Probability and Counting Lesson 4: Permutations and Combinations

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So a lot of our counting involves ordering things without replacement, and we can use the multiplication principle repeatedly, but we can also look at things as permutations and combinations and speed things up.

Finding the Number of Ways to Order n Things

Let's say we have a set A with n elements. Let's say that we're trying to find the set S of possible ways to order the n elements
$S = \{x : x$ is an ordering $(a_1, \ldots, a_n)$ of the n elements of $A\}$
To order it, we can look at S as a n staged experiment with each sample space having one less element than the last (because we're ordering a set by picking an element, noting the order, *without replacement*) What we get is
$n(S) = n \cdot (n-1) \cdot (n-2) \cdot \cdots 2 \cdot 1 = n!$

So that's how we do that, and this line of thinking opens up the door to permutations, then combinations.

Permutations

Example 4.3 (page 87). A four-letter ID code made from the following set of letters: $\{A, B, C, D, E, F, G\}$, where no letter can be used more than once. How many ID codes can be formed.

Let S be the set of ID codes

Way to think about this, is SLOTS. We have four slots
_ _ _ _

Each slot represents a stage in the 4 stage experiment we use to determine the ID code by picking a letter, putting it in the first slot, then repeating the process three more times.

This gives us sample spaces $S_1,\ S_2, S_3,$ and S_4 each with one less number of possibilities than the last, so we use the multiplication principle, and get

$n(S_1) = 7$
$n(S_2) = 6$
$n(S_3) = 5$
$n(S_4) = 4$

$n(S) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ different codes.

This kind of calculation can be generalized to when we have an arbitrary number n of objects when we start selecting, and some $1 \le r \le n$ of selections.

When we select r objects from a collection of n objects in order without replacement, we call that an (n, r)**-permutation**. And we refer to this sequence as simply a **permutation** when the context of n and r is known.

We often want to find $n\big((n, r) \text{ permutations}\big)$ and we use notation $P(n, r)$ talk about that number, so

$$P(n, r) = n\big((n, r) \text{ permutations}\big)$$

Then we end up doing a similar multiplication principle calculation, each stage with one less outcome than the last, and get

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) = \frac{n!}{(n-r)!} \ .$$

Combinations

With permutations, we cared a lot about the order in which objects were selected, in other words "order matters".

With combinations, we disregard the order, and look at the selections as a set where "order doesn't matter".

A (n, r)**-combination** for $1 \le r \le n$ is a r-element subset of an n-element set. We again use the term **combination** when the context of n and r is clear.

Example 5.2. (page 97) List all the different strings of letters of the set $\{A, B, C, D\}$

First, we'll find all the possibilities by hand and count them

Combinations vs. Permutations

$A, B, C \qquad ABC,\ ACB,\ BCA,\ BAC,\ CAB,\ CBA$
$A,\ B, D \qquad ABD,\ ADB,\ BAD,\ BDA,\ DAB,\ DBA$
$A, C, D \qquad ACD,\ ADC,\ CAD,\ CDA,\ DAC,\ DCA$
$B,\ C, D \qquad BCD,\ BDC,\ CBD,\ CDB,\ DBC,\ DCB$

(if I got that miscopied, I'll correct it later)

The point is, there are 4 combinations, every combination can be ordered 6 different ways to form a permutation. The fact that there is a uniform number of permutations for every combination is no coincidence.

Let's generalize the above above to n element set, and finding the number of possible r element subsets. We get $r!$ ways to make any given combination into a permutation because (as mentioned before) there's $r!$ ways to order a combination. So we get

$P(n, r) = r!\ \cdot n\big((n, r)\text{-combintations}\big)$.

We use $C(n, r)$ as notation for $n\big((n, r)\text{-combintations}\big)$, we have
$C(n, r) = n\big((n, r)\text{-combintations}\big)$.

And using the formula we know for $P(n,\ r)$, we get

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}.$$

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Example. How many 4-element subsets can be slected from a set containing 7 elements?

The question is asking how many $(7, 4)$-combinations, so to do that, we use the formula
$$C(7, 4) = \frac{7!}{(7-4)!4!} = \frac{7\ \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

Telling the Difference Between Permutations and Combinations

Example 5.7 a. (page 103)

Compute the Probability of selecting the Ace of hearts, king of hearts, and queen of hearts.

NOTE: We don't care about the order, so we use $(52, 3)$-combinations as our sample space (which are *equally likely*)

$n(E) = n\left(\text{selecting the Ace, king, and queen of hearts}\right) = 1$

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{1}{C(52, 3)}.$$

Let's tweek the wording as follows:
Compute the probability of selecting the Ace of hearts, THEN the king of hearts, THEN the queen of hearts.

Now order matters, and we use *permutations*, and to cut to the chase, we get

$$Pr[E] = \frac{n(E)}{n\left(\text{permutations}\right)} = \frac{n\left(\left\{\left(\text{ace of hearts, kings of hearts, queen of hearts}\right)\right\}\right)}{n(\text{permutations})} = \frac{1}{P(52, 3)}.$$

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Homework 4 and 5 Questions

Homework 4 question 4. If n is any positive whole number, what is $P(n, 1)$?

