Linear Equations Lesson 4: Multi-Variable Systems of Equations and Matrices

10/29

What are Matrices?

Matrices are recutangular arrays of numbers, with varying columns and rows, with each entry containing a number

$$\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 7 & 3 \\ 1 & e & \pi \end{bmatrix}$$

Columns are the horizontal dimension of numbers, so the left most matrix has two columns and the right matrix three columns

Rows are the vertical dimension of numbers, so both matrices above for example have two rows.

When we talk about the amount of columns and rows, i.e. the matrix dimension, we refer to such as a $m \times n$ "m by n" where m is the amount of rows and n is the amount of columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 2 & 3 \end{bmatrix}$$

So A is a 2×2 matrix and B is 3×2 matrix.

For this course, we're not doing any matrix multiplication or matrix addition, but we're doing **reduced row operations**, which correspond to algebra operations.

So matrices are used to model systems of equations in the following way. In a two-variable system of equation set-up

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

corresponds to the following matrix:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

A three-variable equation set-up

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

corresponds to the following matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

The basic idea is for each variable plus the constant to correspond to each column and the system of equations to correspond to each row.

So every every system of equations gives us a $m \times (n+1)$ where m is the amount of equations and n is the amount of variables.

NOTATION NOTE: For the purposes of my lectures, I will set up my matrices as follows. So the book uses square brackets as I did above with a matrix like

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix},$$

but to show that the coefficients $d_1, d_2,$ and d_3 are on the "other side" of the equation, I'll draw matrices as follows, with a vertical line on the last column representing the "other" side of the equation:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

Reduced Row Operations

reduced row operations are operations on the matrix that correspond to algebraic operations that preserve the equality of the system.

For example, remember when we did the addition method when we had the following system:

$$4x + y = 13$$
$$-2x + 3y = -17$$

The first step was to multiply each side by -2

$$-2x + 3y = -17$$
$$\times -2 \qquad \times -2$$
$$4x + \;\; -6y = 34$$

And we end up with the new system

$$4x + y = 13$$
$$4x + \;\; -6y = 34$$

So we can think of this operation as "row multiplication", which is done in this case by taking the second row, and multiplying all the entries by -2. When we do that, we go from

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ -2 & 3 & -17 \end{array}\right)$$

to

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ -2 \cdot -2 & 3 \cdot -2 & -17 \cdot -2 \end{array}\right) = \left(\begin{array}{cc|c} 4 & 1 & 13 \\ 4 & -6 & 34 \end{array}\right)$$

Next, in the addition method, we did "row subtraction" in the following way: We took the first row and "subtracted it from the second row".

$$4x + y = 13$$
$$-(4x - 6y) \quad -34$$
$$7y = \; -21$$

and this gives us the system of equations

$$4x + y = 13$$

$$7y = -21$$

This corresponds to subtracting the second row by the first row to go from the matrix

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ 4 & -6 & 34 \end{array}\right)$$

to the matrix

$$\left(\begin{array}{cc|c} 4-4 & 1-(-6) & 13-34 \\ 4 & -6 & 34 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 7 & -21 \\ 4 & -6 & 34 \end{array}\right)$$

This operation is "row subtraction" or "row addition". Another thing we can do, which is purely semantic, is to look at the system of equations in "reverse order" with originally the second equation on top and the first equation on the bottom, as follows:

$$\begin{aligned} 7y &= -21 \\ 4x + y &= 13 \end{aligned}$$

So in terms of matrices, we go from

$$\left(\begin{array}{cc|c} 0 & 7 & -21 \\ 4 & -6 & 34 \end{array}\right)$$

to

$$\left(\begin{array}{cc|c} 4 & -6 & 34 \\ 0 & 7 & -21 \end{array}\right)$$

by switching the rows.

