Sets Lesson 2: Set Operations

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In the previous section (section 1) we talked about the complement A' of a set A and applying the complement $'$ to a set is a unary operation on that set (similar to the that the negative operation $-$ on a number n gives us a negative number $-n$)

So now we're going to talk about the binary operations union " $\cup$ " and " $\cap$ " and " $\times$ "

Intersection and Union

Intersection: $A \cap B := \left\{x | x \ \in A \text{ and } x \in B\right\}$
So the **intersection** $A \cap B$ of the sets A and B is the set of every element of both A and B, so:

$x \in A \cap B$ precisely when $x \in A$ and $x \in B$

<u>Example 2.1 (page 13):</u> $S = \{1, 2, 6\},\ T = \{1, 3, 5\},\ W = \{2, 6\}$ We shall find

a. $S \cap T$ So to find that, we go through the sets in S that are also in T, so
$1 \in T$, so $1 \in S \cap T$
$2 \notin T$, so $2 \notin S \cap T$
$6 \notin T$, so $6 \notin S \cap T$
So the only element of $S \cap T$ is 1, so $S \cap T = \{1\}$

b. $S \cap W$ We go through the sets in S that are also in W
$1 \notin W$, so $1 \notin S \cap W$
$2 \in W$, so $2 \in S \cap W$
$6 \in W$, so $6 \in S \cap W$
So the elements of $S \cap W$ are 2 and 6, so $S \cap W = \{2, 6\}$

c. $T \cap W$, we go through W and see the elements of W that are also in T: Note that:

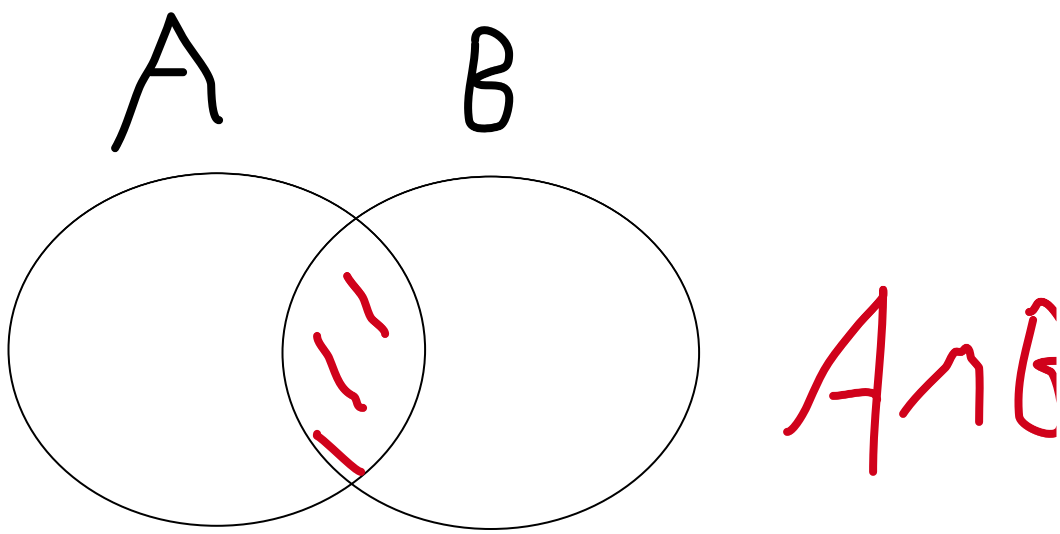
$2 \notin T,\ 6 \notin T,$

so as result, there NO elements in $T \cap W$ since we can't find any elements of T that are also in W. So $T \cap W = \varnothing$, i.e. $T \cap W$ is the "empty set" or "null set".

Note that in Example 2.1 b., we have $S \cap W = W$, which is not always the case (as with example 2.1). We have $S \cap W = S$ or W precisely when $S \subset W$ or $W \subset S$, because $S \cap W \subset S$ and $S \cap W \subset W$ because everything that is in $S \cap W$ is also in S and W.

