, i.e. in the example above, we have two rows and two columns for the product matrix

$$\begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix} \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Step 2. Identify each row of the first matrix; each column of the second, and for every i, j entry of the matrix , for every a number i that is between 1 and the number of rows matrix, for every number j that is between 1 and the number of columns in the second matrix. For the given matrices, for example, we have

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix}, \text{ row } 1 = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}, \text{ row } 2 = \begin{bmatrix} 9 & -7 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix}, \text{ Column } 1 = \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix}, \text{ column } 2 = \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix}$$

For each entry of AB, which as a result of matrix multiplication, we have a set of four product entries, which are made up of arithmetic (to be step three) of the correspond row and column.

$$AB = \begin{bmatrix} \text{row } 1 \times \text{ col. } 1 & \text{row } 1 \times \text{col. } 2 \\ \text{row } 2 \times \text{col } 1 & \text{row } 2 \times \text{col.} 2 \end{bmatrix}$$

i.e., we have

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} & \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} \\ \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} & \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} \end{bmatrix}$$

Step 3. Cross multiply each entry together, i.e., we take the corresponding entries of the rows and columns, multiply them together, and then add those pairs of entries we multiply together, so we have the following entries for AB

$$\text{entry } 1,1 = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} = 1 \cdot 23 + 5 \cdot 67 + 7 \cdot (-90) = 23 + 335 + (-630)$$
$$= -272$$

$$\text{entry } 1{,}2 = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} = 1 \cdot 50 + 5 \cdot 10 + 7 \cdot (-20) = 50 + 50 - 140$$
$$= -40$$

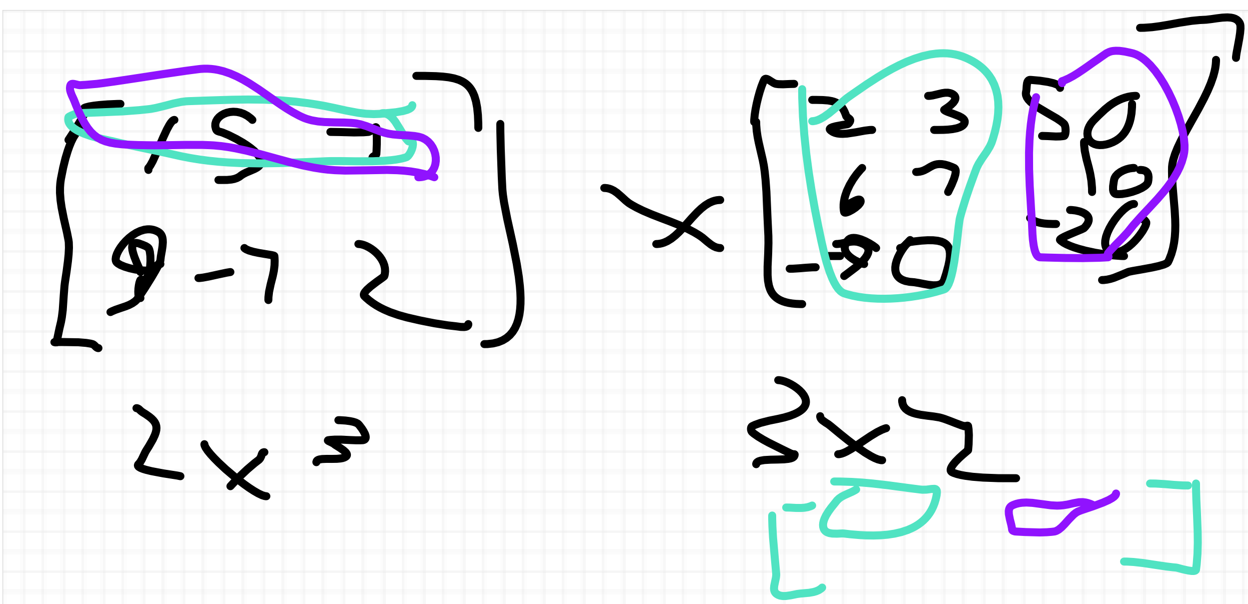
$$\text{entry } 2{,}1 = \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 23 \\ 67 \\ -90 \end{bmatrix} = 9 \cdot 23 + (-7) \cdot 67 + 2 \cdot (-90)$$
$$= 207 + (-469) + (-180) = -442$$

