Sets Lesson 4: Counting Using Venn Diagrams

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Notation for Counting Elements

We use $n(S)$ notation for a given set, which means the "number of elements".

$n(S) =$ "the number of elements in S"

Example 4.3 (page 41) Suppose
$U = \{a, b, c, \ldots, g\},\ A = \{a, d, e\},\ B = \{b, c, d, g\},\ C = \{b, c\}$

Find.

a. $n(A) = 3$
b. $n(B) = 4$
c. $n(U) = 7$
d. $n(B') = 3$
e. $n(A \cup C) = 5$
f. $n(A \cup B)$ Note that $A \cup B = \{a, d, e, b, c, g\}$, there are six elements in the list, so $n(A \cup B) = 6$.

Notice three things:

1. $n(B) + n(B') = n(U)$ (that's not a coincidence)
2. $n(A \cup C) = n(A) + n(C)$, and A and C don't share elements (i.e. they're disjoint $A \cap C = \varnothing$)
3. $n(A \cup B) \neq n(A) + n(B)$, and A and B SHARE elements (i.e., they're not disjoint $A \cap B \neq \varnothing$)

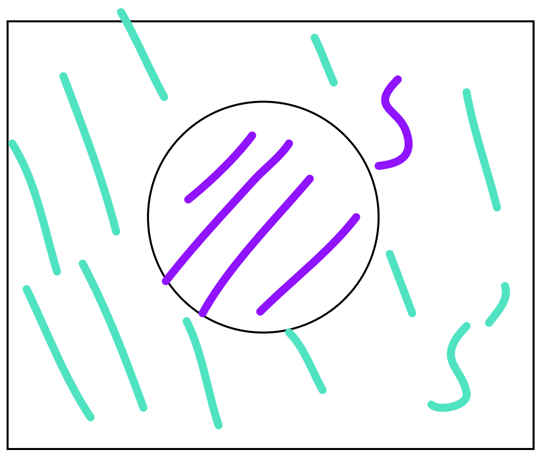
So these ideas are worth keeping in mind as we discuss the formulas for counting sets.

Formulas for Counting Sets

There's four formulas that I want to talk about

Formula 1: The Complement Formula

For any set S,
$n(S) + n(S') = n(U)$



This gives us a way to find the complement $n(S')$ if we know $n(S)$ and $n(U)$, or similarly find $n(S)$ if we know $n(S')$ and $n(U)$

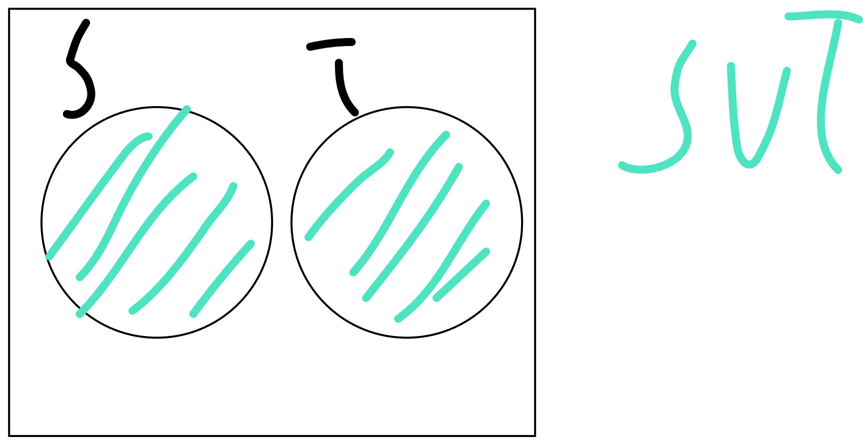
Useful Variations:
$n(S') = n(U) - n(S)$
$n(S) = n(U) - n(S')$

Formula 2: The Disjoint Union Addition Formula

If S and T are disjoint (if $S \cap T = \varnothing$ and they don't have any elements in common), then

$n(S \cup T) = n(S) + n(T)$

This is easy to visualize, since we can draw the two sets in the venn diagram and see that we're not double counting by adding them together since there's no overlap



NOTE:

This is a generalization of the complement formula (formula 1) in the sense that $U = S \cup S'$ and S and S' are disjoint.

