Inequalities Lesson 1: Linear Inequalities

11/3

What are Inequalities?

inequalities are statements indicating that two quantities are unequal or possibly uniequal

There are various different symbols for inequalities, which are as follows

$a \neq b$ which means a and b are not equal
$a < b$ which means a is (strictly) less than b
$a > b$ which means a is (strictly) greater than b
$a \ \leq b$ which means a is less than OR EQUAL to b
$a \geq b$ which means a is greater than OR EQUAL to a

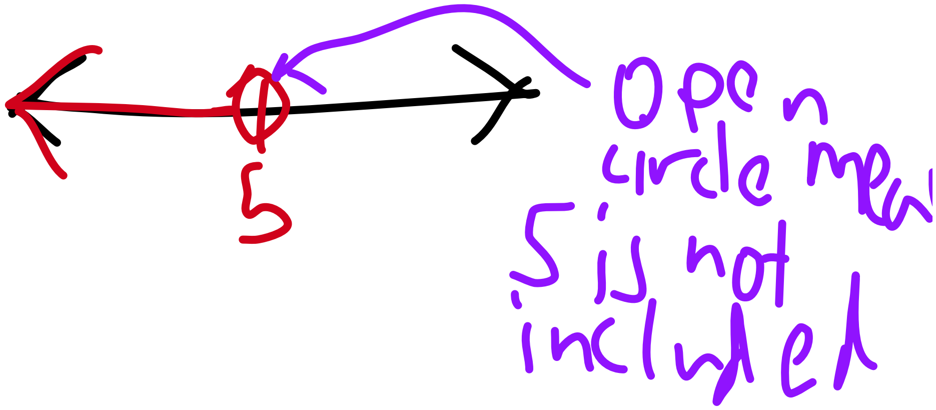
In this module, we'll focus specifically on linear inequalities and their systems. A **linear inequality** in x (a single variable) is any inequality that can be equivalently in one of the following forms (with $a \neq 0$)

$ax + c < 0 \quad ax + c > 0 \quad ax + c \geq 0$ or $ax + c \geq 0$

Linear inequalities are inequalities involving "linear terms" (terms of the form $ax + c$).

As with finding all solutions to linear equations, we find all solutions to sets to linear inequalities like $ax + b \leq cx + d$. These solutions are of the form of what we call "intervals".

So for example a possible solution of $x < 5$ is 4, as well as any number less than 5 (such as $4, 3, 2, 3.59, \ \pi, \ -27,$ etc. In fact, the solution can be illustrated by everything on the number line shaded red as follows:

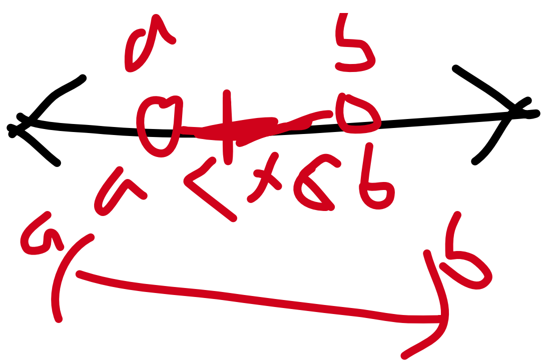


We express sets of numbers that are less or between any given numbers in "interval notation" and call such solution sets **intervals**.

There are many kinds of intervals:

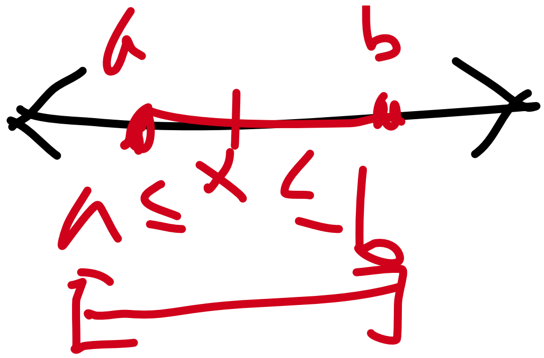
1. Open Intervals

Intervals that we call (a, b) where a and b are numbers such that $a < b$, and these intervals consist of all x that are strictly between a and b, so $a < x < b$.



2. Closed intervals

intervals that we call $[a, b]$ (note the closed parentheses, where a and b are numbers such that $a \leq b$, and these intervals consist of all x that are (not necessarily strictly) in between a and b, so $a \leq x \leq b$



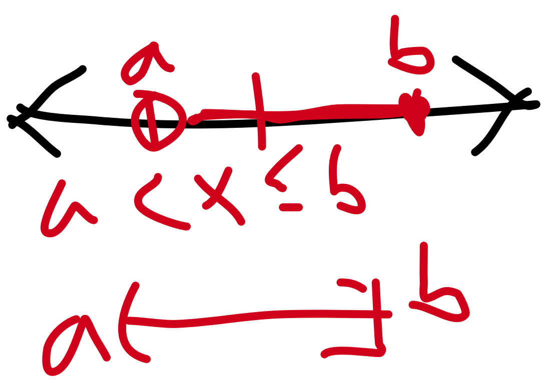
11/4

Last Time:

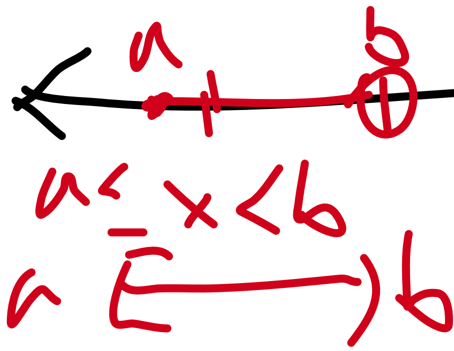
3. Half-Open Intervals

"bounded" intervals that include one (and only one) endpoint. They are denoted $(a, b]$ when the upper endpoint b is included (and a is not included), and they are denoted $[a, b)$ when the lower enpoint a is included (and b is not included).

