Sets Lesson 1: Basic Definitions

11/11

Now, we're going to use the book *Sets and Probability* by Linda McKinley.

So for this general unit, we use a mathematical concept called "sets", which generally speaking describes a collection of objects. We find that these collections open up a new world of arithmetic which allow us (as we'll see in the probability module) to better articulate what exactly probabilities are and how we calculate them.

What Exactly is a Set?

A **set** is a collection of objects, which are referred to as **elements**.

It's a whole new "mathematical language" (in the same way you have arithmetic with numbers, addition, and subtraction)

For this set unit, we're learning how to do math on sets:

How do we mathematically describe a set (i.e., what notation do we use)?

To describe any set, we often use capital letters like $A, B, C,$ and often S.
When we talk about sets in capital letters, we often do that for *arbitrary* sets--in other words, we don't know what the set necessarily is--unless that set is specified (think of these capital letters as "variables")

Set Notation

Now, we'll talk about the specific notation to describe different sets. There's two ways to describe a set using mathematical notation:

1. A bracketed list
2. "Set Buider Notation"

Sets as Bracketed Lists

One way to describe a set is by simply listing all the elements and putting that list in brackets. For example:

$\{1, 2, 3\}$ is the set that contains 1,2, and 3 as elements
$\{a, b, c, d\}$ is the set that contains a, b, c, and d as elements
$\{apple,\ orange,\ bannana\}$ is the set that contains "apple", "orange", and "bannana" as elements.

We can even write an infinite list of numbers using brackets writing first three elements and identifying the pattern:

$\{1, 2, 3, \ldots\}$ refers to the set containing every positive whole number as an element.
$\{2, 4, 6, \ldots\}$ refers to the set containing every positive even number as an element.

Sometimes, especially when the sets are infinite, it may be better to describe the set by describing the condition (in english) for something to be an element.

Set Builder Notation

Another way to describe sets is a bracket with a descriptor as follows in these examples.

Example:
$\left\{x | x \text{ is a positive whole number}\right\}$ is another way to describe the set of positive whole numbers.
$\left\{x | x \text{ is a fruit}\right\}$ describes the set of all fruits
$\left\{x | x \text{ is both at IU campus and not at IU campus}\right\}$ which is a contradictory phrase, which we'll see has no element (when we talk about the "empty set")

Note that a set expressed in set-builder notation could be "equal to" (i.e. the same set as) a set described differently (even if it's described using a list). For example:
$\left\{x | x \text{ is a positive whole number between 1 and 3}\right\} = \{1, 2, 3\}$

This brings us to the question: What does it mean for two sets to be equal? (hence the next section)

Elementhood, Equality of Sets, and Subsets

We have the *following notation* to describe if an object a **is an element of** a set A:

To signify that any a is an element of A, we write "$a \in A$". To signify that a is NOT an element of A, we write $a \notin A$.

Example:

In the set $\{1, 2, 3\}$, we have $1 \in \{1, 2, 3\}$, $2 \in \{1, 2, 3\}$, $3 \in \{1, 2, 3\}$ since those are objects in the set; however 4 is NOT an element of $\{1, 2, 3\}$, so $4 \notin \{1, 2, 3\}$

For any positive whole number $n > 0$, we have $n \in \left\{x | x \text{ is a positive whole number}\right\}$; however, for -1, we have $-1 \notin \left\{x | x \text{ is a positive whole number}\right\}$.

We can define equality in the universe of sets in terms of a set's elements as follows:

Two sets A and B are said to be **equal** (i.e., the same set) if every $a \in A$ is also an element of B and conversely every $b \in B$ is also an element of A. In other words, they are equal if they contain the same elements

Example:
$\left\{x | x \text{ is a positive whole number between 1 and 3}\right\} = \{1, 2, 3\}$,
because we know $1, 2, 3$ are all the elements of
$\left\{x | x \text{ is a positive whole number between 1 and 3}\right\}$
and those are also elements of $\{1, 2, 3\}$, and conversely, we know $1, 2, 3$ are all the elements of $\{1, 2, 3\}$ and those are in $\left\{x | x \text{ is a positive whole number between 1 and 3}\right\}$.

11/17

More Examples:
$\{1, 2, 3\} = \{2, 3, 1\}$ because they have the same elements; order of the list doesn't change what the set is.

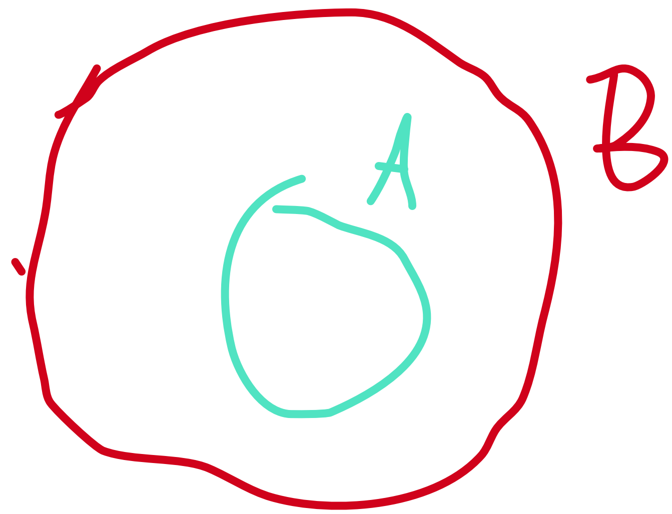
$\{1, 2, 3\} = \{1, 1, 2, 2, 3\}$ because they have the same elements; repitition in a list also doesn't change what the set is.

