Inequalities Lesson 1: Linear Inequalities

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What are Inequalities?

Inequalities are statements indicating that two quantities are unequal.

There are various different symbols for inequalities; they are as follows

$a \neq b$ which means "a is not equal to b".
$a < b$ which means "a is less than b"
$a > b$ which means "a is greater than b"
$a \leq b$ which means "a less than OR equal to b"
$a \geq b$ which means "a is greater than OR equal to b"

In this module, we'll focus mostly on linear inequalities and their systems. A **linear inequality** in x (a single variable) is any equality that can be expressed in one of the following forms (with $a \neq 0$)

$ax + c < 0$ $\quad ax + c > 0$ $\quad ax + c \leq 0$ or $\quad ax + c \geq 0$

Linear inequalities are inequalities involving "linear terms" (terms of the form $ax + c$)

Linear inequalies, like equalities, have possible solutions, which are typically in the form of one or more <u>intervals</u> across the number line, whenever dealing with a single variable

Next Time: We'll talk more about linear inequalities.

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Solutions of inequalities are given in the form of one or more intervals.

intervals are sets of real numbers ranging from certain values.

Many kinds of intervals

Whenever we write open parentheses, that means the bound is EXCLUDED from the set (drawn with an open hole or a open parentheses.

Whenever we write closed parentheses, that means the bound is INCLUDED in the set (drawn with a fully drawn point or closed parentheses.

The points at the end of the interval are called **endpoints** (we call those enpoints "open" if they're excluded from the interval, and "closed" if they're included in the interval)

Kind of intervals:

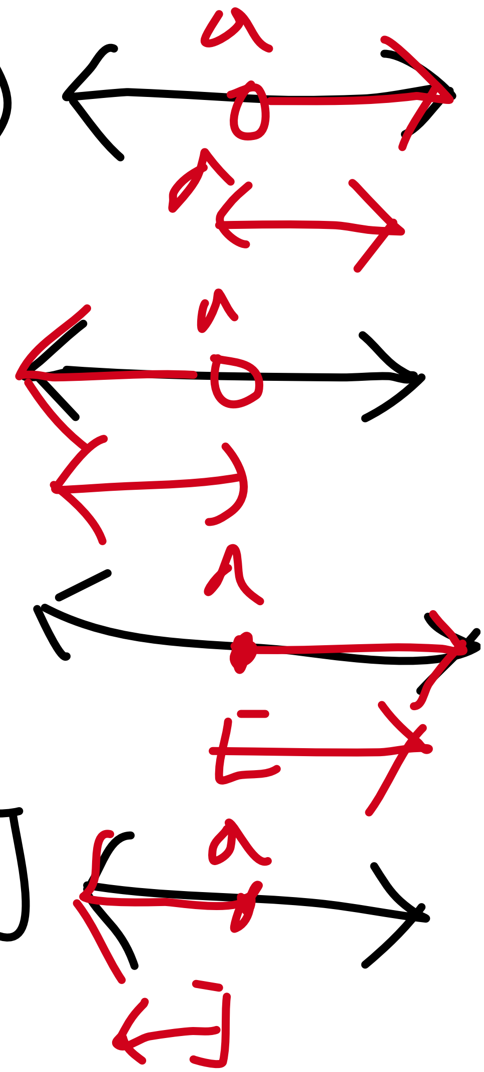
Unbounded intervals

$\{x \mid x > a\} := (a, \infty)$

$\{x \mid x < a\} := (-\infty, a)$

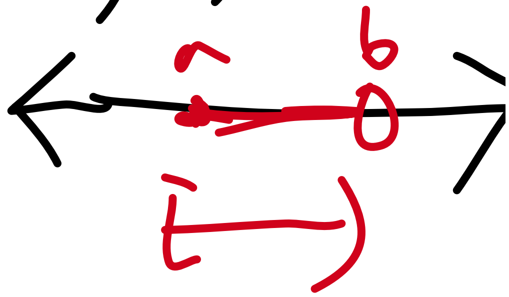
$\{x \mid x \leq a\} := [a, \infty)$

$\{x \mid x \geq a\} := (-\infty, a]$

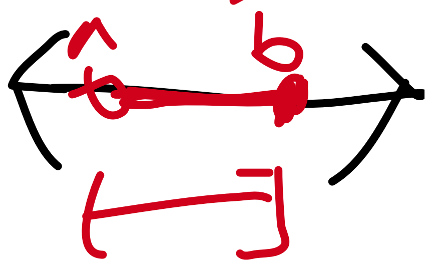


Half-Open Intervals

$\{x \mid a \le x < b\} = [a, b)$

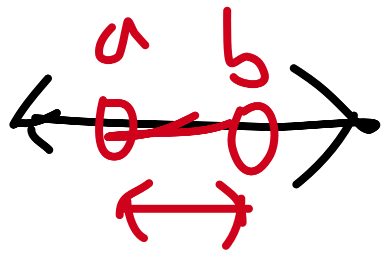


$\{x \mid a < x \le b\} = (a, b]$



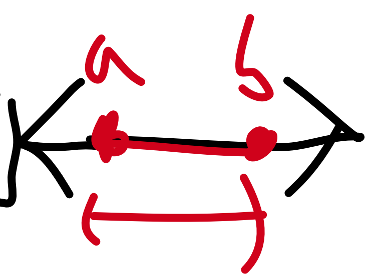
Open Intervals

$\{x \mid a < x < b\} = (a, b)$



Closed Intervals

$\{x \mid a \le x \le b\} = [a, b]$



Finding Solutions to Inequalities?