$(a, b]$ consists of all x such that $a < x \leq b$.



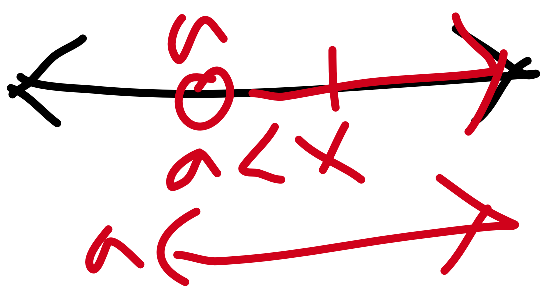
$[a, b)$ consists of all x such that $a \leq x < b$.



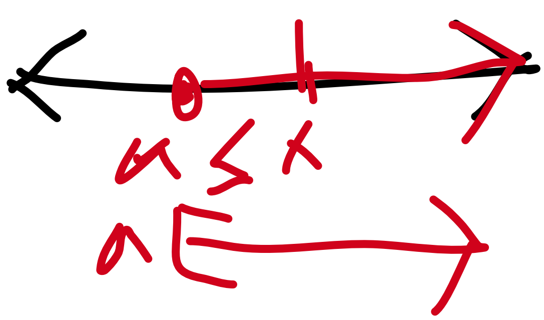
2. Unbounded Intervals

These are intervals denoted by (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$ depending on whether the interval is bounded above or below (but not bounded on the other side) and whether that bound is included (closed parentheses implies inclusion, and open parentheses imply exclusion)

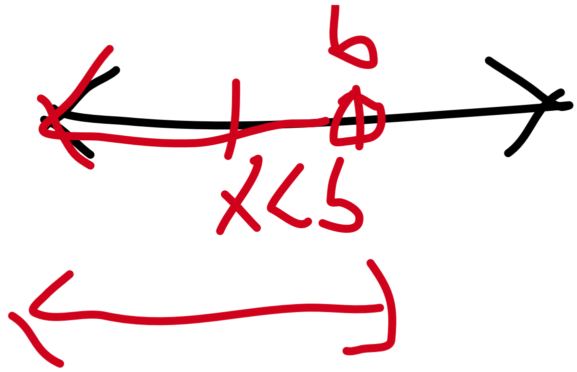
(a, ∞) consists of all x such that $x > a$



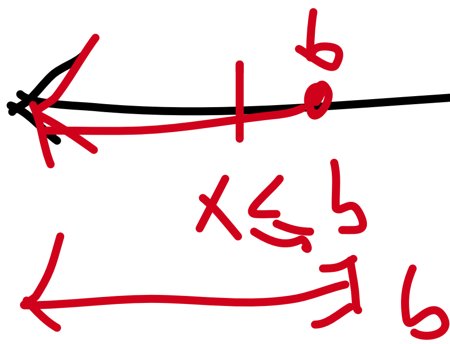
$[a, \infty)$ consists of all x such that $x \geq a$



$(-\infty, b)$ consists of all x such that $x < b$



$(-\infty, b]$ consists of all x such that $x \leq b$



Finding Solutions to Inequalities

The name of the game is taking linear inequalities and figuring out the "solution" to it which consists of one or more such intervals, and we find them out by "solving" for x as an inequality.

So we start with linear inequalities expressed as follows:

$ax + b < c \quad ax + b \leq c$

$a_1x + b_1 < a_2x + b_2 < a_3x + b_3$ (or any variant of that with $\leq$ involved)

$a_1x + b_1 < a_2x + b_2$ or $a_3x + b_3 < a_4x + b_4$

And from that, we need to "solve for x"

To do this, we need to know some algebraic properties that preserve the inequality:

Properties of Inequalities:

1. Trichotomy Property:
for any a and b, $a < b$ or $a = b$ or $a > b$
for a and b, $a \leq b$ or $a \geq b$

2. Transitive Property:
If $a < b$ and $b < c$, then $a < c$
If $a \leq b$ and $b \leq c$, then $a \leq c$

3. Addition/Subtraction Property:
If $a < b$ then $a + c < b + c$
If $a \leq b$ then $a + c \leq b + c$

4. Multipication/Division Property:
(a) If $a < b$ and $c > 0$, then $ac < bc$
If $a \leq b$ and $c \geq 0$, then $ac \leq bc$

(b) If $a < b$ and $c < 0$, then $ac > bc$
If $a \leq b$ and $c \leq 0$, then $ac \geq bc$

5. Multiplicative Inverse Property:
If $a < b$ and $a, b \neq 0$, then $1/a > 1/b$
If $a \leq b$ and $a, b \neq 0$, then $1/a \geq 1/b$

