Spring 2023 Math 151C Chapter 3 Notes

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Concept and Definition of a Derivative

When we look at a derivative, we think of the "instaneous rate of change" of a given point of a function. For a line, recall that this rate of change is constant and is called the **slope**

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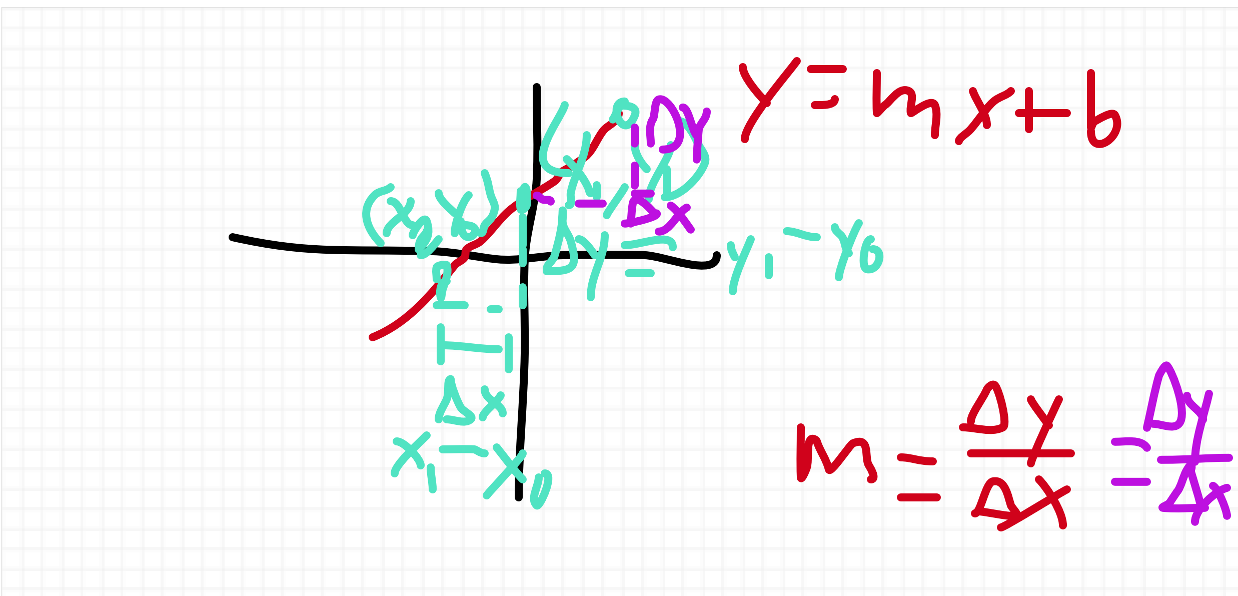
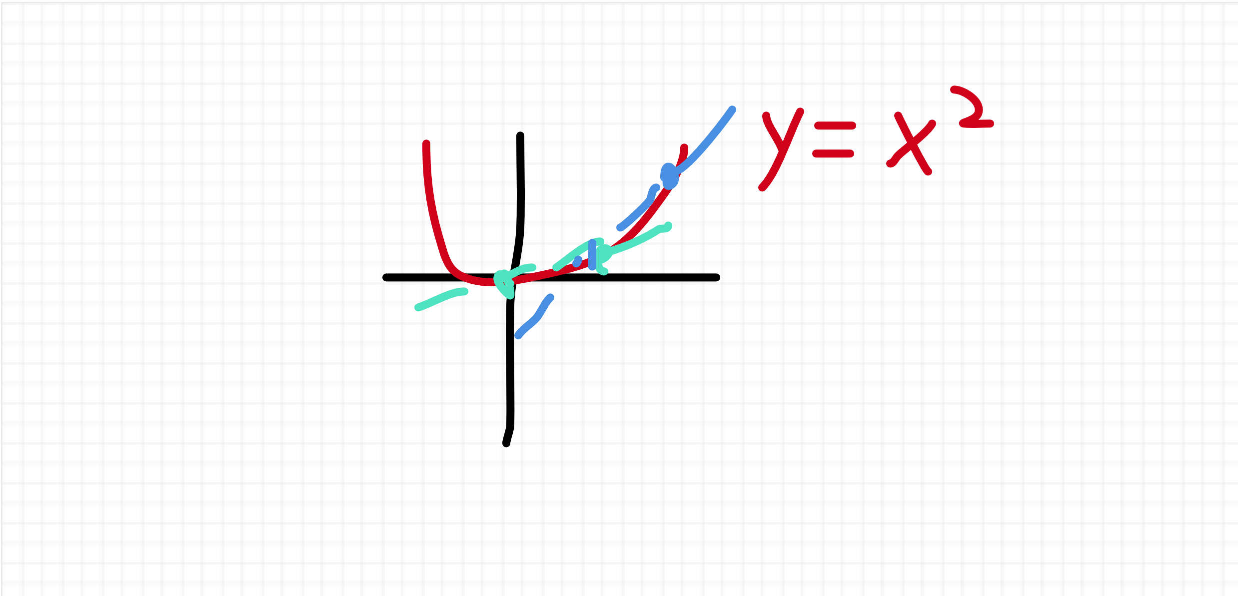


Note that the slope m is going to be the same regardless of which two points $(x_0, y_0),\ (x_1, y_1)$.

Move on to any kind of nonlinear function, this is not the case.



The ratio of the change in x and the change in y for the turquoise is different from the rate of change in the blue between the corresponding x and y values.

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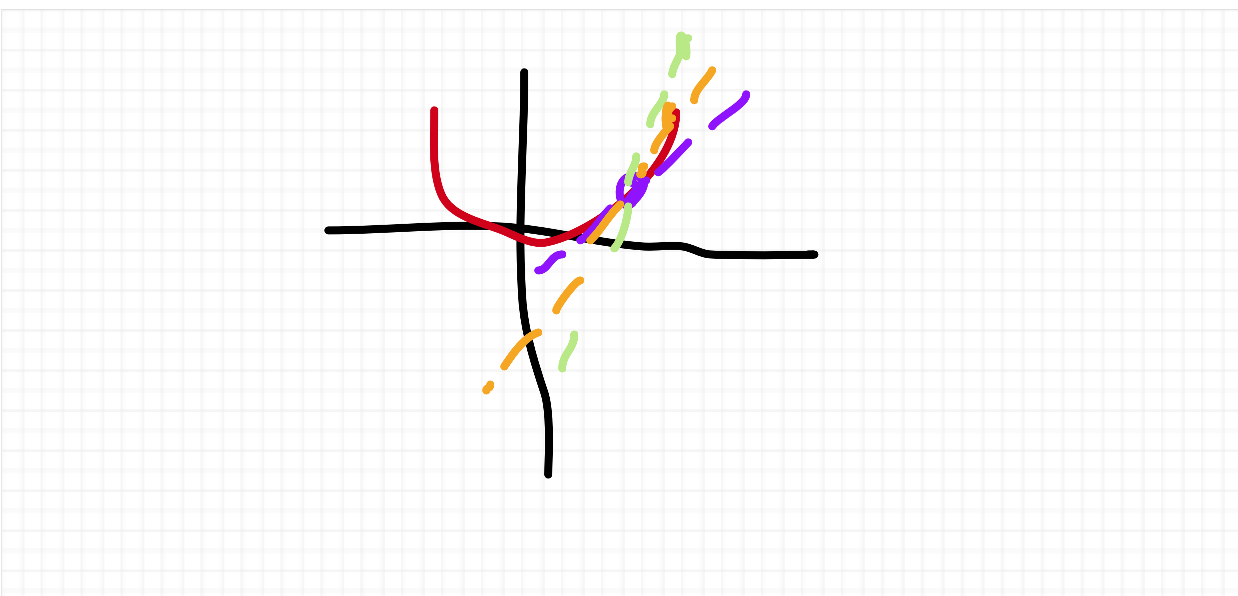


We see that the instantaneous rate of change for what we call "differentiable functions" is something that can be appoximated between rates of change that are closer and closer to the point.

We start with secant lines, but the line that we're ultimately after is the line that is tangent to the curve (we obviously call this line a tangent line); we find this tangent line by approximating the line into finer and finer secant lines, which we can evaluate as a limit.

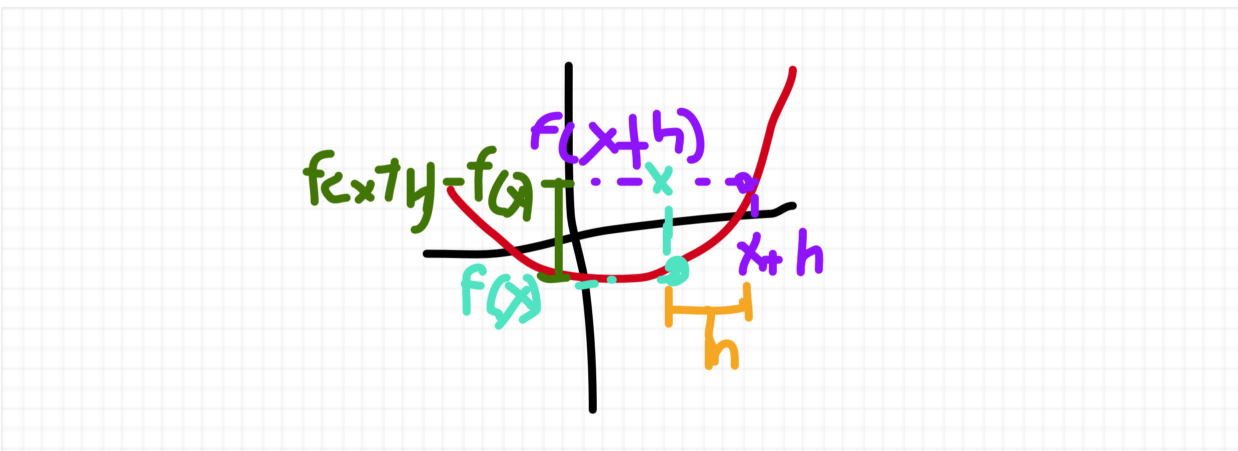
This motivates the first two concepts of a derivative, one where it's a limit of approximations of smaller and smaller rates of change and then a derivative as a tangent line.