The name of the game is taking a linear inequality and figuring the "solution" to it which will consist of a set containing one or more of these kinds of intervals and we find them out by doing "algebra" on inequalities similar to what you do with equalities, though the rules are slightly different.

Properties of Inequalities:

1. Trichotomy Property: $a < b$ or $a = b$ or $a > b$

2. Transitive Property: If $a < b$ and $b < c$ then $a < c$

3. Addition/Subtraction Property: If $a < b$ then $a + c < b + c$

4. Multiplication/Divisiion Property:

(a) If $a < b$ and $c > 0$, then $ac < bc$.
(b) If $a < b$ and $c < 0$, then $ac > bc$

5. Multiplicative Inverse Property: If $a < b$, then $1/a < 1/b$

NOTE: Properties 2-5 also apply to the $\leq$ symbol

Using these properties, we can find the solutions, i.e., the intervals for inequalities.

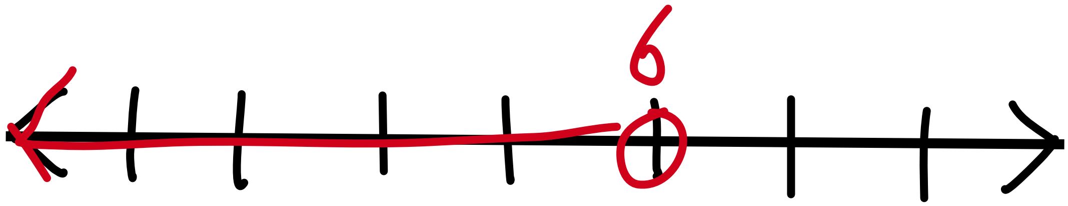
2.1 Example 1 (page 211)

(a) $3(2x - 9) < 9$

First let's use distributive property (similar to what we do in a usual algebra problem) to get x isolated

$$\begin{aligned}
6x - 27 &< 9 \\
+27 \quad & \quad +27 \\
6x \quad &< 36 \\
\div 6 \quad & \quad \div 6 \quad (\text{note } 6 > 0) \\
x \quad &< 6
\end{aligned}$$

The solution consists of the interval $\{x | x < 6\} = (-\infty, 6)$

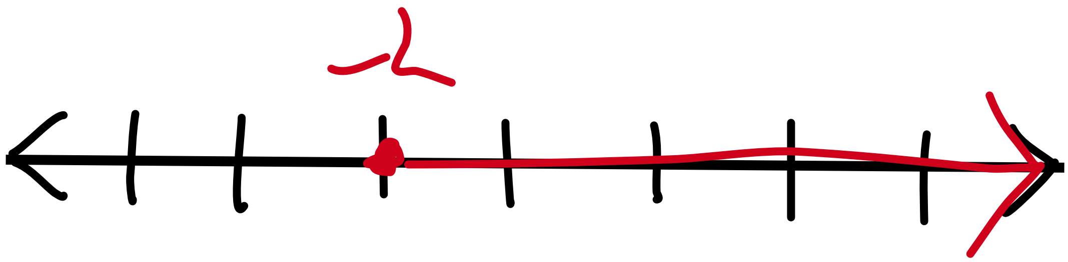


(b) $-4(3x+2) \;\leq 16$

First distribute as before

$$\begin{array}{lll} -12x - 8 & \leq 16 & \\ \quad +8 & +8 & \\ -12x & \leq 24 & \\ \div -12 & \div -12 & \left(\text{note that } -12 < 0\right) \\ \quad x & \geq -2 & \end{array}$$

The solution consists of the interval $\{x|x \geq -2\} = [-2, \infty)$



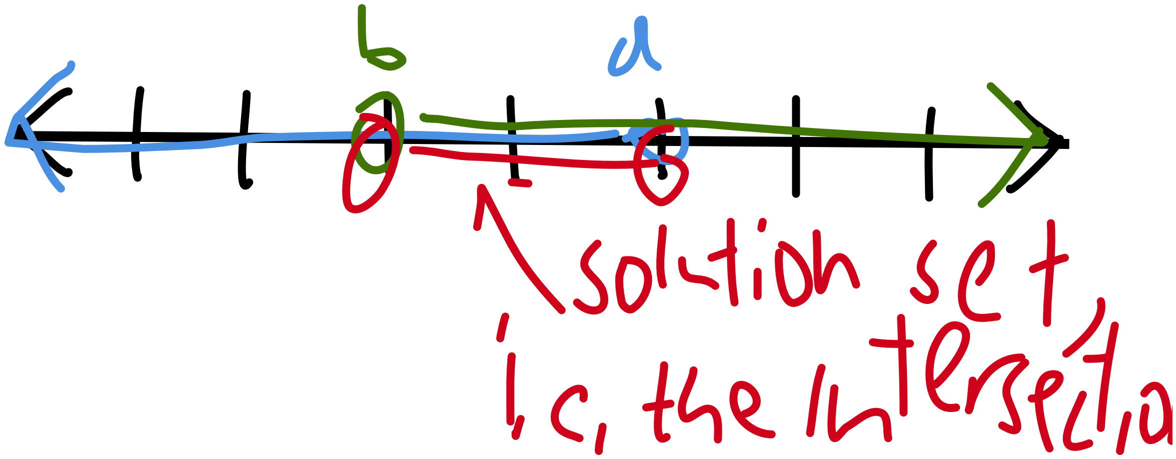
We moreover look at solution sets whenever we "compound" inequalities (using the word "and" or "or" between two inequalities)

<u>Compounding with the Word "and"</u>

What happens with $x < a$ AND $b < x$?

