Probability and Counting Lesson 2: Probability Spaces

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What is a Probability Space?

As a disclaimer, most of the probability spaces we compute have their outcomes equally likely (we don't deal much with unequally likely outcomes)

We define a **probability space** as a "weighted sample space" of outcomes, i.e. each outcome is assigned a positive number that we call the **likelihood value** (means the "likelihood" of that outcome). The numbers are assigned each of the outcomes so that they <u>sum up to one.</u>

The **probability** of an event E is the sum of likelihood values in that outcome. We write $Pr[E]$ to denote this value.

Example. For an "unfair coin flip", we have the sample space
$S = \{heads,\ tails\}$

and we assign likelihood values $1/4$ and $3/4$ to heads and tails respectively. So we get
$Pr[\{heads\}] = 1/4$ (the sum only consists of one outcome)
$Pr[\{tails\}] = 3/4$
$Pr[\{head,\ tails\}] = 1/4 + 3/4 = 1$ (because the event $\{heads,\ tails\}$ consists of both the outcomes $heads$ and $tails$)
$Pr[\varnothing] = 0$ (because the empty set has no outcomes so the sum of nothing is 0)

Example. Next class

Equally Likely Outcomes

A probability/sample space with **equally likely outcomes** is a sample space with a finite number of outcomes, each outcome assigned equal likelihood value.

Three Important Things to Note:

1. In a probability space with equally likely outcomes, each outcome $s \in S$ is necessarily

assigned the likelihood value $1/n(S)$, in other words the probability of $\{s\}$ happening is equal to one over the number of outcomes in S.
$Pr[\{s\}] = 1/n(S)$

NEXT CLASS we'll talk about 2 and 3.

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Last Time: We talked about what a probability space was and went into equally likely outcomes, and we also did lesson 1 on samples spaces

2. In a probability space with equally likely outcomes, if follows from 1. that the probability of any event E can be computed as follows:
$Pr[E] = Pr[\{s_1\}] + Pr[\{s_2\}] + \cdots + Pr[\{s_n\}],$
where $n = n(E)$ and $s_1, \ldots, s_n$ are the distinct outcomes in E, so $Pr[E] = nPr[\{s_1\}]$ because the outcomes are equally likely, and we get
$Pr[E] = \dfrac{n(E)}{n(S)}.$
Important formula for probabilities of equally likely outcomes.

3. In this class, we generally assume (as I've mentioned before) <u>that any probability space we use has equally likely outcomes</u> (unless otherwise stated).

Example 2.1 (page 67) Suppose that a fair coin is tossed twice and the side showing "up" is recorded after each toss.

a. What probability should be assigned to each outcome?
$Pr[\{s\}] = 1/4$, for any $s \in S$ since $S = \{HH, HT, TH, TT\}$, so $n(S) = 4$

What is the probability that heads will come up on exactly one toss?
Let's use the formula
$Pr[E] = n(E)/n(S)$
Let's set $E =$ exactly one head, which is shorthand for
$E = \left\{x : x \text{ is an outcome where exactly one head pops up}\right\}$
$Pr\left[\text{exactly one head}\right] = \dfrac{n\left(\text{exactly one head}\right)}{n(S)}$
To help see what's going, let's write the event and the sample space S in list notation (let's find all the outcomes)

exactly one head $= \{TH, HT\}$

$S = \{HH,\ TH,\ HT,\ TT\}$

We have $n\left(\text{exactly one head}\right) = 2$, and $n(S) = 4$, so we conclude

$$Pr\left[\text{exactly one head}\right] = \frac{n\left(\text{exactly one head}\right)}{n(S)} = 2/4 = 1/2.$$

Example 2.2 (page 68) Your friend tells you that he has written down a whole number that is more than 9 and less than 100 and asks you to guess what it is. What is the probability that you win?

To do this example, let's translate this from english to math:
First question to ask is "What is the sample space?" The sample space is the possible numbers one can guess, which are between 9 and 100
$S = \left\{\text{any whole number between 9 and 100}\right\} = \{9, 10,\ \ldots, 100\}$
Important to note that there are 91 outcomes in the sample space $n(S) = 91$.
Our desired event is $E =$ the correct number

NOTE: We don't what the correct number is, but we know that only a single outcome is the correct number, so $n(E) = 1$

We find that

$$Pr\left[\text{we win}\right] = Pr\left[\text{correct number}\right] = \frac{n\left(\text{correct number}\right)}{n(S)} = 1/91.$$

Takeaway: It often to rephrase your desired as a phrase where you know what the outcomes are.

