Linear Equations Lesson 2: Graphs of Linear Equations

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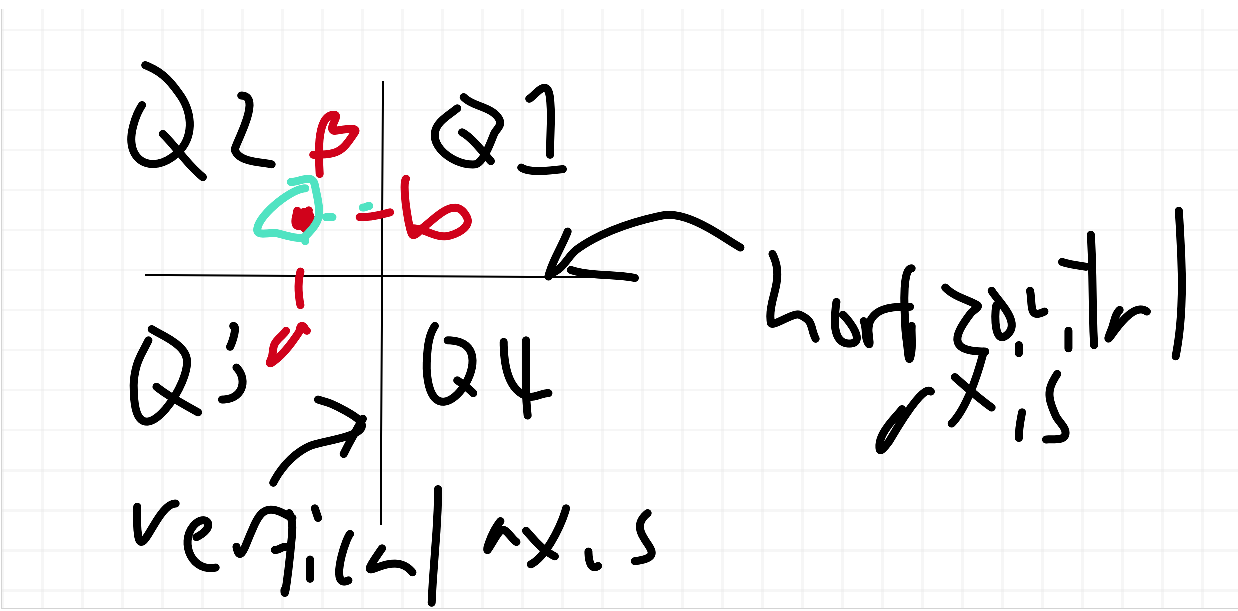
Graphing Lines from Points

So in this course, we graph a lot of lines, and we do it in (2 dimensional) rectangular coordinates.

In a **rectangular cooridinate system (the cartesian plane)**, we designate two lines, one vertical and horizontal that we look at as **axes** and we call:

The axis that runs horizontally we call the **horizontal axis (x-axis)**



The axis that runs vertically we call the **vertical axis (y-axis)**

We have four "quadrants", which are the four different regions that form by two axes of the plane

How do we talk about points (like P illustrated above) We do it in terms of its x value and y value

Coordinate system Notation

We can locate the point in terms of what x value and y value it has, for example, the point P has x value a and y value b.

Note that once we know this information, we can locate P. This is because the cartesian plane can viewed as a product set $\mathbb{R} \times \mathbb{R}$ of two real number lines (in particular, the two axes, which can be viewed as a number line going vertically and number line going horizontally)

Next Time: I'll finish this thought, and then we'll graph some lines!

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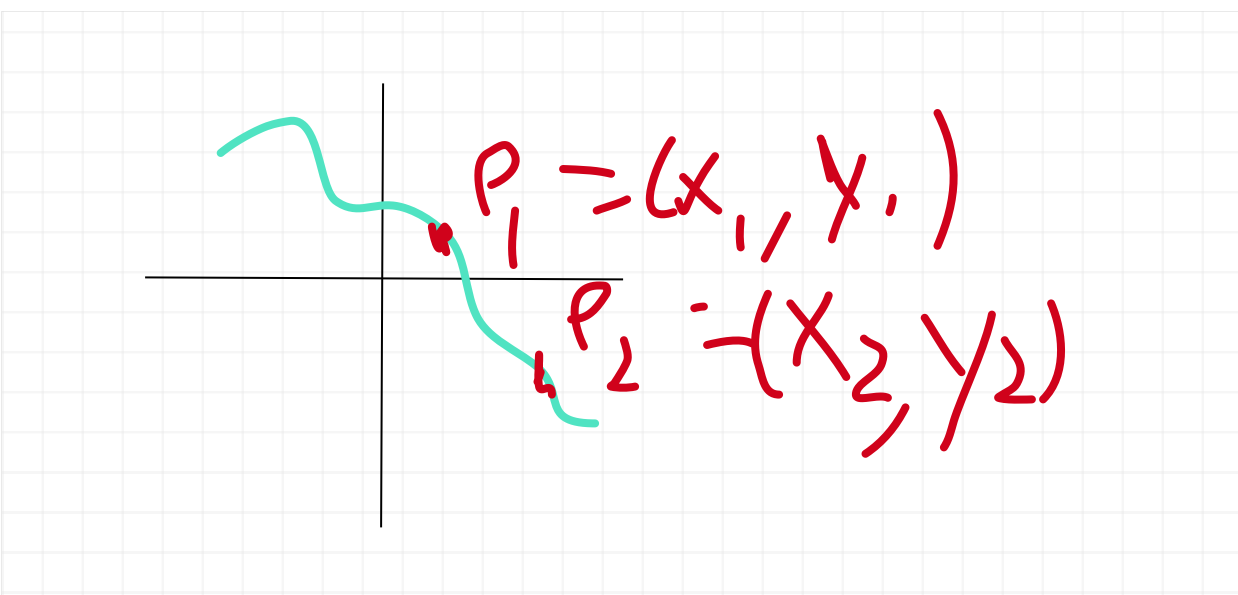
Last Time: Finished 1.5, and went over coordinate system notation in the cartesian plane (the book calls it the rectangular coordinate system)

So when we write (a, b) to refer to a location in the cartesian, we're referring to where it is on the horizontal axis (x-axis) for the a value and the where it is on the vertical axis (y-axis) for the b value.

So in this chapter, we talk about graphs of lines on the cartesian plane.

We often take equations (of lines) and graph them. A **graph of an equation** is a drawing of all the points (x, y) that when plugged in satify the linear equation



