Linear Equations Lesson 3: Solutions to Systems of Linear Equations

9/29

Systems of Equation Overview

Before, we looked a single linear equation, whether that equation has one variable or two variables, we only looked at one of them.

So a **system of equations** in general is a list of multiple equations
$a_1 = b_1,\ a_2 = b_2,\ \ldots, a_n = b_n$. For 3.1 and 3.2, we focus on a system of linear equations of the form
$A_1x + B_1y = C_1$
$A_2x + B_2y = C_2$

Later, we'll do more systems of equations with as much as three variables, so a system of linear equations of the form
$A_1x + B_1y + C_1z = D_1$
$A_2x + B_2y + C_2z = D_2$
$A_3x + B_3y + C_3z = D_3$

Two-Variable Systems of Equations

There are three ways to solve for two-variable systems of equations

1. The Graphing Method-you graph the two lines, and find their intersection(s) (if any) (comprises of section 3.1)

2. The Subsitution Method-Solving one of the equations as one variable, and then substituting that what the variable is found to be equal to into the other equation and getting a single variable equation.

3. The Addition Method-Use the other equation to cancel out one variable and then solve the other variable. and then do the cancelling out for both equations to get both solutions. (explained in more detail in the "Three-Variable Systems of Equations" subsection)

Here are the steps for the **The Graphing Method**

1. On a single set of coordinate axes, graph each equation.
2. Find the coordinates fo the point(s) where the graphs intersect; any intersection forms a solution.
3. If the graphs have no points in common, then the system has no solutions.
4. If the graphs of the equations coincide, every possible combination of $(x,\ y)$ is a solution.
5. Check the solution in both of the original equations (plug in the values that you claim to be the solution) (an optional step, but good to make sure you're right)

3.1 Example 1. (page 153) We have the system

$x+2y=4$
$2x-y=3$

We shall solve the system using the graphing method.

The book has a chart, but really we only need two points for the line

$x+2y=4$

plugging in 0 for x and y to get the intercepts, we get

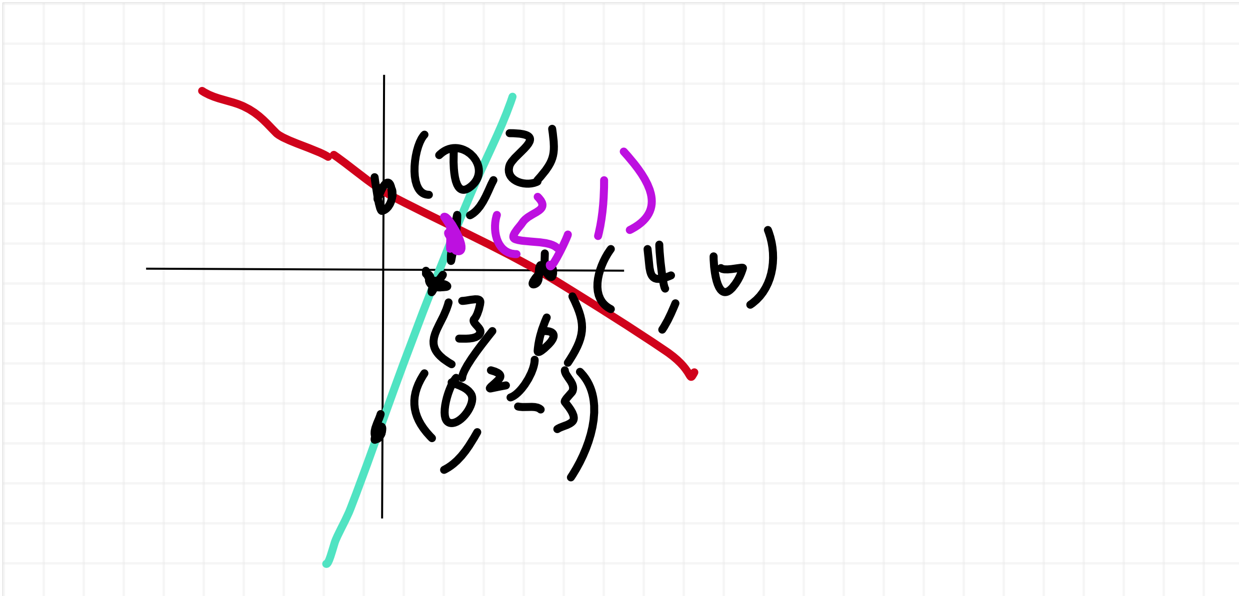
$x+2\cdot 0=4$
$x \qquad =4$

$0+2y=4$
$2y=4$
$\div 2 \quad \div 2$
$y=2$

x	y	$(x,\ y)$
4	0	$(4,0)$
0	2	$(0,\ 2)$

$2x-y=3$

x	y	(x,y)
$\frac{3}{2}$	0	$\left(\frac{3}{2},0\right)$
0	-3	$(0,-3)$



NOTE: Getting the precise intersection requires good drawing (better than these notes)