$$\text{entry } 2, 2 = \begin{bmatrix} 9 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 10 \\ -20 \end{bmatrix} = 9 \cdot 50 + (-7) \cdot 10 + 2 \cdot (-20) = 450 - 70 - 40$$
$$= 340$$

and we have

$$AB = \begin{bmatrix} 1 & 5 & 7 \\ 9 & -7 & 2 \end{bmatrix} \begin{bmatrix} 23 & 50 \\ 67 & 10 \\ -90 & -20 \end{bmatrix} = \begin{bmatrix} -272 & -40 \\ -442 & 340 \end{bmatrix}$$

What are Matrices? (Cont.)



Let's do a few light examples

Example 1. Let's say we have

$$A = \begin{bmatrix} 5 & 3 \\ 6 & 9 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0 \\ 3 & 0 \end{bmatrix}$$

What is AB? Note that A has two columns and B has two rows, so the columns of A agree with the rows of B and there is a solution, which is a 2×2 matrix. We get

$$AB = \begin{bmatrix} 5 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -5+9 & 0+0 \\ -6+27 & 0+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 21 & 0 \end{bmatrix}$$

Note that we often do matrix operations in succession, so it's important to know the order we do them

Matrix Operation Order of Operations (so Far):

1. Parentheses (operations inside them, first)
2. Matrix Multiplication/Scalar Multiplication
3. Matrix Addition

Next, we'll do some examples with not computing the actual matrix values, but with figuring out the demensions of the matrix determined by the operations done on arbitrary matrices (or whether a solution exists).

Example 2. Suppose that A is a 2×2 matrix, B is a 3×2 matrix, C is a 2×2 matrix, and D is a 2×3 matrix, and E is a 3×2 matrix. Let's figure out whether the following matrix operations are defined, and if they're defined, state the dimensions of the matrix

a. $A + B$

no solution, since A and B have different rows.

b. AC

2×2

c. BE

no solution

d. EB

no solution

e. BA

3×2

f. $BE + CA$

no solution

g. $ED + AC$

no solution

ED is a 3×3, and AC is a 2×2, and they don't agree in columns or rows, so you can't add them, and we have no solution

h. $DE + CA$

DE is a 2×2 and CA is a 2×2, so the dimensions agree and $DE + CA$ is a 2×2

Tuesday Assignment

Algebra 2 Unit 3 Week 2 Tuesday Assignment

9/28

Teacher PD, no class today

9/29

Shortened period; finished off the Tuesday assignment

9/30

Previously...

Warm up

Find the answer for the following; if no solution exists, write no solution

1. $\begin{bmatrix} 1 & 5 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 10 \\ -3 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

4. $\begin{bmatrix} 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix}$

What are Matrices? (Cont.)

Repeated Multiplication, Matrix Exponentiation and Inverses

Something we might wonder with multiplying matrices is does it matter the order that we multiply the matrices? The answer, as demonstrated by the next example, is the order DOES NOT matter, i.e. multiplying matrices is **associative**:

$(AB)C = A(BC)$

Example 3. (Note with this example that matrices with only the diagonal entries as possibly nonzero are known as **diagonal matrices**; these matrices happen to multiply entry-by-entry)

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

The big idea with **matrix exponentiation** is it's like regular exponentiation except for matrices. Matrix exponentiation is repeated matrix multiplication

$$A^n = \underbrace{A \cdot A \cdot \ \cdots \ \cdot A}_{n \text{ times}}$$

Exponentiation only makes sense for matrices with the same rows and columns, which we call **square matrices**.