As we remembered from sets, the solution set consists of
$\left\{x|x < a \text{ and } b < x\right\} = \{x|x < a\} \cap \{x|b < x\} = (-\infty, a) \cap (b, \infty) = (a, b)$

So first off, let's assume $a < b$, so what does that look like



Next Time: We'll finish off combounding inequalities and talk about two-variable linear equations in 4.3.

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Last Time: Talked about linear inequalities, how to find their solutions, and what their solutions looked like (finite combinations of intervals), we left off at what solutions to compound inequalities look like.

IMPORTANT ADDITIONAL NOTE:

1. $a < x < b$ means the same thing as "$a < x$ and $x < b$"
2. In the situation that $a \geq b$, but we are looking the solution to $a < x < b$, then no such x exists, because a is NOT less than b, so there's no such x greater than a but less than b, so then the solution is the empty set $\varnothing$. In other words
$\{x|a < x < b\} = \varnothing$

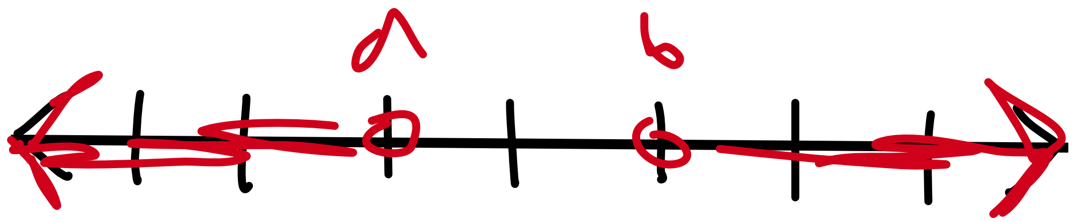
For an example of how to solve two simultaneous inequalities of the form (with possibly $\leq$ used in place of one of the $<$ signs), refer to question 28 of the homework 1 questions:

$a_1x + b_1 < a_2x + b_2 < a_3x + b_3$

<u>Compounding with the Word "or"</u>

What happens with $x < a$ OR $b < x$? Well the solution ends up being the union of the the two solution sets for $x < a$ and $b < x$.

Note that $x < a$ is an interval $(-\infty, a)$, and so is $b < x$; the interval is (b, ∞), so the solution set is
$(-\infty, a) \cup (b, \infty)$

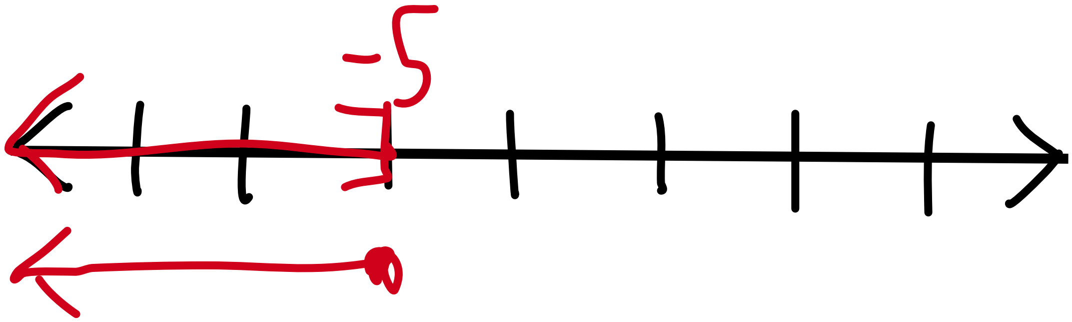


Questions on Homework 1

Question 24. (page 218) Find the solution and graph

$$\begin{array}{lll} -2x+6 & \geq 16 & \\ \quad -6 & \;\; -6 & \\ -2x & \geq 10 & \\ \div -2 & \div -2 & \left(-2 < 0 \text{ dividing by a negative switches the inequality sign}\right) \\ \quad x & \leq -5 & \end{array}$$

The solution is the interval $(-\infty, 5]$, so note that 5 is included, so to draw the solution, we make sure to draw it with either a closed point (a fully drawn point) or a SQUARE bracket



Question 28. $-3 \ \leq 3x < 12$

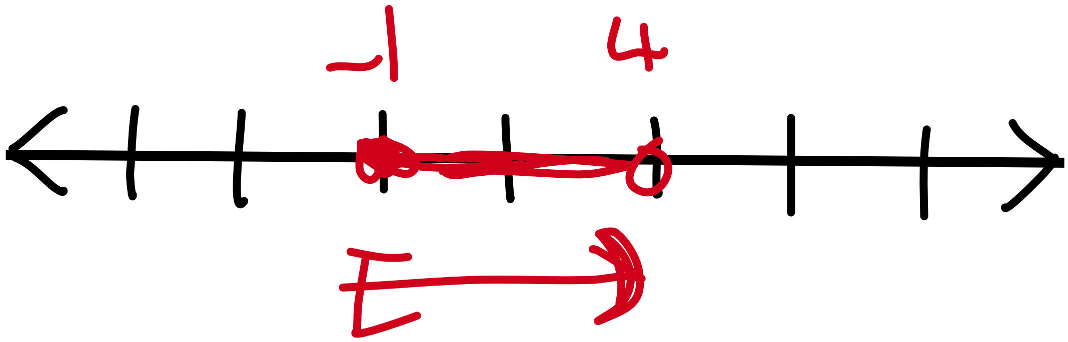
We do algebra for each of the inequalities individually, so we solve it for $-3 \ \leq 3x$ and $3x < 12$ (and we could hypothetically do it simultaneously, but in general, doing it separate will work most generally)

$-3 \ \leq 3x$
$\div 3 \quad \div 3 \ (3 > 0)$
$-1 \ \leq x$

next we do $3x < 12$

$3x < 12$
$\div 3 \quad \div 3$
$x < 4$

So the solution is both of those inequalities put together $-1 \ \leq x < 4$, which gives us the half-open interval $[-1, 4)$ (also can be viewed as the intersection of $[-1, \infty)$ and $(-\infty, 4)$)



Inequalities in Two Variables

A **linear inequality** in two variables x and y is any inequality that can be written in the following form

$ax + by < c$ or $ax + by > c$ or $ax + by \le c$ or $ax + by \ge c$

The solution sets of such linear inequalities can be graphed in the cartesian plane.