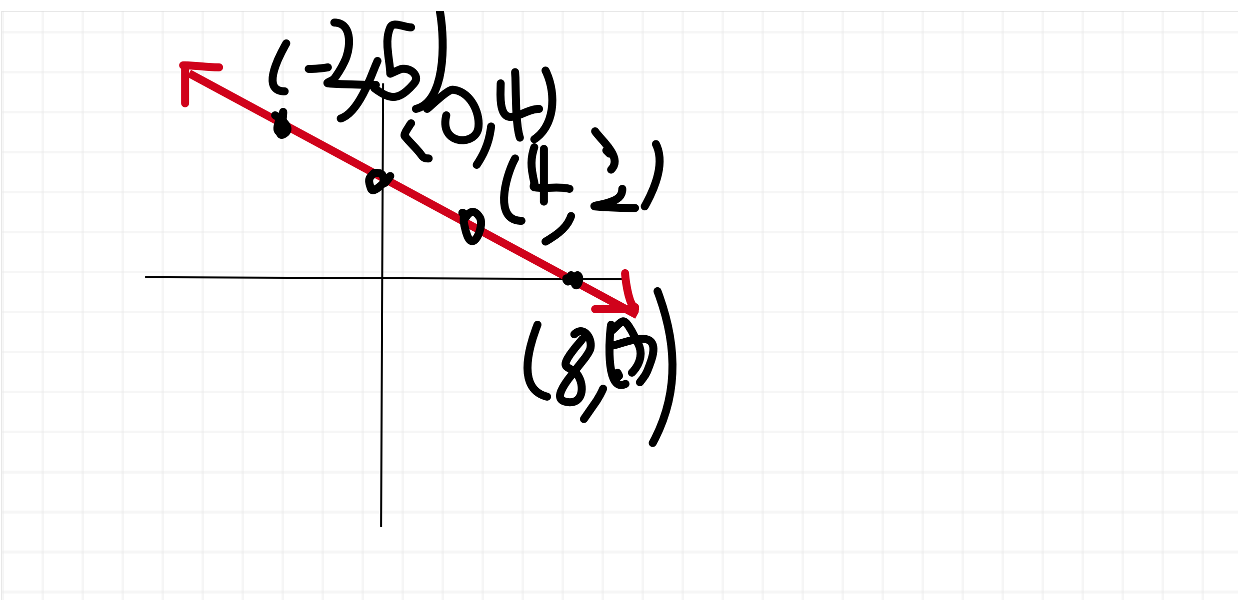
So let's say we have some equation $f(x) = y$ and it represents the blue curve, then $f(x_1) = y_1$ for the $P_1 = (x_1, y_1)$ and $f(x_2) = y_2$ for P_2.

Remember earlier when talked one-variable linear equations, which are of the form
$ax + b = 0$
a and b are arbitrary real numbers

In this course specifically, we graph **two-variable linear equations** are equivalent to equations of the form
$ax + by + c = 0$
$a, b,$ and c are arbitrary real numbers and we treat y and x as variables.

2.1 Example 2 .(page 88) We'll show a graph of $y = -1/2x + 4$

This line is the line that represents that equation, and it is that line because you pick any point that satisfies the equation and plot it and it'll be in that line (as the black points are that I drew)



How do we draw a line represting an equation or between two points?
There's quite a few that we'll cover

One way to graph a line representing an equation is to plot some points in the cartesian plane, and then connect the dots. We'll do this for Example 3.

2.1 Example 3. (page 88-89) Graph $3x + 2y = 12$

1. plot some points

We do this by plugging the x value into the equation, and getting a y value (by solving for y)

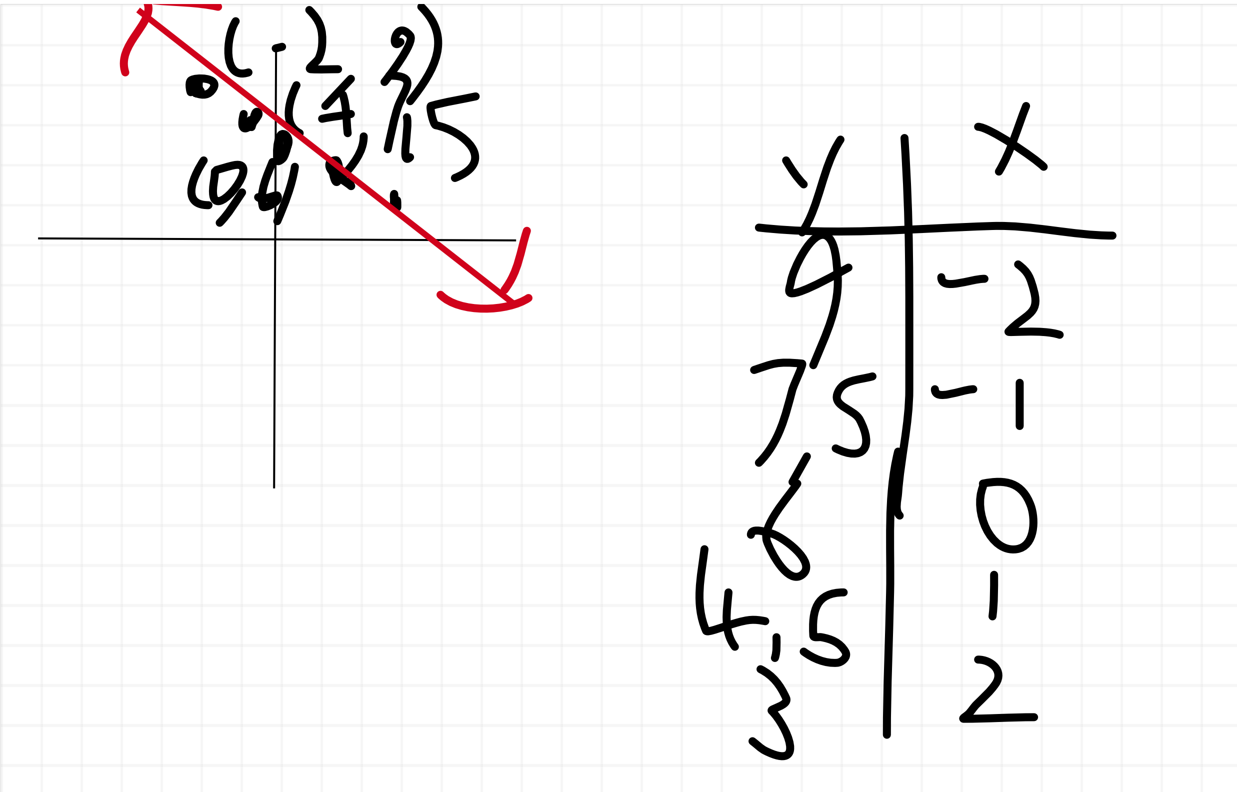
Let's start with -2 We get the equation

$$
\begin{aligned}
3(-2) + 2y &= 12 \\
-6 + 2y &= 12 \\
+6 \quad & \quad +6 \\
2y &= 18 \\
\div 2 \quad & \quad \div 2 \\
y &= 9
\end{aligned}
$$

$$
\begin{aligned}
3(-1) + 2y &= 12 \\
-3 + 2y &= 12 \\
+3 \quad & \quad +3 \\
2y &= 15 \\
\div 2 \quad & \quad \div 2 \\
y &= 7.5
\end{aligned}
$$

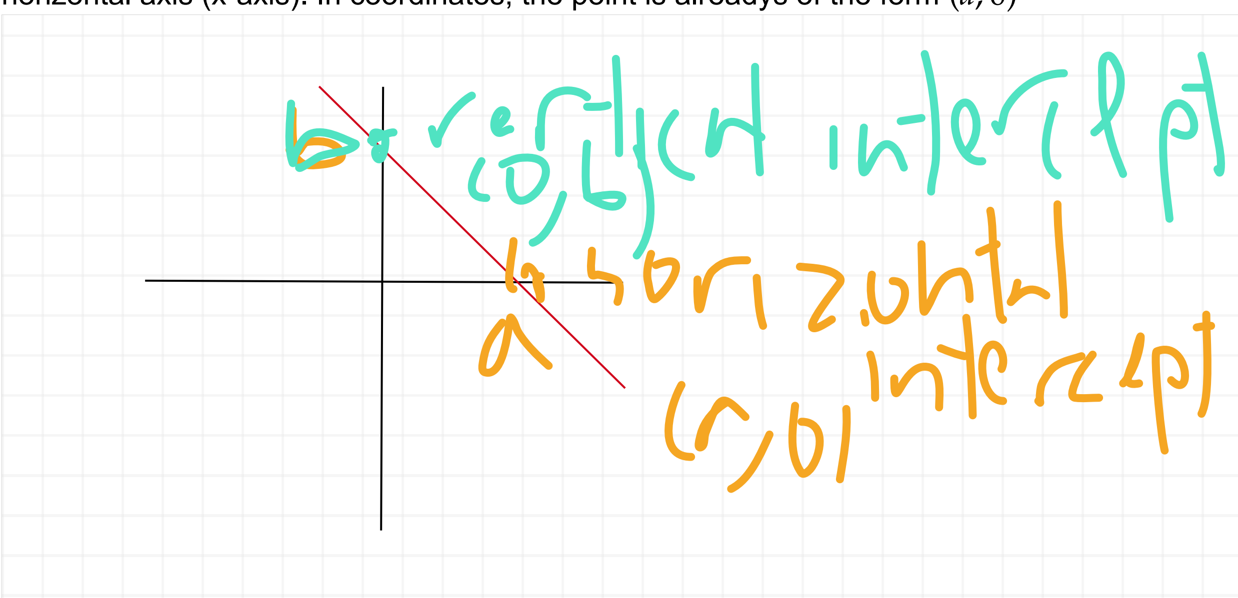
NOTE: There's a bit of a shortcut to all the algebra here, which is solving for y in terms of x and then we can just plug in numbers into the formula