<u>2.1 Example 1 (page 211)</u> We want to find the solution set for the following:

(a) $3(2x - 9) < 9$

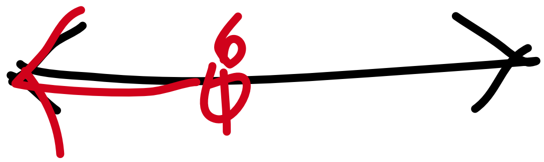
First, we can use distributive property on the left side since that preserves the equality of the left side, so we get

$$6x - 27 = 3 \cdot 2x - 3 \cdot 9 = 3(2x - 9) < 9$$

$6x - 27 < 9$
$+27 \quad + 27$ addition property
$6x \quad < 36$
$\div 6 \quad \div 6$ division property (a), because $6 > 0$

$x \quad < 6,$

so we have all x such that $x < 6$ for the solution, which gives us $(-\infty, 6]$ as the solution set, which can be illustrated as follows:



(b) $-4(3x+2) \leq 16$

First, we distribute as before

$-4(3x+2) = -4 \cdot 3x + -4 \cdot 2 = -12x - 8,$

so we have

$-12x - 8 \leq 16$
$\quad +8 \quad +8$ addition property
$-12x \quad \leq 24$
$\div -12 \quad \div -12$ division property (b), because $-12 < 0$
$\quad x \quad \geq -2,$

we have all x such that $x \geq -2$, which gives us $[-2, \infty)$ as the solution set, which can illustrated as follows:

What to do with the words "and" and "or"

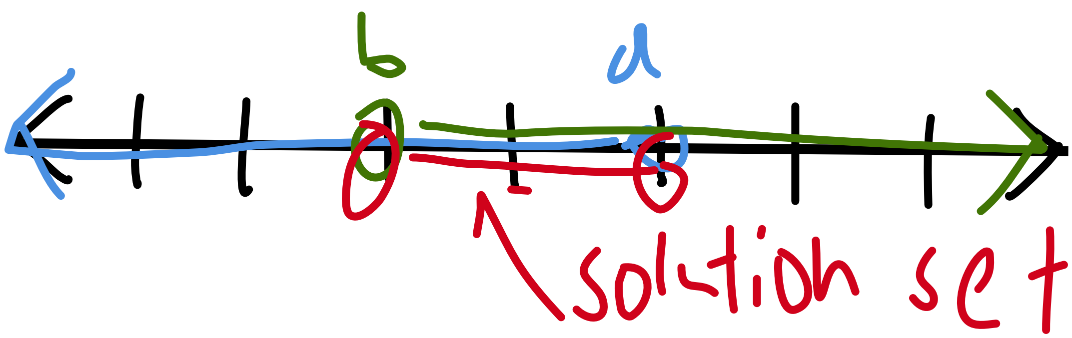
As mentioned previously, sometimes we're given equations of the form

$a_1x + b_1 < a_2x + b_2$ and $a_2x + b_2 < a_3x + b_3$ or equivalently,
$a_1x + b_1 < a_2x + b_2 < a_3x + b_3$ (or any variant of that with $\leq$ involved)

$a_1x + b_1 < a_2x + b_2$ or $a_3x + b_3 < a_4x + b_4$

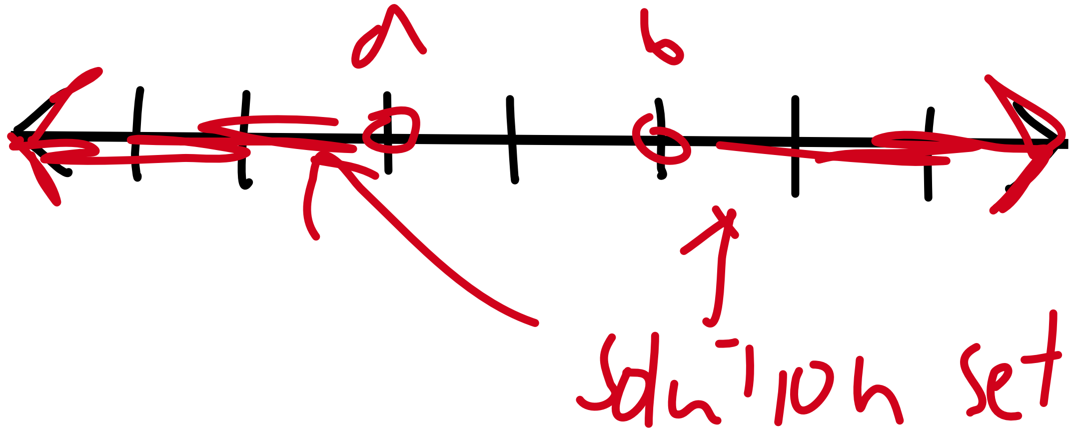
So to deal with these compound inequalities, we do two things:
1. Solve for both inequalities and find the solutions for each (which will always be unbounded intervals)
2.
-If joined by "and", then the interval consists of all x that satisfies BOTH condtions

As illustrated below, if $b < x$ and $a > x$ then the solution set is all x that satisfies both conditions (the analogous idea holds when $\leq$)



-If joined by "or", then the interval consists of all x that satisfies EITHER conditions