Derivatives as a Limit

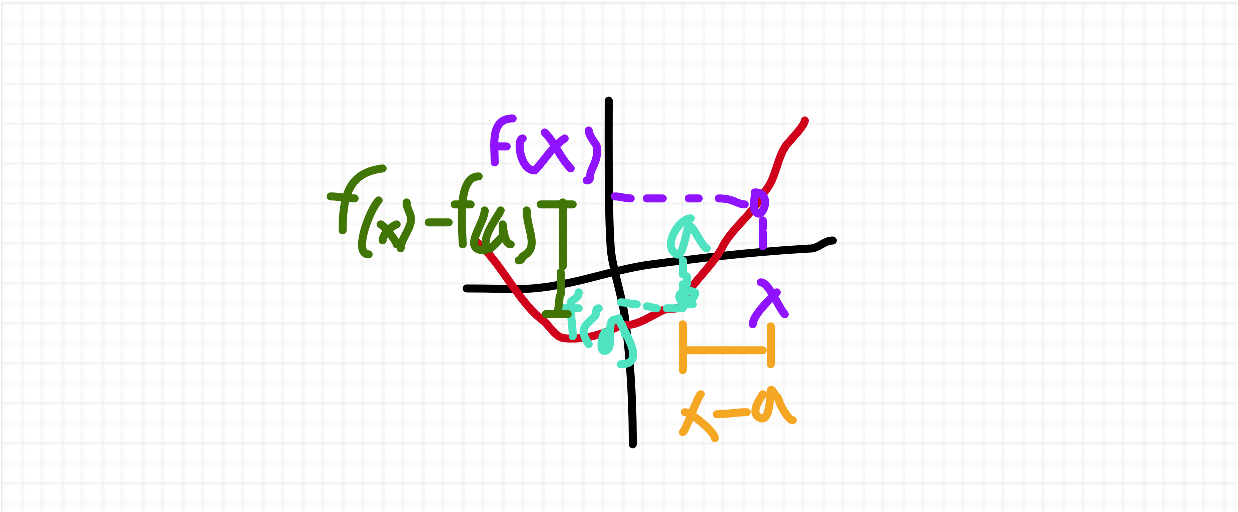
Let's call $f'(x)$ the so-called value of the **derivative**. We can think of the derivative as a limit in two ways:

Definition of a Derivative.

1. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$



2. $f'(x) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$



Example 1. *(from Example 1 on page 104)* Find $f'(2)$, i.e., the derivative at $a = 2$ for $f(x) = x^2$. To do this, we evaluate the limit

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$$
$$= \lim_{h \to 0} 4 + h = 4.$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2} x + 2 = 4.$$

Example 2. Find $f'(5)$ for the function $f(x) = x^3 + 3$

$$\begin{aligned}
f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} &= \lim_{h \to 0} \frac{(5+h)^3 + 3 - \left(5^3 + 3\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(5^2 + 10h + h^2\right)(5+h) + 3 - 5^3 - 3}{h} \\
&= \lim_{h \to 0} \frac{\left(5^3 + 50h + 5h^2\right) + \left(5^2 h + 10h^2 + h^3\right) - 5^3}{h} \\
&= \lim_{h \to 0} \frac{\left(50h + 5h^2\right) + \left(25h + 10h^2 + h^3\right)}{h} \\
&= \lim_{h \to 0} \frac{75h + 15h^2 + h^3}{h} \\
&= \lim_{h \to 0} 75 + 15h + h^2 \\
&= 75.
\end{aligned}$$

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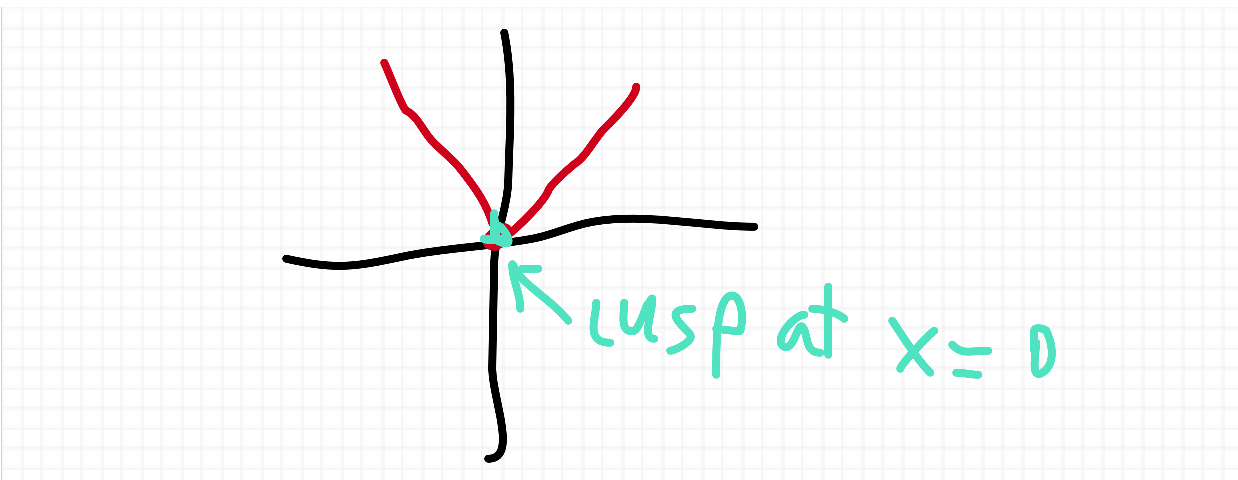
NOTE: Not every point of every function has a derivative, i.e.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ may not exist at some point x.

Example 3. Define $f(x) = |x|$, now we can look at

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Clearly at any point $x \neq 0$, the derivative exists, and is equal to the following

$$f'(x) = \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} = \frac{x}{|x|}$$

However, at the point $x = 0$ the derivative doesn't exist; note that

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{(x+h) - x}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{-(x+h) - (-x)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = -1$$

hence the one-sided limits don't agree and $f'(0)$ does not exist.

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We shall now find $f'(5)$ using the other limit definition using $x \to a$. To do this, we need the following formula below that allows us to simplify $x^{n+1} - a^{n+1}$:

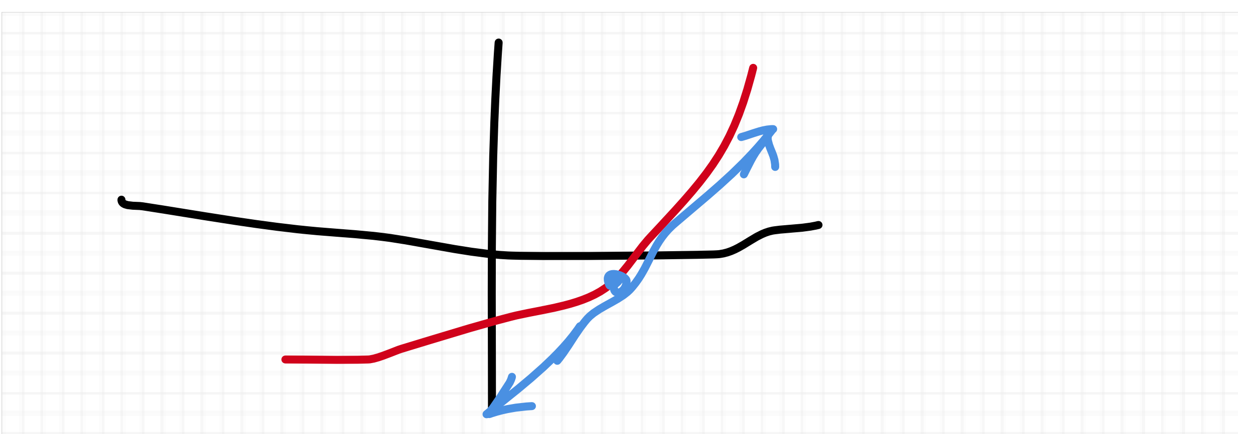
$$x^{n+1} - a^{n+1} = (x-a)\left(x^n + ax^{n-1} + \cdots + a^{n-1}x + a^n\right)$$

Note that the above formula is the generalization of the formula $x^2 - a^2 = (x-a)(x+a)$. Doing this, we get:

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x-5} = \lim_{x \to 5} \frac{\left(x^3+3\right) - \left(5^3+3\right)}{x-5} = \frac{x^3 - 5^3}{x-5} = \lim_{x \to 5} \frac{(x-5)\left(x^2+5x+25\right)}{x-5}$$
$$= \lim_{x \to 5} \left(x^2+5x+25\right) = 5^2 + 5(5) + 25 = 75.$$

Derivatives as a Tangent Line

One can think of derivatives as a tangent line, as opposed to simply the limit of an average rate of change.



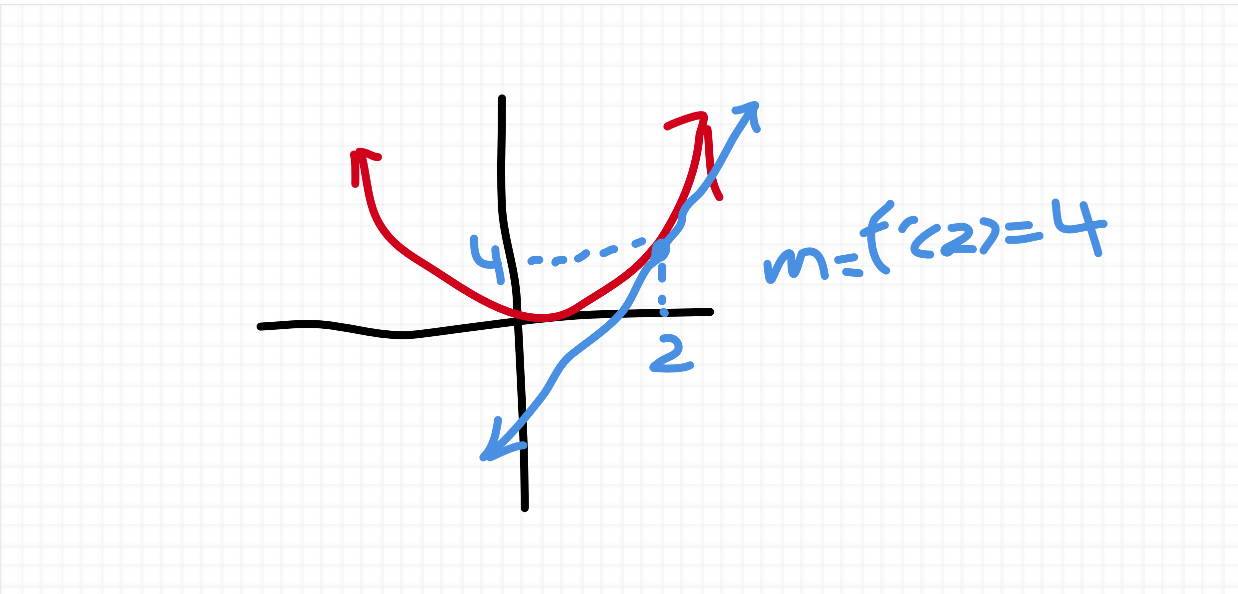
How do we go about finding this tangent line?

We can use the point slope formula

$$y - y_0 = m(x - x_0)$$
$$y = m(x - x_0) + y_0$$ where m is the slope and (x_0, y_0) is a point on the line

To find the tangent of a given point, we find the slope and the point.