$$
\begin{aligned}
3x + 2y &= 12 \\
-3x \quad & \quad -3x \\
2y &= 12 - 3x \\
\div 2 \quad & \quad \div 2 \\
y &= 6 - 1.5x
\end{aligned}
$$



A **vertical intercept (y-intercept)** of a line s the point of the line that intersects the vertical axis (y-axis). In coordinates, the point is alreadys of the form $(0, b)$

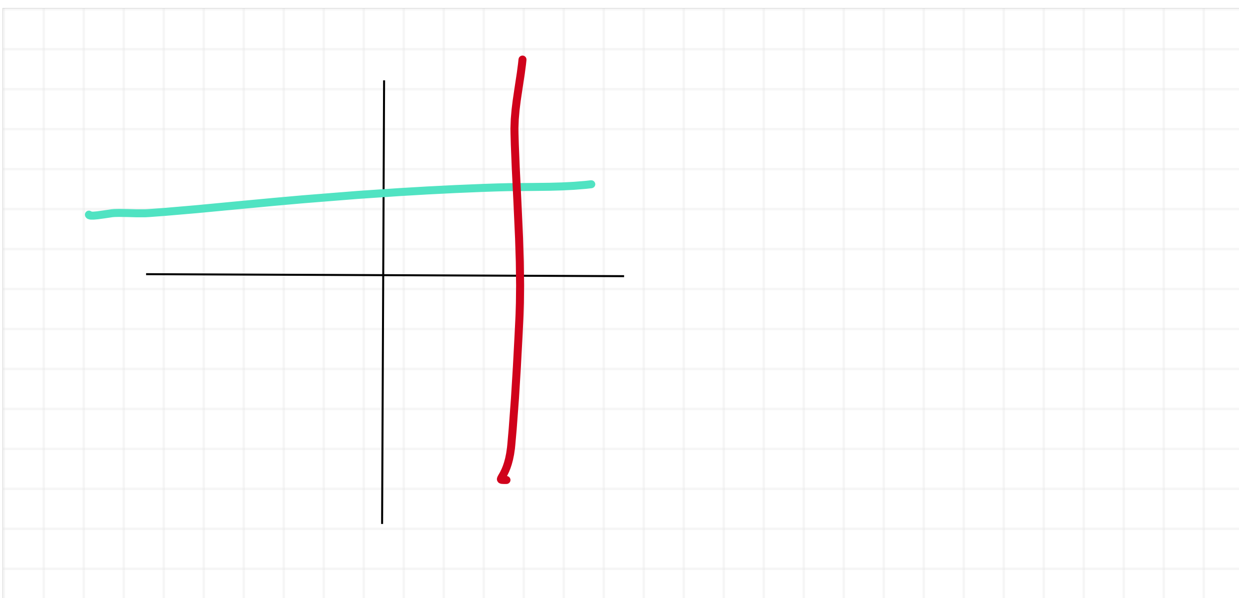
A **horizontal intercept (x-intercept)** of a line s the point of the line that intersects the horizontal axis (x-axis). In coordinates, the point is alreadys of the form $(a, 0)$



So to draw a line, it ends up being a lot easier than plotting a lot of points and connecting the dots. We know that in geometry, we connect two points and we get the line we want. So then all we have to do to plot a line is two points and connect them.

In a situation where the line intersects with the vertical axis and the horizontal axis, we can then draw the line through plotting those two points and connecting the dots together (as demonstrated above)

Note that lines don't always intersect both axes, in particular, lines might be vertical and horizontal.



Vertical lines and horizontal can be drawn in other ways. In particular, their equations are always of the following form:
$x = a$ for a **vertical line**
$x = b$ for a **horizontal line**

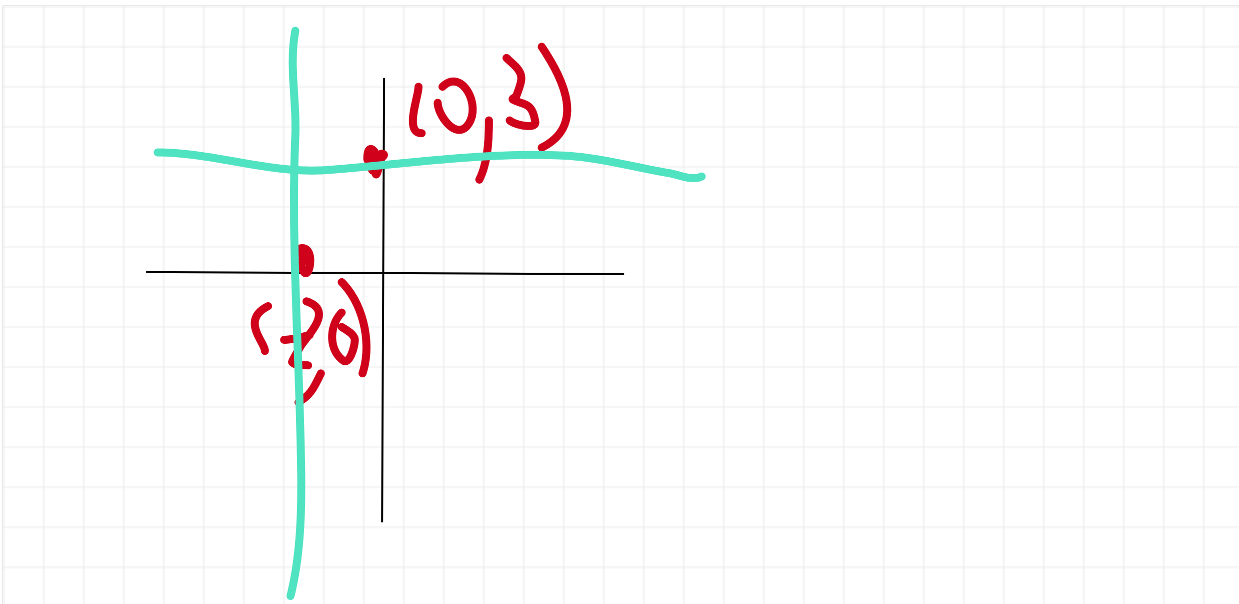
To draw such a line you only need to know one of its points.