Here are the steps for the **The Subsitution Method**:
1. If necessary, solve one equation for one variable
2. Substitute the resulting expression for the variable obtained in Step 1 into the other equation and solve that equation.
3. Find the value for the other variable by once again substituting the solution for the variable found in step 2.
4. State the solution and (optionally) check by plugging that solution into both equations.

3.2 Example 1. (page 161)

$4x + y = 13$

$-2x + 3y = -17$

Note that we already have y with the coefficient of 1 in the first equation, so we can get y by itself as follows:

$$\begin{aligned} 4x + y &= 13 \\ -4x \quad &\quad -4x \\ y &= 13 - 4x \end{aligned}$$

Next, for Step 2, plug in $13 - 4x$ in place of y in the second equation. So we get

$$\begin{aligned} -2x + 3(13 - 4x) &= -17 \\ -2x + 39 - 12x &= -17 \\ 39 - 14x &= -17 \\ -39 \quad &\quad -39 \\ -14x &= -56 \end{aligned}$$

$$\div -14 \quad \div -14$$
$$x = 4$$

We're not quite done yet. For step 3, we plug in 4 for x into the solution we found in terms of x for y to get x

$y = 13 - 4(4) = 13 - 16 = -3$

We can check if we want.

9/30

Last Time: Talked about the graphing and substitution method.

We have one more method left, and that's the "addition method" (in some ways, that's the most important method because it's segway to solutions by matrices, aka, "Gaussian Elimination" which is 3.4).

3.2 Example 2 (page 162) Solve the system using the substitution method.

$$\frac{4}{3}x + \frac{1}{2}y = -\frac{2}{3}$$
$$\frac{1}{2}x + \frac{2}{3}y = \frac{5}{3}$$

We have to solve for one of the variables in terms of the other, so let's just pick the first equation and solve for y. So first, we want to get rid of the fractions by multiplying by a common denominator (6)

$$\frac{4}{3}x + \frac{1}{2}y = -\frac{2}{3}$$
$$\times 6 \qquad \times 6$$
$$8x + 3y = -4$$
$$-8x \qquad -8x$$
$$3y = -4 - 8x$$
$$\div 3 \quad \div 3$$
$$y = \frac{-4 - 8x}{3}$$

What next? Solve for x by substitution our solution for y in terms of x into our other equation

$$\frac{1}{2}x + \frac{2}{3}\left(\frac{-4 - 8x}{3}\right) = \frac{5}{3}$$

To deal with this, we want to deal with the parenthesis first

$\frac{1}{2}x + \frac{-8-16x}{9} = \frac{5}{3}$

Now we want to get rid of the fractions

$$\begin{aligned}&\frac{1}{2}x + \frac{-8-16x}{9} = \frac{5}{3}\\ &\times 18 \qquad\qquad\qquad \times 18\\ &9x + (-16-32x) = 30\end{aligned}$$

Next, we combine like terms

$-23x - 16 = 30$

Now we have the equation to where we can solve for x directly

$$\begin{aligned}-23x - 16 &= 30\\ +16 \quad &+16\\ -23x \quad &= 46\\ \div -23 \quad &\div -23\\ x \quad &= -2\end{aligned}$$

Finally, we plug in the solution for x into our equation for y, and then get a numeric solution for y:

$y = \frac{-4-8(-2)}{3} = \frac{12}{3} = 4$

Check if you're confident in it (I am confident in this, though!)