In general, we graph the inequality by graphing the line first (either we graph the line in bold, when we want to include the line in the set, or we graph the line partially if it's not included in the solution set)

We signify that the line is *included* in the solution set if the line is *fully drawn*. We signify that the line is *not included* in the solution set if the line is *traced* (and not fully drawn).

Next, we shade the region that is part of the solution to the linear inequality. It will either be above or below the line. How we we know which one it is?

To know which one it is, we want the line in slope intercept form, then it will be more obvious how it's drawn. In slope intercept form, the possibilities are as follows.

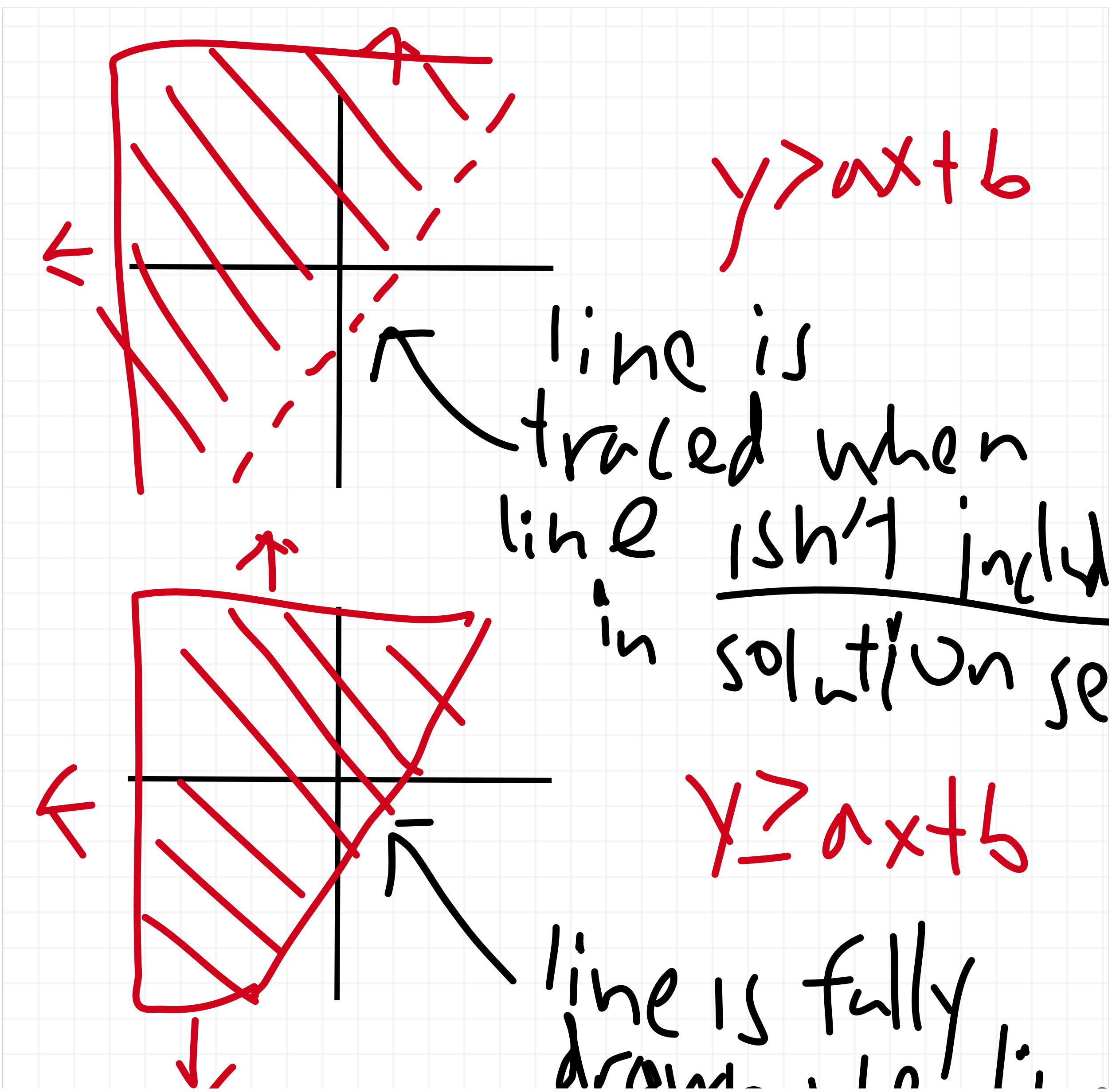
$y > ax + b,$
$y \ge ax + b,$

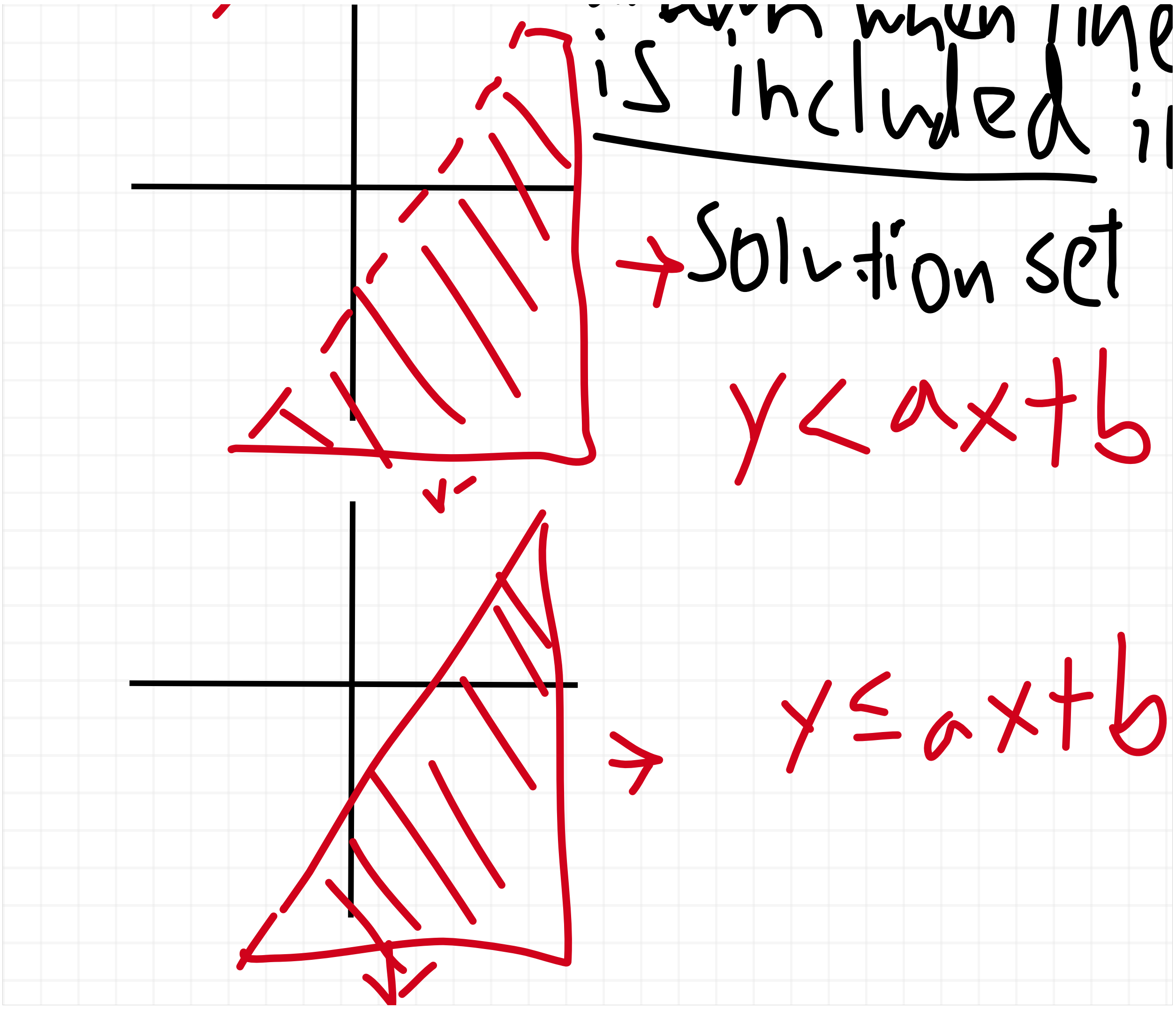
$y < ax + b,$
$y \leq ax + b.$

In the first two situations of $y > ax + b$ and $y \geq ax + b$, the solution set ends up being every (x, y) point where y is (equal or) *greater* than $ax + b$, which consists of y *above the line.*

Similarly, in the next two situations of $y < ax + b$ and $y \leq ax + b$, the solution set ends up being every (x, y) point where y is (equal or) *less* than $ax + b$, which consists of y *below the line.*

The different graphs look as follows:





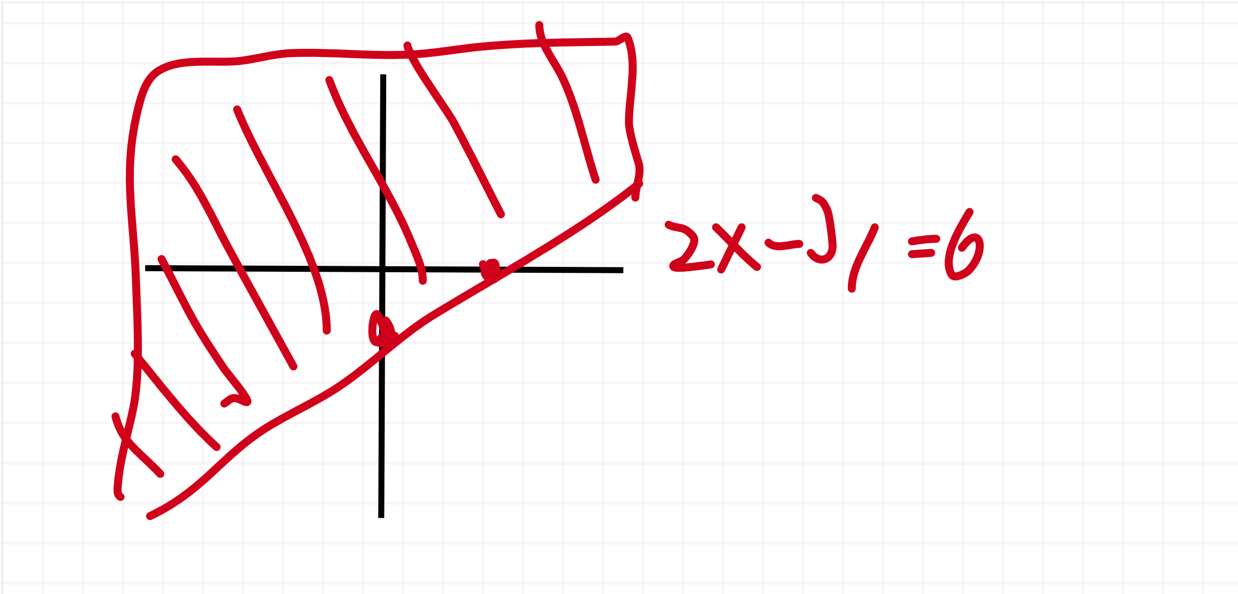
We'll do 4.3 Examples 1 and 2 (page 235-236)