Example 5. Graph

a. $y = 3$
b. $x = -2$

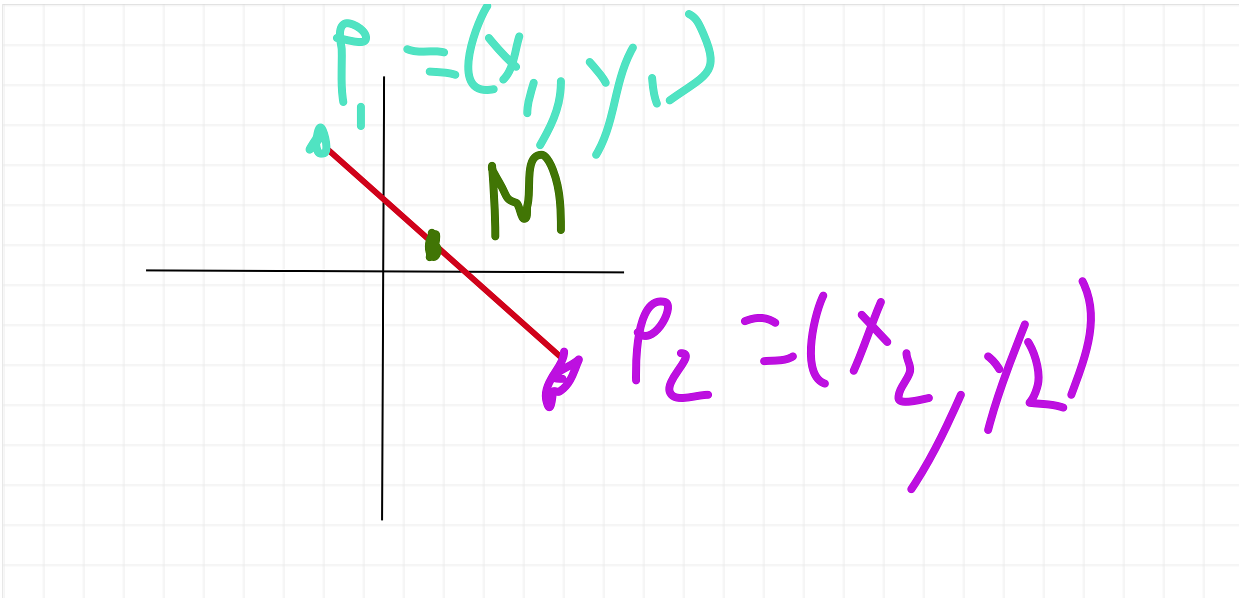
To graph those, we know that $(0,\ 3)$ for $y = 3$ and $(-2,\ 0)$ for $x = -2$ are points on that line, so we plot those points.

Afterwards, we know that $y = 3$ is horizontal so we draw the horizontal line going through that point
And we know that $x = -2$ is vertical so we plot the vertical line going through that point.



We end this section by talking about midpoints between two points.

A **segment** between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is the part of the line between those two points that is between P_1 as the starting point and P_2 as the endpoint. We
label the segment P_1P_2



The **midpoint** M of P_1P_2 is the starting point P_1 and the endpoint P_2.

There is a **midpoint formula** to determine the value of the midpoint M, which is simply average out the vertical and horizontal coordinates of $P_1 = (x_1, y_1),\ P_2 = (x_2, y_2)$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Next Time: We'll go over slopes and I'll demonstrate some homework problems.

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Last Time: We talked about more about the cartesian plane. We talked about lines and how to plot them based on linear equations. And then ended the class by talking about the midpoint formula.

2.1 Example 8. Find midpoint of the line segment joining $P = (-2, 3)$ and $Q = (7, -5)$

So we have (x_1, y_1) the points of P , i.e., $x_1 = -2, y_1 = 3$
We have (x_2, y_2) the points of Q, i.e. $x_2 = 7, y_2 = -5$

We plug in these numbers into the formula to get

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2+7}{2}, \frac{3+(-5)}{2}\right) = \left(\frac{5}{2}, \frac{-2}{2}\right) = (5/2, -1)$$

Questions on Homework 2

39. (page 96)

Plot some points (we only need 2)

$y = -x + 4$

$y = 0$

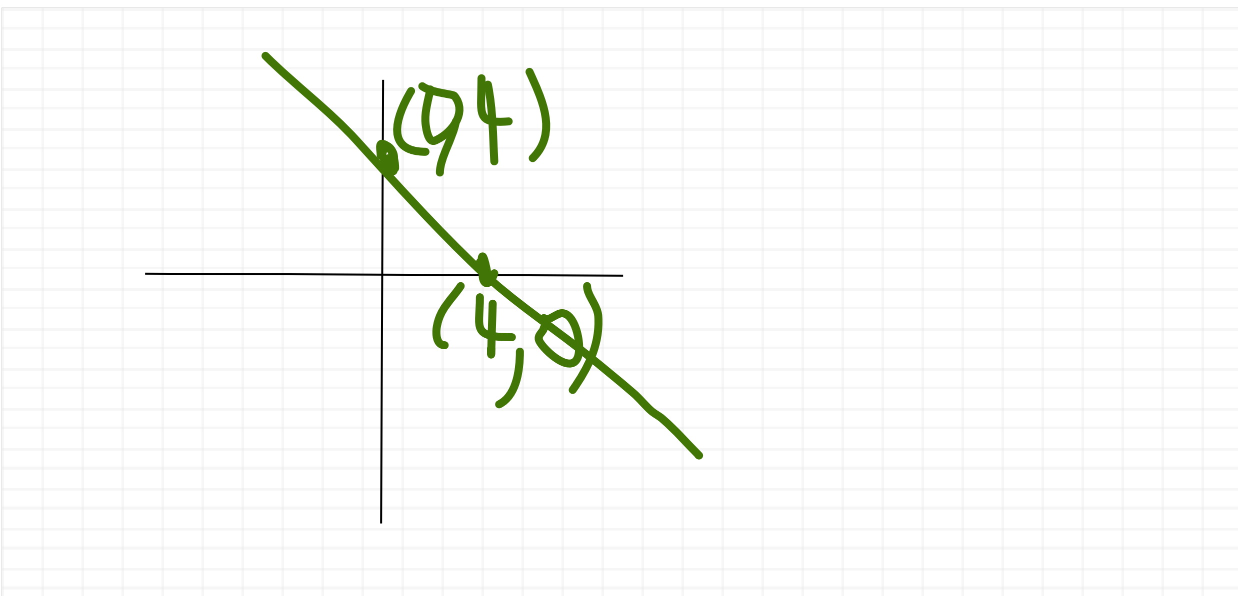
$0 = -x + 4$

$+x \quad +x$

$x = 4$

$x = 0$

$y = -(0) + 4 = 4$



57. (page 97) Want to find the midpoint of PQ

$P = (x_1, y_1) = (0, 0),\ Q = (x_2, y_2) = (6, 8)$

$x_1 = 0,\ x_2 = 6$

$\dfrac{x_1 + x_2}{2} = \dfrac{0 + 6}{2} = 3$

$y_1 = 0,\ y_2 = 8$

$\dfrac{y_1 + y_2}{2} = \dfrac{0 + 8}{2} = \dfrac{8}{2} = 4$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = (3, 4)$$

