Probability and Counting Lesson 3: Product Sample Spaces

9/8

Often probability experiments are done in multiple stages, such as rolling a die twice, flipping a coin 100 times, or even rolling a die then flipping a coin.

To make sense of the sample of these kinds of probability experiements, we have a helpful concept of product sample spaces and the "multiplication principle" (I'll get to the full multiplication principle in the next class)

Product Sets and Product Sample Spaces

First, let's define a product set. Given two sets A and B, we define the set
$A \times B = \left\{x : x \text{ is an ordered pair } (a,\ b) \text{ with } a \in A \text{ and } b \in B\right\},$
to be the **product set**

Given two probability experiments with sample spaces S_1 and S_2 respectively the sample space of the two-staged probability experiment where the outcomes are to do the first experment then the second one and record each outcome of S_1 then S_2 in the order that it happens is called the **product sample space** $S_1 \times S_2$.

Example. (homework 1 problem 3) What is the sample space of the probability experiment where one rolls a dice then flips a coin.

So here, we have two-staged experiment where
$S_1 = \{1, \dots, 6\}$ (because we're rolling a dice)
$S_2 = \{H, T\}$ (because we're flipping a coin)

The outcomes of this experiment are the ordered pairs (s_1, s_2) where s_1 is a whole number between 1 and 6 and s_2 is either H or T
$S_1 \times S_2 = \{(1,\ H),\ (2,\ H),\ (3,\ H),\ \dots, (6,\ H),\ (1,\ T),\ \dots, (6,\ T)\}$

Next Time: Talk about finding the number of outcomes of these large sample spaces using the multiplication rule, and (if time) we'll get into permutations (lesson 4).

9/9

Last Time: Talking about products of two sets, and two staged probability experiments, which
are modeled by product sample spaces.

Now we'll generalize this idea to k staged probability experiments, which gives rise to "The Multiplication Principle", which will be a helpful tool to compute large sample spaces (and events).

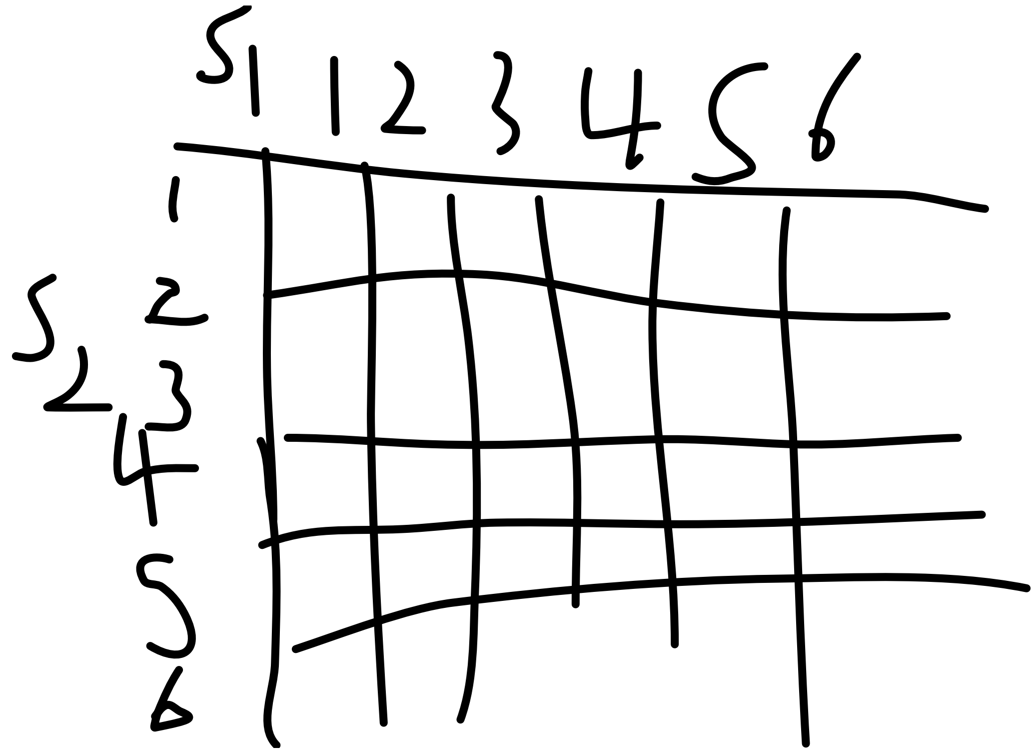
Let's continue off of our previous example. In particular, with rolling a dice, then flipping, and recording the results (in an ordered pair) what is the total number of outcomes in that sample space S?

