Probability and Counting Lesson 4: Permutations and Combinations

12/8

Finding the Number of Ways to Order n Things

Let's say that we have a set A with n elements. Let's say we're trying find the set S of possible ways to order the n elements

$$S = \left\{x : x \text{ is an ordering } (a_1, \dots, a_n) \text{ (with each } a_n \text{ distinct) of the } n \text{ elements of } A\right\}$$

To order it, we look at S as an n-staged experiment with each sample space having one less element than the last (because we're ordering a set by picking an element, noting the order, *without replacement*). What we get is, using the mulplication principle:

$$n(S) = n \cdot (n-1) \cdot (n-2) \cdot \ \cdots \ \cdot 2 \cdot 1 = n!$$

(remember that $n!$ is multiplying the succession of n and $n-1$ all the way down to 1)

So that's how we do that, and that opens the door to permutations and combinations.

Permutations

Example 4.3 (page 87)

A four letter ID code made from the following set of letters:

$$\{A, B, C, D, E, F\}$$

where no letter can be used more than once. How many ID codes can be formed?

The way to think about this is slots. We have four flows

$\_ \_ \_ \_$

Each slot represents a stage in the 4 stage experiment we use to determine the ID code by picking a letter, putting it in the first slot, then repeating the process three more times.

This gives us sample spaces S_1, S_2, S_3, S_4 each one less number of possibilities than the last, and we have

$n(S_1) = 7,\ n(S_2) = 6,\ n(S_3) = 5,\ n(S_4) = 3$

Using the multiplication principle, we get

$n(S) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ different codes.

This kind of calculation can be generalized to when we have an arbitrary number n of objects when we start selecting, and some number r between 1 and n $(1 \leq r \leq n)$ of selections.

When we select r objects from a collection of n objects in order without replacement, we call that an (n, r)**-permutation**. And we refer to this sequence simply as a **permutation** when the context of n and r is known.

We often want to find $n\big((n, r)\text{-permutations}\big)$ and we use notation $P(n, r)$ to talk about that number, so

$P(n, r) := n\big((n, r)\text{-permutations}\big)$

Then we end up doing a similar multiplication principle calculation to the previous example, each stage with one less outcome than the last, and we get the following formula:

Permutation Formula:

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot\ \cdots\ \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

Combinations

With permutations, we cared a lot about the order in which objects were selected, in other words, "order matters".

With combinations, we disregard the order, and look at the selections as a set where "order doesn't matter".

A (n, r)**-combination** for $1 \leq r \leq n$ is a subset of a set with n elements that has r elements:

$\{a_1, \ldots, a_r\} \subset \{b_1, \ldots, b_n\}$
with $a_1, \ldots, a_r$ are distinct from each other
and $b_1, \ldots b_n$ are distinct from each other

We use the term **combination** when the context of n and r is clear.

Example 5.2. (page 97)

List all the different strings of letters of the set $\{A, B, C, D\}$

First, we'll find all the possibilities by hand and count them.

Combinations Vs.	Permutations
$\{A, B, C\}$	$ABC, ACB, BCA, BAC, CAB, CBA$
$\{A, B, D\}$	$ABD, ADB, BAD, BDA, DAB, DBA$
$\{B, C, D\}$	$BCD, BDC, CBD, CDB, DBC, DCB$
$\{A, C, D\}$	$ACD, ADC, CAD, CDA, DAC, DCA$

The point is, there are four combinations, every combination can be ordered 6 different ways to form a permutation. The fact that there is a uniform number of permutations for every combination is no coincidence.

Let's generalize the above to an n element set, and finding the number of possible subsets with r elements. We have $r!$ ways to make any given combination into a permutation because (as mentioned before) there's $r!$ ways to order a combination. So we get

$$P(n, r) = r! \cdot n\big((n, r)\text{-combinations}\big)$$

We use $C(n, r)$ as shorthand for $n\big((n, r)\text{-combinations}\big)$ (i.e. $C(n, r) := n\big((n, r)\text{-combinations}\big)$), and using the formula for $P(n, r)$, we get the following formula:

Formula for Combinations:

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1)}{r \cdot (r-1) \cdot \cdots \cdot 1}.$$

Example.