If we have "$x < a$ or $b > x$", then the solution set is all x that satisfies either (but not necessarily both) conditions, as illustrated below (again, the analogous idea holds for $\leq$)



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11/5

<u>4.1 Example 3 (page 214).</u> Solve the inequality $-3 \le 2x + 5 < 7$

To solve that, we solve for the inequalities $-3 \le 2x + 5$ and $2x + 5 < 7$

$-3 \le 2x + 5$
$-5 \qquad -5$
$-8 \le 2x$
$\div 2 \quad \div 2$
$-4 \le x$

$2x + 5 < 7$
$\quad -5 \quad -5$
$2x \qquad < 2$
$\div 2 \qquad \div 2$
$x \qquad < 1$

Our solution is $-4 \le x < 1$, so we end up with the half-open interval $[-4, 1)$

We can also solve for this inequality in one fell swoop as follows

$-3 \leq 2x + 5 < 7$
$-5 \quad\quad -5 \quad\quad -5$
$-8 \leq 2x < 2$
$\div 2 \quad\quad \div 2$
$-4 < x < 1$

But in the next example, we need to individually.

<u>4.1 Example 4 (page 214).</u> Solve $x + 3 < 2x - 1 < 4x - 3$.

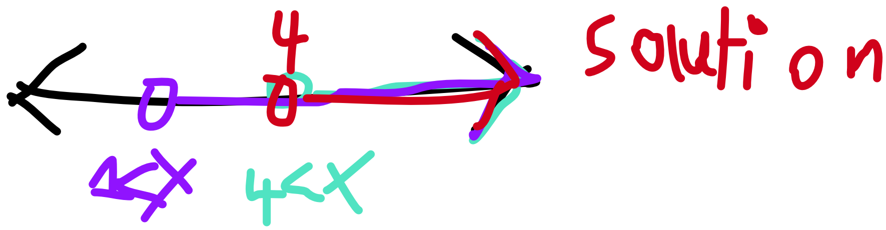
First we solve for $x + 3 < 2x - 1$

$x + 3 < 2x - 1$
$-x \quad\quad -x$
$3 < x - 1$
$+1 \quad\quad +1$
$4 < x$

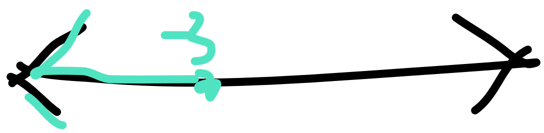
Next we solve for $2x - 1 < 4x - 3$

$2x - 1 < 4x - 3$
$-2x \quad\quad -2x$
$-1 < 2x - 3$
$+3 \quad\quad +3$
$2 < 2x$
$\div 2 \quad \div 2$
$1 < x$

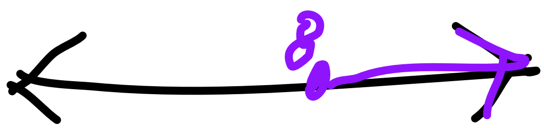
We conclude that the solution is $(4, \infty)$.



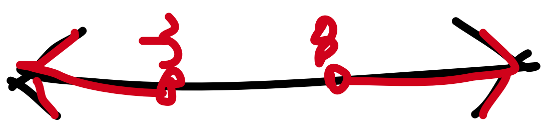
4.1 Example 5 (page 215). To find the solution to $x \leq -3$ or $x \geq 8$, we know that the solution set for $x \leq -3$ is $(-\infty, -3]$



and we also know the solution set to $x \geq 8$ is the interval $[8, \infty)$



So the solution set for $x \leq -3$ or $x \geq 8$ is every x that is either in the interval $(-\infty, -3]$ or $[8, \infty)$



