Probability and Counting Lesson 3: Product Sample Spaces

12/8

Product Sets and Product Sample Spaces

Let's recall what a product set is. For any two sets A and B, we define

$$A \times B = \left\{ x : x \ = (a, b),\ \text{where } a \in A \text{ and } b \in B \right\}$$

Recall that we can define a product set for "n-tuples" as follows for sets $A_1, \ldots, A_n$:

$$A_1 \times \ \cdots \ \times A_n = \left\{ x : x = (a_1, \ldots, a_n) \text{ where } a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \right\}$$

Now we'll talk about product sample spaces.

Given two probability experiments with sample spaces S_1 and S_2 respectively the sample space of the two-staged probability experiment where the outcomes are to do the first experiment then the second one and record each of the outcomes of S_1 then S_2 in the order that it happens. This is called the **product sample space** with the sample space as the product set $S_1 \times S_2$.

We can generalize this idea to a **k-staged product sample space** where we do some $k \geq 1$ different probability experiments with sample spaces $S_1, \ldots, S_k$ and record each of the outcomes in the order that it happens. The sample space consists of the k-product set $S_1 \times \ \cdots \ \times S_k$.

Example (homework 2 problem 9). An exeriment consists of tossing a fair die and recording the number then flipping a fiar coin and recording the result of that.

The sample space of this experiment is the product sample space $S = \{1, \ldots, 6\} \times \{H, T\}$.

Example. Flipping a coin 6 times. We have the sample space

$$S = \underbrace{\{H, T\} \times \ \cdots \ \times \{H, T\}}_{\text{6 times}} = \{H, T\}^6$$

NOTE: We like to write $S = T^k$ if S is the repeatedly doing the same probability experiment k times

$$S = \underbrace{T \times \cdots \times T}_{\text{n times}}$$

Example. Picking a random assortment of clothes from a wardrobe in the following way: Picking 3 possible pairs of pants, 2 shirts pairs of shoes, and 5 possible pairs of shirts. If each of the possible pairs of clothes are done in a sequence at random then the sample space is as follows

$$S = 3 \text{ possible pants} \times 2 \text{ possible shoes} \times 5 \text{ possible pairs of shirts}.$$

The Multiplication Principle

Two Staged Multiplication Principle:

For two sample space S_1 and S_2, the number of elements for the product space $S_1 \times S_2$ is computed using the product formula:

$$n(S_1 \times S_2) = n(S_1) \cdot n(S_2)$$

We even gave a general idea for sets $A_1, \ldots, A_k$ for the product formula as

$$n(A_1 \times \cdots \times A_k) = n(A_1) \cdot \cdots \cdot n(A_k).$$

This leads to the general multiplication principle, which is as follows:

General Multiplication Principle:

If we have a k-staged probability experiment with sample spaces $S_1, \ldots S_k$, with $n(S_1) = n_1, n(S_2) = n_2, \ldots, n(S_k) = n_k$. We find for the k-staged experiment $S = S_1 \times \cdots \times S_k$, we get

$$n(S) = n_1 \cdot \cdots \cdot n_k.$$

Calculating Probabilities in a Multi-Staged Experiment

Example 3.2 (page 76). Lauren gives three dresses, five scarves, four pairs of shoes, and three hats. Lauren picks a dress, a scarf, and a hat. How many outfits can she make?

What is the size of the multistaged experiment of picking a dress, then a scarf, then shoes, then a hat. We use the multiplication principle and get

$$n(S) = n\left(\text{dresses}\right) \cdot n\left(\text{scarves}\right) \cdot n\left(\text{shoes}\right) \cdot n\left(\text{hats}\right) = 3 \cdot 5 \cdot 4 \cdot 3 = 180.$$

Let's say **one** of the three dresses is red and **two** of the four pairs of shoes are white. What is the probability of wearing a red scarf and a white pair of shoes?

Let E be the event where that happens, and note that it is a product set of the number of red dresses, whire shoes, and any hats and scarves in general (since the event doesn't impose any condition on the hats or dresses)

$$Pr[E] = \frac{n(E)}{n(S)}$$

We want to find $n(E)$. To find $n(E)$ we use the multiplication principle (on product sets) as well.

$$n(E) = n\left(\text{red dresses}\right) \cdot n\left(\text{scarves}\right) \cdot n\left(\text{white shoes}\right) \cdot n\left(\text{hats}\right) = 1 \cdot 5 \cdot 2 \cdot 3 = 30$$

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{30}{180} = \frac{1}{6}.$$

<u>Example 3.3 (page 77).</u> A multiple choice quiz consists of <u>8 questions</u>, each with **five** possible answer choices: A, B, C, D or E. How many different sets of answers are possible?

So note that the sample space is the 8-staged experiment of picking a answer on a question

$$S = \{A, B, C, D, E\}^8$$

And using the multiplication principle, we find

$$n(S) = n(\{A, B, C, D, E\})^8 = 5^8.$$

What is the probability that someone taking the quiz gets question 1,3, and 8 correct; let's assume that each question <u>has only one correct answer</u>. So let E be the event that question 1,3, and 8 are correct. Note that this is following product set

$$E = \Big\{\text{correct answer for question } 1\Big\} \times \Big\{\text{any answer}\Big\} \times \Big\{\text{correct answer for question } 3\Big\}$$
$$\times \Big\{\text{any answer}\Big\} \times \Big\{\text{any answer}\Big\} \times \Big\{\text{any answer}\Big\} \times \Big\{\text{any answer}\Big\}$$
$$\times \Big\{\text{correct answer for question } 8\Big\},$$

So note that

$$n\Big(\Big\{\text{correct answer for a given question }\Big\}\Big) = 1$$
$$n\Big(\Big\{\text{any answer}\Big\}\Big) = 5$$

Using the multiplication principle, we have

$$n(E) = n\Big(\Big\{\text{correct answer for question } 1\Big\}\Big) \cdot n\Big(\Big\{\text{any answer}\Big\}\Big) \cdot n\Big(\Big\{\text{correct answer for question } 3\Big\}\Big)$$
$$\cdot\, n\Big(\Big\{\text{any answer}\Big\}\Big) \cdot n\Big(\Big\{\text{any answer}\Big\}\Big) \cdot n\Big(\Big\{\text{any answer}\Big\}\Big) \cdot n\Big(\Big\{\text{any answer}\Big\}\Big)$$
$$\cdot\, n\Big(\Big\{\text{correct answer for question } 8\Big\}\Big)$$
$$= 1 \cdot 5 \cdot 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 1 = 5^5$$

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{5^5}{5^8} = \frac{1}{5^3} = \frac{1}{125}.$$