So to conclude, there are many ways to name the same set (using set builder notation or bracketed lists).

Let's continue by defining a subset

We say that A is a subset (we write " $\subset$ " or " $\subseteq$ ") of B if for every $a \in A$ is also element of B; in other words, any element of B you can find in A as well.

In a Venn-Diagram (we'll talk more about Venn Diagrams in lesson 3), we symbolize $A \subset B$ by drawing the set A completely inside the set B



<u>Examples:</u>

$\{1, 2\} \subset \{1, 2, 3\}$ because 1 and 2 of $\{1, 2\}$ are both in $\{1, 2, 3\}$ as well.
$\left\{x | x \text{ is an even positive number}\right\} \subset \left\{x | x \text{ is a positive integer}\right\}$ because every even positive number is also a positive number.

One way to think about it:
$\left\{x | x \text{ is an even positive number}\right\} = \{2, 4, 6, 8, \dots\}$
$\left\{x | x \text{ is a positive integer}\right\} = \{1, 2, 3, 4, \dots\}$

So any number n that an even number in $\{2, 4, 6, 8, \dots\}$ is also a positive number in $\{1, 2, 3, 4, \dots\}$, so we conclude
$\left\{x | x \text{ is an even positive number}\right\} \subset \left\{x | x \text{ is a positive integer}\right\}$

In general $A \not\subset B$ when there is $a \in A$ that is not an element of B (in other words $a \notin B$)

<u>Non-Examples:</u>

$\{1,2,4\} \not\subset \{1,2,3\}$ because $4 \in \{1,2,4\}$, but not an element of $\{1,2,3\}$
$\{x|x \text{ is a business major in the U.S.}\} \not\subset \{x|x \text{ is an IU student}\}$, because if x is a business major in Purdue, then $x \in \{x|x \text{ is a business major in the U.S.}\}$ but $x \notin \{x|x \text{ is an IU student}\}$.

In general, $A \subset B$ doesn't necessarily imply that they're equal. As I said before $\{1,2\} \subset \{1,2,3\}$ but $\{1,2\} \neq \{1,2,3\}$ since they don't have the same elements, since $3 \in \{1,2,3\}$ but $3 \notin \{1,2\}$.

However, $A \subset B$ and $B \subset A$ implies that they're equal, because then every $a \in A$ is also in B and every $b \in B$ is also in A, so they have the same elements; in other words, there's no element I can name from one set that isn't in the other.

Disclaimer: " $\subseteq$ " and " $\subset$ " mean the same thing (they mean it's a subset and it *could be equal*), but we write " $\subsetneq$ " or " $\subsetneqq$ " to mean that a set is a subset but not equal, e.g. $\{1,2\} \subsetneqq \{1,2,3\}$, and we call a subset that is not equal a **strict subset**.

The Universal Set, the Empty Set, and Complements

Whenever we talk about sets, we do it to a point of reference in a possible "universe" of elements.

For example:

The universe of whole numbers consists of $\cdots, -2, -1, 0, 1, 2, \cdots$
The universe of real numbers consist of any real number (rational or irrational) such as $1, \pi, 3/2, \pi/e$, way more than we can name or list
The universe of fruits which consist of all possible categories of fruits (apples, bannanas, strawberries)
The universe of all ~50,000 IU students. (Correction: 45,000)

A **univerval set** is defined to be the "reference set" that contains all the elements in a given universe.

We always use the letter "U" when talking about the universal set.

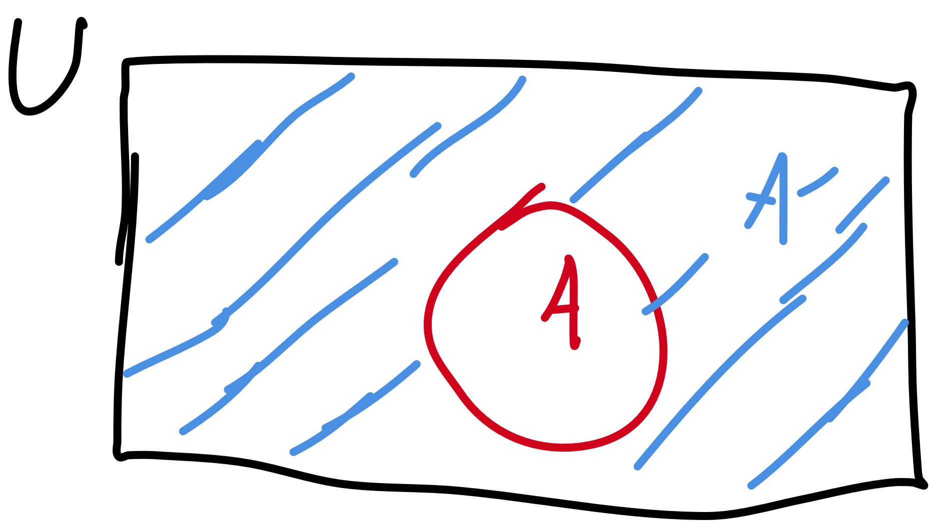
<u>Examples:</u>

In the universe of whole numbers
$U = \left\{x|x \text{ is a whole number}\right\} = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$.
In the universe of real numbers $U = \left\{x|x \text{ is a real number}\right\}$
The universe could hypothetically be just be the first five letters of the alphabet, which would give us $U = \{a, b, c, d, e\}$.