<u>4.3 Example 1.</u> Graph $2x - 3y \ \leq 6$

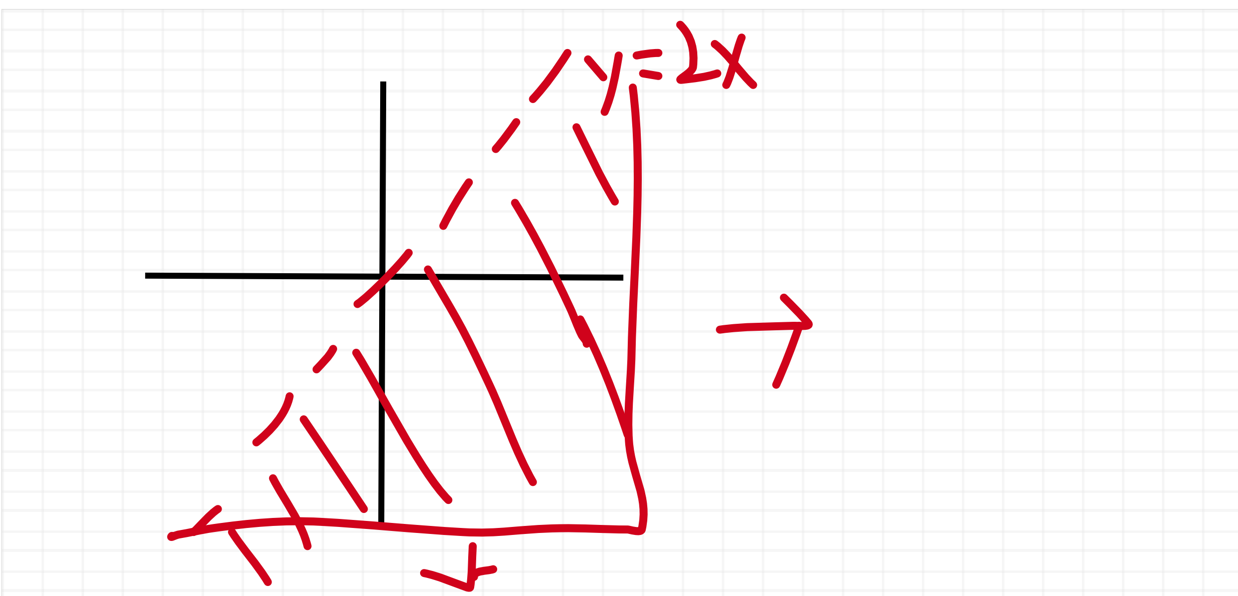
$2x - 6 \ \leq 3y$
$\div 3 \qquad\quad \div 3$ $\big($note that 3 is positive, so the sign stays the same$\big)$
$2x - 2 \ \leq y$

The values of y are all those that are equal or greater to $2x - 6$ (i.e., the line below)

To graph the solution set to this inequality, we first want to graph the line $2x - 3y = 6$ (graph the line in your favorite way)



4.3 Example 2. Graph the solution set for $y < 2x$. Here we already have the inequality in slope-intercept form, so we see that the solution is all y that is less than $2x$, so we draw it as follows



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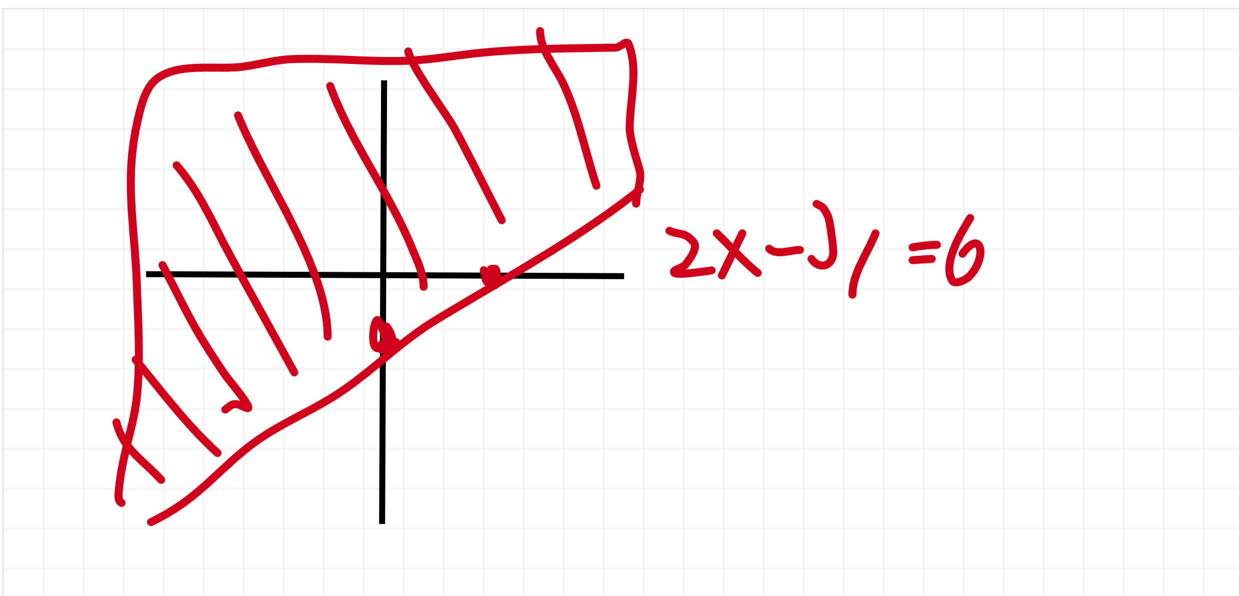
IMPORTANT NOTE: We can also draw an inequality through isolating x instead of y, and that works just as well, and sometimes that's necessary when the line is vertical.