$n(S) = 12$ Why is that?

	1	2	3	4	5	6
H	(1, H)	(2, H)	(3, H)	(4, H)	(5, H)	(
T	(1, T)	(2, T)	(3, T)	(4, T)	(5, T)	(

Observe that we have a rectangle when we find the ordered pair outcomes of a two staged experiment.

This is also true for rolling a dice two times (Example 1.6) page 60. Here, we have
$S_1 = S_2 = \{1, \ldots, 6\}$



As a result, we have 6 by 6 grid of outcomes, and finding the number of outcomes becomes finding the area of a 6 by 6 square, so $n(S) = 36$.

In general, for a two-staged experiment with S_1 and S_2 as the sample space of each staged experment, to find the amount of outcomes of $S = S_1 \times S_2$ it boils to finding the area of a $n(S_1)$ by $n(S_2)$ rectangle, so we have:

Two Staged Multiplication Principle:
$n(S_1 \times S_2) = n(S_1) \cdot n(S_2)$

This also true for any two sets A and B in general (they need not be sample spaces)
$n(A \times B) = n(A) \cdot n(B)$

How does this work in general? Let's look at what a k-staged experiment entails.

The Multiplication Principle

In general, with k sets $A_1, A_2, \ldots, A_k$ we can define the **product of those sets** to be
$A_1 \times A_2 \times \cdots \times A_k = \{x : x$ is a ordered sequence $(a_1, a_2, \ldots, a_k)$ with $a_1 \in A_1,\ a_2 \in A_2,$
$\ldots, a_k \in A_k\}$

With that, we can define the sample space S of a k-staged probability experiment with sample spaces $S_1, \ldots, S_k$ in each corresponding stage as the **product sample space** $S = S_1 \times S_2 \times \cdots \times S_k$

Examples.

1. Flip a coin k times

2. Roll a dice k times

3. Grab a ball from a bag of balls, record the color, put it back in, then grab the ball again, record it, and do the same thing a third time.

How do we count these complicated sample spaces!?!?!?

Well, doing so is in a way basically the same as doing it for a two staged space.

Example. Roll a dice twice (let S_1 and S_2 indicate each of those dice stages), then flip a coin (Let S_3 indicate that stage)

We know from the 2-staged multiplication principle that
$n(S_1 \times S_2) = 36$
So we can use the two staged multiplication principle again
$n(S_1 \times S_2 \times S_3) = n(S_1 \times S_2) \cdot n(S_3) = 36 \cdot 2 = 72$

The Takeaway: We can use the two-staged multiplication principle repeadly to count the sample space of a 3 staged experiment. We can do this repeatedly for an arbitrary number k of experiments and for arbitrary sizes of $S_1, \ldots, S_k$

The General Multiplication Principle

set $n(S_1) = n_1, n(S_2) = n_2, \ldots, n(S_k) = n_k$. For the k staged experiment involving $S_1, \ldots, S_k$ we get
$n(S) = n_1 \cdot n_2 \cdots n_k$

NOTE: Included this part of the notes after class:
In addition, for any set $A_1, \ldots, A_k$, we get
$n(A_1 \times \cdots \times A_k) = n(A_1) \cdot n(A_2) \cdot \cdots \cdot n(A_k)$

Calculating Probabilities in a Multi-Staged Experiment

Example 3.2 (page 76). Lauren gives three dresses, five scarves, four pairs of shoes, and three hats. Lauren picks a dress, a scarf, and a hat. How many outfits can she make?

What is the size of the multistaged experiment of picking a dress, then a scarf, then shoes, then a hat
$n(S) = n\left(\text{dresses}\right) \cdot n\left(\text{scarves}\right) \cdot n\left(\text{shoes}\right) \cdot n\left(\text{hats}\right) = 3 \cdot 5 \cdot 4 \cdot 3 = 180$.