So the universal set gives us context for talking about the "complement"

The **complement** of a set A, which we write as A', means everything in the universal set U that is *not in* A.
$A' = \left\{x|x \in U \text{ but } x \notin A\right\}$

A note about Venn Diagrams: When we draw a Venn Diagram, we treat the frame of the Venn
Diagram (the black rectangle below) as the universal set U (so the universe is the entire canvas of a drawing, so the set U consists of the entire canvas), and the complement of A in a Venn Diagram, A', is everything shaded outside of the set A (in the illustration below, the blue shaded region consists of A')



Examples:

1. Let's have $U = \left\{x | x \text{ is a whole number}\right\}$.

If $A = \left\{x | x \text{ is an even number}\right\}$, then $A' = \left\{x | x \text{ is an odd number}\right\}$

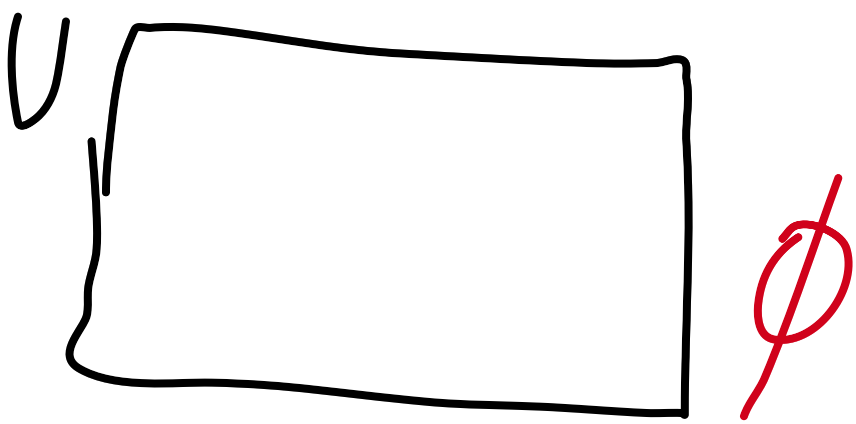
2. Let's have $U = \{a, b, c, d, e\}$. We find $\{a, e\}' = \{b, c, d\}$.

The **empty set** (or the "null set") is a set without *any* element, and we write the empty set as $\varnothing$, but like with sets in general, there are many ways to write the empty set.

We can write the empty list $\{\}$ which has no elements, so $\{\} = \varnothing$
We can also write contradictory phrases in set-builder notation, which have no element since they're contradictory, and hence is equal to $\varnothing$, e.g.
$\left\{x | x \text{ is both at IU campus and not at IU campus}\right\} = \varnothing$

11/19

General Fact: For the universal set U, it's always true that $U' = \varnothing$ because there's nothing that is both in U but not in U, and note that
$U' = \left\{x | x \in U \text{ and } x \notin U\right\} = \varnothing$



We shade nothing for the empty set, since the empty set contains nothing

Sets Homework 1 Questions

Question 13 (page 11)

$S = \{2, 3, 4\}$ Find S' if

a. $U = \{1, 2, 3, 4, 5\}$
$S' = \{1, 5\}$

b. $U = \{1, 2, 3, \dots, 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$S' = \left\{x | x \in U \text{ and } x \notin S\right\}$
$= \{1, 5, 6, 7, 8, 9, 10\}$

c. $U = \{2, 3, 4\} = S$
$S' = \emptyset$

d. $U = \left\{x | x \text{ is an integer greater than zero}\right\}$
$S' = \{1, 5, 6, 7, \dots\} = \{1\} \cup \{5, 6, 7, \dots\} = \left\{x | x = 1 \text{ or } x \text{ is an integer greater than } 4\right\}$

<u>Question 23 (page 12)</u> Suppose U is the set of all students enrolled in M014. Suppose E are are students in U that are enrolled in English W131 and P consists of all students in U that are enrolled in Psych 101

$E = \left\{x | x \text{ is enrolled in W131 }\right\}, \; P = \left\{x | x \text{ is enrolled in Psych 101}\right\}$

Translate the following (I'll leave parts b. and d. to you guys):

a. $E' = \emptyset$
"all students not enrolled in W131 are no students at all"
We can be a bit more efficient with how we phrase such sentences in the problem and we can write
"no students are not enrolled in W131"
"all students are enrolled W131"

c. $P \subset E$
"all students enrolled in Psych 101 are enrolled in W131"