Example 4. *(from Example 1 on page 104)* Recall previously in *Example 1* that we found $f'(2) = 4$, and now we want to find the tangent line at $a = 2$. Observe that this tangent line intersects with x^2 at $x = 2$, so it contains the point $(2, 4)$



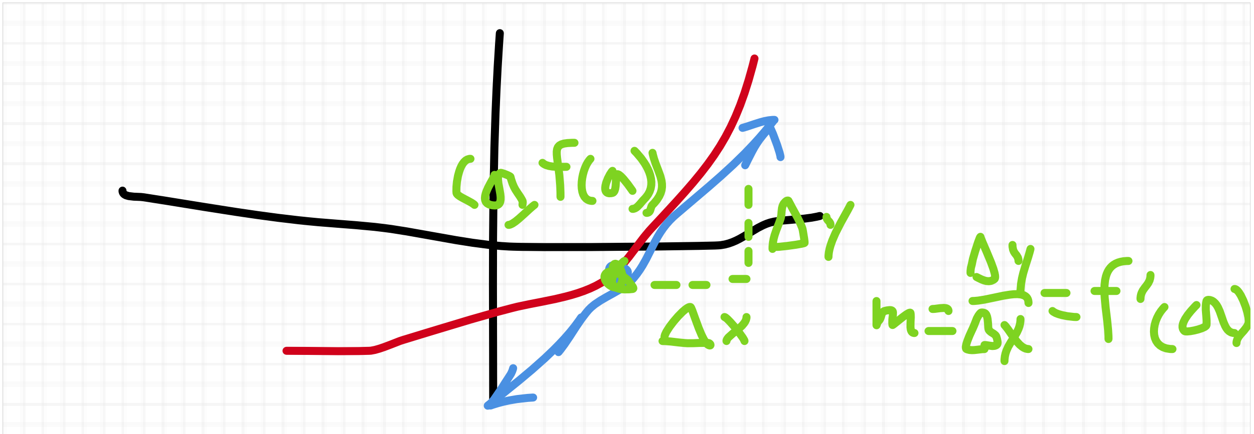
Using the point-slope formula, we have

$$\begin{aligned} y &= f'(2)(x-2) + f(2) \\ &= 4(x-2) + 4 \end{aligned}$$

In general, when we compute the tangent line of a given value a of a function f, we do the following step-by-step process:

Step 1: Compute the derivative $f'(a)$

Step 2: Use the point slope formula with the point $(a, f(a))$ and slope $f'(a)$ to get the following equation:



$y = f'(a)(x-a) + f(a)$

Example 5. *(from Example 2 from page 105)* Find the tangent line at $a = 3$ for the function $f(x) = x^2 - 8x$.

First, we do step 1 and compute the slope $f'(3)$

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{\left[(3+h)^2 - 8(3+h)\right] - \left[3^2 - 8(3)\right]}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 24 - 8h - 9 + 24}{h} \\ &= \lim_{n \to 0} \frac{-2h + h^2}{h} = \lim_{h \to 0} -2 + h = -2 \end{aligned}$$

Next, we do step and use the point slope formula with the point $x_0 = a = 3$ and $y_0 = f(3) = 3^2 - 8(3) = 9 - 24 = -15$ and slope $f'(3) = -2$ and we have

$y = f'(3)(x-3) + f(3) = -2(x-3) - 15.$

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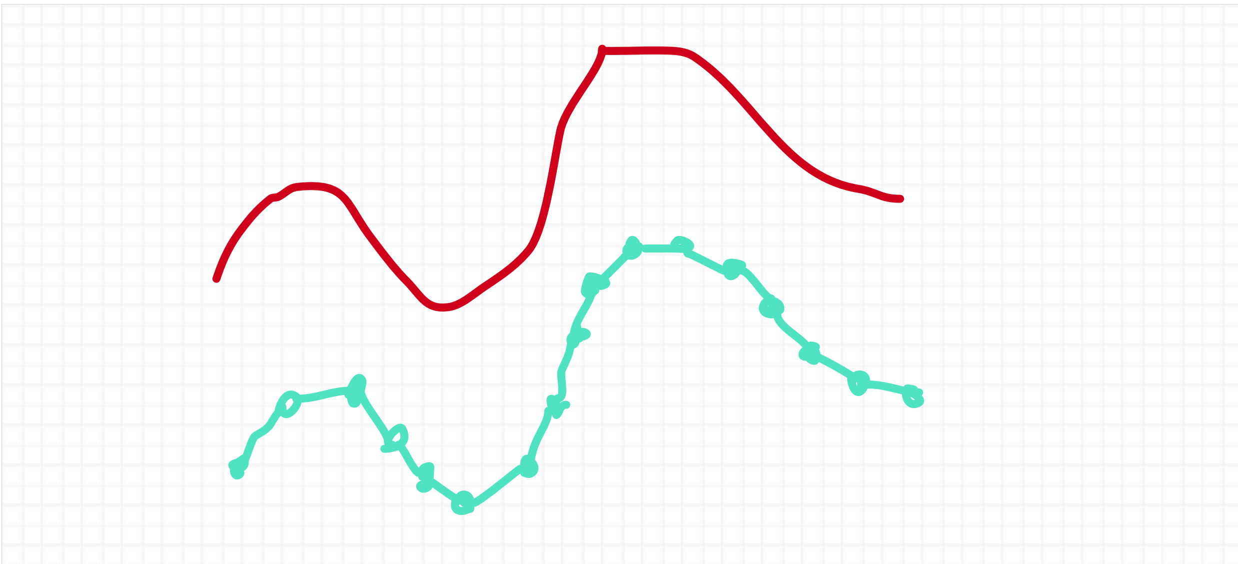
Takeaway Properties from Differentiable Functions

Define $\approx$ as follows: Given two functions f, g

$f(x) \approx g(x)$ as $x \to a$ precisely when $g(x) \neq 0$ as $x \to a$ $\lim\limits_{x \to a} \dfrac{f(a)}{g(a)} = 1$

We talk about differentiable functions as locally linear



Theorem 1. Lets assume that f is differentiable at a and $f'(a) \neq 0$

1. *(Local Linearity)* $f(x) \approx f(a) + f'(a)(x - a)$, i.e.

$$\lim_{x \to a} \frac{f(x)}{f(a) + f'(a)(x - a)} = 1$$

2. *(Differentiability implies Continuity)* f is continuous as $x = a$.

Proof.

1. Note first that

$$\lim_{x \to a} \frac{f(x)}{f(a) + f'(a)(x - a)} = 1$$

algebraically follows from the definition of the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2.

$$\lim_{x \to a} f(x) = f(a) - \lim_{x \to a}(f(x) - f(a)) = f(a) - \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right]$$
$$= f(a) - \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right)\left(\lim_{x \to a}(x - a)\right)$$
$$= f(a) - f'(a) \cdot 0$$
$$= f(a),$$

and the definition of continuity is met. □

Derivatives as a "Slope" Function

The next way to look at derivatives is as a "slope function". It may be that the derivative may exist for not just one value of x but x over a large domain, maybe even the entire real number line. This allows us to look at the derivative of f as a function (we shall call this function f') going from the original domain where the derivative exists to potentially anywhere on the real number line

Leibiniz Notation. From this point on, in addition to using what we "prime notation", i.e., using $f'(x)$ to talk about the derivative, we can use another form of notation called "Leibiniz Notation" where the derivative is expressed as

$$f'(x) := \frac{d}{dx}[f(x)]$$

Example 1. Let's find $\frac{d}{dx}[f(x)]$ $f(x) = x^{-1}$.

$$\frac{d}{dx}\left(x^{-1}\right) = \lim_{h \to 0} \frac{(x + h)^{-1} - x^{-1}}{h} = \lim_{h \to 0}\left[\frac{1}{h(x + h)} - \frac{1}{hx}\right] = \lim_{h \to 0} \frac{1}{h(x + h)} \cdot \frac{x}{x} - \frac{1}{hx} \cdot \frac{x + h}{x + h}$$
$$= \lim_{h \to 0} \frac{x - (x + h)}{hx(x + h)} = \lim_{h \to 0} \frac{-h}{hx(x + h)} = \lim_{h \to 0} \frac{-1}{x(x + h)} \equiv -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(x^{-1}\right) = -\frac{1}{x^2}$$

Example 2. Let's find $\frac{d}{dx}\left(\sqrt{x}\right)$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0}\left[\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right]$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

Differentiation Rules

Note that we don't always have to go through the tedious procedure of evaluating limits to find derivatives when we always know the rules of derivatives of arrangements of functions, and this allows us to find derivatives of functions without at all evaluating the limit.

We know that the derivative of any line is just its slope

$$\frac{d}{dx}(mx+b) = m$$

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proof.

$$\lim_{h \to 0} \frac{m(x+h) + b - (mx+b)}{h} = \lim_{h \to 0} \frac{mh}{h} = m.$$

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We can conclude from this that

$$\frac{d}{dx}(x) = 1, \ \frac{d}{dx}(c) = 0$$
$\uparrow$ constant rule 1

for any constant c. We can additionally recall from Example 1 and Example 2 that

$$\frac{d}{dx}\left(x^{-1}\right) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2},$$
$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

Theorem 2. Here are some basic rules, letting f and g be differentiable functions at x

1. *(addition/subtraction rule)*

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) = f'(x)+g'(x),$$
$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)) = f'(x)-g'(x).$$

2. *(constant rule 2)* Let c be a constant:

$$\frac{d}{dx}(cf(x)) = c\left(\frac{d}{dx}f(x)\right) = cf'(x).$$

3. *(power rule)* For any positive integer $n > 0$, we have

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

4. *(power rule for real numbers)* For any constant a, we have

$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}.$$

We'll wait until next Tuesday to prove these rules.

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Proof.

1.

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]\\ &= f'(x)+g'(x),\end{aligned}$$

and the subtraction rule can be proved similarly.

2.

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) = c\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\\ &= cf'(x).\end{aligned}$$

3. We'll start to prove the power rule after the product rule. □

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Example 1. Recall previous examples that we did using the definition:

(a) x^2

We just use the power rule and find that

$$\frac{d}{dx}\left(x^2\right) = 2x.$$

(b) x^3+1

Using the addition rule

$$\frac{d}{dx}\left(x^3+1\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(1)$$

Next, using the power rule and constant rule, respectively, we get

$$\frac{d}{dx}\left(x^3\right) = 3x^2,\ \frac{d}{dx}(1) = 0.$$

Then we can plug in what we have above and conclude that

$$\frac{d}{dx}\left(x^3+1\right) = 3x^2 + 0 = 3x^2.$$

Example 2. *(from Example 3, page 116)* Set $f(t) = t^3 - 12t + 4$ and find $f'(t)$

$f'(t) = \frac{d}{dt}\left(t^3\right) - \frac{d}{dt}(12t) + \frac{d}{dt}(4)$ repeated use of addition/subtraction rule

$= 3t^2 \qquad - 12\frac{d}{dt}(t) \qquad + 0$

↑ power rule ↑ constant rule 2 ↑ constant rule 1

$= 3t^2 - 12 \cdot 1$

↑ power rule $\frac{d}{dt}(t) = \frac{d}{dt}\left(t^1\right) = 1 \cdot t^0 = 1$

$= 3t^2 - 12$

Example 3. *(from exercise Exercise 26, page 121)* We shall calculate $\frac{d}{dt}\left(6\sqrt{t} + \frac{1}{\sqrt{t}}\right)$.