73. The table gives the amount y (in dollars) that a student can earn for working x hours.

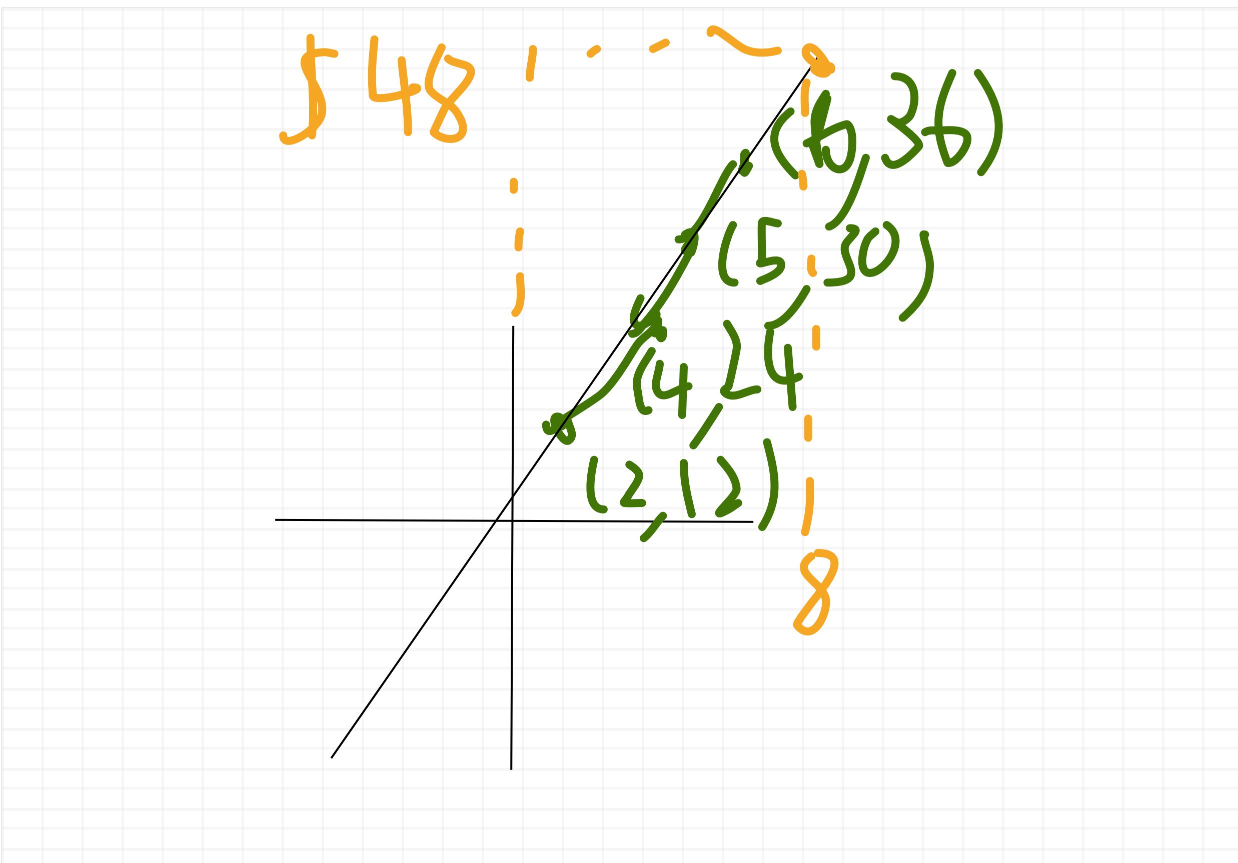
x	2	4	5	6
y	12	24	30	36

Plot the ordered pairs and estimate the amount for 8 hours

Step 1. Let's rewrite the points on the table in terms of coordinates:
$(2, 12),\ (4,\ 24),\ (5, 30),\ (6, 36)$

Step 2. Connect the dots, and drawing out the line that contains dots

Step 3. Find the point 8 and look at approximately what that point is.



Slopes of a Line

The **rate of change** from a point $P_1 = (x_1, y_1)$ to a point $P_2 = (x_2, y_2)$ is the ratio of the vertical change and the horizontal change.

rate of change from P_1 to P_2 $= \dfrac{y_2 - y_1}{x_2 - x_1}$

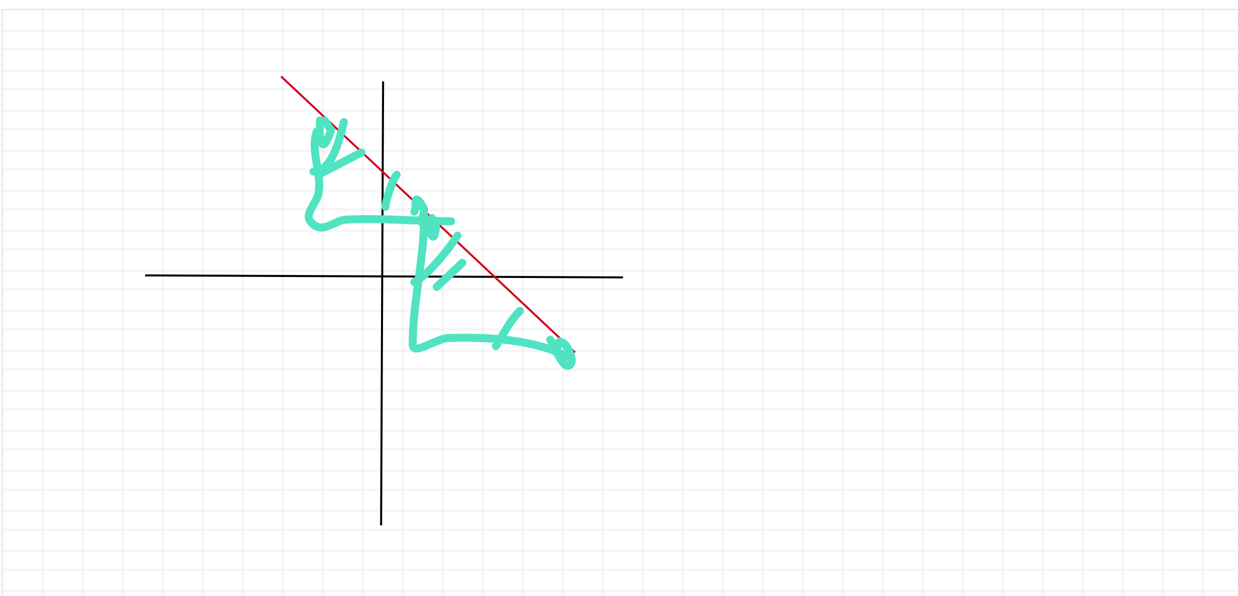
We like to call the quantity $y_2 - y_1$ that measures the difference, i.e., the change in y to get from P_1 to P_2 Δy, and similarly, we call $x_2 - x_1$ Δx

So one another way to talk about the rate of change formula is

rate of change from P_1 to P_2 $= \dfrac{\Delta y}{\Delta x}$.

NOTE: The rate of change does not always exist, and it doesn't exist when $x_2 = x_1$, i.e. $\Delta x = x_2 - x_1 = 0$ (you can't divide by zero)

In a line, we find that the rate of change between any two point is constant when going from any point $P_1 = (x_1, y_1)$ to $P_2 = (x_1, y_2)$



This is true since a line gives us similar triangles when looking at the ratio of the rise over run.

We shall call this constant rate of change the **slope** of the line, whenever the line is nonvertical.

We often use the letter "m" to refr to the slope of a given line, and we get the formula

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{"rise"}}{\text{"run"}},$$

for ANY two points (doesn't matter which) (x_1, y_1) and (x_2, y_2).