Questions on Homework 1

None

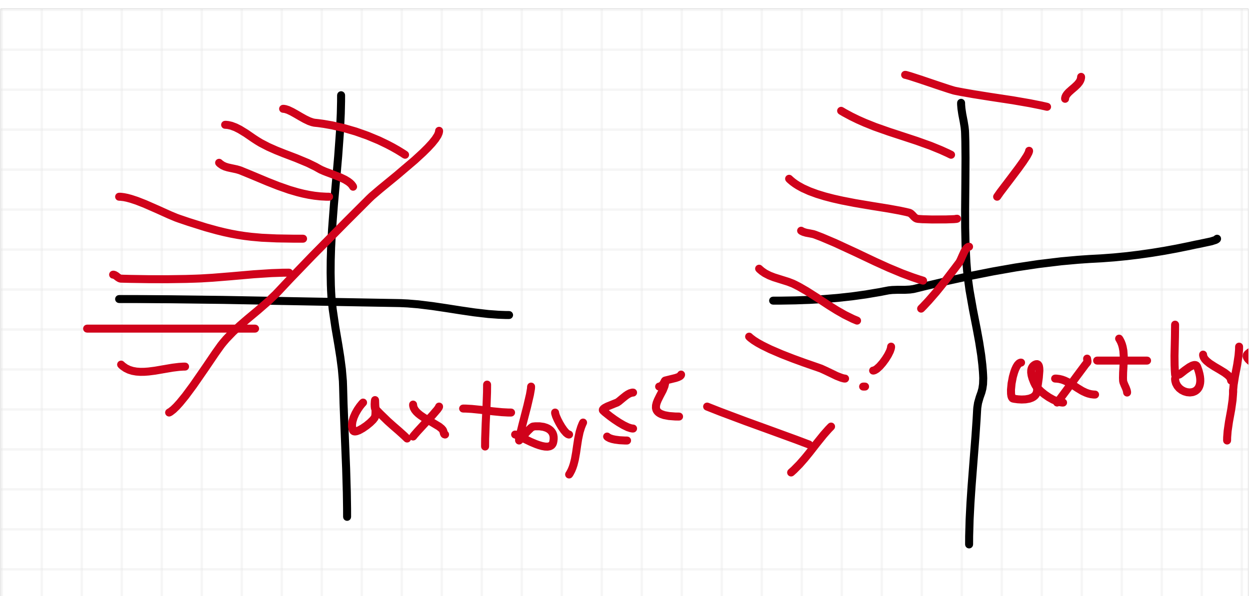
Inequalities in Two Variables

A **linear inequality** in two variables x and y is any inequality that can be equivalently written in the following form
$ax + by < c$
$ax + by > c$
$ax + by \leq c$
$ax + by \leq c$

The solution sets of such inequalities can be graphed in the cartesian plane, as we shall demonstrate.

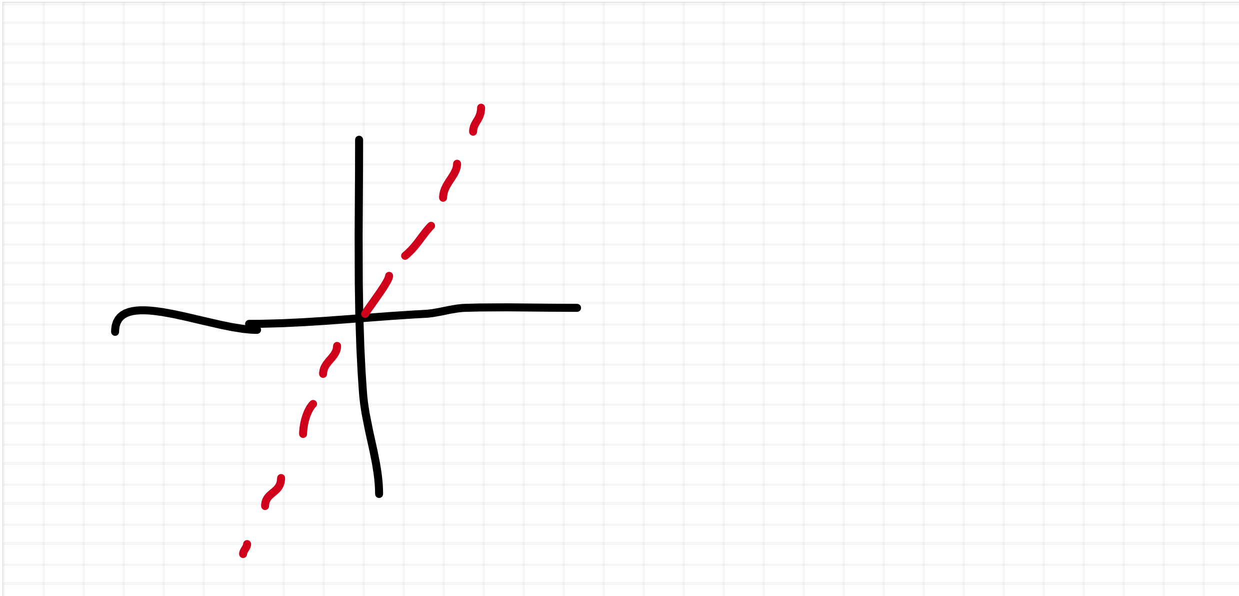
In general, we graph the inequality by graphing the line first that we obtain by switching the inequality to an equality, and then finding the region that will be either above or below that line

IMPORTANT CONVENTION: Note that if the inequality is inclusive, then we draw the WHOLE LINE, and if the inequality is strict, then we TRACE the line.