Using the fact that $\sqrt{t} = t^{1/2},\ \frac{1}{\sqrt{t}} = t^{-1/2}$, we find that

$$\frac{d}{dt}\left(6\sqrt{t} + \frac{1}{\sqrt{t}}\right) \overset{\text{addition rule}}{=} \frac{d}{dt}\left(6\sqrt{t}\right) + \frac{d}{dt}\left(\frac{1}{\sqrt{t}}\right)$$

$$\overset{\text{constant rule 2}}{=} 6\frac{d}{dt}\left(\sqrt{t}\right) + \frac{d}{dt}\left(\frac{1}{\sqrt{t}}\right)$$

$$
\begin{aligned}
&= 6\frac{d}{dt}\left(t^{1/2}\right) + \frac{d}{dt}\left(t^{-1/2}\right) \\
&\overset{\text{power rule}}{=} \quad 6 \cdot \left[\frac{1}{2}t^{1/2-1}\right] + \left[-\frac{1}{2}t^{-1/2-1}\right] \\
&= 3t^{-1/2} - \frac{1}{2}t^{-3/2}.
\end{aligned}
$$

Product Rule

Theorem 1. *(Product Rule)* if f and g are differentiable functions at x, then the function $f \cdot g$ (i.e.$(f \cdot g)(x) = f(x)g(x)$) is also differentiable at x and

$$
\begin{aligned}
(f \cdot g)'(x) = \frac{d}{dx}(f(x)g(x)) &= \left[\frac{d}{dx}(f(x))\right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx}(g(x))\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}
$$

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Proof.

$$
\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x)) + (f(x+h) - f(x))g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h \to 0} \frac{(f(x+h) - f(x))g(x)}{h} \\
&= \lim_{h \to 0}[f(x+h)] \cdot \left[\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right] + \left[\lim_{h \to 0} \frac{(f(x+h) - f(x))}{h}\right] g(x) \\
&= f(x) \cdot g'(x) + f'(x)g(x) \\
&= f'(x)g(x) + f(x) \cdot g'(x)
\end{aligned}
$$

□

We have a general product rule, which states that

$$
\begin{aligned}
\frac{d}{dx}(f_1(x) \cdot \ \cdots \ \cdot f_n(x)) = {} & f_1{}'(x) \cdot f_2(x) \ \cdots \ \cdot f_n(x) + f_1(x) \cdot f_2{}'(x) \cdot f_3(x) \ \cdots \ f_n(x) + \\
& + \ \cdots \ + f_1(x) \ \cdots \ f_{n-1}(x) f_n{}'(x)
\end{aligned}
$$

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Proof. Use the product rule repeatedly as follows

$$
\begin{aligned}
\frac{d}{dx}(f_1(x)\cdot \ \cdots \ \cdot f_n(x)) &= f_1{}'(x)[f_2(x)\ \cdots \ f_n(x)] + f(x)\frac{d}{dx}[f_2(x)\ \cdots \ f_n(x)] \\
&= \cdots \\
&= f_1{}'(x)\cdot f_2(x)\ \cdots \ \cdot f_n(x) + f_1(x)\cdot f_2{}'(x)\cdot f_3(x)\ \cdots \ f_n(x) \\
&\quad + \cdots + f_1(x)\ \cdots \ f_{n-1}(x) f_n{}'(x) \qquad \square
\end{aligned}
$$

Proof of the power rule for positive integers $n > 0$. We use the general product rule for $f_i(x) = x$ for $1 \le i \le n$

$$
\begin{aligned}
\frac{d}{dx}\left(x^n\right) = \frac{d}{dx}(\underbrace{x\ \cdots \ x}_{n \text{ times}}) &= 1\cdot \underbrace{x\ \cdots \ x}_{n-1 \text{ times}} + x\cdot 1\cdot \underbrace{x\ \cdots \ x}_{n-2 \text{ times}} \\
&\quad + \cdots + \underbrace{x\ \cdots \ x}_{n-1 \text{ times}}\cdot 1 \\
&= \underbrace{x^{n-1} + x^{n-1} + \ \cdots \ + x^{n-1}}_{n \text{ times}} \\
&= nx^{n-1}. \qquad \square
\end{aligned}
$$

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Example 1. *(from Example 1 on page 125)* Find the derivative of $h(x) = x^2(9x+2)$.

In this situation, we could just multiply the polynomials and use the power rule

$$
\begin{aligned}
\frac{d}{dx}(h(x)) &= \frac{d}{dx}\left(9x^3 + 2x^2\right) = \frac{d}{dx}\left(9x^3\right) + \frac{d}{dx}\left(2x^2\right) = 9\frac{d}{dx}\left(x^3\right) + 2\frac{d}{dx}\left(x^2\right) = 9\cdot 3x^2 + 2\cdot 2x \\
&= 27x^2 + 4x,
\end{aligned}
$$

but we could use the product rule and find the derivative in less steps:

$$
\begin{aligned}
\frac{d}{dx}(h(x)) &= \left[\frac{d}{dx}\left(x^2\right)\right]\cdot(9x+2) + x^2\cdot\left[\frac{d}{dx}(9x+2)\right] \\
&= 2x\cdot(9x+2) + x^2\cdot(9) \\
&= 18x^2 + 4x + 9x^2 \\
&= 27x^2 + 4x.
\end{aligned}
$$

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Example 2. find $\frac{d}{dx}\left[\left(2+x^{-1}\right)\left(x^{3/2}+1\right)\right]$.

Set $h(x) = \left(2+x^{-1}\right)\left(x^{3/2}+1\right)$. Note that $h(x)$ is a product of $f(x) = 2+x^{-1}$ and $g(x) = x^{3/2}+1$. And so we'll use the product rule as follows:

$$\begin{aligned}\frac{d}{dx}\left[\left(2+x^{-1}\right)\left(x^{3/2}+1\right)\right] &= \frac{d}{dx}[f(x)g(x)] \\ &= \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))\end{aligned}$$

$$f'(x) = \frac{d}{dx}\left(2+x^{-1}\right) = \frac{d}{dx}(2) + \frac{d}{dx}\left(x^{-1}\right) = 0 + (-1)x^{-1-1} = -x^{-2},$$

$$g'(x) = \frac{d}{dx}\left(x^{3/2}+1\right) = \frac{d}{dx}\left(x^{3/2}\right) + \frac{d}{dx}(1) = (3/2)x^{3/2-1} + 0 = \frac{3}{2}x^{1/2},$$

$$= \left(-x^{-2}\right)\left(x^{3/2}+1\right) + \left(2+x^{-1}\right)\left(\frac{3}{2}x^{1/2}\right).$$

$$\frac{d}{dx}\left[\left(2+x^{-1}\right)\left(x^{3/2}+1\right)\right] = \left(-x^{-2}\right)\left(x^{3/2}+1\right) + \left(2+x^{-1}\right)\left(\frac{3}{2}x^{1/2}\right).$$

Chain Rule

Recall that if you have two functions f, g, we define the composition function $f \circ g$ to be

$$(f \circ g)(x) := f(g(x))$$

Theorem 1. *(Chain Rule)* If g is differentiable at some x and f is differentiable at $c = g(x)$, then $(f \circ g)'$ exists and

$$\begin{aligned}&\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \\ &(f \circ g)'(x) = (f' \circ g)(x) \cdot g'(x)\end{aligned}$$

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Proof.

%ELABORATE THIS CASE
Case 1. If $g(c) = g(a)$ for some c arbitrarily close, then we $f(g(c)) = f(g(a))$ for c arbitrarily close, then long story short, we have $g'(a) = 0$, and we have $(f \circ g)'(a) = 0$, so

$$(f \circ g)'(a) = 0 = f'(g(a)) \cdot g'(a).$$

Case 2. If $g(c) \neq g(a)$ for any c arbitrarily close. Then set $y = g(x)$ and $b = g(a)$ and observe that

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a)}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a)}{g(x) - g(a)} \right) \cdot \lim_{x \to a} \left(\frac{g(x) - g(a)}{x - a} \right) \\
&= \lim_{y \to b} \left(\frac{f(y) - f(b)}{y - b} \right) \cdot \lim_{x \to a} \left(\frac{g(x) - g(a)}{x - a} \right) \\
&= f'(b) \cdot g'(a) \\
&= (f' \circ g)(a) \cdot g'(a). \qquad \square
\end{aligned}$$

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Example 1. $h(x) = x^{-2} = \left(x^2\right)^{-1}$; in other words, we have that $h(x)$ is the composition of $f(x) = x^{-1}$ and $g(x) = x^2$, so $h(x) = (f \circ g)(x)$. So using the chain rule, we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}, \\
g'(x) &= \frac{d}{dx}\left(x^{2}\right) = 2x, \\
h'(x) &= (f' \circ g)(x) \cdot g'(x) = -(g(x))^{-2} \cdot 2x = -\left(x^2\right)^{-2} \cdot 2x = -x^{-4} \cdot 2x = -2x^{-4+1} = -2x^{-3}.
\end{aligned}$$

Generalizing Example 1, we have what is called in the book as the "general power rule":

$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$ for any integer n

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Proof of "general power rule".

The general power rule (for positive integers) follows immediately by the chain rule and the power rule for positive integers, and then the power rule for negative integers $-n$ for $n > 0$ follows by using the general power rule (for positive integers) on the function x^{-1}

$$\frac{d}{dx}\left(x^{-n}\right) = \frac{d}{dx}\left(\left(x^n\right)^{-1}\right) = \frac{d}{dx}\left(x^{-1} \circ x^n\right) = -\frac{1}{\left(x^n\right)^2} \cdot nx^{n-1} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

It now remains to find the power rule for rational numbers followed by real numbers, which we'll wait until a future lecture to talk about. ◻

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Utilizing the chain rule on a function h consists of finding the "outer-function" f and "inner-function" g such that

$$h(x) = (f \circ g)(x) = f(g(x))$$

and then using the Chain Rule to find the derivative.

Example 2. Set $h(x) = \left(x^5 + 3x\right)^{-4}$, and find $h'(x)$

Outer function: $f(x) = x^{-4}$,
Inner function: $g(x) = x^5 + 3x$.

So we have $h(x) = (f \circ g)(x)$

$$f'(x) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5}$$

$g'(x) = \frac{d}{dx}\left(x^5 + 3x\right) = 5x^4 + 3$
$h'(x) = (f' \circ g)(x) \cdot g'(x) = -4\left(x^5 + 3x\right)^{-5} \cdot \left(5x^4 + 3\right)$

Note that sometimes we use the chain rule multiple times to find a derivative

Example 3. Calculate $\frac{d}{dx}\left(\sqrt{1 + \sqrt{x^2 + 1}}\right)$. First, let's identify the inner-function and the outer function.

Outer function: $f(x) = \sqrt{x}$,
Inner function: $g(x) = 1 + \sqrt{x^2 + 1}$.