3.1 Example 2. (p. 153) Try to solve the system

$2x + 3y = 6$

$4x + 6y = 24$

Let's try to use the substitution method

$$\begin{aligned}2x + 3y &= 6\\ -3y \quad &-3y\\ 2x \quad &= 6 - 3y\\ \div 2 \quad &\div 2\\ x \quad &= 3 - \frac{3}{2}y\end{aligned}$$

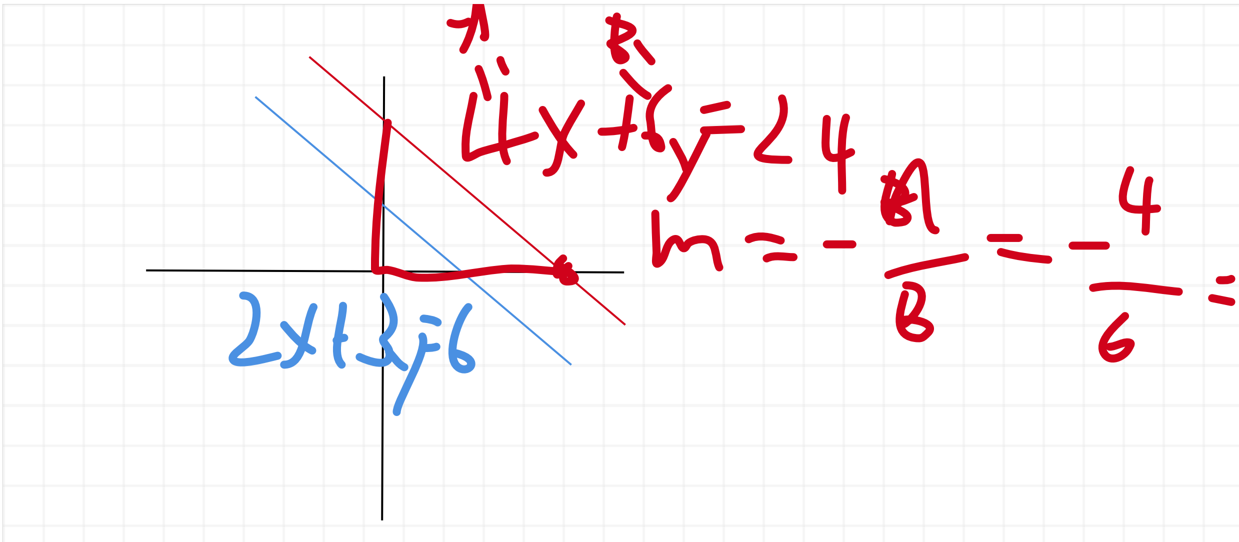
Substitute this into the other equation

$4\left(3 - \frac{3}{2}y\right) + 6y = 24$

$12 - 6y + 6y = 24$

$12 \qquad = 24$

Something's fishy! Because $12 \neq 24$. We'll see that we have a contradiction. We can show this "contradiction" graphically as follows:



The Takeaway: Inconsistent systems of equations are graphed by parallel lines

In general, a **inconsistent system** of equations is one with no solution; in particular, solving the system of equations will lead to a contradiction (as explained in 1.5)

There's another type of system of equations in two variables with infinitely many solutions, and that is called **dependent equations**

2.1 Example 3. (page 154)

$2y - x = 4$
$2x + 8 = 4y$

Let's solve it using substitution

$$
\begin{aligned}
2y - x &= 4 \\
-2y \qquad &\; -2y \\
-x &= 4 - 2y \\
\div -1 \quad &\div -1 \\
x &= -4 + 2y
\end{aligned}
$$