This formula can be generalized with multiple disjoint $S_1, \dots, S_n$ for arbitrary n (any combination of $S_i \cap S_j = \varnothing$ when $i \neq j$) as follows:
$n(S_1 \cup \cdots \cup S_n) = n(S_1) + \cdots + n(S_n)$

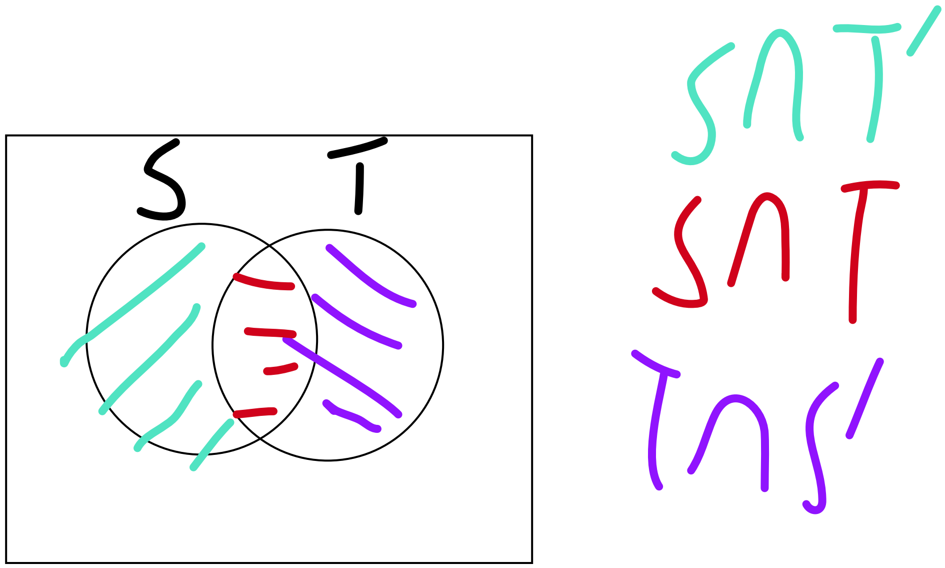
It's not as pretty when there is overlap, as we'll see with formula 3

Formula 3: The Intersection-Union Addition Formula

The most general addition formula (this formula is a generalization of the previous disjoint union addition formula)

$n(S \cup T) = n(S) + n(T) - n(S \cap T)$

This takes care of the overlap issue, because we subtract the time where we possibly count the overlap twice when we just add $n(S) + n(T)$ if there is no overlap



We can see that it generalizes the disjoint union formula since if S and T are disjoint, then $S \cap T = \varnothing$, and $n(S \cap T) = 0$, so

$$n(S \cup T) = n(S) + n(T)$$

This also gives us a formula for the intersection $S \cap T$ as follows

Useful Variant:

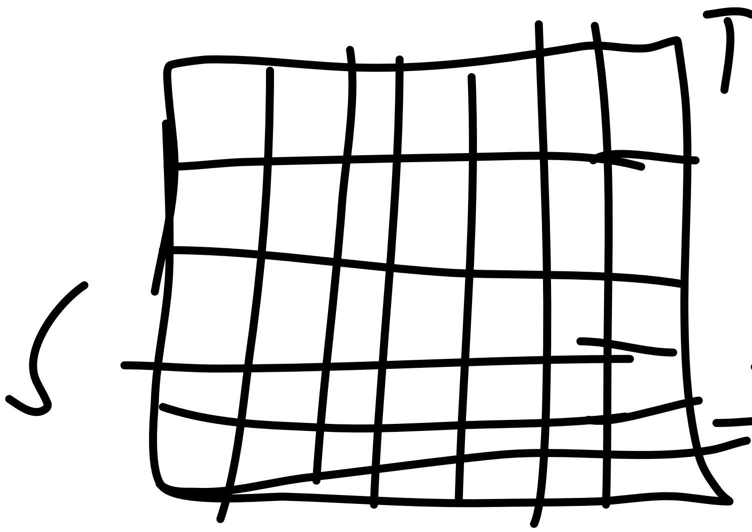
$$n(S \cap T) = n(S) + n(T) - n(S \cup T)$$