Example 2.1 c. is part of a more general phenomenon where the intersection of two sets T and W could contain no elements at all, i.e. $T \cap W = \varnothing$. We call T and W **disjoint** whenever $T \cap W = \varnothing$, or when they have no elements in common.

So in terms of a Venn-Diagram, the intersection visually is the region of the two sets A and B that overlap.



Union: $A \cup B := \left\{x | x \ \in A \ \text{ or } x \in B\right\}$

So the **union** $A \cup B$ of the sets A and B is the set of every element in either A or B (or both).

Note: When we mention "or" in the union, we mean the "inclusive or" where we include elements of both A and B, in other words, we have $A \cap B \subset A \cup B$.

$x \in A \cup B$ precisely when $x \in A$ or $x \in B$ (or $x \in A$ and $x \in B$).

<u>Example 2.2 (page 14).</u> Suppose $X = \{1, 2, 3, 5\}, \ Y = \{2, 4, 8\}, \ \ Z = \{3, 9\}$. Find:

a. $Y \cup Z$

So to find that, we make a list that includes all the elements of Y and then all the elements of Z that are left over; in other words we combine the lists

Y has elements $2, 4, 8$
Z has elements $3, 9$

So we end up with a combined list $2, 4, 8, 3, 9$ of all the elements of $Y \cup Z$, and we have

$Y \cup Z = \{2, 4, 8, 3, 9\}$.

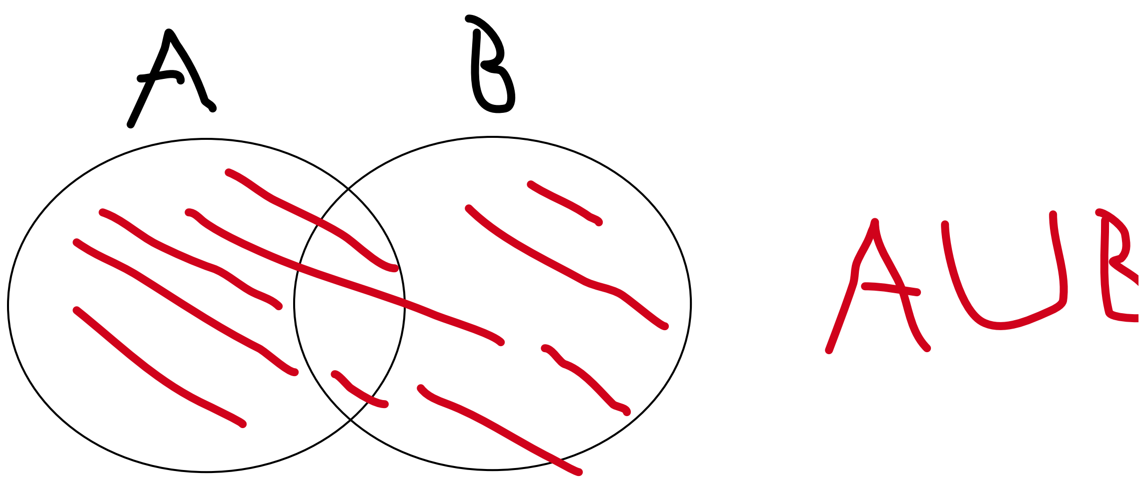
b. $X \cup Y$

X has elements $1, 2, 3, 5$
Y has elements $2, 4, 8$

So we end up with a combined list $1, 2, 3, 5, 4, 8$

$X \cup Y = \{1, 2, 3, 5, 4, 8\}$

In terms of the Venn Diagram for $A \cup B$, we shade everything in both regions to get the following combined region:



Translating the Language of Sets to English (and Vice Versa)

Note that the paradigm of sets that we're talking is a mathematical language with operations $\cup, \cap, ', \times$ (just like the language of arithmetic with $+, -, \cdot$), and a useful one for making sense of finding probabilities.