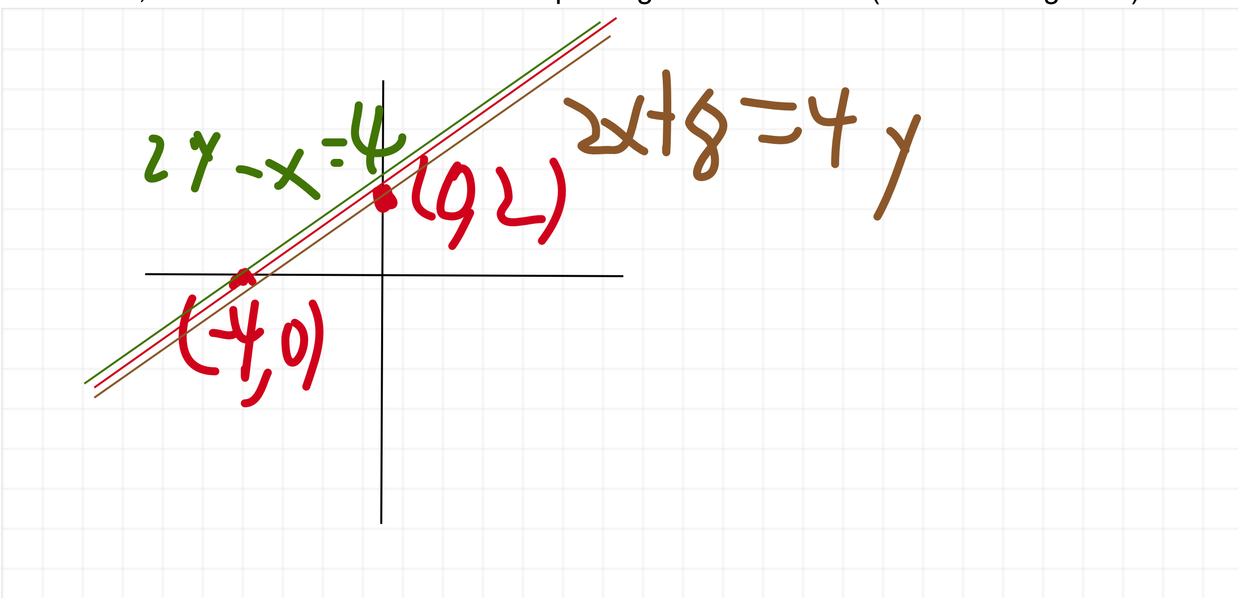
Let's substitute our solution for x in terms of y

$2(-4 + 2y) + 8 = 4y$

$$
\begin{aligned}
-8 + 4y + 8 &= 4y \\
4y &= 4y
\end{aligned}
$$

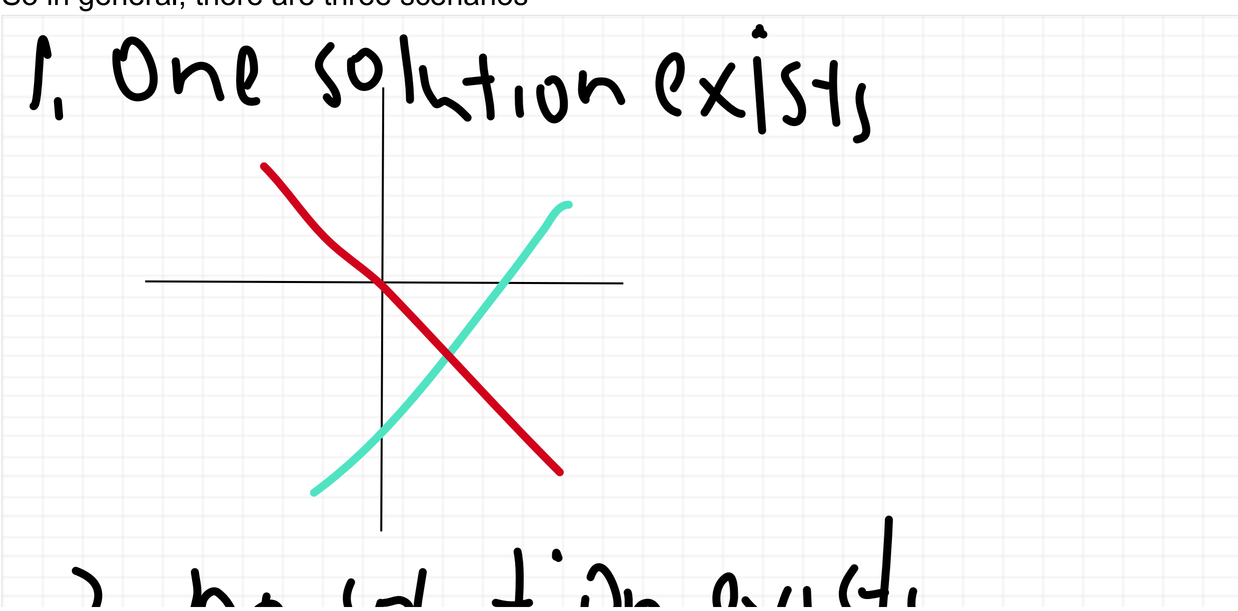
We got an identity after substituting. Note that y ends up being free and any y works (as long as x follows the equation $x = -4 + 2y$ we get the solution of $(-4 + 2y,\ y)$. Let's draw this this solution graphically