Let's say **one** of the three dresses is red and **two** of the four pairs of shoes are white. What is the probability of wearing a red scarf and a white pair of shoes?

let E be the event where that happens, and note that it is a product of the number of red dresses, white shoes, and any hats and scarves in general (since the event doesn't impose any condition on the hats or dresses)

$$Pr(E) = \frac{n(E)}{n(S)}$$

We want to find $n(E)$. To find $n(E)$, we use the multiplication principle as well.

$$n(E) = n\left(\text{red dresses}\right) \cdot n\left(\text{scarves}\right) \cdot n\left(\text{white shoes}\right) \cdot n\left(\text{hats}\right) = 1 \cdot 5 \cdot 2 \cdot 3 = 30$$
$$Pr(E) = \frac{30}{180} = \frac{1}{6}.$$

NOTE: This example had a lot of typos when I originally typed it, but they've been fixed now.

9/9

Last Time:

Example 3.7 (page 80).
Example 3.8 (page 79).

Probability Homework 3 Questions:

Question 7.

Four parts are available.
Six girls and 4 boys are trying out

Dorothy and the lion must be played by girls
The wizard and scarecrow must be played by boys

How many different casts are possible?

We use the multiplication principle (of course in the following way)

We have one stage with casting girls, another stage with casting boys.
We set S_1 and S_2 to be the sets in the respective stages
We compute $n(S_1)$ by dividing it further into two substages. First substage, the Dorothy is selected, in the second substage the lion is selected. And with each of these substages, we know the number of people that can be selected. Let $S_{1,1}$ and $S_{1,2}$ be the two substages.
$n(S_{1,1}) = 6,\ n(S_{1,2}) = 5$
$n(S_1) = 5 \cdot 6 = 30$ (using the multiplication principle)

$n(S_{1,2})$ is 5 because one person already got selected by Dorothy, so that means there's only 5 left for the Lion.

Similarly, let $S_{2,1},\ S_{2,2}$ be the substages for S_1 where the first substage is where the wizard is selected,

$n(S_{2,1}) = 4,\ n(S_{2,2}) = 3$
$n(S_2) = 4\ \cdot 3 = 12$ (using the multiplication principle again)

Then for a final time, we use the multiplication principle for S_1 and S_2, and we get
$n(S) = n(S_1)\ \cdot n(S_2) = 30\ \cdot 12 = 360$ (final answer!)

NOTE: The way we did this was pretty inefficient (as I'm sure you can see), so let's make some of these calculations simpler by learning about permutations and combinations.

I'll answer the other two questions (question 3 and 8 after class in the notes)

9/11

More Probability Homework 3 Questions:

Question 3. Staffing the interns is made up of three stages:
S_1 = pediatrics
S_2 = obstetrics

S_3 = emergency services
We assign one student in pediatrics, then one student in obstetrics, then one student in emergency services. It results that each student is picked in each stage *without replacement*, and the number of students in each stage is one less than the previous. We start out with ten students, so we have
$n(S_1) = 10$. It follows that $n(S_2) = 9$ and $n(S_3) = 8$. Using the Multiplication Principle, we get

$$n\left(\text{staffing possibilities}\right) = n(S_1) \cdot n(S_2) \cdot n(S_3) = 10 \cdot 9 \cdot 8 = 720 .$$

ADDITIONAL NOTE: As we think about Lesson 4, note that the different individual assignments of positions creates an order of slots, and specifically a *permutation*. It's totally valid as an alternative (and quicker way), then to use the permutation formula from section 4--and note that the assignment is a $(10, 3)$-permutation--to get

$$n\left(\text{staffing possibilities}\right) = P(10, 3) = \frac{10!}{7!} = 720.$$

Question 8.

a. To find the outcomes, we have six swimmers, and three stages that are the place that each swimmer ends up in (set S_1 to be the sample space of first place winners, and define S_2 and S_3 accordingly for second and third place). Note that after a swimmer finishes in a given place, there is one less amount of possible swimmers that finishes the next place. So we have
$n(S_1) = 6,\ n(S_2) = 5,\ n(S_3) = 4$
Using the multiplication principle, we get
$n\left(\text{possible places}\right) = 6 \cdot 5 \cdot 4 = 120.$

NOTE also that it's totally valid to view the assigned placings as a $(6, 3)$-permutation (which we would do in Lesson 4). Doing it this way, we get

$$n\left(\text{possible places}\right) = P(6, 3) = \frac{6!}{3!} = 120.$$

b. Let E be the event in the sample space S of possible places where Sahsa wins. To do this, we use the multiplication principle and count each of the possible outcomes in stages 1,2, and 3 of this experiment. Where Sasha concievably comes in first. We can think of this visually as the slots:

first place   second place   third place

We do this as follows:

Let E_1 be the event in S_1 containing all the first place outcomes in S_1 where Sasha conceivably comes in first place. There's only *one* such outcome (Sasha), so we get $n(E_1) = 1$.