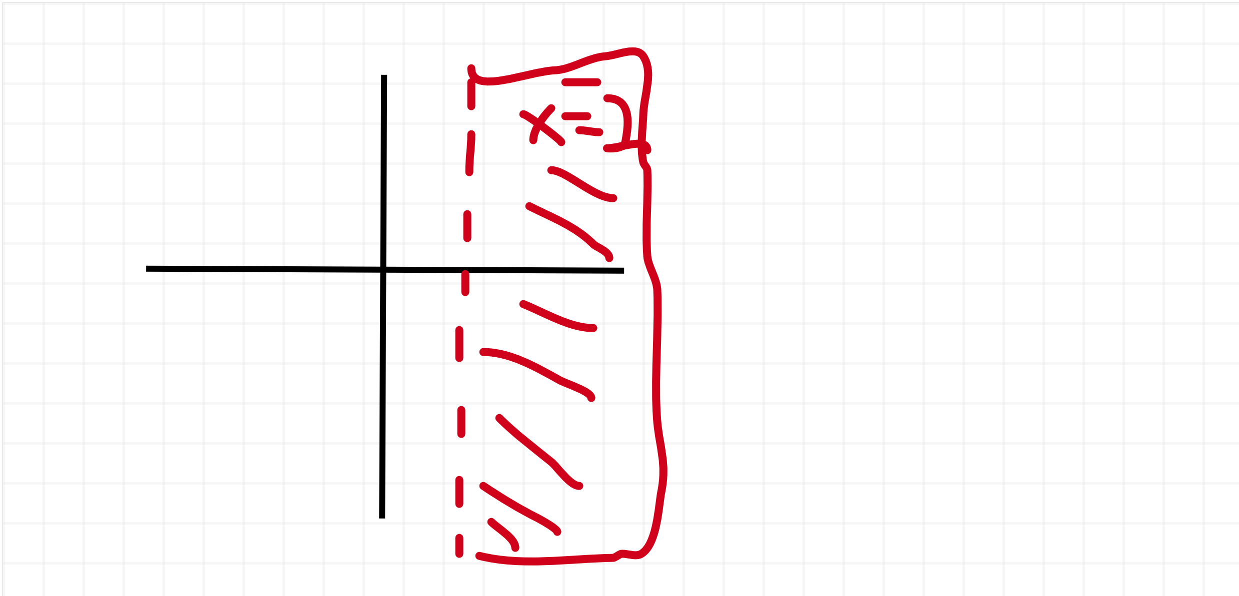
We can do 4.2 Example 1 by solving for x instead of y as follows:

$$\begin{array}{ll}
2x - 3y & \leq 6 \\
\quad +3y & \quad +3y \\
2x & \leq 6 + 3y \\
\div 2 & \quad \div 2 \\
\quad x & \leq 6 + \frac{3}{2}y
\end{array}$$

Once we have x by itself, we can either shade the region to the left (if $\leq$) or the right (if $\geq$) of the line depending on the inequality. Note that in 4.2 Example 1, we go to the left



Example. Let's say we have $x > 2$. Then we *trace* the vertical line $x = 2$



Previously Incorrect Notes

Note that these notes are incorrect because it depends on whether the constant b in the linear inequalities of the form

$$\begin{aligned} ax + by &< c \\ ax + by &> c \\ ax + by &\le c \\ ax + by &\ge c, \end{aligned}$$

Is positive or negative. More specifically, if $b > 0$, then we do the following algebra (which we shall do for only the case for $ax + by < c$, since the other cases derived analogously):

$$\begin{aligned} ax + by &< c \\ -ax \qquad &\quad - ax \\ by &< c - ax \\ \div b \quad &\quad \div b \\ y &< -\frac{a}{b}x + \frac{c}{b} \end{aligned}$$

and we have the following corresponding linear inequalities with the relation between y and x in slope intercept form:

$$y < -\frac{a}{b}x + \frac{c}{b}$$

$$y > -\frac{a}{b}x + \frac{c}{b}$$
$$y \geq -\frac{a}{b}x + \frac{c}{b}$$
$$y \leq -\frac{a}{b}x + \frac{c}{b},$$

and the region is drawn *exactly* how I drew it in the illustration below.