How many 4-element subsets can be selected from a set containing 7 elements?

The question is asking how many $(7, 4)$-combinations, so to do that, we use the formula

$$C(7,4) = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{6} = 7 \cdot 5 = 35.$$

Telling the Difference Between Permutations and Combinations

Example.

There are 9 books on a shelf, including the new Twighlight book, in how many ways can you select 4 out of the 9 books to take with you on vacation?

Is it permutations or combinations?

Order doesn't matter, so we do combinations.

What type of combinations do we have?

We are selecting 4 books out 9 books so we're trying to find a 4 element subset of a 9 element subset

$S =$ (9,4)-combinations

$$n(S) = C(9,4) = \frac{9!}{(9-5)!4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{8 \cdot 3} = 9 \cdot 7 \cdot 2 = 126$$

Example.

There are seven track runners in the finals of the 100 meter dash. Let's say the top five score points, so in how many ways can

a. there be a top 5?

Permutations or combinations? Combinations, because we're talking about the subset of the top five

What type of combinations? $(7, 5)$-combinations

$$n(S) = C(7,5) = \frac{7!}{(7-5)!5!} = \frac{7 \cdot 6}{2 \cdot 1} = 21$$

b. The number of possible scoring arrangements.

Permutations or combinations? Permutations
What type of Permutation? $(7,5)$-permutations

$$n(S) = P(7,5) = \frac{7!}{(7-5)!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 42 \cdot 60 = 2520$$

NOTE: I ended the lecture at this point, but I want to provide notes for examples where you calculate the probability of an event using permutations and combinations

(AFTER 12/8 LECTURE)

Computing Probabilities with Permutations and Combinations

We often compute probabilties by using permutations and combinations by using the formula $Pr[E] = n(E)/n(S)$ and using permutations and combinations to count $n(E)$ and $n(S)$. Here are some examples where we do this:

<u>Example 5.7 a. (page 103)</u>

Compute the Probability of selecting the Ace of hearts, king of hearts, and queen of hearts.

NOTE: We don't care about the order, so we use $(52,3)$-combinations as our sample space (which are *equally likely*)

$$n(E) = n\big(\text{selecting the Ace, king, and queen of hearts}\big) = 1$$

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{1}{C(52,3)}.$$

Let's tweek the wording as follows:
Compute the probability of selecting the Ace of hearts, THEN the king of hearts, THEN the queen of hearts.

Now order matters, and we use *permutations*, and to cut to the chase, we get

$$Pr[E] = \frac{n(E)}{n\left(\text{permutations}\right)} = \frac{n\left(\left\{\left(\text{ace of hearts, kings of hearts, queen of hearts}\right)\right\}\right)}{n(\text{permutations})} = \frac{1}{P(52,3)}.$$

<u>Example.</u>

Let's say we select a red ball, 2 blue balls, 2 green balls, and 3 orange balls. Let's select 3 balls

a. What is the probability that you select 3 orange balls?

$S =$ (8,3)-combinations, because we don't care about the order we select the balls (and we're finding 3 items from a collection 8)

$E =$ (3,3)-combinations, because we're selecting three orange balls out of three orange balls

$$n(S) = C(8,3) = \frac{8!}{5!3!} = \frac{8\ \cdot 7 \cdot 6 \cdot 5!}{5!3!} = \frac{8\ \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$$

$$n(E) = C(3,3) = \frac{3!}{(3-3)!3!} = \frac{3!}{0!3!} = \frac{3!}{1\ \cdot 3!} = 1$$

$Pr[E] = n(E)/n(S) = 1/56$

b. Probability that a primary color is selected?

The event E is the possible blue and red balls selected (remember that there are 1 red and 1 2 blue), so there are three such balls total, so
$E =$ (3,3)-combinations
$n(E) = C(3,3) = 1$
$Pr[E] = n(E)/n(S) = 1/56$