$f'(x) = \frac{1}{2\sqrt{x}}$,

$g'(x)$ we need to compute using the chain rule again, since it is composed of:

Outer function: $a(x) = 1 + \sqrt{x}$,
Inner function: $b(x) = x^2 + 1$.

so we have $g(x) = (a \circ b)(x)$, hence

$a'(x) = \frac{d}{dx}\left(1 + \sqrt{x}\right) = \frac{1}{2\sqrt{x}}$,
$b'(x) = \frac{d}{dx}\left(x^2 + 1\right) = 2x$,

$$g'(x) = (a' \circ b)(x) \cdot b'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}.$$

$$\frac{d}{dx}\left(\sqrt{1 + \sqrt{x^2 + 1}}\right) = (f' \circ g)(x) \cdot g'(x) = \frac{1}{2\sqrt{1 + \sqrt{x^2 + 1}}} \cdot \frac{x}{\sqrt{x^2 + 1}}.$$

Quotient Rule

Theorem 1. *(Quotient Rule)* If f and g are differentiable at x, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}(f(x))\right] \cdot g(x) - f(x) \cdot \left[\frac{d}{dx}(g(x))\right]}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

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proof. Follows from using the product rule, along with the chain rule as so

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) = \frac{d}{dx}(f(x))\frac{1}{g(x)} + f(x) \cdot \frac{d}{dx}\left((g(x))^{-1}\right) \\
&= \frac{f'(x)}{g(x)} - f(x)\left[\frac{1}{g(x)^2} \cdot g'(x)\right] \\
&= \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\end{aligned}$$

□

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Example 1. *(from Example 1 on page 127)* Find the derivative of $\dfrac{x}{1+x^2}$

Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{\frac{d}{dx}(x)\left(1+x^2\right) - x\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} = \frac{(1)\left(1+x^2\right) - x(2x)}{\left(1+x^2\right)^2} = \frac{1+x^2-2x^2}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}.$$

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Example 2. find the derivative of $h(x) = \dfrac{3x^3+x-2}{\sqrt{x^3+1}}$. The function consists of a function $f(x) = 3x^3 + x - 2$ on the numerator and $g(x) = \sqrt{x^3+1}$ on the denominator, and so we use the quotient rule

$$\frac{d}{dx}\left(\frac{3x^3+x-2}{\sqrt{x^3+1}}\right) = \frac{\frac{d}{dx}\left(3x^3+x-2\right)\left(\sqrt{x^3+1}\right)-\left(3x^3+x-2\right)\frac{d}{dx}\left(\sqrt{x^3+1}\right)}{\left(\sqrt{x^3+1}\right)^2}$$

$$\frac{d}{dx}\left(3x^3+x-2\right) = \frac{d}{dx}\left(3x^3\right)+\frac{d}{dx}(x)-\frac{d}{dx}(2) = 9x^2+1$$

$$\frac{d}{dx}\left(\sqrt{x^3+1}\right) = \frac{d}{dx}\left(\left(x^3+1\right)^{1/2}\right)$$

to find this derivative, we use the Chain Rule as so :

outer function: $x^{1/2}$
Inner function : x^3+1

$$\frac{d}{dx}\left(x^{1/2}\right) = 1/2x^{1/2-1} = \frac{1}{2}x^{-1/2}$$
$$\frac{d}{dx}\left(x^3+1\right) = 3x^2$$

$$\begin{aligned}\frac{d}{dx}\left(\left(x^3+1\right)^{1/2}\right) &= \left(\frac{d}{dx}\left(x^{1/2}\right)\circ\left(x^3+1\right)\right)\cdot\frac{d}{dx}\left(x^3+1\right)\\ &= \left(\left(\frac{1}{2}x^{-1/2}\right)\circ\left(x^3+1\right)\right)\cdot 3x^2\\ &= \frac{1}{2}\left(x^3+1\right)^{-1/2}\cdot 3x^2\end{aligned}$$

$$\frac{d}{dx}\left(\frac{3x^3+x-2}{\sqrt{x^3+1}}\right) = \frac{\left(9x^2+1\right)\left(\sqrt{x^3+1}\right)-\left(3x^3+x-2\right)\left(\frac{1}{2}\left(x^3+1\right)^{-1/2}\cdot 3x^2\right)}{x^3+1}$$

NOTE: When finding derivatives using these rules, you DON'T HAVE to simplify, but you won't be penalized for simplifying (though be aware that simplifying might run the risk of making mistakes in your final answer so it's perhaps wise to not simplify).

Derivatives of Trig. Functions

We first note that the derivatives of $\sin x$ and $\cos x$ exist and we can compute them:

Theorem 1.

$$\frac{d}{dx}(\sin x) = \cos x, \ \frac{d}{dx}(\cos x) = -\sin x$$

Proof. Recall the trig. identities

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\cos(a+b) = \cos a \cos b + \sin a \sin b$$

and recall also that

$$\lim_{h\to 0}\left[\frac{1-\cos h}{h}\right] = 0, \ \lim_{h\to 0}\left[\frac{\sin h}{h}\right] = 1$$

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{[\sin x \cos h + \sin h \cos x] - \sin x}{h}\\
&= \lim_{h\to 0}\frac{[\sin x(\cos h - 1) + \sin h \cos x}{h}\\
&= \lim_{h\to 0}\frac{[\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\sin h \cos x}{h}\\
&= -\sin x \lim_{h\to 0}\left[\frac{1-\cos h}{h}\right] + \cos x \lim_{h\to 0}\left[\frac{\sin h}{h}\right]\\
&= -\sin x\cdot 0 + \cos x \cdot 1\\
&= \cos x,
\end{aligned}$$

and $\frac{d}{dx}(\cos x) = -\sin x$ is a similar derivation using the other trig. identity that I'll cover in the revision. □

Corollary 1.

$$\frac{d}{dx}(\tan x) = \sec^2 x, \ \frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x,\ \frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

Proof. We'll prove $\frac{d}{dx}(\tan x) = \sec^2 x,\ \frac{d}{dx}(\sec x) = \sec x \cdot \tan x$ and let the others follow from similarlity. Using the quotient rule and the Pythagorean Identity

$$\cos^2 x + \sin^2 x = 1$$

we have

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Using the chain rule, we have

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left((\cos x)^{-1}\right) = -\frac{1}{\cos^2 x} \cdot (-\sin x) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \cdot \tan x. \quad \square$$

Example 1. *(from example 2 on page 148)*

(a) Find $\frac{d}{dx}(x \cos x)$.

First, we use the product rule to get

$$\frac{d}{dx}(x \cos x) = \frac{d}{dx}(x)\cos x + x\frac{d}{dx}(\cos x) = 1 \cdot \cos x + x \cdot (-\sin x) = \cos x - x \sin x.$$

(b) Find $\frac{d}{dx}\left(\frac{d}{dx}(x \cos x)\right)$

$$\begin{aligned}\frac{d}{dx}\left(\frac{d}{dx}(x \cos x)\right) &= \frac{d}{dx}(\cos x - x \sin x) = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)\\ &= -\sin x - [1 \cdot \sin x + x(\cos x)]\\ &= -[\sin x + \sin x + x(\cos x)]\\ &= -[2 \sin x + x \cos x].\end{aligned}$$

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Example 2. find $\frac{d}{dx}\left(\sin\left(x^2\right)\right)$

outer function: $\sin x$
inner function: x^2

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}\left(x^2\right) = 2x$$

$$\begin{aligned}\frac{d}{dx}\left(\sin\left(x^2\right)\right) &= \left(\frac{d}{dx}(\sin x) \circ x^2\right) \cdot \frac{d}{dx}\left(x^2\right) \\ &= \left(\cos x \circ x^2\right) \cdot 2x \\ &= \cos\left(x^2\right) \cdot 2x\end{aligned}$$

To recall.

$$\tan x = \frac{\sin x}{\cos x},\ \sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x},\ \csc x = \frac{1}{\sin x}$$

Example 3. find $\frac{d}{dx}(\tan x \sec x)$.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

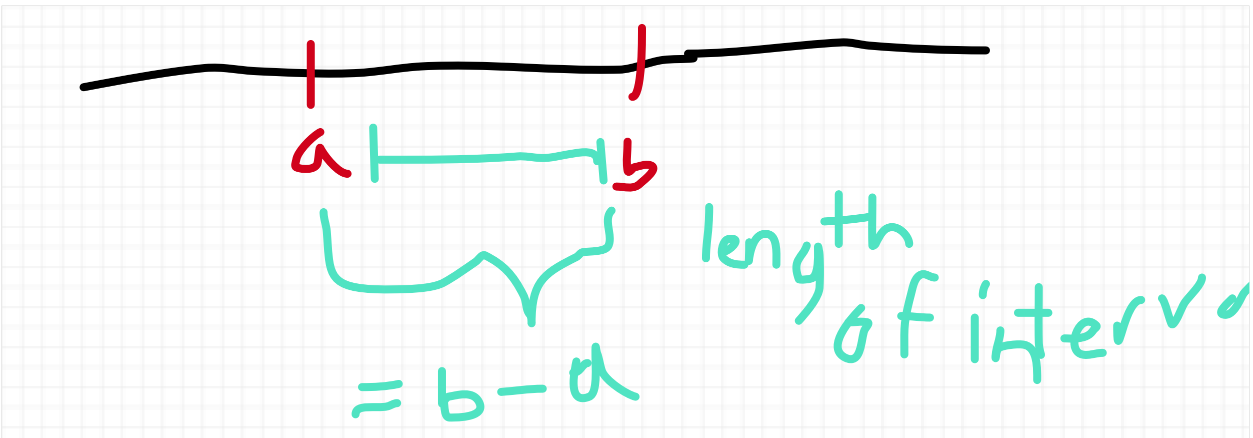
$$\begin{aligned}\frac{d}{dx}(\tan x \sec x) &= \frac{d}{dx}(\tan x)\sec x + \tan x\frac{d}{dx}(\sec x) \\ &= \sec^3 x + \tan^2 x \sec x\end{aligned}$$

Exam 2 Review

2.8 Exercise 6. Use the **Intermediate Value Theorem (IVT)** to find an interval of length $\frac{1}{2}$ containing a root of $f(x) = x^3 + 2x + 1$.

A root of a function (not to be confused wtih square root, cubic root, nth root) is the values for which that function is equal to 0, i.e., it is the solution to the equation $f(x) = 0$; we also call a root a "zero" a "zero value" aka a value of function that equals zero.

The length of an interval (a, b), $[a, b]$, etc. is the distance between the endpoints, i.e. $|b - a|$



We want to show that there is a solution to $x^3 + 2x + 1 = 0$

For $x = 0$, we have $f(0) = 0^3 + 2(0) + 1 = 1$

$$f(-1/2) = \left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right) + 1 = -\frac{1}{8} - 1 + 1 = -\frac{1}{8}$$

Noting that

$f(-1/2) < 0 < f(0)$, we find by the intermediate value theorem that there exists c in the interval $\left(-\frac{1}{2}, 0\right)$ (which note is an interval of length $\frac{1}{2}$) $-\frac{1}{2} < c < 0$ such that $f(c) = 0$, which is a root of f.

2.8 Exercise 4. Given

$$f(x) = \frac{x^2}{x^7 + 1}$$

we show that there exists c such that $f(c) = 0.4$. Since $f(x)$ is continuous on all points outside of -1, we want to show that there exists values a, b on an interval that $f(x)$ is continuous (i.e., not containing -1) such that $f(a) < 0.4 < f(b)$, and the existence will follow immediately by the intermediate value theorem. Observe that for $a = 0$, we have

$$f(0) = \frac{(0)^2}{(0)^7 + 1} = \frac{0}{1} = 0 < 0.4$$

$$f(1) = \frac{1^2}{1^7 + 1} = \frac{1}{2} = .5 > 0.4$$

Then we have $f(0) = 0 < 0.4 < 0.5 = f(1)$, so we conclude there exists c in the interval $(0, 1)$ such that $f(c) = 0.4$.