Example 4. Let's find

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Before we get into inverses, we'll touch on the **additive identity** matrix and **multiplicative identity** matrix (and we often multiplicative identity the **identity**)

For just numbers, recall that the additive identity is the number e such that

$$x + e = x = e + x$$

and the multiplicative identity is just the number f such that

$$x \cdot f = x = f \cdot x$$

And we know that $e = 0$ and $f = 1$

The additive identity for an $n \times m$ matrix turns out to be

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & & \\ \vdots & & \ddots & \\ a_{n,1} & & & a_{n,m} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

The multiplicative identity of an $n \times m$ matrix turns out be of the form

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \underbrace{\ddots}_{n \text{ times}} & \\ 0 & & & 1 \end{bmatrix}$$

$$
\begin{aligned}
I_1 &= 1 \\
I_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
I_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\cdots \text{ and so on}
\end{aligned}
$$

This identity is only a complete identity if $n = m$ and we have square matrices. Whenever n and m are different, we have a **left multiplicative identity**, which is I_n, so $I_n A = A$, and a **right multiplicative identity** I_m, so $AI_m = A$

Example 5. Let's say

$$
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
$$

Then

$$
I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
$$

and we get

$$
\begin{aligned}
AI_3 &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
I_2 A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\end{aligned}
$$

but note that if we try to switch either matrix, we'll get no solution.

Example 6. Let's say

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},
$$

Then the right identity and the left identity are both I_2, we have

$I_2A = AI_2$.

This only happens more generally when A is a square matrix.

Recall that additive inverses for regular numbers are when given a number x, you have some number y such that

$$x + y = 0.$$

Recall that multiplicative inverses for regular numbers are when given a number x, you have some number y such that

$$x \cdot y = 1.$$

For matrices, we have a similar idea, where for additive inverses for an $n \times m$ matrix A, we have the **additive inverse** of A as the matrix B such that

$$A + B = 0_{n \times m},$$

where

$$0_{n \times m} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}.$$

The **left multiplicative inverse** of an $n \times m$ matrix A is an $m \times n$ matrix B such that

$BA = I_m$ (recall that I_m is the right multiplicative identity)

The **right multiplicative inverse** of an $n \times m$ matrix A is a $m \times n$ matrix B such that

$AB = I_n$ (recall that I_n is the left multiplicative identity)

For square matrices (i.e. when $n = m$), the **multiplicative inverse** is the matrix B such that

$$AB = I_n = BA,$$

or equivalently, both the left and right multiplicative inverse.

Example 7. Let's say we have

$$A = \begin{bmatrix} 3 & -3 \\ 4 & 5 \end{bmatrix}.$$

Then the additive inverse of A is

$$B = \begin{bmatrix} -3 & 3 \\ -4 & -5 \end{bmatrix},$$

since $A + B = 0_{2\times 2}$.

More generally, for any matrix A, the additive inverse will always be of the form

$$(-1) \cdot A,$$

since it's straightforward to verify (entry-by-entry) that

$$A + (-1) \cdot A = 0_{n\times m}.$$

Example 8. Let's say we have

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -9 \end{bmatrix}.$$

Then the inverse of A turns out to be

$$B = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/9 \end{bmatrix},$$

since

$$A \cdot B = \begin{bmatrix} 2/2 & 0 \\ 0 & -9/-9 \end{bmatrix} = I_2.$$

Properties of Matrices

Additive Properties

- $A + 0_{m\times n} = A = 0_{m\times n} + A$
- $A + B = B + A$
- $A + (-1)A = 0_{m\times n} = (-1)A + A = 0_{m\times n}$
- $c(A + B) = cA + cB$
- $(c + d)A = cA + dB$

Multiplicative Poperties

- $A(BC) = (AB)C$ (refer to **Example 3** of the last section)
- We don't always have $AB = BA$ (though sometimes we do)
- $I_m A = A,\ AI_n = A$, and for square matrices, we have $AI_n = A = I_n A$

Next Time...

Go over the updated Order of Operations and we'll talk about determinants and how to compute the inverse (for a 2×2 matrix).