All in all, there are three reduced row operations, which are as follows:

1. Row Switching

Any two rows of a matrix can be switched, we symbolize this operation by

$$R_i \ \leftrightarrow R_j$$

where i and j are the rows being. So for example we use $R_1 \leftrightarrow R_2$ to get:

$$\left(\begin{array}{cc|c} 0 & 7 & -21 \\ 4 & -6 & 34 \end{array}\right)$$

$R_1 \leftrightarrow R_2$

$$\left(\begin{array}{cc|c} 4 & -6 & 34 \\ 0 & 7 & -21 \end{array}\right)$$

2. Row Addition

Any row i of a matrix can be changed by adding a nonzero constant $c \neq 0$ multiple of another row j to it as follows, which we symbolize by

$R_i + cR_j$,

or if the constant is $-c$ for some $c \neq 0$, we symbolize the operation $R_i + (-c)R_j$ by

$R_i - cR_j$

For example we use $R_1 - 4R_2$

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ 4 & -6 & 34 \end{array}\right)$$

$R_1 - 4R_2$

$$\left(\begin{array}{cc|c} 0 & 7 & -21 \\ 4 & -6 & 34 \end{array}\right)$$

3. Row Multiplication

Any row i of a matrix can be multiplied by a nonzero constant $c \neq 0$. We symbolize this operation by
$c \times R_i$

So for example, we use $-2 \times R_2$

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ -2 & 3 & -17 \end{array}\right)$$

$-2 \times R_2$

$$\left(\begin{array}{cc|c} 4 & 1 & 13 \\ 4 & -6 & 34 \end{array}\right)$$

Next Time: We'll show how the exact step by step process to solve a system of equations using the "matrix method". It's similar to the addition method, but a little different.

11/2

Last Time: We talked about what matrices are, what they represent, and what row operations are, and what they represent.

Now, we want to go over the process of "Gaussian Elimination" to find solutions and then we'll demonstrate this idea on some homework problems.

Using Gaussian Elimination to Find Solutions

What we want to do is go through each column from the first column to the next column until we've gone through all the columns, and then do row operations so that we end up with a 1 on the diagonal i, i entry and 0 elsewhere.

We start out with a matrix like this:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

and we want to do row operations so that we get the "identity matrix" on the left side:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & e_1 \\ 0 & 1 & 0 & e_2 \\ 0 & 0 & 1 & e_3 \end{array}\right)$$

Note that this matrix represents the equation
$x = e_1$
$y = e_2$
$z = e_3$

In summary, we do this by doing the necessary operations to make column 1 into $(1, 0, 0)$:

$$\left(\begin{array}{ccc|c} 1 & b_{1,2} & c_{1,2} & d_{1,2} \\ 0 & b_{2,2} & c_{2,2} & d_{2,2} \\ 0 & b_{3,2} & c_{3,2} & d_{3,2} \end{array}\right)$$

NOTE 1: Each of the letters with commas on them are arbitrary coefficients (possibly different from the previous coefficients)

NOTE 2: To get the 1 coefficent, oftentimes we "divide" (divide by the multiplicative reciprical) pf a nonzero constant on one of the rows. Then we use row addition (if necessary) to cancel out the other constants.

We then more row operations to make the second column into $(0,1,0)$:

$$\left(\begin{array}{ccc|c} 1 & 0 & c_{1,3} & d_{1,3} \\ 0 & 1 & c_{2,3} & d_{2,3} \\ 0 & 0 & c_{3,3} & d_{3,3} \end{array}\right)$$

Finally, we do some more row operations to make the third column into $(0,0,1)$ to give us the solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & e_1 \\ 0 & 1 & 0 & e_2 \\ 0 & 0 & 1 & e_3 \end{array}\right).$$

There's a pattern in doing these operations, and there's a more specific step-by-step process that I describe on my complete guide linked below:

https://www.mathcha.io/editor/9w3zBtEWS0JcDkNeNcM6kq2KTJ7D8MzSPO6qnD

Questions on Homework 7

<u>Question 20 (page 192)</u>

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right)$$

So note that we already have the desired 1 coefficient in the $1,1$ entry. So we proceed to cancel the $2,1$ entry by subtracting the second row by the first row