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1}{(n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1}$$

Note that there are terms at the top cancelling out with terms at the bottom, in particular

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n.$$

The answer is n (i.e., the positive integer n you started with) .

Homework 4 question 7. Production of Peter Pan with 4 parts:

Peter   John   Michael   Captain Hook

Eight boys try out for the part, how many different casts are possible (we assume one person can only be assigned one part)

Let's identitfy the permutation. We are selecting 8 boys to be filled in 4 slots, so we have an $(8, 4)$-permutation

$$n\left(\text{possible casts}\right) = P(8,4) = \frac{8!}{(8-4)!} = \frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 8 \cdot 210 = 1680$$

Homework 4 question 6. (assume they aren't chosen more than once) Seven students. Three of seven will be chosen to head subcomittees on:

scholarship   intramural sports   entertainment

We $(7, 3)$-permutations

$$n\left(\text{possible assignments}\right) = P(7,3) = \frac{7!}{(7-3)!} = 210$$

NOTE: There's a difference between the permutation and combination formula. In particular

$$P(n,\ r) = \frac{n!}{(n-r)!} \quad C(n,r) = \frac{n!}{(n-r)!r!}$$

With $P(7,3)$, you plug it into the formula and get

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!}$$

Homework 5 question 10. A bag with nine balls numbered $1,\ \ldots, 9$. You select a set of 3 balls. How many different sets are possible?

In this problem, note that it states we select a *set of 3 balls*, in particular, order doesn't matter, and this is combinations, so in this case, we use the combinations formula.

We end up with $(9, 3)$-combinations because we end up counting the possible subsets of three items we get from a 9-item set

What to look for: What is the set n we're picking from to get the subsets and what is the amount of items r in each subset. In this situation, we have (n, r)-combinations

$$n\left(\text{possible set of 3 balls}\right) = C(9,3) = \frac{9!}{(9-3)!3!} = \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 3 \cdot 4 \cdot 7 = 84\,.$$

Let's change the wording of homework 5, question 10, so that we end up with permutations:

Let's say that instead of selecting a "set" of three balls, we instead select one ball, then record the number, and select two more and record the numbers *without replacement*. What ends up being the number of possible recordings here?

Here, order does matter, because we're counting the balls in an ordered sequence, and not an unordered set.

$n\left(\text{possible recordings}\right) = P(9,3) = \dfrac{9!}{(9-3)!} = \dfrac{9!}{6!} = 9 \cdot 8 \cdot 7 = 84 \cdot 6 = 504.$

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More Homework 4 and 5 Questions

Homework 5 Question 13. 7 CDs, including three by Barenaked Ladies, two by the Beatles, and two by Clay Aiken.

Randomly select two of them to listen on your way to class.

What is the probability that neither is a Beatles CD?
$S = (7,2)$-combintations
The sample is $(7,2)$-combinations because we're selecting two CDs out of seven CDs, so we're looking at all the possible subsets (of CD's) with 2 elements of a seven element set (the CD's)

$n(S) = C(7,2) = \dfrac{7!}{(7-2)!2!} = \dfrac{7 \cdot 6 \cdot 5!}{5!2!} = \dfrac{7 \cdot 6}{2 \cdot 1} = 21$

E = neither is a Beatles CD = all the combinations that are not beatles CDs =
There are five CDs total that are not beatles CD's, and our event is a selection of 2 CD's of those that are not beatles CDs, and the selection doesn't care about the order, so we again end up with $(5,2)$-combinations, so we get
E = (5,2)-combinations
So we have

$n(E) = C(5,2) = \dfrac{5!}{(5-2)!2!} = \dfrac{5 \cdot 4 \cdot 3!}{3!2!} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

which CD's are not beatles CD's? five of seven are not beatles CD's

So we have
$Pr(E) = n(E)/n(S) = 10/21$

NOTE: Often to count the outcomes for an event in a sample space of combinations, you also count combinations for that event.

IN GENERAL: Whenever your sample space ends up being (n, r)-combinations of some kind (so it may not literally be stated in the problem as $(n,\ r)$-combinations, but the problem when you read and interpret it might be in a way where the sample space is such combinations (because remember it's a reading class not a math class), then we compute the sample using the $C(n, r)$, formula, so

$$n(S) = C(n, r) = \frac{n!}{(n-r)!r!}.$$

Homework 5 Question 15. NOTE: It's (rather implicitly) assumed that the jumping contest is a random selection of two "top finishers".

In this problem we have order in the form of top six places, so order matters. There are six frogs and we are considering the possible places that each of the six frogs can finish in the jump race, so the sample space

$S =$ the possible order that each of the six frogs finish in the race

$\quad =$ (6,6)-permutations

We find that

$$n(S) = P(6,6) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1.$$

The event E is the number of places where the frogs Romeo and Goliath will be among the top two finishers. E is done in six stages, with six slots, where we can compute the number of outcomes using the multiplication principle. Each of the slots contain the following

place 1: Romeo or Goliath
place 2: one of Romeo or Goliath (whichever remains)
place 3: one of the other four frogs
place 4: one of the other three frogs that remain of the four
place 5: one of the other two frogs that remain of the four
place 6: the last frog remaining of the four.