And this language of sets has an "english translation", and it works as follows

First, we look a given "atomic set", i.e. sets in terms of set builder notation, which uses an english phrase to describe all its elements, and it translates in english to that english phrase

$\left\{x|x \text{ is a whole number}\right\}$ translates to the english phrase "the whole numbers"

$\left\{x|x \text{ is an IU student}\right\}$ translates to the english phrase "the IU students"

$\left\{x|x \text{ is a fruit}\right\}$ translates to the english phrase "the fruits"

Note that sometimes sets are given as a list, such as the set $\{1, 2, 3\}$, so then we have to find a phrase that describes the set in set-builder notation and that's how we translate that set, so since

$\{1, 2, 3\} = \left\{x|x \text{ is a whole number between 1 and 3}\right\}$,

we find that:

$\{1, 2, 3\}$ translates to "the whole numbers between 1 and 3

intersection, union, and complement translate to the words "and", "or", "not" respectively that modifies/combines atomic phrases.

$\left\{x|x \text{ is a fruit}\right\} \cup \left\{x|x \text{ is a vegetable}\right\}$ translates to "fruits or vegetables"

$\left\{x|x \text{ is a fruit}\right\} \cap \left\{x|x \text{ is a vegetable}\right\}$ translates to "fruits and vegetables", but not necessarily the collection that consists of "all fruits and vegetables" (that's what the union $\cup$ describes). What ends up being described is objects that are both fruits and vegetables

$U = \left\{x|x \text{ an IU student}\right\}$

$S = \left\{x|x \text{ is a Kelly Student}\right\}$

S' translates to "Non-kelly students".

$U = \left\{x|x \text{ is an integer (positive or negative)}\right\}$

$S = \left\{x|x \text{ is a nonnegative integer}\right\} = \{0, 1, 2, 3, \dots\}$

S' translates to "the nonnegative integers"
The negation symbol cancels out with "nonnegative" and replaces it with "negative"
$S' = \left\{x|x \text{ is a negative integer}\right\}$ translates to "the negative integers"

In general, applying $'$ to a set S translates to the negation phrase of the translation of S

In conclusion: We can think about sets as "noun phrases" in the english language.

Sentences in the Language of Sets:

Sentences involve objects and relations: the relations are $\in$, $\subset$, $=$.

In the elementhood relation $\in$ we link it to an object and a set. When we have $x \in S$ that translates in general to "x is [insert translation of S here]"

Example.

$x \in \left\{x|x \text{ is a fruit}\right\} \cap \left\{x|x \text{ is a vegetable}\right\}$ translates to "x is a fruit and a vegetable".

"$A \subset B$" that translates to "all [insert translation of A] are [insert translation of B]"

Example:

$A = \left\{x|x \text{ is an integer}\right\},\ B = \left\{x|x \text{ is a rational number}\right\}$

"$A \subset B$" translates to "all integers are rational numbers".

"$A = B$" that translates to "all [insert translation of A] are *precisely* the [insert transation of B]"

Example:

$A = \left\{x|x \text{ fractions of integers}\right\},\ B = \left\{x|x \text{ is a rational number}\right\}$

"$A = B$" translates to "all fractions of integers are precisely the rational numbers"

Order of Operations

In practice, like with arithmetic, we use multiple operations for multiple sets, such as $A \cup B \cap C'$ and $(A \cap B)' \cup C$ (and we often use parenthesis as well), so to do the order correctly, we have (like with regular arithmetic) and order of operations to do set operations, since it could be that we do the set operations in different order and we don't necessarily get the same result

Examples. *(Corrected from before!)*

$A = \{3\},\ B = \{3, 4, 6\},\ C = \{1, 2, 3, 4, 5\}$

We find that "$A \cap B \cup C$" is different depending on the order we do the result. We find

$(A \cap B) \cup C = \{3\} \cup C = \{1, 2, 3, 4, 5\},$ but

$A \cap (B \cup C) = A \cap \{1, 2, 3, 4, 5, 6\} = \{3\},$

so the order is different in that situation. Moreover, In the example with
$U = \{1, 2, 3\}$
$A = \{1, 2\}$
$B = \{3\}$
that we explain below, we find $A \cup B'$ is different depending on whether we do the union or intersection first.