All the solutions are in the red line; in particular, the dark green line completely overlaps the brown line, and hence the solutions end up being all of both lines (as the red signifies)

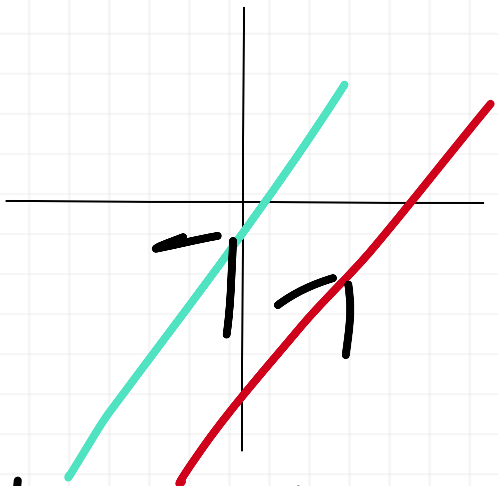


The takeaway: Any dependent equation has lines that completely overlap, both equations have the same line representing it.

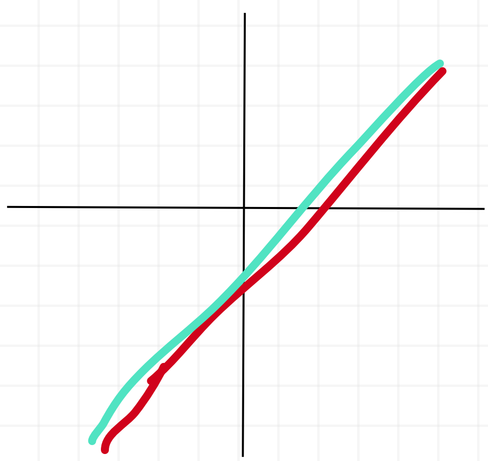
So in general, there are three scenarios



2. No solution exists
(inconsistent system)



3. infinitely many solution exist
(dependent equation)



NOTE: On homework 5 (due tomorrow night), you have to solve any problem within 15-36 by graphing.

Three-Variable Systems of Equations

The substitution method is great intuitively and for solving 2 variables, but it becomes a hassle with 3 variables. Let's observe Example 1 of 3.3 (p. 177).

$2x + y + 4z = 12$
$x + 2y + 2z = 9$
$3x - 3y - 2z = 1$

And note that the substitution (generalized to three variable) becomes an extreme trainwreck, but there's another method of solving these equations called the "addition method", which becomes "Gaussian elimination" that ends up translating "decently" to three variable systems of equations.

Next Time: We'll go over the addition method in more detail, and solve equations in three variables.

10/1

Last Time: We talked about the substitution and how it's great for two variables, not so much for more than two variables. If we try and substitute, we get
$y = 12 - 2x - 4z$
$x = 9 - 2y - 2z$
Then we have to plug in $y = 12 - 2x - 4z$ in order to get x in terms of z, and then we have to plug x and y into the third equation to get the solution.

So the point is that substitution is intuitive but isn't nice computationally. So we have a different method which is good for more than two variables called the **Addition Method**.

Here are the steps for the **The Addition Method**:
1. Write all equations of the system in general form
2. Multiply the terms of one or both the equations by constants chosen to make the coefficients of either x, y, or z one.
3. Add all the other systems by the coefficient multiple of the other equation (the one that we factored with coefficient 1) to cancel out that variable in the other equations.
4. Repeat the process (repeat step 2-5) for each variable until we get the solution.
5. Check our answer by plugging in the variables (if you want to)

3.2 Example 3. (page 163) Solve
$4x + y = 13$
$-2x + 3y = -17$

We have step 1. completed since all the equations are in general form.