3.1 Exercise 39. Find $\frac{d}{dx}\left(\sqrt{x+4}\right)$ using the limit definition and then find the tangent of $a = 1$.

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(1+h)+4} - \sqrt{5}}{h} \cdot \frac{\sqrt{(1+h)+4} + \sqrt{5}}{\sqrt{(1+h)+4} + \sqrt{5}} \\
&= \lim_{h \to 0} \frac{[(1+h)+4] - [5]}{h\left(\sqrt{(1+h)+4} + \sqrt{5}\right)} \\
&= \lim_{h \to 0} \frac{5 + h - 5}{h\left(\sqrt{(1+h)+4} + \sqrt{5}\right)} \\
&= \lim_{h \to 0} \frac{h}{h\left(\sqrt{(1+h)+4} + \sqrt{5}\right)} \\
&= \lim_{h \to 0} \frac{1}{\left(\sqrt{(1+h)+4} + \sqrt{5}\right)}
\end{aligned}$$

$$= \lim_{h \to 0} \frac{1}{\left(2\sqrt{5}\right)}$$
$$= \frac{1}{2\sqrt{5}}$$

IMPORTANT NOTE: Finding the derivative IS NOT ITSELF finding the tangent line. The derivative itself is the slope of the tangent line. To find the tangent line, we need to use the point slope formula to plug in the point and the slope.

$$y = f'(a)(x-a) + f(a) = \frac{1}{2\sqrt{5}}(x-1) + \sqrt{(1)+4} = \frac{1}{2\sqrt{5}}x + \left(\sqrt{5} - \frac{1}{2\sqrt{5}}\right)$$

NOTATION NOTE: For 3.3 review problems 5, 9, and 21. we set

$$\frac{df}{dx} := \frac{d}{dx}(f(x))$$
$$\left.\frac{df}{dx}\right|_{x=a} := \frac{d}{dx}(f(a)) = f'(a)$$

3.3 Exercise 9. Set $g(t) = \dfrac{t^2+1}{t^2-1}$, and find

$$\frac{dg}{dt} = \frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right) = \frac{2t\left(t^2-1\right) - \left(t^2+1\right)2t}{\left(t^2-1\right)^2}$$
$$\left.\frac{df}{dt}\right|_{t=0} = \frac{2(0)\left((0)^2-1\right) - \left((0)^2+1\right)2(0)}{\left((0)^2-1\right)^2} = \frac{0-0}{1} = 0.$$

3.6 Exercise 20. Find the derivative of $f(\theta) = \theta \tan\theta \sec\theta$

Note that

$$f(\theta) = \theta \cdot [\tan\theta \sec\theta] = [\theta \tan\theta] \cdot \sec\theta = [\theta \sec\theta] \cdot \tan\theta$$

To find the derivative, we use the product rule (and we're going to use it twice starting with one of the individual functions multiplied by the other two).

$$\begin{aligned} f'(\theta) &= \frac{d}{d\theta}(\theta) \cdot \tan\theta \sec\theta + \theta \frac{d}{d\theta}[\tan\theta \sec\theta] \\ &= 1 \cdot \tan\theta \sec\theta + \theta\left[\sec^3\theta + \tan^2\theta \sec\theta\right] \\ &= \tan\theta \sec\theta + \theta\left[\sec^3\theta + \tan^2\theta \sec\theta\right] \end{aligned}$$

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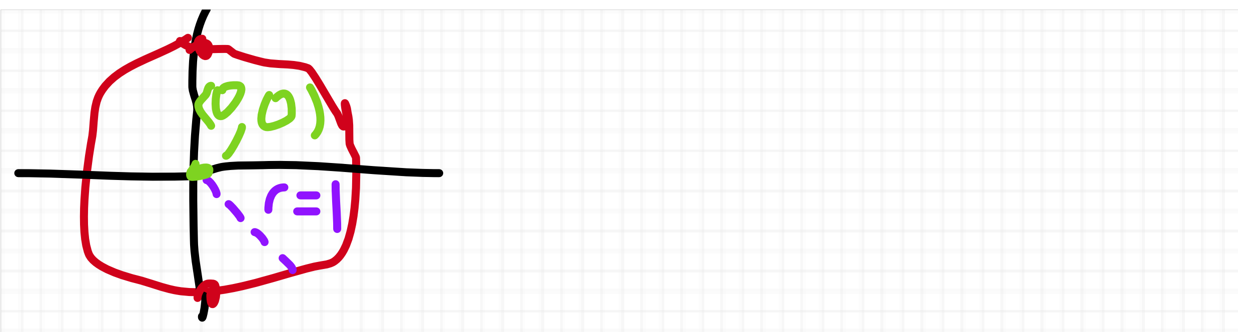
Implicit Differentiation

Implicit differentiation is another way of finding the derivative, and it's useful for functions/curves defined implicitly

Example 1. Let's talk about the equation of the unit circle, i.e., the equation $x^2 + y^2 = 1$



We want to find $\frac{dy}{dx}$ of that implicitly defined function y.

There's two ways to do this, for the circle. We could solve for y

$$\begin{aligned} &x^2 + y^2 = 1 \\ &-x^2 \qquad -x^2 \\ &y^2 = 1 - x^2 \\ &\sqrt{y^2} = \pm\sqrt{1 - x^2} \\ &y = \pm\sqrt{1 - x^2} \end{aligned}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \pm\left[\frac{d}{dx}\left(\sqrt{x}\right)\circ\left(1-x^2\right)\right]\cdot\frac{d}{dx}\left(1-x^2\right)\\
&= \pm\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\\
&= \frac{-x}{\pm\sqrt{1-x^2}}
\end{aligned}
$$

For the second way, we look at y as a differentiable function of x, and in doing so write $y = y(x)$

$$
\begin{aligned}
&\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(1)\\
&\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y(x)^2\right) = 0\\
&2x \quad + \left(\frac{d}{dx}\left[x^2\right]\cdot y(x)\right)\cdot\frac{d}{dx}(y(x)) = 0\\
&2x \quad + 2y(x)\cdot\frac{dy}{dx} \qquad = 0
\end{aligned}
$$

So now we have

$$2x + 2y\frac{dy}{dx} = 0,$$

and we can solve for $\frac{dy}{dx}$ as so:

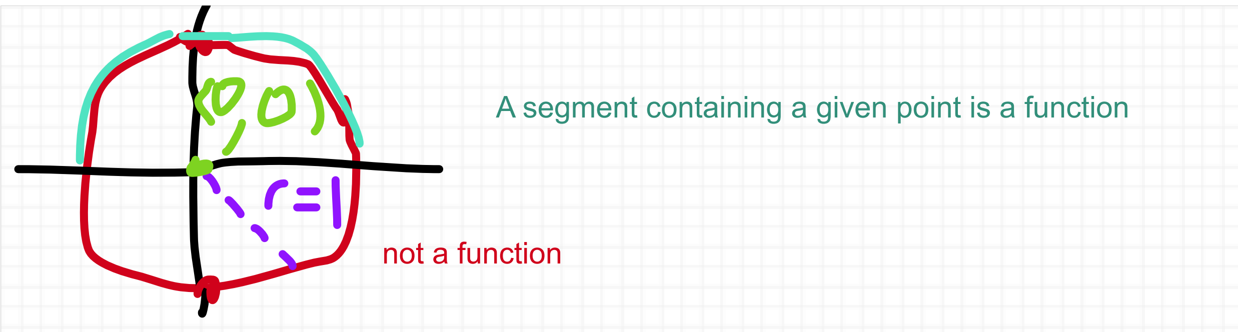
$$
\begin{aligned}
2x + 2y\frac{dy}{dx} &= 0\\
-2x \qquad &\quad -2x\\
2y\frac{dy}{dx} &= -2x\\
\div 2y \qquad &\quad \div 2y\\
\frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}
\end{aligned}
$$

Noting that as before, we can set $y = \pm\sqrt{1-x^2}$. we have that

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-x}{\pm\sqrt{1-x^2}}$$

IMPORTANT NOTE:

1. Functions that are implicitly defined are not necessarily functions in the entirety of the diagram of the curve, though locally they are.



2. While sometimes we can explicitly talk about y, we sometimes can't find a clean value for y and we just leave the derivative in terms of x and y, and we can still use the derivative just as practically as before.

3. Note that we are making a few CRUCIAL assumptions that I WILL NOT EVEN ATTEMPT TO JUSTIFY (though maybe if you ever take a differential equations class with me...maybe I will!). The first being that any function $y(x)$ implicitly defined by equations such as $x^2 + \cos xy = \frac{x}{y}$ is a *differentiable function* of x, let alone exists.

For someone (like myself) who doesn't like to assume results unless they are proven, this does not sit well, as I'm sure you can imagine. But given that the justification of why we can do that--aka, the **Implicit Function Theorem**--entails understanding material that goes wayyy beyond the scope of this class, let alone, most Calc. II courses that someone takes after that, I am left with NO CHOICE but to ask you guys to take it on good faith that $y(x)$ exists such that the implicit equation is satisfied and $y'(x)$ exists

Example 2. *(from Example 2 on page 160)* Find the tangent line at the point $(1, 1)$ for the implicitly defined curve given by the following equation

$$y^4 + xy = x^3 - x - 2$$

First, we find $\frac{dy}{dx}$ using the following steps.

Step 1. Write $y = y(x)$ as a differentiable (locally defined) function of x, and note that $\frac{d}{dx}(y(x)) = \frac{dy}{dx}$.

$$y(x)^4 + xy(x) = x^3 - x - 2$$

Step 2. Differentiate on both sides and use the chain rule to multiply out $\frac{dy}{dx}$ as appropriate.

$$\frac{d}{dx}\left[y(x)^4 + xy(x)\right] = \frac{d}{dx}\left[x^3 - x - 2\right]$$

$$\frac{d}{dx}\left[y(x)^4\right] + \frac{d}{dx}[xy(x)] = 3x^2 - 1$$

$$4y(x)^3 \cdot \frac{dy}{dx} + \left[(1)y(x) + x\frac{dy}{dx}\right] = 3x^2 - 1$$

$$4y^3\frac{dy}{dx} + y + x\frac{dy}{dx} = 3x^2 - 1$$

Step 3. Factor out $\frac{dy}{dx}$ and then solve for it algebraically

$$4y^3\frac{dy}{dx} + y + x\frac{dy}{dx} = \left(4y^3 + x\right)\frac{dy}{dx} + y$$

$$\begin{aligned}
\left(4y^3 + x\right)\frac{dy}{dx} + y &= 3x^2 - 1 \\
-y \quad &\quad -y \\
\left(4y^3 + x\right)\frac{dy}{dx} \quad &= 3x^2 - 1 - y \\
\div\left(4y^3 + x\right) \quad &\quad \div\left(4y^3 + x\right) \\
\frac{dy}{dx} \quad &= \frac{3x^2 - 1 - y}{4y^3 + x}
\end{aligned}$$