2.2 Example 2. (page 102)

Find the slope of the line determined by $3x - 4y = 12$

Find two points of the line (your favorite ones!) and find the rate of change

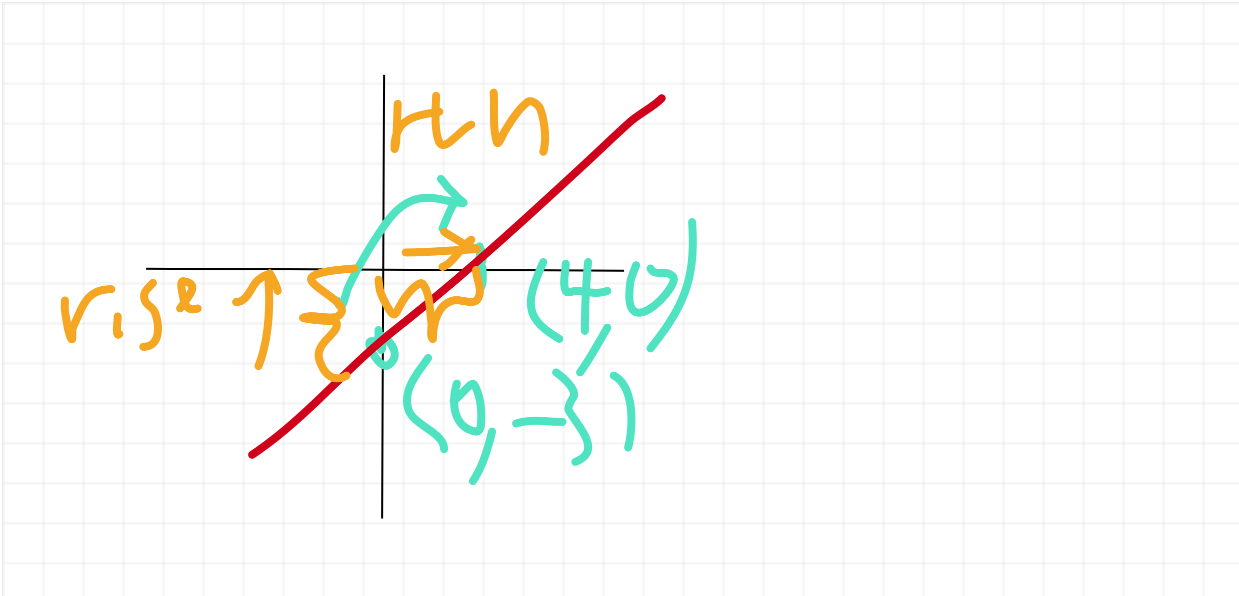
My favorite points are the horizontal and vertical intercepts.

$x = 0$
$3(0) - 4y = -4y$
$-4y = 12$
$\div -4 \quad \div -4$
$y \quad = -3$

$y = 0$
$3x - 4(0) = 3x$
$3x = 12$
$\div 3 \quad \div 3$
$x \quad = 4$

Once we have two have those two points, we can find the rate of change

$y_1 = -3\ y_2 = 0$
$x_1 = 0,\ x_2 = 4$
$m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 - (-3)}{4 - 0} = \dfrac{3}{4}$
So we get $m = 3/4$



Many applied problems involve equations of lines and their slopes.

2.2 Example 4 (page 103). It takes a skier 25 min. to complete the course as showin in page 103. Find the average rate of descent.

We're going $y = 12000$ ft in hight to $y = 8500$ ft. We go from $t = 0$ to $t = 25$

We use the rate of change formula to calculate the "average of descent"

In general, we gave the rate of change formula

$$\frac{\Delta y}{\Delta t} = \frac{8500 - 12000}{25 - 0} = \frac{-3500}{25} = -140$$

NOTE: Be careful not to get the signs mixed up, in particular, be consistent with which value is the initial point P_1 and which value is the final point P_2

Next time: End section 2.1 by talking about parallel and perpendicular lines and their slopes.

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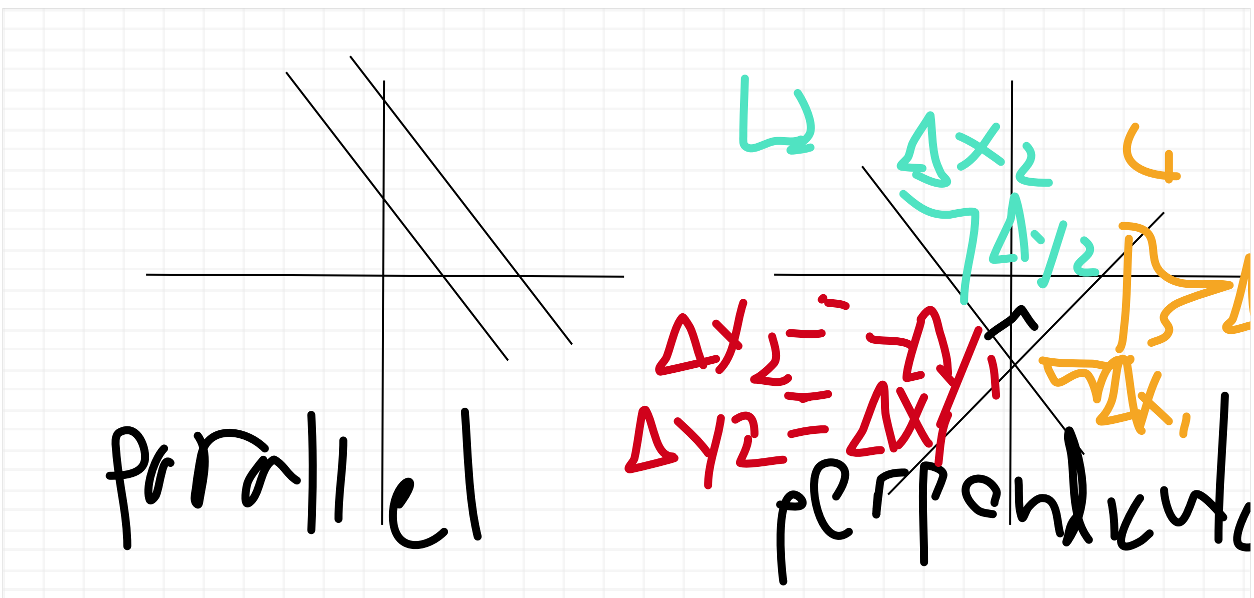
Last time: We defined what a slope was and did some exampes with calculating it.

Today we'll talk about the slpe of parallel and perpendicular lines, and then we'll talk about finding the equations of lines based on information about the slopes and points. (finish off what we'll do in Ch. 2)

NOTE: Only nonvertical have slopes. Horizontal lines always have slope zero (the "rise" is always going to be zero), and vertical lines have no defined slope (in particular, any two points do not change the x value so the rate of change would involve dividing by zero, which we can't do).

Slopes allow us to talk about parallel and perpendicular lines in a quantifiable way. In particular, we can define parallel and perpendicular lines in terms of their slope.

For those who need to recall basic geometry, parallel lines are lines that never intersect. Perpendicular lines are ones that inersect at a 90 degree angle.



So as we can see above, parallel lines necessarily have the same slope, and any two lines that have the same slope are parallel. So for two parallel lines $L_1,\ L_2$ with slopes m_1, m_2, we have

$m_1 = m_2$

What about perpendicular lines?