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right)$$

$R_2 - R_1$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1-(1) & -1-(1) & -1-(3) \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

Next, we want the second row to contain 1 on the diagonal $2,2$ entry, so we divide the second row by -2 (i.e., we multiply it by $-1/2$)

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

$-1/2 \times R_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 \times -1/2 & -2 \times -1/2 & -4 \times -1/2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

Finally, we want the $1,2$ entry to be 0, which will give us the 2×2 identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on the left side. We do this by subtracing the first row by the second row as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

$R_1 - R_2$

$$\left(\begin{array}{cc|c} 1-0 & 1-1 & 3-2 \\ 0 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right),$$

and we have the solution $(x, y) = (1, 2)$.

<u>Question 32 (page 192)</u> Using gaussian elimination, we'll solve for the equation:

$$2x + 3y - z = -8$$
$$x - y - z = -2$$
$$-4x + 3y + z = 6.$$

In matrix form, we have

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -8 \\ 1 & -1 & -1 & -2 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

First there's our desired 1 for the first column in the second row, so we'll switch the first and second row

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -8 \\ 1 & -1 & -1 & -2 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_1 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2 & 3 & -1 & -8 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

Next, we want to make the first column entries of the second and third row into 0 by subtracting by 2 times the first row on the second row and -4 times the first row on the third row as follows:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2 & 3 & -1 & -8 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_2 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2-2(1) & 3-2(-1) & -1-2(-1) & -8-2(-2) \\ -4 & 3 & 1 & 6 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ -4 & 3 & 1 & 6 \end{array}\right)$$

$R_3 + 4R_1$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ -4+4(1) & 3+4(-1) & 1+4(-1) & 6+4(-2) \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ 0 & -1 & -3 & -2 \end{array}\right)$$

Next Time: We'll finish this homework problem.

11/3

We'll start this lesson by finishing where we left off with Question 20 (page 192).

Next, since in the second column, we have -1 in the $3,2$ entry, which is easier to divide, we switch row two and row three and we get

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 5 & 1 & -4 \\ 0 & -1 & -3 & -2 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & -1 & -3 & -2 \\ 0 & 5 & 1 & -4 \end{array}\right).$$

Next, we multiply row 2 by -1, which gives us

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & -1 & -3 & -2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$-1 \times R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ -1 \times 0 & -1 \times -1 & -1 \times -3 & -1 \times -2 \\ 0 & 5 & 1 & -4 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right).$$

Next, we cancel out the first row and the third row with the second to get our desired $(0,1,0)$ on the second column. We do this by adding 1 times the second row to the first row and by subtracting 5 times the second row to the third row, to get

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$R_1 + R_2$

$$\left(\begin{array}{ccc|c} 1+0 & -1+1 & -1+3 & -2+2 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & -4 \end{array}\right)$$

$R_3 - 5R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 5-5(1) & 1-5(3) & -4-5(2) \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -14 & -14 \end{array}\right).$$

Next, we take the third row and divide it by -14 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

$-1/14 \times R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ (-1/14)\times 0 & (-1/14)\times 0 & (-1/14)\times -14 & (-1/14)\times -14 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Finally, we cancel out the first and second row by the third row to get a third column of $(0, 0, 1)$ as follows: We subtract 2 times the third row to the first row and we subtract 3 times the third row to the second row, which gives us

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$R_1 - 2R_3$

$$\left(\begin{array}{ccc|c} 1-2(0) & 0-2(0) & 2-2(1) & 0-2(1) \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$R_2 - 3R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0-3(0) & 1-3(0) & 3-3(1) & 2-3(1) \\ 0 & 0 & 1 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This gives us the solution
$x = -2$
$y = -1$
$z = 1$
so we have the unique solution of $(x, y, z) = (-2, -1, 1)$.

Last Time: We talked moare about about how to find solutions to systems of equations using the technique of matrix row reduction and the method of Gaussian Elimination.