Next, we want to find the tangent line at the point $(x, y) = (1, 1)$, so we can plug in $x = 1$ and $y = 1$ and get a value for the slope of the tangent line

$$\left.\frac{dy}{dx}\right|_{x=1,y=1} = \frac{3(1)^2 - 1 - (1)}{4(1)^3 + 1} = \frac{1}{5}$$

and then we use the point slope formula as so

$$\begin{aligned} y - 1 &= \frac{1}{5}(x - 1) \\ y &= \frac{1}{5}x + \frac{4}{5}. \end{aligned}$$

Example 3. *(from Example 3 on page 161)* Calculate $dy\,/\,dx$ at the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ on the curve

$$\sqrt{2}\cos(x + y) = \cos x - \cos y$$

As given in the step-by-step process, we start by thinking of y as a "function" of x and then differentiating on both sides

$$\begin{aligned} \frac{d}{dx}\left[\sqrt{2}\cos(x + y)\right] &= \frac{d}{dx}[\cos x - \cos y] \\ \left(\frac{d}{dx}\left[\sqrt{2}\cos(x)\right] \circ (x + y)\right) \cdot \frac{d}{dx}[x + y] &= -\sin x - \left(\frac{d}{dx}[\cos x] \circ y\right) \cdot \frac{dy}{dx} \\ -\sqrt{2}\sin(x + y) \cdot \left(1 + \frac{dy}{dx}\right) &= -\sin x - \left[-\sin(y) \cdot \frac{dy}{dx}\right] \end{aligned}$$

$$\begin{aligned} -\sqrt{2}\sin(x + y) - \sqrt{2}\sin(x + y)\frac{dy}{dx} &= \sin(y)\frac{dy}{dx} - \sin x \\ -\sin(y)\frac{dy}{dx} \qquad\qquad &\qquad -\sin(y)\frac{dy}{dx} \\ -\sqrt{2}\sin(x + y) - \sqrt{2}\sin(x + y)\frac{dy}{dx} - \sin(y)\frac{dy}{dx} &= -\sin x \\ +\sqrt{2}\sin(x + y) \qquad\qquad &\qquad +\sqrt{2}\sin(x + y) \\ \left[-\sqrt{2}\sin(x + y) - \sin(y)\right]\frac{dy}{dx} &= \sqrt{2}\sin(x + y) - \sin x \\ \div\left[-\sqrt{2}\sin(x + y) - \sin(y)\right] &\qquad \div[-\sin(x + y) - \sin(y)] \end{aligned}$$

$$\frac{dy}{dx} = \frac{\sqrt{2}\sin(x+y) - \sin x}{-\sqrt{2}\sin(x+y) - \sin(y)}$$

We found in general that

$$\frac{dy}{dx} = \frac{\sqrt{2}\sin(x+y) - \sin x}{-\sqrt{2}\sin(x+y) - \sin(y)}$$

and it remains to plug in the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ to get the derivative at the specific point. Note that

$$\sin\left(\frac{\pi}{2}\right) = 1,\ \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)}{-\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}\cdot 1 - \frac{1}{\sqrt{2}}}{-\sqrt{2}\cdot 1 - \frac{1}{\sqrt{2}}}$$

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Example 4. *(from exercise 13 on page 163)* Calculate the derivative with respect to x of the other variable appearing on the other equation, involving the equation

$$x^2y + 2x^3y = x + y$$

Step 1. Treat y as a function of x, so we have $y = y(x)$

$$x^2y(x) + 2x^3y(x) = x + y(x)$$

Step 2. Differentiate on both sides, assuming that y is differentiable

$$\begin{aligned}\frac{d}{dx}\left[x^2y(x) + 2x^3y(x)\right] &= \frac{d}{dx}\left[x^2y(x)\right] + \frac{d}{dx}\left[2x^3y(x)\right] \\ &= \left[x^2\frac{d}{dx}[y(x)] + \frac{d}{dx}\left[x^2\right]y(x)\right] + \left[\frac{d}{dx}\left[2x^3\right]y(x) + 2x^3\frac{d}{dx}[y(x)]\right] \\ &= x^2y'(x) + 2xy(x) + 6x^2y(x) + 2x^3y'(x)\end{aligned}$$

$$= x^2\frac{dy}{dx} + 2xy(x) + 6x^2y(x) + 2x^3\frac{dy}{dx}$$

$$\frac{d}{dx}]x + y(x)] = 1 + y'(x) = 1 + \frac{dy}{dx}$$

Step 3. Factor in the $\frac{dy}{dx}$ term and solve for $\frac{dy}{dx}$ as so:

$$x^2\frac{dy}{dx} + 2xy(x) + 6x^2y(x) + 2x^3\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$-\frac{dy}{dx} \qquad\qquad\qquad\qquad\qquad\qquad -\frac{dy}{dx}$$

$$x^2\frac{dy}{dx} + 2xy(x) + 6x^2y(x) + 2x^3\frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\left(x^2 + 2x^3 - 1\right)\frac{dy}{dx} + 2xy + 6x^2y = 1$$

$$-2xy - 6x^2y \qquad -2xy - 6x^2y$$

$$\left(x^2 + 2x^3 - 1\right)\frac{dy}{dx} = 1 - 2xy - 6x^2y$$

$$\div\left(x^2 + 2x^3 - 1\right) \qquad \div\left(x^2 + 2x^3 + 1\right)$$

$$\frac{dy}{dx} = \frac{1 - 2xy - 6x^2y}{x^2 + 2x^3 - 1}$$

How to do the quotient rule through implicit differentiation: Let's say that

$$y = h(x) = \frac{f(x)}{g(x)}$$

for $g(x) \neq 0$, then y can be defined implicitly as

$g(x)y = f(x)$
%EXPLAIN THIS IN BETTER DETAIL

and then we can use implicit differentiation to get the quotient rule (I'll add the detailed derivation in the moodle version of these notes)
%PLACE THIS IDEA IN THE FUTURE

Proof of the power rule for rational numbers $\frac{p}{q}$.

$y = x^{p/q}$
$y^q = \left(x^{p/q}\right)^q = x^p,$

and we can find the derivative using what we know about the power rule for natural numbers as follows:

$$\frac{d}{dx}\left(y(x)^q\right) = \frac{d}{dx}\left(x^p\right)$$
$$qy(x)^{q-1} \cdot y'(x) = px^{p-1}$$
$$qy^{q-1} \cdot \frac{dy}{dx} = px^{p-1}$$
$$\div\, qy^{q-1} \qquad \div\, qy^{q-1}$$
$$\frac{dy}{dx} = p\frac{x^{p-1}}{q\left(x^{p/q}\right)^{q-1}} = \frac{p}{q}x^{(p-1)-(p/q)(q-1)} = \frac{p}{q}x^{(p-1)-p+\frac{p}{q}} = \frac{p}{q}x^{p/q-1}.$$

For any real number r, we have $x^{p/q} \to x^r$ as $p\,/\,q \to r$ and so we have

$$\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{p/q-1} \to rx^{r-1} = \frac{d}{dx}\left(x^r\right). \quad \square$$

%MAKE CAVEAT ABOUT WHY THIS THEORETICAL TRICK WORKS

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Derivatives as a Rate of Change

Rates of Change

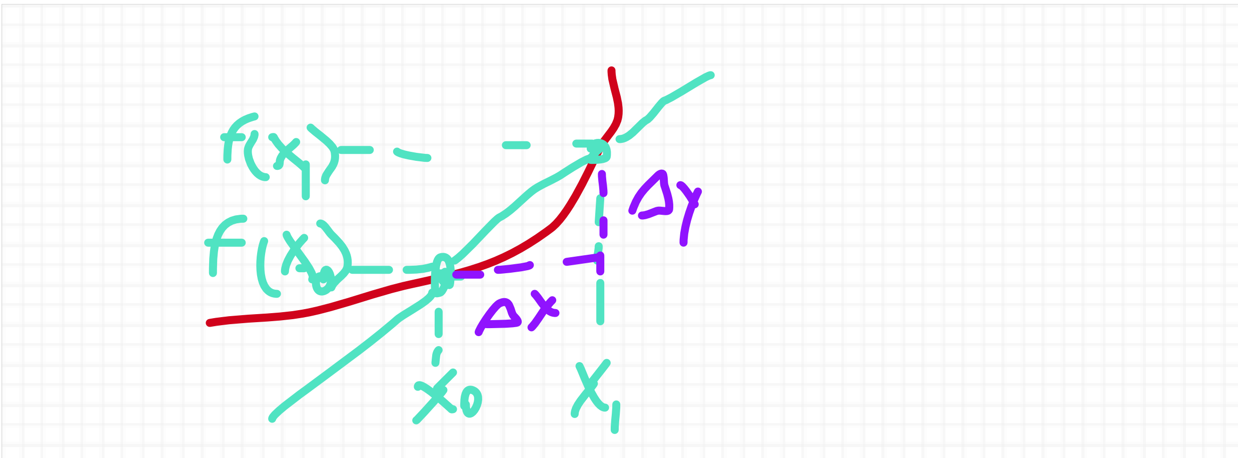
We talked about derivatives as a slope, tangent, and function, but we can also look at derivatives as a "instantaneous rate of change".

We can first look at the average rate of change of a function $y = f(x)$ from initial value x_0 to the final value x_1, and we get

$\Delta y =$ change in $y = f(x_1) - f(x_0)$

Δx = change in $x = x_1 - x_0$

average rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$



We can think of the derivative as the **instantaneous (average) rate of change**, since the average rate of change as the values get smaller and smaller fit the definition of a derivative

instantaneous rate of change $= f'(x_0) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim_{x_1 \to x_2} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

Recall that in linear motion, we have what's called the **position/displacement** (the one-dimensional location) as a function of time, which is the location $(t, s(t))$, and the **velocity** $v(t)$, which is defined to be the *instantaneous rate of change of position*, so we have the relationship

velocity $= v(t) = \dfrac{ds}{dt}$

Example 1. *(from Example 4 on page 134)* A truck enters off-ramp of a highway at $t = 0$. Its position on the ramp after t seconds is $s(t) = 25t - 0.3t^3 m$ for $0 \le t \le 5$ seconds

How fast is the truck going at the moment it enters off the ramp?

Next time: We're going to use differentiation to talk more about that.

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To answer that question, note that the truck starts at $x = 0$

Example 4. *(from Example 4 from page 134)*

$v'(0) = s'(0) = 25 - 0.9t^2 \mid_{t=0} - 0.$

Note that in economics, The number of x unity is called the prodcution caust. the rate of change of the cost (i.e., the rate of change between the dollar costs per unit of a good) the **marginal cost**. Given a cost function C, we have

$C'(x) =$ marginal cost

We also have revenue R, which is also a function quantity

$R = pq,$

where p the price chosen, and q is the supply function of p. Assuming that q is of the linear form

$q(p) = m(p - p_0) + q(p_0),$

and then pugging that in for q leads to being able to find the derivative to get **marginal revenue**

$$R(x) = pq(x) = mp^2 + (-mp_0 + q(p_0)p$$
$$R'(x) = \frac{d}{dx}[mp^2 + (-mp_0 + q(p_0)p] = 2mp - mp_0 + q(p_0)$$

Example 3 *(from example 3 on page 135)* Company data suggest that when there are 50 more passengers, the total dollar cost of a certain flight is about
$C(x) = 0.00005x^3 + 0.38x^2 + 120x.$

marginal cost $= C'(x) - 0.00005 + .76x + 120$

The question also asks whether it's more expensive to add a passenger when $x = 150$ or when $x = 200$?