Since we have $\Delta x_2 = -\Delta y_1,\ \Delta y_2 = \Delta x_1$, we get

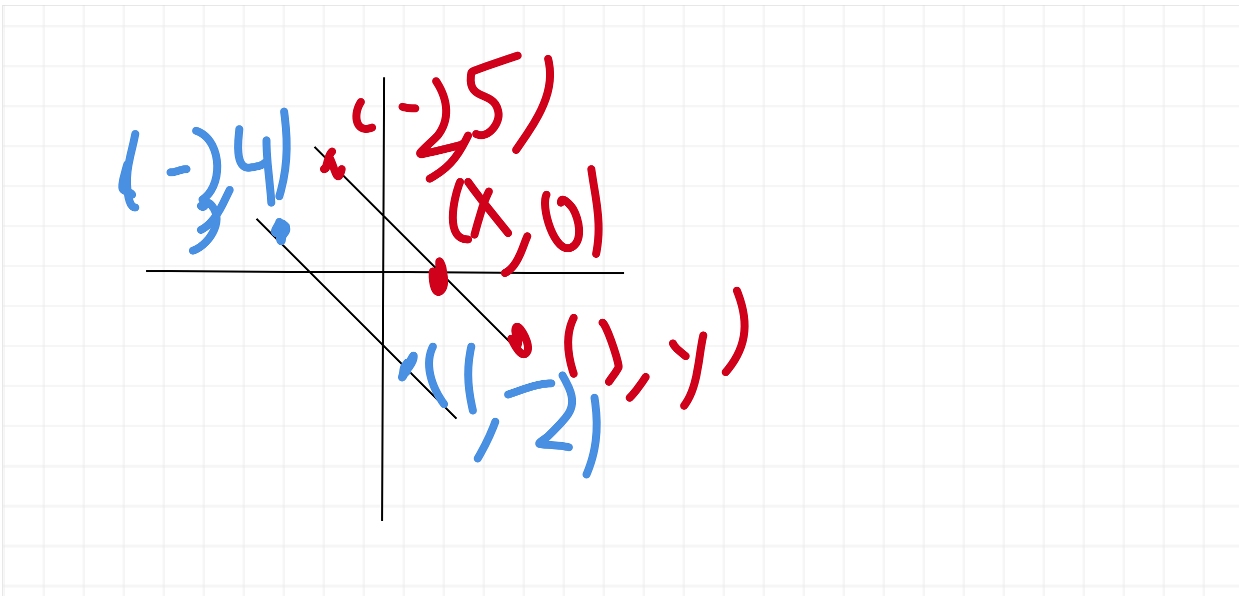
$$m_1 = \frac{\Delta y_1}{\Delta x_1} = \frac{-\Delta x_2}{\Delta y_2} = -1/(\Delta y_2/\Delta x_1) = -\frac{1}{m_2},$$

$$m_1 = -\frac{1}{m_2}$$

for perpendicular lines.

From these properties, we can caclulate parallel and perpendicular lines as follows

2.2 Example 5 (page 105) Figure 2-24



We know two points in the line with the blue points, so let's find m

$$m = \frac{y_2 - y_1}{x_2 - x_1} \frac{-2-4}{1-(-3)} = \frac{-6}{4} = -\frac{3}{2}$$

we know that the slopes equal in these lines, so we can set up an equation using the slope formula.

$$-\frac{3}{2} = m = \frac{0-(-2)}{x-5},$$

$$-\frac{2}{3} = \frac{x-5}{2}$$

$$\begin{aligned}
&\times 6 \quad \times 6 \\
&-4 = 2x - 10 \\
&+10 \qquad +10 \\
&6 = 2x \\
&3 = x
\end{aligned}$$

We can do a similar thing to get y by setting

$$-\frac{3}{2} = m = \frac{y - (-2)}{3 - 5}$$

and then solve for y

2.2 Example 6 (page 106) Are the lines illustrated below perpendicular?

L_1 has points $(0, 0)$ and L_2 point $(9, 4)$ and they intersect at $(3, -4)$.

Let's find their rate of change and see if one is the negative reciprical of the other.

$$m_1 = \frac{-4 - 0}{3 - 0} = -\frac{4}{3}$$

$$m_2 = \frac{-4 - 4}{3 - 9} = \frac{-8}{-6} = \frac{4}{3} = -\frac{1}{m_1},$$

so the answer is yes the lines are perpendicular.

Questions on Homework 3

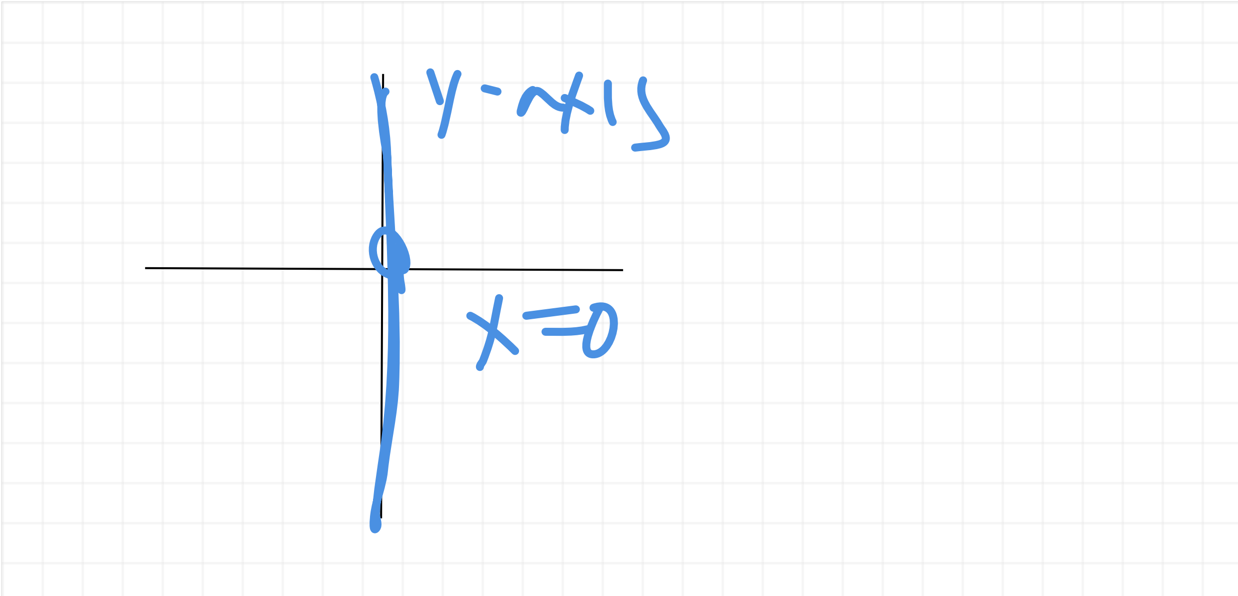
34. $x = y$; find the slope.

How we find the slope of a line is the rate of change between any two points (we can pick our favorite).

One of my favorite points is the origin $(0, 0)$, Let's plug in 1 for x to get another point $(1,\ 1)$.
Then use the rate of change formula

$$m = \frac{1 - 0}{1 - 0} = 1.$$

64. (page 106)
$x = 0$
No slope



Writing Equations of Lines

To write an equation of a line, the following information suffices:

1. point and the slope (gives us the slope-intercept formula, i.e. "$mx + b$")