<u>Question 16 (page 192)</u>

We want to figure out the reduced row operation that gets us from

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ & 1 & 5 \end{bmatrix}$$

How did we get from -2 to 1 and also from 2 to 5? It's clear that we didn't multiply row 2 by some constant becuase row 2 of the second matrix is not a factor of the first

$(-2,\ 3)\ \neq t(1, 5)$

$(-2, 3) = -1/2(1, -6)\ \neq (1, 5)$

The first row in the second matrix is the same as the first row in the first matrix, so no row switching happened.

So we're left with row addition. In other words, we did the operation $R_2 + tR_1$ to get from one matrix to the other. In other words

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$R_2 + tR_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1+t(-1) & -2+t(3) & 3+t(2) \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ & 1 & 5 \end{bmatrix},$$

so we have $(1-t, -2+3t, 3+2t) = (x, 1, 5)$, so for the equality of row 2 to happen, that's the system of equations

$1 - t = x$
$-2 + 3t = 1$
$3 + 2t = 5.$

We want to solve for t first, and then we can calculate x. So we know $-2+3t = 1$, so

$$\begin{aligned} -2 + 3t &= 1 \\ +2 \quad &\quad +2 \\ 3t &= 3 \\ \div 3 \quad &\quad \div 3 \\ t &= 1. \end{aligned}$$

Moreover, we know $1 - t = x$, so we plug in $t = 1$ into that equation and get
$x = 1 - (1) = 0,$

so the missing entry is 0 and the row operation is $R_2 + R_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$R_2 + 1R_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1+(-1) & -2+(3) & 3+(2) \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 5 \end{bmatrix}.$$

<u>Question 18 (page 192)</u>

We want to figure out the reduced row operation that gets us from

$$\begin{bmatrix} 2 & 1 & -3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & \\ 2 & 6 & 1 \end{bmatrix}$$

Here, we check to see if $t(2,1) = (6,3)$. Then $2t = 6$, so you'd have $t = 3$, and if we plug in $t = 3$, to $t = 3$ we get

$3(1) = 1,$

so $t = 3$ actually gives us $t(2,1) = (6,3)$ since $(6,3) = 3(2,1)$. So the operation we have is $3\ \times R_1$, which gives us

$$\begin{bmatrix} 2 & 1 & -3 \\ 2 & 6 & 1 \end{bmatrix}$$

$3\ \times R_1$

$$\begin{bmatrix} 6 & 3 & -9 \\ 2 & 6 & 1 \end{bmatrix}$$

<u>Question 22 (page 192)</u>

$2x - 3y = 16$
$-4x + y = -22$

$$\left(\begin{array}{cc|c} 2 & -3 & 16 \\ -4 & 1 & -22 \end{array}\right)$$

$R_1 + R_2$

$$\left(\begin{array}{cc|c} -2 & -2 & -6 \\ -4 & 1 & -22 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & -3 & 16 \\ -4 & 1 & -22 \end{array}\right)$$

$1/2 \times R_2$

$$\left(\begin{array}{cc|c} 1 & -3/2 & 8 \\ -4 & 1 & -22 \end{array}\right)$$

$x - 3/2y = 8$
$-4x + y = -22$

$R_2 + 4R_1$

$$\left(\begin{array}{cc|c} 1 & -3/2 & 8 \\ -4+4(1) & 1+4(-3/2) & -22+4(8) \end{array}\right) = \left(\begin{array}{cc|c} 1 & -3/2 & 8 \\ 0 & -5 & 10 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -3/2 & 8 \\ 0 & -5 & 10 \end{array}\right)$$

$-1/5 \times R_2$

$$\left(\begin{array}{cc|c} 1 & -3/2 & 8 \\ 0 & 1 & -2 \end{array}\right)$$

$R_1 + 3/2R_2$

$$\left(\begin{array}{cc|c} 1 & -3/2+3/2(1) & 8+3/2(-2) \\ 0 & 1 & -2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 8-3 \\ 0 & 1 & -2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right)$$

$x = 5$
$y = -2$

So with the second row, we find that