For step 2, we divide the first equation by 4 to get an equation for x with coefficient 1:

$4x + y = 13$
$\div 4 \qquad \div 4$
$x + \frac{1}{4}y = \frac{13}{4}$
We now have
$x + \frac{1}{4}y = \frac{13}{4}$
$-2x + 3y = -17$
For step 3, we cancel out x in the second equation by adding 2 times the first equation
$-2x + 3y = -17$
$+2\left(x + \frac{1}{4}y\right) \quad +2\left(\frac{13}{4}\right)$
$-2x + 3y + 2\left(x + \frac{1}{4}y\right) \; = \; -17 + 2\left(\frac{13}{4}\right)$
$0x + \frac{7}{2}y \qquad\qquad = \frac{-34}{2} + \frac{13}{2} = \frac{-21}{2}$
Note that $x + 1/4y = 13/4$, so we're adding the same thing on each side

We now have
$x + \frac{1}{4}y = \frac{13}{4}$
$\frac{7}{2}y = \frac{-21}{2}$
Now we repeat step 2 for y to get y by itself, so we divide y by $7/2$ to get
$\frac{7}{2}y = \frac{-21}{2}$
$\div 7/2 \quad \div 7/2$
$y = \frac{-21}{2} \cdot \frac{2}{7} = -3$
We now have
$x + \frac{1}{4}y = \frac{13}{4}$
$y = -3$
And then we subtract by $1/4$ of the bottom equation on each side to cancel out the y, giving us x
$x + \frac{1}{4}y = \frac{13}{4}$
$-\frac{1}{4}y \quad -\frac{1}{4}(-3)$

$x + 0y = \dfrac{13}{4} + \dfrac{3}{4} = \dfrac{16}{4} = 4$,
so we have the solution $(x,\ y) = (4, -3)$.

3.3 Example 1 (page 177)

$2x + y + 4z = 12$
$x + 2y + 2z = 9$
$3x - 3y - 2z = 1$

Note that the equations are already in general form (they're not always in general form)

Actually, we have x with coefficient 1, so we can simply take $x + 2y + 2z = 9$ to be the top equation
$x + 2y + 2z = 9$
$2x + y + 4z = 12$
$3x - 3y - 2z = 1,$
so step 2 applied to the variable x has been completed. It remains to do step 3 and cancel the other variables out.
$2x + y + 4z = 12$
$-2(x + 2y + 2z) \ - 2(9)$
$0x + (y - 4y) + (4z - 4z) = 12 - 18$
$\quad (-3)y + 0z = \ -6$

We want to do the same thing and cancel out x for the third equation, so
$3x - 3y - 2z = 1$
$-3(x + 2y + 2z) \ \ - 3(9)$
$0x + (-3y - 6y) + (-2z - 6z) = \ -26$
$\quad (-9)y + (-8)z \qquad = \ -26$

Now we have
$x + 2y + 2z = 9$
$(-3)y \qquad = \ -6$
$(-9)y + (-8)z = \ -26,$

We're done with steps 2 and 3 for x and we do the same thing for y. First, we take the second equation and divide by -3
$(-3)y = \ -6$
$\div\!-3 \quad \div\!-3$
$\quad y = 2$
We now have
$x + 2y + 2z = 9$

$$y \quad = 2$$
$$(-9)y + (-8)z = -26$$
Now we cancel the y's in the other equation. So in the top equation, we subtract by 2 times the equation $y = 3$ and we get
$$x + 2y + 2z = 9$$
$$-2y \quad -2(2)$$
$$x + 0y + 2z = 5$$
We cancel out the bottom equation by adding 9 times the equation $y = 3$ and we get
$$(-9)y + (-8)z = -26$$
$$+9y \quad +9(2)$$
$$0y \quad + (-8)z = -8$$
So we end up with
$$x \quad + 2z = 5$$
$$y \quad = 3$$
$$(-8)z = -8$$
It remains to do steps 2 and 3 one more time for z, so we third equation now, and divide by -8 to get
$$(-8)z = -8$$
$$\div -8 \quad \div -8$$
$$z = 1$$
So we then have the system
$$x \quad + 2z = 5$$
$$y \quad = 3$$
$$z = 1$$
And it remains to cancel the z out in the top equation, and then we have the full solution. To do that we subtract by 2 times the bottom equation to get
$$x + 2z = 5$$
$$-2z \quad -2(1)$$
$$x \quad = 3$$
Our solution is
$$x = 3$$
$$y = 2$$
$$z = 1$$

The Final Takeaway: So with the addition method, it is often helpful to think of these equations as matrices, because they effectively model what we're doing with the addition method. In other words with 3.2 Example 3, we can think of
$$4x + y = 13$$
$$-2x + 3y = -17$$
as
$$\begin{bmatrix} 4 & 1 & 13 \\ -2 & 3 & -17 \end{bmatrix}$$

and then the operations for the addition method end up being reduced row operations.

Questions on Homework 5 and 6

No questions of note