Probability and Counting Lesson 2: Probability Spaces

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What is a Probability Space?

As a disclaimer, most of the probability spaces we compue have their outcomes "equally likely" (we don't deal much with unequally likely outcomes)

We define a **probability space** as a "weighted sample space" of outcomes, i.e. each outcome is assigned a positive number (a "weight") that we call the **liklihood value** (means the "likelihood" of that outcome). The numbers are assigned each of the outcomes so that they sum up to one.

Let's say we roll a six-sided die. We assign each of the outcomes (which remember are $S = \{1, \ldots, 6\}$) the likelihood value $1/6$. For equally likely outcomes (such as rolling a die), we generally give each outcome the same likelihood value.

The **probability of an event** E is the sum of likelihood values in that outcome. We write $Pr[E]$ to denote the value

Example. For an "unfair coin flip", we have the sample space $S = \{H, T\}$, and we assign likelihood values $1/4$ and $1/3$ to heads and tails respectively. So we get

$Pr[\{H\}] = 1/4$ (the sum only consists of that outcome)
$Pr[\{T\}] = 3/4$
$Pr[\{H, T\}] = 1/4 + 3/4 = 1$ (because the event $\{H, T\}$ consists of both outcomes H and T)
$Pr[\varnothing] = 0$ (because the empty set has no outcomes so the sum of nothing is 0)

Equally Likely Outcomes

A probability/sample space with **equally likely outcomes** is a sample space with a finite number of outcomes, each outcome assigned equal likelihood value.

Three Important Things to Note:

1. In a probability space with equally likely outcomes, each outcome $s \in S$ is necessarily assigned the likelihood value $1/n(S)$, in other words the probability of $\{s\}$ happening is equal to one over the number of outcomes in S.

$$Pr[\{s\}] = \frac{1}{n(S)}.$$

2. In a probability space with equally likely outcomes, it follows from **1.** that the probability of any event $E = \{s_1, \dots, s_n\}$ can be computed as follows:

$$Pr[E] = Pr[\{s_1\}] + Pr[\{s_2\}] + \cdots + Pr[\{s_n\}],$$

where $n = n(E)$, so

$$Pr[E] = nPr[\{s_1\}] = n(E) \cdot Pr[\{s_1\}] = \frac{n(E)}{n(S)},$$

so the probability of an event in a sample space of equally likely outcomes is always the ratio of the number of outcomes $n(E)$ in the event over the number of outcomes in the $n(S)$:

$$Pr[E] = \frac{n(E)}{n(S)}.$$

3. In this class, we generally assume (as mentioned before) that any probability space we use has equally likely outcomes. (unless otherwise stated)

Example 2.1 (page 67) Suppose that a fair coin is tossed twice and the side showing "up" is recorded after each toss.

$$S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$$

a. What probability should be assigned to each outcome?

$$Pr[\{s\}] = \frac{n(\{s\})}{n(S)} = \frac{1}{4} \quad \text{for any } s \in S$$

b. What is the probability that heads will come up after exactly one toss?
Let's again use the formula

$$Pr[E] = \frac{n(E)}{n(S)},$$

Let's set $E = \{x : x \text{ has exactly one head}\}$, or for short, we just write the phrase
$E = $ exactly one head.

$E = $ exactly one head $= \{HT, TH\}$

$$Pr[E] = \frac{n(E)}{n(S)} = \frac{n\left(\text{exactly one head}\right)}{n(S)} = \frac{2}{4} = \frac{1}{2}.$$

Example 2.2 (page 68). Your friend tells you that he has written down a whole number that is more than 9 and less than 100 and asks you to guress what it is. What's the probability that you win?

To do this example, let's translate this from english to math:

First question to ask is "what is the sample space?" The sample space is the possible numbers one can guess, which are between 9 and 100, so
$S = $ any whole number between 9 and 100 $= \{9, 10, 11, \ldots, 100\}$

Important to note that there are 91 outcomes $n(S) = 91$.

Our desired event is $E = $ the correct number

NOTE: We don't know what the correct number is, but we know that only a single outcome is the correct number, so $n(E) = 1$, so sometimes the event may not be something we know explicitly, but we know the number of outcomes in that event, so we know how to compute the probability.

We find

$$Pr\left[\text{we win}\right] = Pr\left[\text{correct number}\right] = \frac{n\left(\text{correct number}\right\}}{n(S)} = \frac{1}{91}.$$

The Takeaway: "It's a reading class, not a math class."

Some Probability Formulas

Just like in Sets lesson 4, there were some formulas. Analogously, probability formulas hold

Formula 1: The Complement Probability Formula

For any event E and sample space S, we have
$Pr[E'] = 1 - Pr[E]$
$Pr[E] = 1 - Pr[E']$

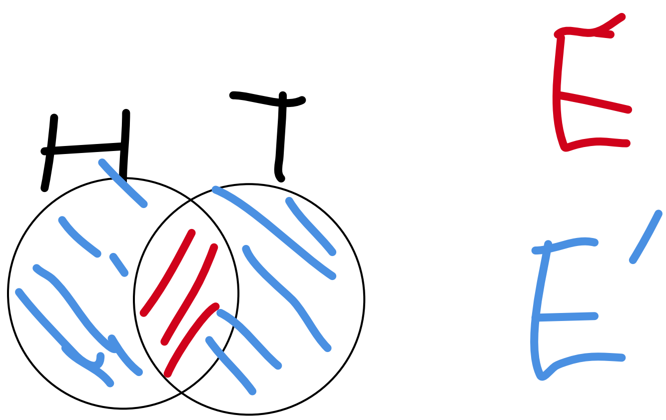
This formula is useful, because sometimes its easier to find the number of outcomes in the complement event.

Example 2.7 (page 72) A fair coin is tossed six times and the face showing "up" is recorded after each toss. Find the probability that both "heads" and "tails" come up.

The desired event is
E = both heads and tails come up
It's a tough event to count by itself, especially because

$$n(S) = n\left(\{H,T\}^6\right) = n\Big(\underbrace{\{H,T\}\times \cdots \times\{H,T\}}_{\text{6 times}}\Big) = n(\{H,T\})^6 = 64,$$

and its hard to pinpoint all the outcomes where both heads and tails come up. But what about E'?



E' = both heads and tails don't come up = either heads or tails doesn't come up

In particular, the drawing above shows that $E' = H' \cup T'$; there are two outcomes that fit the

discription
$HHHHHH, TTTTTT$
either all heads come or all tails come up. So using the formula

$$Pr[E] = 1 - Pr[E'],$$

we get
$$Pr[E'] = \frac{n(E')}{n(S)} = 2/64 = 1/32$$
$$Pr[E] = 1 - Pr[E'] = 1 - (1/32) = 31/32 .$$

Formula 2: The Disjoint Union Addition Probability Formula

Given two disjoint events E_1 and E_2 in a probability space, we have
$Pr[E_1 \cup E_2] = Pr[E_1] + Pr[E_2]$

More generally, for k disjoints $E_1, E_2, \ldots, E_k$, we have

$$Pr[E_1 \cup \cdots \cup E_k] = Pr[E_1] + Pr[E_2] + \cdots + Pr[E_k]$$

Formula 3: The Intersects-Union Probability Formula

Most generally, for any event E_1 and E_2 in a probability space, we have

$$Pr[E_1 \cup E_2] + Pr[E_1] + Pr[E_2] - Pr[E_1 \cap E_2].$$

Note that these formulas are analogous to the formulas for computing the number of elements in **Sets Lesson 4**.

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Probability Homework 2 Questions