Let E_2 be the event in S_2 containing all the second place outcomes in S_2 where Sasha conceivably comes in first place. Note that *any* outcome in S_2 could result with Sasha in first place; in other words, what happens with second place (and third place) is independent of what happens with first place. We get $n(E_2) = 5$

Let E_3 be the event in S_3 containing all the third place outcomes in S_3 where Sasha concievably comes in first place. As with E_2, note that *any* outcome in S_3 could result with Sasha in first place. We get $n(E_3) = 4$.

Using the multiplication principle, we get
$n(E) = n(E_1) \cdot n(E_2) \cdot n(E_3) = 1 \cdot 5 \cdot 4 = 20$.

c. To compute the probability of the event E that Sahsa wins, note that every outcomes in the the sample space S of possible places is assumed to be equally likely (note that in the real world, Sasha may be a faster or slower swimmer than most, but just bear with me as I explain this problem). Using the equally likely formula and part a and b, we get
$Pr[E] = \dfrac{n(E)}{n(S)} = \dfrac{20}{120} = \dfrac{1}{6}$.

d. Let E be the event that Sasha comes first, second, or third. For this one, we assume the outcomes are equally likely, use the formula $Pr[E] = n(E)/n(S)$, and count the event E where Sasha comes first, second, or third. There's two ways to do this problem, and I'll do it both ways, but highly recommend the second way.

The first way: Count E directly. We can do this by splitting E into individual disjoint outcomes that are easy to count, and then adding them up as follows.
Set
E_1 = Sasha comes first
E_2 = Sasha comes second
E_3 = Sasha comes third
We find these events are disjoint from each other, since Sasha coming in any place means that she doesn't come in any of the others, so we get $E = E_1 \cup E_2 \cup E_3$, and
$n(E) = n(E_1) + n(E_2) + n(E_3)$
In part c., we've already counted E_1 and know that $n(E_1) = 20$. Counting E_2 and E_3 are

similar to counting E_1 except note that in E_2 and E_3, there are *five* outcomes (not all six) in S_1 where E_2 or E_3 still happens, and there are *all four outcomes* in S_2 where E_3 concievably happens.

To cut to the chase, we use the multiplication principle and get
$n(E_2) = 5 \cdot 1 \cdot 4 = 20$
$n(E_3) = 5 \cdot 4 \cdot 1 = 20.$

We then get
$n(E) = 20 + 20 + 20 = 60,$
and we find
$Pr[E] = \dfrac{60}{120} = 1/2.$

The second way: Use the complement formula $Pr[E] = 1 - Pr[E']$ and calculate the probability $Pr[E']$ of the complement by counting E' where Sasha doesn't finish *any place*. To count E' we can count in terms of stages, then use multiplication principle. Let F_1, F_2 and F_3 be the events in stages S_1, S_2, and S_3 respectively containing the outcomes in the respective stages where E' conceivably happens (i.e., Sasha isn't included, since she doesn't finish any of those places). We get
$n(F_1) = 5,\ n(F_2) = 4,\ n(F_3) = 3,$
and we use the multiplication principle to get
$n(E') = n(F_1) \cdot n(F_2) \cdot n(F_3) = 5 \cdot 4 \cdot 3 = 60,$
$Pr[E'] = \dfrac{60}{120} = 1/2,$
$Pr[E] = 1 - Pr[E'] = 1 - 1/2 = 1/2.$

NOTE: We can also count E' as $(5, 3)$-permutations.

9/13

Example. Let's say that we flip a coin k times. What is the number of outcomes? Note that this experiment is a k-staged experiment consisting of coin flips in each stage.

Let S be the sample space. We use the multipication principle to get
$n(S) = \underbrace{2 \cdot 2 \cdot \ \cdots \ \cdot 2}_{k\ times} = 2^k$

Example. Let's say that we roll a dice k times. What is the number of outcomes? Similarly, it's

a k-staged experiment where in each stage we roll a dice, and we get for the sample space S
$n(S) = 6^k$