The order of operations are as follows

Order of Operations

1. Work inside innermost parenthesis

Example. $(\{1, 2\} \cap \{1\})' \cup \{3\}$

We do $\{1, 2\} \cap \{1\}$ first since it's in parenthesis

2. Complements $'$ before intersections $\cup$ or unions $\cap$

Example.

$U = \{1, 2, 3\}$
$A = \{1, 2\}$
$B = \{3\}$
$(A \cup B)' \equiv \{1, 2, 3\}' = \varnothing$, since we do parenthesis first
But if we have $A \cup B' = \{1, 2\} \cup \{1, 2\} = \{1, 2\} = A$

Example. 2.4 b (page 17-18). $U = \{1, 2, 3, \ldots, 9\},\ A = \{2, 3, 7, 8, 9\},\ C = \{2, 3\}$

$C' = \{1, 4, 5 \ldots, 9\}$
$A \cap C' = A \cap \{1, 4, 5, \ldots, 9\} = \{7, 8, 9\}$

3. find intersections and unions from left to right.

Example 2.4 c (page 17-18).
$U = \{1, 2, 3, \ldots, 9\},\ A = \{2, 3, 7, 8, 9\},\ B = \{1, 5, 9\},\ C = \{2, 3\}$

$A \cap B = \{9\}$
$A \cap B \cup C = \{9\} \cup C = \{9, 2, 3\}$

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Example 2.4 a and d (page 17-18).

$U = \{1, 2, 3, \ldots, 9\},\ A = \{2, 3, 7, 8, 9\},\ B = \{1, 5, 9|,\ C = \{2, 3\}$

a. $A \cap C'$

First, we find the complement of C
$C' = \{1, 4, 5, \ldots, 9\}$

$A \cap C' = \{7, 8, 9\}$

b. $(B \cap A) \cup (C \cap A)$

First, we do parentheses

In parentheses are $B \cap A$ and $C \cap A$

$B \cap A = \{9\}$
$C \cap A = \{2\}$

Next, we take the union of $B \cap A$ and $C \cap A$ to get

$(B \cap A) \cup (C \cap A) = \{9, 2\}$.

The Cartesian Product

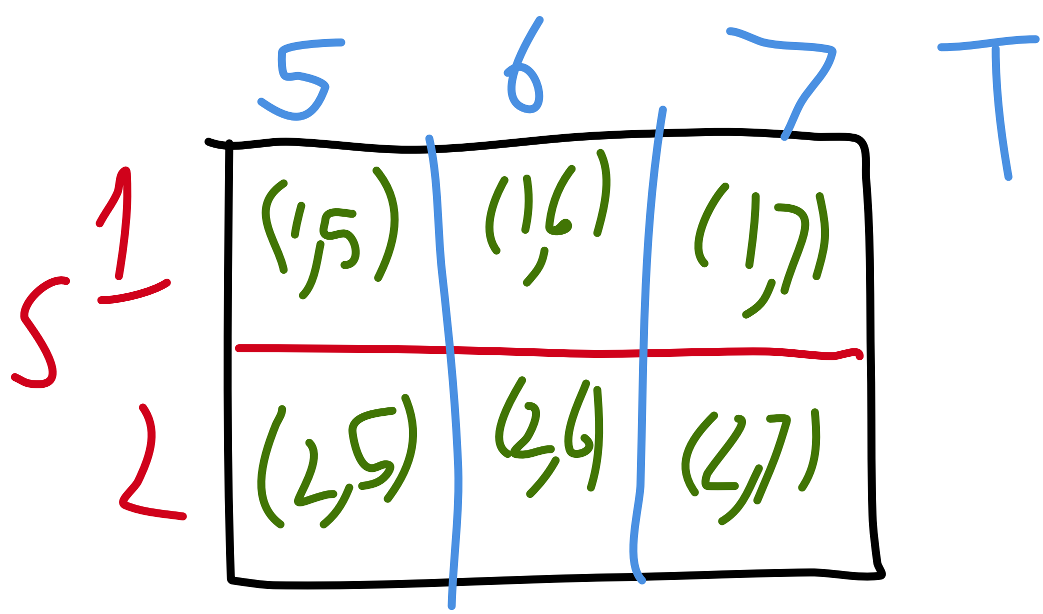
For two sets S and T, we define the **cartesian product** $S \times T$ (using " $\times$ " as the binary operation symbol) of S and T to be the sets of ordered pairs (s, t) for $s \in S$ and $t \in T$. In other words