Formula 4: The Multiplication Principle with Two Sets

With sets S and T

$$n(S \times T) = n(S) \cdot n(T),$$

because we can visualize products (as we did yesterday in lesson 3) as a rectangle and the elements of $S \times T$ correspond to the individul grid points. Remember the visualization of rolling two six sided dice:



$n(S \times T)$
$= n(S) \cdot n(T$

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Example 4.4 page 44. S has 20 elements (means $n(S) = 20$), T has 15 elements (means $n(T) = 15$), and 6 elements "in common" (this means $n(S \cap T) = 6$). Find how many elements are in the union.

To do this, we use formula 3 (the intersection-union formula)

$n(S \cup T) = n(S) + n(T) - n(S \cap T) = 20 + 15 - 6 = 35 - 6 = 29$.

Different Example (not in the book).

S has 30 elements, T has 10, and $S \cup T$ has 35 elements. What is the number elements in $S \cap T$?

$n(S \cap T) = n(S) + n(T) - n(S \cup T) = 30 + 10 - 35 = 40 - 35 = 5$

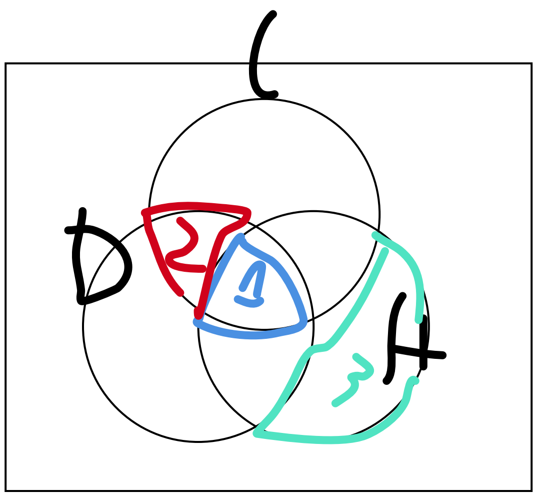
Solving Counting Problems Given in English

Example 3.3 (page 27).

C = all who drink Classic Coke

D = all respondents who drink Diet Coke
H = all respondents who drink Cherry Coke

Describe in english the respondents represented by each number.



Region 1: "all people who drink all three types of coke"

In the language of Sets, we write $C \cap D \cap H$

Region 2: "People who ONLY drink diet coke and classic coke"

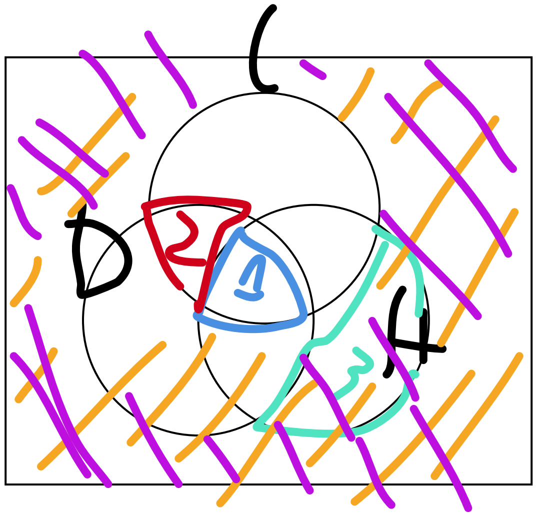
"People who drink diet coke and classic coke but not Cherry Coke"

In the language of Sets, we write $C \cap D \cap H'$

Region 3: "People who ONLY drink Cherry Coke."

"People who Cherry Coke and don't drink diet coke and don't drink classic coke."