Some Probability Formulas

Just like in Sets Lesson 4, there were some set formulas. Analogous probability formulas hold

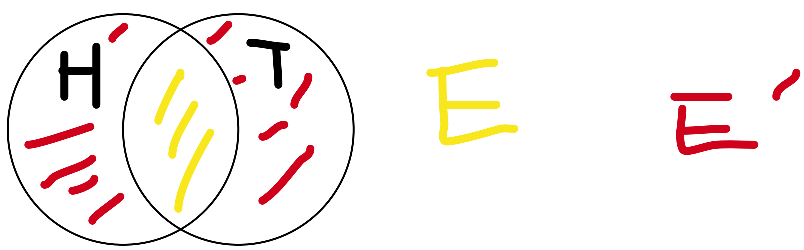
Formula 1: The Complement Probability Formula

For any event E and sample space S, we have
$Pr[E'] = 1 - Pr[E]$
$Pr[E] = 1 - Pr[E']$

This formula is useful, because sometimes it's easier to find the number of outcomes in the complement event.

Example 2.7. (page 72) A fair coin is tossed six times and the face showing "up" is recorded after each toss. Find the probability that both "heads" and "tails" come up.

The desired event is
E = both heads and tails come up
It's a tough event to count by itself, especially because $n(S) = 64$ (we'll find out a good way to show that $n(S) = 64$ later) and it's hard to pinpoint all the outcomes where both heads and tails come up. But what about E'



E' = both heads and tails don't come up = either heads or tails doesn't come up

In particular, the drawing above shows that $E' = H' \cup T'$
There are two outcomes that fit that description:
$HHHHHH,\ TTTTTTT$
either all heads come or all tails come up. So using the formula
$Pr[E'] = \dfrac{n(E')}{n(S)} = 2/64 = 1/32$
Then we compute the probability of E using E' and the complement formula
$Pr[E] = 1 - Pr[E'] = 1 - (1/32) = 31/32$.

Formula 2: The Disjoint Union Addition Probability Formula

Given two disjoint events E_1 and E_2 in a probability space, we have
$Pr[E_1 \cup E_2] = Pr[E_1] + Pr[E_2]$

More generally, for k disjoint events $E_1, E_2, \dots, E_k$, we have
$Pr[E_1 \cup \cdots \cup E_k] = Pr[E_1] + Pr[E_2] + \cdots + Pr[E_k]$.

Formula 3: The Intersection-Union Probabiity Formula

Even more generally, with any two events E_1 and E_2 (they need not be disjoint), we have $Pr[E_1 \cup E_2] = Pr[E_1] + Pr[E_2] - Pr[E_1 \cap E_2]$.

These formulas are analogous to the formulas for the formulas in Sets Lesson 4 involving the number of elements in a set.

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More Probability Homework 2 Questions:

1. A box contains the following:
four red balls numbered 1,3,5,7
five white balls numbered 2,4,5,8, and 10

Find the probability that:

a. The ball is red
$E =$ the ball is red
$S =$ all the balls (this sample space has equally likely outcomes)
$Pr[E] = n(E)/n(S)$
In order to solve the probability, we now have to count E and S
$n(S) = 9$ (because there are nine balls total)
$n(E) = 4$
$Pr[E] = 4/9$

NOTE: In an experiment where the outcomes are the color, then no it's not equally likely

But in an experiment where the outcomes are the specific individual balls of the nine balls, yes it is equally likely

b. The ball has an even number
$E =$ has an even number
$Pr[E] = n(E)/n(S)$
$n(S) = 9$ (again)
$n(E) = n(\{2, 4, 6, 8, 10\}) = 5$
$Pr[E] = 5/9$

c. The ball has a number less than 4.

$E =$ number less than 4
Same formula as the above parts
$n(E) = n(\{1, 2, 3\}) = 3$
$Pr[E] = 3/9 = 1/3$

NOTE: You can simplify the answer if you want to but no need to.

d. The ball is red and has an even number
$E =$ the ball is red and has an even number
$Pr[E] = 0$; why is that?
No red balls are even so we have $E = \emptyset$ and the empty set has no outcomes, so $n(E) = 0$
$Pr[E] = n(E)/n(S) = 0.$