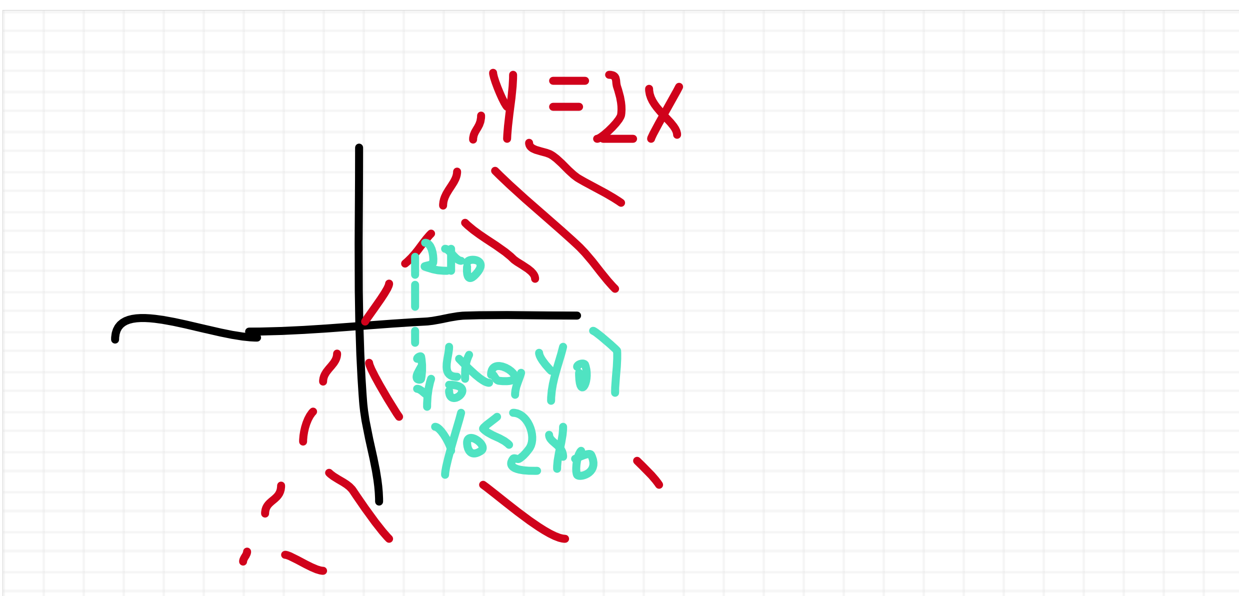


4.3 Example 2 (page 235-236). Graph the inequality $y < 2x$.

To graph this inequality, first we want to find the line $y = 2x$, and trace the line.



After tracing the line, we want to figure out whether we go above or below it. We will talk about two ways to graph above or below the line, but note that with $y < 2x$, what that means is ALL COORDINATES (x_0, y_0) such that y_0 is STRICTLY LESS THAN $2x_0$ are what is included. Note that this consists of everything below the line



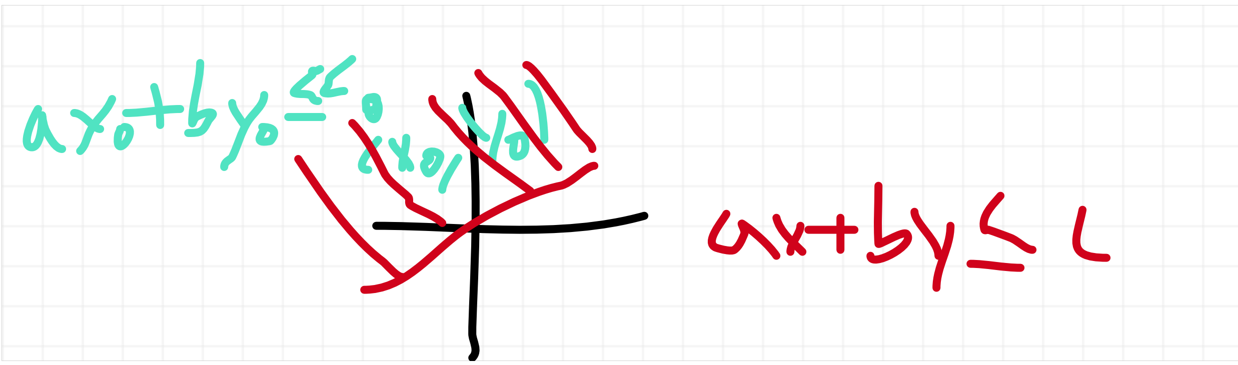
So this idea gives a general step-by-step process to graph an inequality.

Method 1:

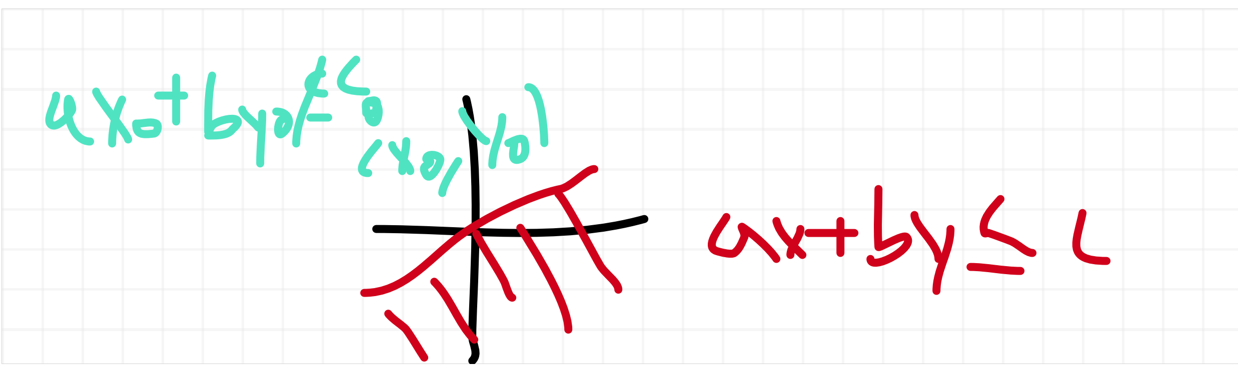
Step 1: Draw or trace the line that forms if you replace the inequality as an equality.

Step 2: Find a point (x_0, y_0) (any point) outside the line and plug x_0 and y_0 into the inequality. If the inequality matches shade the region with the point.

If the inequality matches ($ax_0 + by_0 \leq c$), then shade the region with the point.