$H \cap D' \cap C'$

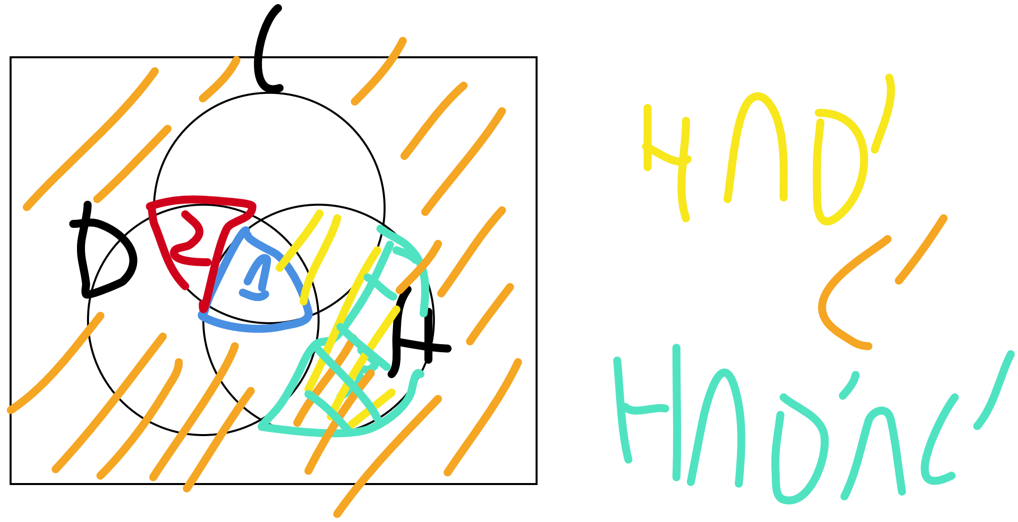


C'

D'

H

H∩D'



<u>Example 4.1 (page 37)</u>

Of 100 students enrolled in an algebra course, 65 were also taking English composition and 50 were also taking Psych. If 35 of the algebra students were enrolled in both English and Psych, how many were not enrolled in either English or Psych?

So to solve a problem like this, first we interpret the english.
U = all the students taking algebra
E = all the students who take English composition
P = all the students who take Psych.

Note that
$E \cap P$ = the students enrolled in both english and psych

$E' \cap P' = (E \cup P)' =$ not enrolled in either English or Psych

We know from the problem explanation that
$n(U) = 100$
$n(E) = 65$
$n(P) = 50$
$n(E \cap P) = 35$

To figure out $E \cup P$, we use the intersection-union formula
$n(E \cup P) = n(E) + n(P) - n(E \cap P) = 65 + 50 - 35 = 115 - 35 = 80$

So to figure out $(E \cup P)'$, we use complement formula
$n((E \cup P)') = n(U) - n(E \cup P) = 100 - 80 = 20$

Using Diagrams to Solve Counting Problems

So we can do problems like Example 4.1 in a different way, i.e., using Venn Diagrams. To use the Venn diagrams, we figure each of the component partition regions of the Venn Diagram, so with two sets, these are sets of the form

$S \cap T,\ S' \cap T,\ T' \cap S,\ T' \cap S'$

With three sets, these are

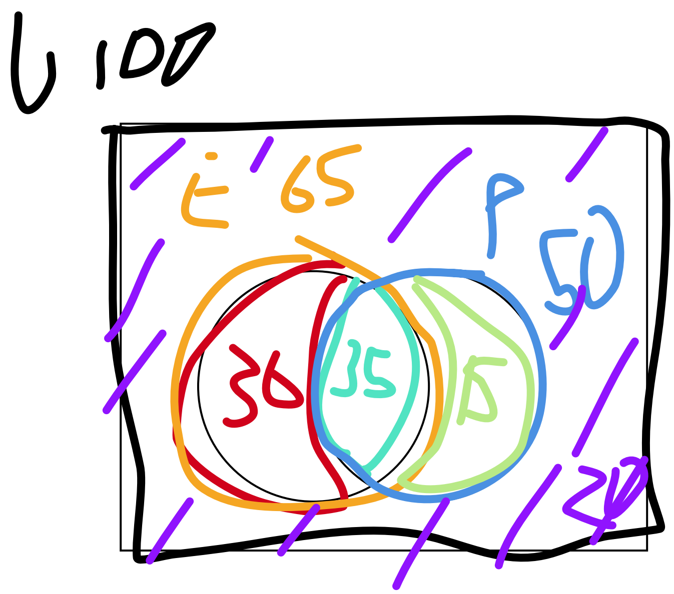
$S \cap T \cap W,\ S' \cap T \cap W',\ S' \cap T' \cap W',$ and so on...