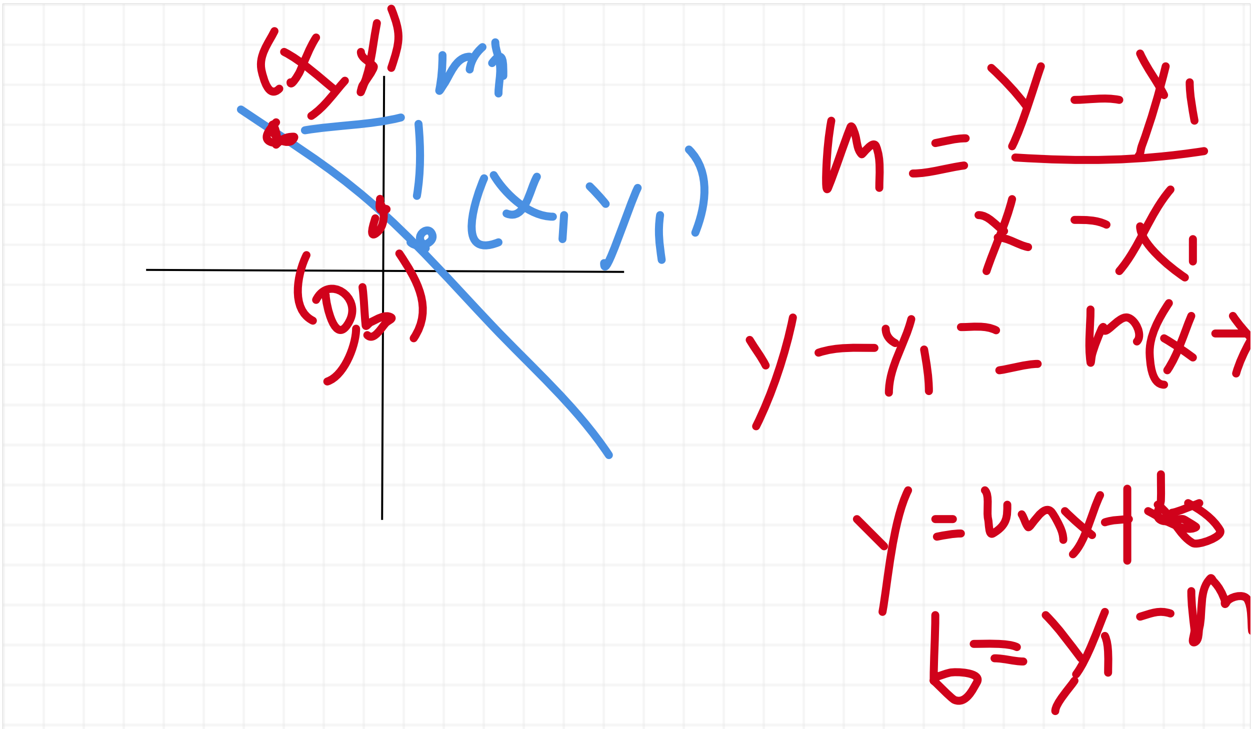
2. two points (gives us the point-slope form)

3. general linear equation

Here, I'll derive each of these formulas.

1. slope intercept formula

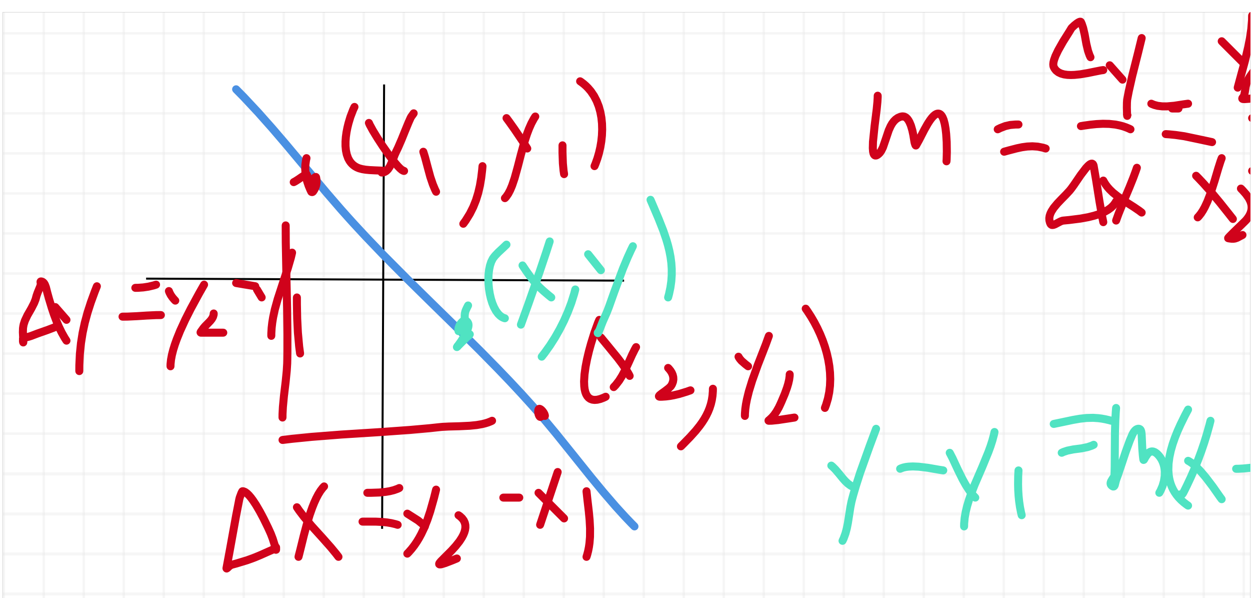
If we have a point (x_1, y_1) and its slope m



We get
$y = mx + b,$
where $b = y_1 - mx_1$.

2. point slope formula

If we have a point (x_1, y_1) and (x_2, y_2) then we can get the point-slope formula of a line as follows



We get
$y - y_1 = m(x - x_1)$
where
$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

3. the general form
Such a formula is a two variable linear equation of the form $Ax + By = C$, where A, B, C are real numbers. If we know that and we know that the line is nonvertical (i.e. $B \neq 0$), we get $m = -\dfrac{A}{B}$ and the y-intercept is $\left(0, \dfrac{C}{B}\right)$.

2.3 Example 3. (page 111-112) Use the slope-intercept form to write the equation of the line with $m = 4$ that passes through $(5, 9)$
We need to know b since we know that
$y = 4x + b$

and we know $b = y_1 - mx_1 = 9 - 4 \cdot 5 = -11$, so we have
$y = 4x - 11.$

2.3 Example 2. (page 110) Write the equation of the line passing through $(-5, 4)$ and $(8, -6)$.

So we first need to find m by finding the rate of change as follows:
$m = \dfrac{-6 - 4}{8 - (-5)} = \dfrac{-10}{13} = -10/13$

Next, choose the point $(-5,\ 4)$ (note that $(8,\ -6)$ works as well) and then we get

$y - 4 = -\dfrac{10}{13}(x - (-5))$

$y - 4 = -\dfrac{10}{13} \cdot (x + 5)$

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Last time: Went over parallel and perpendicular lines and formulas for how to calculate the equation of a line

2.3 Example 7. Write the equation of the line passing through $(-2, 5)$ and parallel to the line $y = 8x - 3$.

One of the points is $(-2, 5)$ and we can calculate the slope of the line we want from knowing it's parallel to $y = 8x - 3$. So based on that information, how do we compute the slope?

$y = mx + b$ $m = 8,\ b = -3$

We have $m = 8$ and we have $(-2, 5)$ so how do we calculate the equation?

$m = 8$
$b = y_1 - mx_1 = 5 - 8 \cdot (-2) = 5 + 16 = 21$

So we have
$y = mx + b = 8x + 21$

2.3 Example 8. Write the equation of the line passing through $(-2, 5)$ and perpendicular to the line $y = 8x - 3$

We can calculate it like the previous example except to the slope formula for the perpendicular line
$m_1 = 8$
$m_2 = -\dfrac{1}{m_1} = -\dfrac{1}{8}$
$b = y_1 - m_2x_1 = 5 - \left(-\dfrac{1}{8}\right) \cdot (-2) = 5 - \dfrac{2}{8} = \dfrac{20}{4} - \dfrac{1}{4} = \dfrac{19}{4}$
$y = -\dfrac{1}{8}x + \dfrac{19}{4}.$

2.3 Example 9. Show that lines represented by $4x + 3y = 7$ and $3x - 4y = 12$ are perpendicular

To do that, note that these lines are both in the general form $Ax + By = C$, so the slope m_1 representing $4x + 3y = 7$ is given by the equation

$m_1 = -\dfrac{A}{B} = -\dfrac{4}{3},$
since $A = 4$ and $B = 3$ for the equation of the first line. For the second line we have

$m_2 = -\frac{3}{-4} = \frac{3}{4}$.
$m_1 = -\frac{1}{m_2} = -\frac{1}{(3/4)} = -\frac{4}{3}$
it fits the criteria for being perpendicular since the slopes are negative recipricals of each other.

Questions on Homework 4

None.