Linear Equations Lesson 4: Multi-Variable Systems of Equations and Matrices

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What are Matrices?

Matrices are rectangular arrays of numbers, with varying columns and rows, with each entry containing a number

$$\begin{bmatrix} 4 & 1 & 13 \\ -2 & 3 & -17 \end{bmatrix} \qquad \begin{bmatrix} 5 & 7 \\ 7/2 & \pi \\ 1 & e \end{bmatrix}$$

For this course, we're not doing too much with matrices beyond using them to model big systems of equations. In particular, we look at an equation like

$$A_1x + B_1y + C_1z = D_1$$
$$A_2x + B_2y + C_2z = D_2$$
$$A_3x + B_3y + C_3z = D_3$$

as

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{bmatrix}$$

For my notes, I'll put a line in front of the right ride to represent the equation (just for aesthetic purposes, the book just draws the matrices as above

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right)$$

Using Gaussian Elimination to Find Solutions

Gaussian Elimination is a process where we put all linear equations in general form, and make a matrix of coefficients, and then perform **elementary row operations** representing the
addition method to solve the matrices.

The elementary row operations include the following:

1. Any two rows i, j of a matrix can be interchanged. We symbolize this operation by $R_i \leftrightarrow R_j$

2. Any row i of a matrix can be multiplied by a nonzero constant $c \neq 0$. We symbolize this operation by
$c \times R_i$
3. Any row i of a matrix can be changed by adding a nonzero constant $c \neq 0$ multiple of another row j to it. We symbolize this operation by
$R_i + cR_j$,
or if the constant is $-c$, for some $c \neq 0$, we symbolize the operation by
$R_i - cR_j$

These elementary row operations were the kinds algebra techniques we were doing to solve the systems of equations back in lesson 3 for 3.2 Example 3 and 3.3 Example 1.

Next Time. I'll show more on how what we were doing with the addition method corresponds to Gaussian elimination, and then we'll get our hands dirty with some examples.

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Last Time.

In general, when solving a system of equations in matrix form, we do the following general procedure.

We go through each column from the first column to the next to last column (or until we don't have any rows below left, whichever comes first), and we try to make the column i have a 1 coefficient in the diagonal i, i entry and zeros elsewhere, and if we can't do that (i.e., we have zeros in every entry of that column) then we move on.

NOTE: When we do reduced row operations, <u>we're guided by the addition method.</u>

We try to make the column into what we want doing the two steps I outlined in my guide.

Inquire the following link for my guide:
https://www.mathcha.io/editor/zNQE5IElSdyhDDEGjzcJom3MwhLOO8LotegWYkn

Questions on Homework 7

<u>Question 16 (page 192).</u>

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ & 1 & 5 \end{bmatrix}$$

How did we get from -2 to 1 and also from 2 to 5? It's clear that we didn't multiply row 2 by some constant because row 2 of the second matrix is not a factor of the first. So we know that we have the row operation

$R_2 + tR_1$,

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$R_2 + tR_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ & 1 & 5 \end{bmatrix}$$

and we want to solve for t. So we solve the following equality

$1 = -2 + t \cdot 3$
$+2 \quad +2$
$3 = t \cdot 3$
$\div 3 \quad \div 3$
$1 = t$

So $t = 1$ and the row operation is $R_2 + R_1$, so we added row 2 by row 1 (let's check and the determine the blank)

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$R_2 + R_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 5 \end{bmatrix},$$

so the $2, 1$ entry is 0.

Question 20 (page 192).

$x + y = 3$
$x - y = -1$

First we write the equation in matrix form

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right)$$

Next, we do reduced row operations to solve for x and y. First, observe that we already have the desired 1 coefficient in the $1,1$ entry, so we can go straight to cancelling out row 2 with row 1 as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right)$$

$R_2 - R_1$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1-1 & -1-1 & -1-3 \end{array}\right),$$
$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

Next, we want to get the $2,2$ entry to be 1. We do that by multiplying row 2 by $-1/2$ as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

$-(1/2) \times R_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

Finally, we want to cancel the 1 on the $1,2$ entry, because then the 1 in the $1,1$ entry (i.e. the x variable is by itself)

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

$R_1 - R_2$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right),$$

so we now have the equation

$$x = 1$$
$$y = 2,$$

and our solution is $(x, y) = (1, 2)$.

<u>Question 32 (page 192).</u>

$$2x + 3y - z = -8$$
$$x - y - z = -2$$
$$-4x + 3y + z = 6.$$

In matrix form, we have

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -8 \\ 1 & -1 & -1 & -2 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_1 \ \leftrightarrow \ R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2 & 3 & -1 & -8 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_2 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2-2 & 3-2(-1) & -1-2(-1) & -8-2(-2) \\ -4 & 3 & 1 & 6 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_3 + 4R_1$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ 0 & -1 & -3 & -2 \end{array}\right)$$

$R_2 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & -1 & -3 & -2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$-1 \times R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$R_1 + R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$R_3 - 5R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

$-1/14 \times R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$R_1 - 2R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$R_2 - 3R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

We get the reduced equation of
$x = -2$
$y = -1$
$z = 1$
so we have the unique solution of $(x, y, z) = (-2, -1, 1)$.

More Questions on Homework 7

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$$\begin{bmatrix} 2 & 1 & -3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & - \\ 2 & 6 & 1 \end{bmatrix}$$

$6 = t2$
$\div 2 \quad \div 2$
$3 = t$

$(6, 3) = t(2, 1)?$
$(6, 3) = 3(2, 1)?$
$(6, 3) = (3 \cdot 2, 3 \cdot 1) = 3 \cdot (2, 1)$

$$\begin{bmatrix} 2 & 1 & -3 \\ 2 & 6 & 1 \end{bmatrix}$$

$3 \times R_1$

$$\begin{bmatrix} 6 & 3 & -3 \cdot 3 \\ 2 & 6 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 6 & 3 & -9 \\ 2 & 6 & 1 \end{bmatrix}$$