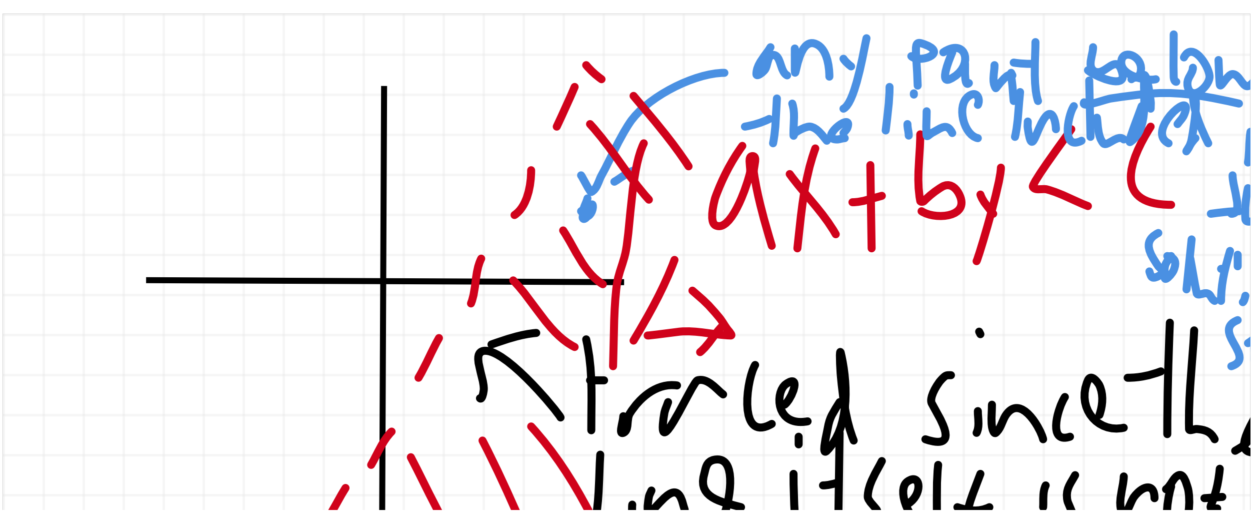
However, if $b < 0$ then the inequality sign reverses when doing the algebra:

$$
\begin{aligned}
ax + by &< c \\
-ax \quad & \quad -ax \\
by &< c - ax \\
\div b \quad & \quad \div b \\
y &> -\frac{a}{b}x + \frac{c}{b}
\end{aligned}
$$

and we have the following corresponding linear inequalities expressed in slope-intercept form

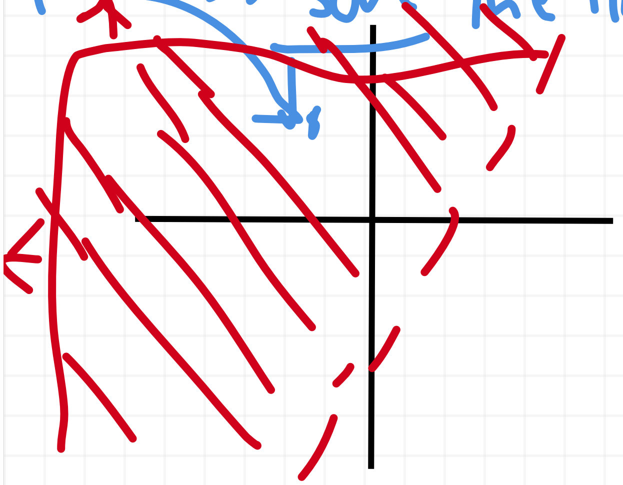
$$y > -\frac{a}{b}x + \frac{c}{b}$$
$$y < -\frac{a}{b}x + \frac{c}{b}$$
$$y \leq -\frac{a}{b}x + \frac{c}{b}$$
$$y \geq -\frac{a}{b}x + \frac{c}{b},$$

which is completely different than how I drew it below.



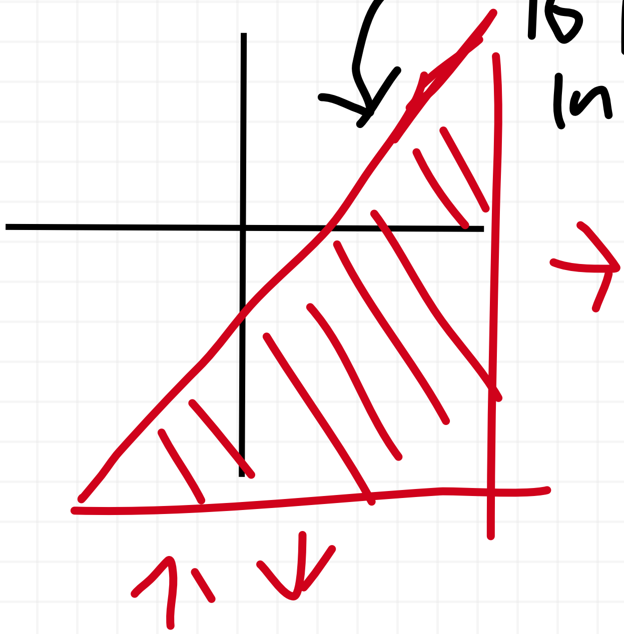
included

points above the line are included

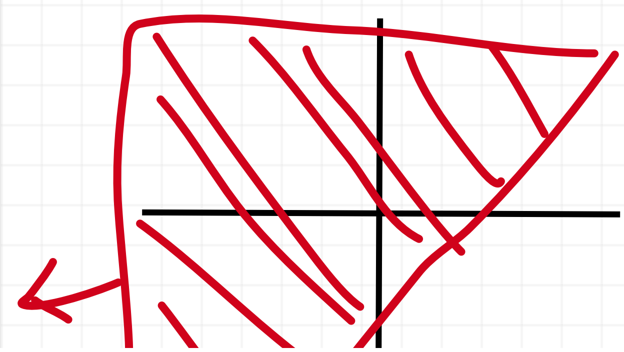


$ax+by>C$

we draw the full line to include the line in the solution



$ax+by\leq C$



$ax+b\geq C$

Questions on Homework 2