Note that as we place a frog into a slot from either the pair consisting of Romeo or Goliath or the four frogs, so as we go down through the places, there is one less frog available every time we go down through the places. We find that

$$n(E) = 2 \ \cdot 1 \ \cdot 4 \ \cdot 3 \cdot 2 \cdot 1$$

We compute

$$Pr[E] = \frac{2\ \cdot 1\ \cdot 4\ \cdot 3 \cdot 2\ \cdot 1}{6\ \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{2}{30} = \frac{1}{15}.$$

Homework 5 Question 16. The jelly beans are being selected as a collection, so *order does not matter*. More specifically, our sample space S is made up of $(10,4)$-combinations, since we are randomly selecting five balls as a subset of ten balls total, we have

$$n(S) = C(10,4) = \frac{10!}{(10-4)!4!} = \frac{10\ \cdot 9\ \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210.$$

The event E is the event that all the balls selected being red, i.e., a subcollection of 4 balls are selected out of the 5 red balls available, so we have $(5,4)$-combinations, hence

$$n(E) = C(5,4) = \frac{5!}{(5-4)!4!} = \frac{5\ \cdot 4!}{1!4!} = 5,$$

so we have

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{5}{210} = \frac{1}{42}.$$

Homework 5 Question 17. A code is an *ordered* sequence of letters, hence order matters. The sample space S is made up of $(7,4)$-permutations, since we are selecting 4 letters out of a set of 7. We find

$$n(S) = P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4$$

The event E is the string of letters ending in AZ. So in the slots of letters, we have the final two slots determined with A and Z and there are five possible letters going to the first two slots. So the slots are as follows:

slot 1: Any letter of four letter
slot 2: Any letter of four letter (after
slot 3: The letter A
slot 4: The letter Z

We find by the multiplication principle that

$$n(E) = 5\ \cdot 4 \cdot 1\ \cdot 1,$$

and it follows that

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{5 \cdot 4 \cdot 1 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4} = \frac{1}{7 \cdot 6} = \frac{1}{42}$$

Telling the Difference Between Permutations and Combinations Part 2

REMEMBER: This formula is the probability for an event E in a sample space S with equally

likely outcomes (in general, unless otherwise spedified, **any probability experiment has equally likely outcomes**)
$Pr[E] = n(E)/n(S)$
E is an event of the sample space S.

Example. There are 9 books on a shelf, including the new Twighlight book, In how many many ways can you slect 4 out of the 9 books to take with you on vacation?

Is it permutations or combinations?

Order doesn't matter, so we do combinations. What type of combinations do we have to select 4 books out of all the books total

$S =$ (9,4)-combinations

$$n(S) = C(9,4) = \frac{9!}{(9-5)!4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{9 \cdot 2 \cdot 7}{1} = 126,$$
because the 8 calcels out with the 4, and the 6 cancels with the 3 and the 2.

Example. There are seven track runners in the finals of the 100 meter dash. Let's say the top five score points, so in how many ways can

1. there be a top 5?

Permutations or combinations? combinations, because we're talking about the subset of the top five
What type of combinations are we counting for S? $(7,5)$-combinations

$$n(S) = C(7,5) = \frac{7!}{(7-5)!5!} = \frac{7 \cdot 6 \cdot 5!}{2!5!} = \frac{7 \cdot 6}{2 \cdot 1} = 21$$ (the $5!$ cancels out)

2. The number of possible scoring arrangements.

Permutations or combinations? Permutations? What type of permutation $(7,5)$-permutations

The $2 \cdot 1$ cancels out
$$n(S) = P(7,5) = \frac{7!}{(7-5)!} = \frac{7!}{2!} = \frac{7 \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 7 \cdot 6 \cdot 5 \cdot \cdots \cdot 3 = 42 \cdot 60$$
$$= 240 + 120 = 360.$$

Example. Let's say we select a red ball, 2 blue balls, 2 green balls, and 3 orange balls. Let's select 3 balls

a. What is the probability that you select 3 orange balls?

S = (8,3)-combinations, because we don't care about the order we select the balls (and we're finding 3 items from a collection 8)

E = (3,3)-combinations, because we're selecting three orange balls out of three orange balls

$$n(S) = C(8,3) = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$$
$$n(E) = C(3,3) = \frac{3!}{(3-3)!3!} = \frac{3!}{0!3!} = \frac{3!}{1 \cdot 3!} = 1$$

$Pr[E] = n(E)/n(S) = 1/56$

b. Probability that a primary color is selected?

The event E is the possible blue and red balls selected (remember that there are 1 red and 1 2 blue), so there are three such balls total, so
E = (3,3)-combinations
$n(E) = C(3,3) = 1$
$Pr[E] = n(E)/n(S) = 1/56$