If the inequality doesn't match ($ax_0 + by_0 \not\leq c$) then we shade the region <u>without </u>the point.



Method 2:

Step 1: Draw or trace the line that forms if you replace the inequality as an equality.

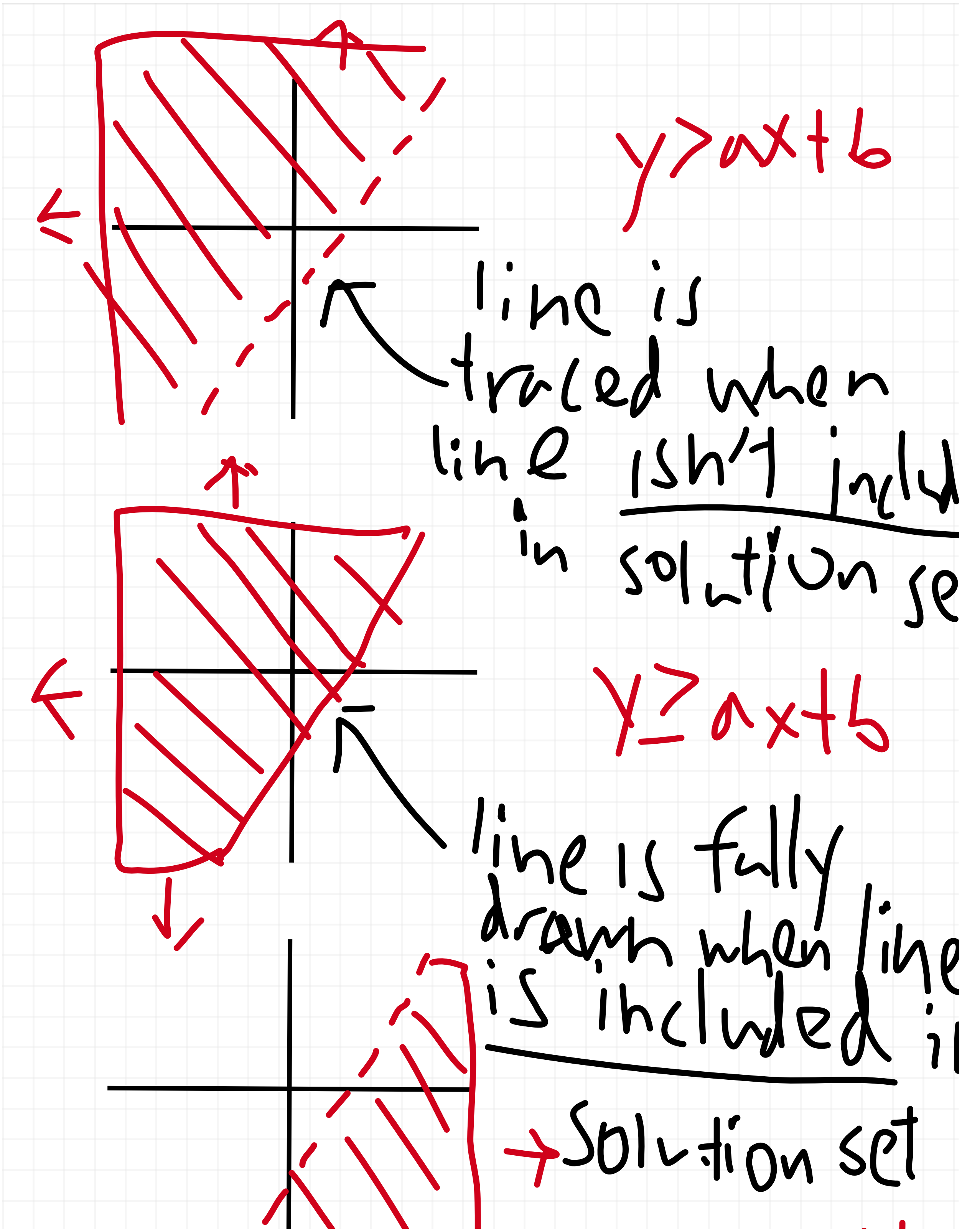
Step 2: Solve the inequality for x or y, i.e. we use algebra to get the inequality to be in the following form