And we figure it out through addition in our head and remembering the number of elements in every set.

With Two Sets:

We'll Compute Example 4.1 in this way. Recall that we're given

$n(U) = 100$
$n(E) = 65$
$n(P) = 50$
$n(E \cap P) = 35$



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With Three Sets:

An advantage of solving problems using the Venn diagram method can generalize easily to three sets.

Example 4.7 (page 46)

There's email E, word processing W, and C computerized instruction
63 used email $n(E) = 63$
70 used word processing $n(W) = 70$
44 used computerized instruction $n(C) = 44$
20 had used email and computerized instruction $n(E \cap C) = 20$
34 had used word processing and computerized instruction $n(W \cap C) = 34$
38 had used word processing and email $n(W \cap E) = 38$
14 had used all three services $n(E \cap W \cap C) = 14$

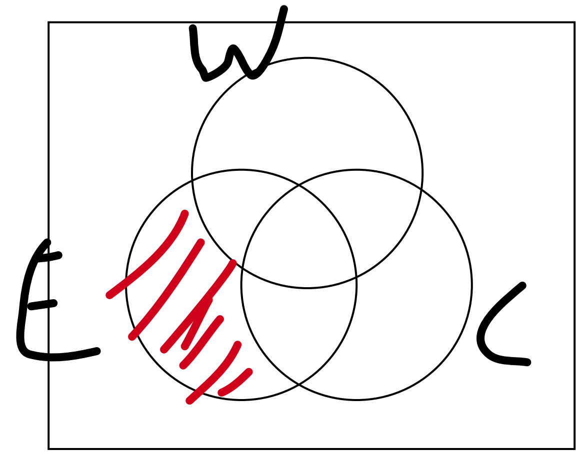
a. How many students surveyed used email but not word processing or computerized instruction

First, let's just translate what is given

What we want to find is

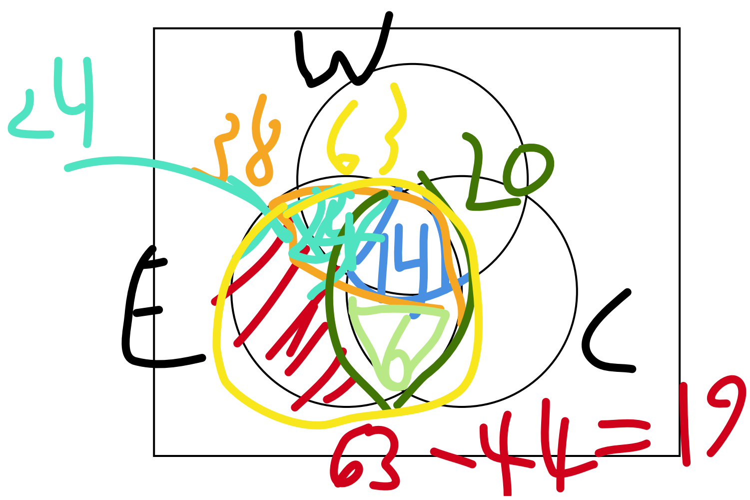
$E \cap (W \cup C)' = E \cap W' \cap C'$

So we're trying to find $n(E \cap W')$



$E \cap W' \cap C$

We want find the number in every component of the diagram as follows



$E \cap W' \cap C$

$n(E \cap W' \cap C') = 19$

Questions on Homework 4