$S \times T = \left\{(s, t) : s \in S \text{ and } t \in T\right\}$

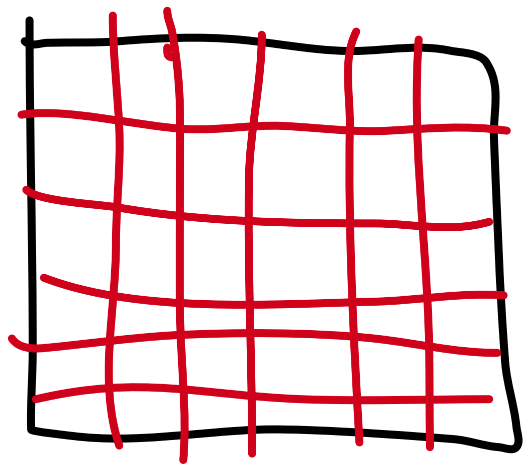
<u>Example.</u> Let's say $S = \{1, 2\}$ and $T = \{5, 6, 7\}$. Then the set $S \times T$ consists of all combinations of elements of S and elements of T:

$(1, 5),\ (1, 6),\ (1, 7)$
$(2, 5),\ (2, 6),\ (2, 7)$

When S and T are finite, we can visualize $S \times T$ as the rectangle with every square corresponding to the ordered pair



Example. Let's look at the set of all ordered combinations when rolling two six sided dice. We can look at that as the product set $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} = \{1,2,3,4,5,6\}^2$, which the grid helps us look at:

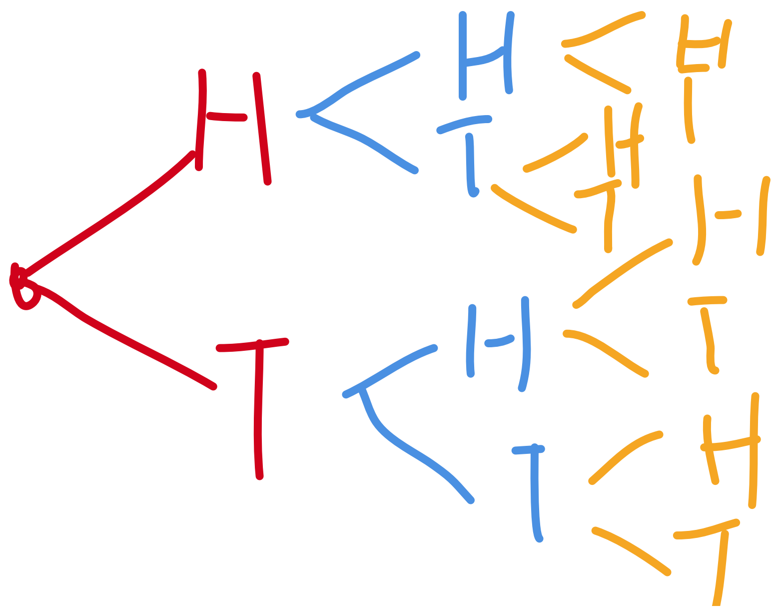


NOTE: We define $S^2 := S \times S$

Example. Let's look at the set of coin faces $\{H,T\}$ for three different coin flips. This set is

$(\{H,T\} \times \{H,T\}) \times \{H,T\} := \{H,T\} \times \{H,T\} \times \{H,T\}$

We can draw such sets with more than one products, using tree diagrams as follows:



How do we reconcile $\times$ with the order of operations?

We always have parentheses between the sets $(-)\times(-)$ that we take the cartesian product over that we take as the cartesian product, and we always figure out the cartesian product before applying intersections, unions and complements.

$X \cap (A \cap B) \times (S \cap T) \cup Y$

I'll make a more general and specific order of operations guide in the future, where I talk about this more specifically.