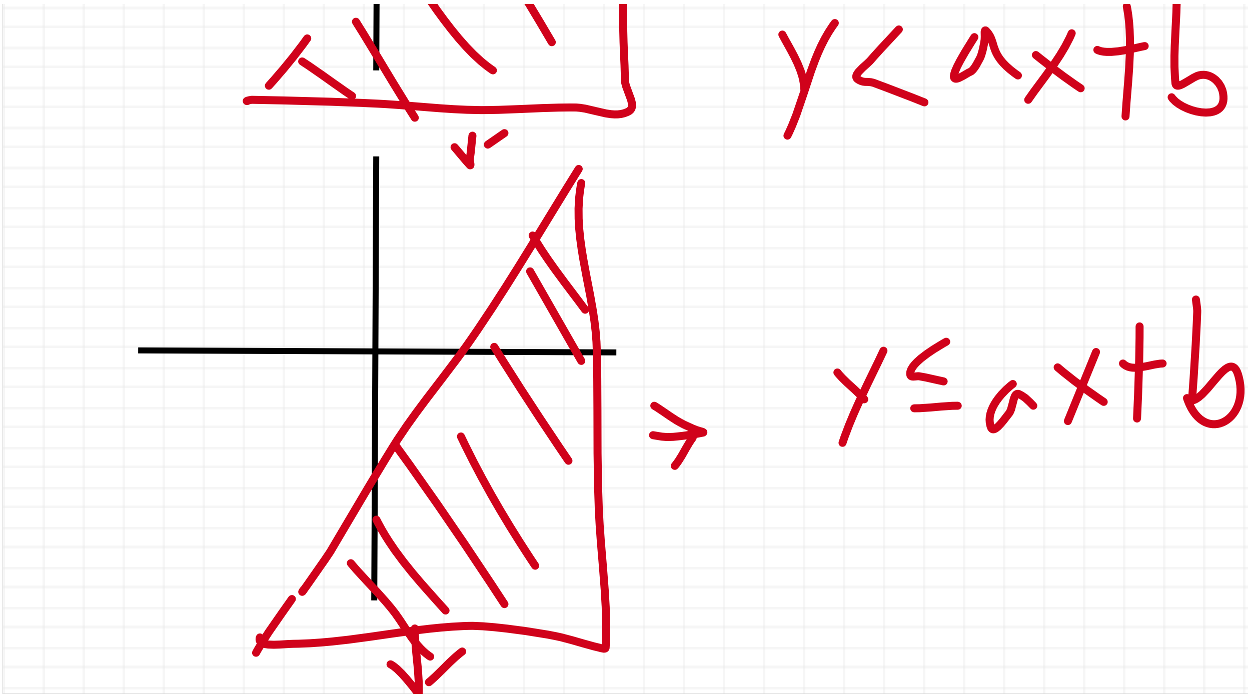
$y < ax + b, \quad y \leq ax + b, \quad y > ax + b, \quad y \leq ax + b$

$x < ay + b, \quad x \leq ay + b, \quad x > ay + b, \quad x \geq ay + b$

If y is by itself, then we shade everything above or below y depending on the direction of the inequality

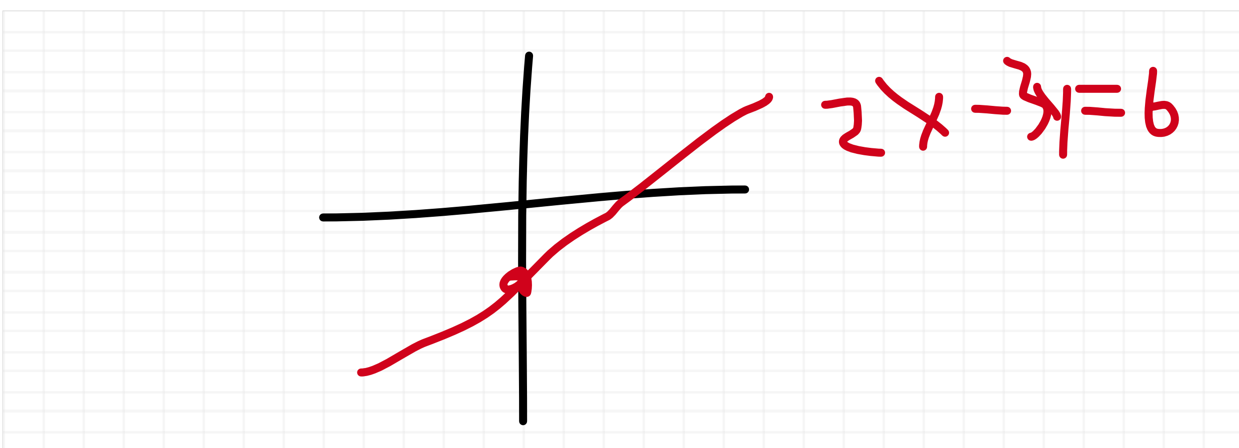
If x is by itself, then we shade everything to the left or to the right of the line, depending on the direction of the inequality (as illustrated below)





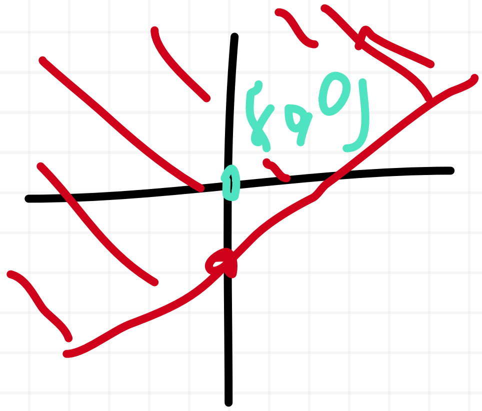
4.3 Example 1 (page 235). Graph $2x - 3y \leq 6$. We'll do method 1 (generally people find this method easier than method 2). First, we draw the line $2x - 3y = 6$.

We know from the general form equation that the slope is $2/3$ and the y-intercept is -2



Next, we pick a point $(0, 0)$ and figure out if the inequality holds

Does $2(0) - 3(0) \leq 6$? We find that holds since $2(0) = 3(0) = 0$ and $0 \leq 6$. So we plug in $(0, 0)$ and graph the region above the line containing it.



$2x - 3y \leq 6$