Plug in $x = 150$ and $x = 200$

$C'(150) = -0.000015(150) + .76(150) + 120 \approx 233$
$C'(200) = -0.000015(150) + .76(150) + 120 \approx 271$

It's more expensive to add a single passenger at $x = 150$ because the marginal cost is higher.

Example 4 *(from Example 2 on page 133)* Let $A = \pi r^2$ be the area of the circl of radius r

Compute $\frac{dA}{dr}$ at $r = 2$ and $r = 5$

$$\left.\frac{dA}{dr}\right|_{r=2} = \left.\frac{d}{dr}\left(\pi r^2\right)\right|_{r=2} = 2\pi r)|_{r=2} = 4\pi$$
$$\left.\frac{dA}{dr}\right|_{r=5} = \left.\frac{d}{dr}\left(\pi r^2\right)\right|_{r=5} = 2\pi r)|_{r=5} = 10\pi$$

Related Rates

Related rates allow us to apply rates of change (usually) to rates that relate to each other

Example 1 *(similar to Example 2 on page 133)* Let's say that have a circle with radius r and we know the change $\left.\frac{dr}{dt}\right|_{r=2} = 3$ and $r = 2$. Then what is the rate of change of area $A = \pi r^2$ at $r = 2$

So note that A as a function of time is equal to $A(t) = \pi r(t)^2$, so we find

$$\left.\frac{dA}{dt}\right|_{r=2} = \pi\left[\frac{d}{dr}\left(r^2\right) \circ r(t)\right] \cdot \left.\frac{dr}{dr}\right|_{r=2} = \pi 2r(t) \cdot \left.\frac{dr}{dt}\right|_{r=2} = \pi \cdot 2 \cdot 2 \cdot 3 = 12\pi.$$

Next Time: Go more in depth for a step-by step process on finding related rates, and shortly after that, we'll start on a new chapter.

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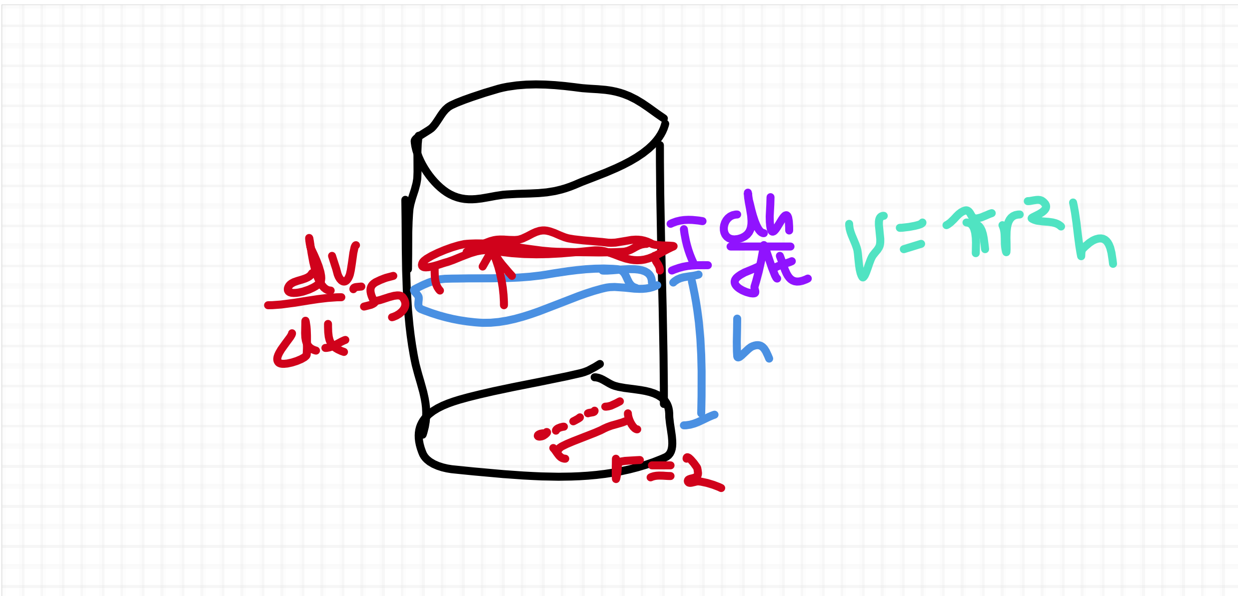
Example 2. *(from Example 1 on page 166)* Water flows into a cylindrical container at a rate of $5\text{in}^3/s$. Assume that the container has a height of 6in and a base of 2in. At what rate is the water level rising in the container.

Let V represent the volume of the water in the container in in^3. Let h be the height of the water. We draw a cylinder, label it, and find how the rates relate



We then find that

$$V(t) = \pi r(t)^2 h(t)$$

and then we use implicit differentiation as follows:

$$\begin{aligned}\frac{d}{dt}(V(t)) &= \frac{d}{dt}\left(\pi r(t)^2 h(t)\right)\\ \frac{dV}{dt} &= \pi 2r(t) \cdot \frac{dr}{dt} \cdot h(t) + \pi r(t)^2 \cdot \frac{dh}{dt}\end{aligned}$$

$$\frac{dh}{dt} = \frac{\frac{dV}{dt} - 2\pi r(t)\frac{dr}{dt}h(t)}{\pi r(t)^2}$$

note that $\frac{dr}{dt} = 0$

$$\frac{dh}{dt} = \frac{5 - 2\pi 2(0)h(t)}{\pi(2)^2} = \frac{5}{4\pi}$$

We did this by doing the following three steps:

Step 1. Identify the variables and the rates that are related.

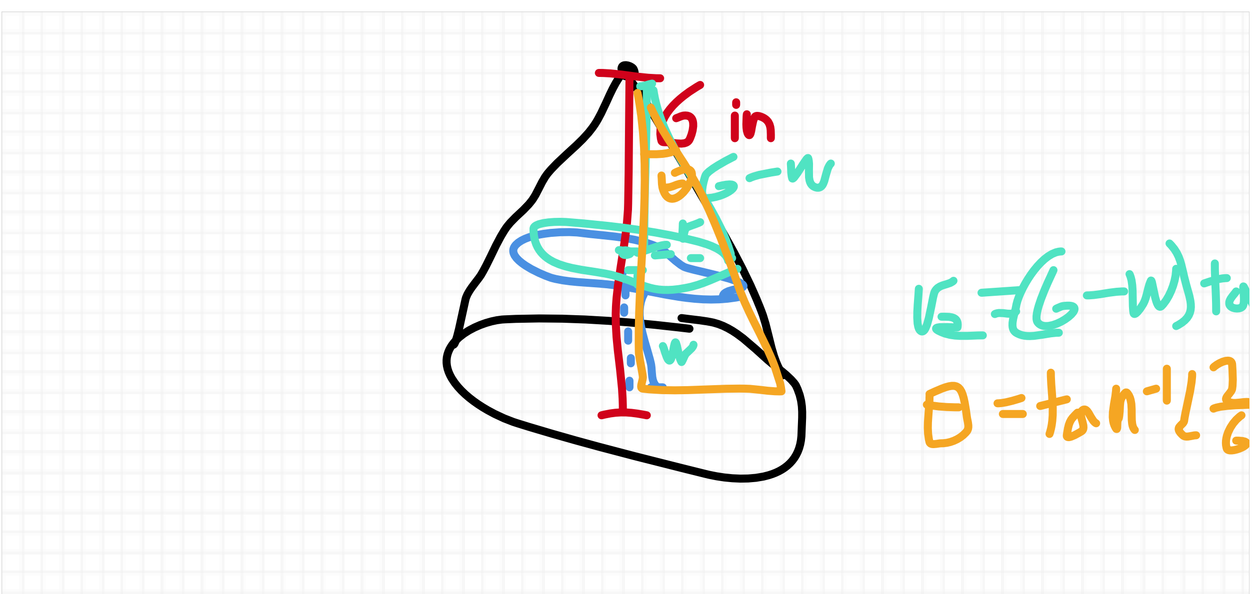
Step 2. Find an equation relating the variables and (maybe implicitly) differentiate it.

Step 3. Use the given differentiated equation to solve for the desired rate.

Example 3. *(from Example 2 on page 166-167)* Water flows into a conical container at a rate of $5\text{in}^3/s$. Assume that the container has a height of 4in and base radius of 2in. Show that the rate that the water level is rising depends on the level of the water in the container, rising faster the higher the water level.

Step 1.

$$\frac{dV}{dt} = 5\text{in}^3/s,\ h = 6\text{in}\ \ r = 2\text{in}\ \ \frac{dr}{dt} = 0$$



Step 2 and 3.

$$r_2(t) = (6 - w)\tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right) = (6 - w)\frac{1}{3}$$

$$\begin{aligned}
V(t) &= V_{\text{container}}(t) - V_{\text{cone from the water point}}(t) = \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_2(t)^2 \\
&= \frac{1}{3}\pi\left(2^2 \cdot 6 - (6 - w)\frac{1}{3}\right) = \frac{1}{3}\pi\left(24 - 2 + \frac{1}{3}w\right) = \frac{1}{3}\pi\left(22 + \frac{1}{3}w\right)
\end{aligned}$$

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{1}{3}\pi\left(22 + \frac{1}{3}w\right)\right] = \frac{1}{3}\pi\left(\frac{1}{3}\frac{dw}{dt}\right) = \frac{\pi}{9}\frac{dw}{dt}$$

which is the relation that we desire to show.

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Second Derivatives

If f is differentiable and if f' is differentiable, the **second derivative** is the derivative of f'. We often this second derivative f'' (instead of $(f')'$).

If f is differentiable n times over, i.e., we have f' existing and f' has a derivative f'', and that derivative has a derivative, and so on (n times), then we call that function the ***n*th derivative** and write $f\underbrace{'' \cdots '}_{n \text{ times}}(x)$ or $f^{(n)}(x)$.

Example 1. *(from Example 2 on page 141)* Calculate $f'''(-1)$ for $f(x) = 3x^5 - 2x^2 + 7x^{-2}$.

First, we calculate $f'(x)$

$$\begin{aligned}
f'(x) &= 15x^4 - 4x - 14x^{-3} \\
f''(x) &= \frac{d}{dx}(f'(x)) = 60x^3 - 4 + 42x^{-4} \\
f'''(x) &= \frac{d}{dx}(f''(x)) = 180x^2 - 168x^{-5}
\end{aligned}$$

So then we plug in -1 and get

$$f'''(-1) = 180(-1)^2 - 168(-1)^{-5} = 180 + 168 = 348$$

Example 2. Find f' and f'' and f''' of the following functions:

1. $f(x) = 14x^2$

$f'(x) = 28x$
$f''(x) = 28$
$f'''(x) = 0$

2. $f(x) = 7 - 2x$

$f'(x) = -2$
$f''(x) = 0$
$f'''(x) = 0$