Linear Equations Lesson 3: Solutions to Systems of Linear Equations

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Systems of Equation Overview

A **system of equations** is a list of multiple equations $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

A **system of linear equations** is a list of multiple linear equations.

In **one-variabled** list would be $a_1x + b_1 = 0,\ a_2x + b_2 = 0,\ \ldots, a_nx + b_n = 0$
A t**wo-variabled** list woud be
$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0,\ \ldots, a_nx + b_ny + c_n = 0$
We can go on for three-variables or more.

In this course, we specifically focus on two variabled systems of equations and three-variabled variable systems of equations.

In section 3.1 and 3.2, we focus on two-variable systems of equations. In the end of this lesson, I'll give an overview of three-variabled systems of equations and then we go on to 3.4 to find their solutions.

A **solution** to a system of linear equations are a set of values (x_0, y_0) (this would be (x_0, y_0, z_0)) such that when you plug them in, each of the equalities hold.

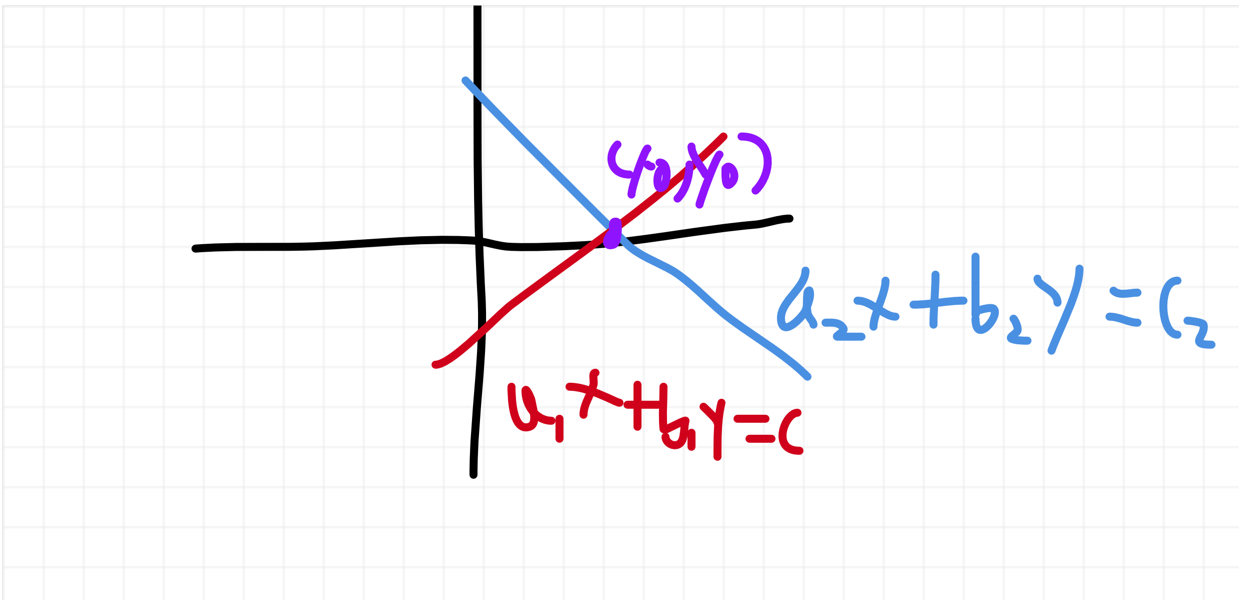
Methods of Solving Two-Variable Systems Equations

So in section 3.1 and 3.2, we give three methods of solving a system of equations.

1. the drawing method
2. substitution method
3. addition method.

1. the drawing method

In the situation of a system of two equations $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$, then we can plot the graphs of both equations and find their point of intersection (x_0, y_0) (if that solution exists), which is the solution, since plugging the intersection satisfies the equation of both lines.



2. substitution method

The basic idea is we solve for one variable in terms of another (either x in terms of y or y in terms of x) in the first equation. Then we plug in the solution in the second equation and get a single-variable equation. Then we can find the solution of that other variable in the single-variable equation, plug the solution of the first variable into our answer for the second in the previous equation, and then get the solution in general.

<u>3.2 Example 1.</u> (page 161)

$4x + y = 13$
$-2x + 3y = -17$

$4x + y = 13$
$-4x \qquad -4x$
$\qquad y = 13 - 4x$

And then we plug in $y = 13 - 4x$ into the other equation to get

$-2x + 3(13 - 4x) = -17$
$-2x + 3 \cdot 13 - 3 \cdot 4x = -17$
$-2x + 39 - 12x = -17$

$$
\begin{aligned}
39 - 14x &= -17 \\
-39 \qquad & \quad -39 \\
-14x &= -56 \\
\div -14 \quad & \div -14 \\
x &= 4
\end{aligned}
$$

And we can plug in $x = 4$ into $y = 13 - 4x$ to get $y = 13 - 4(4) = -3$, so the solution is $(x_0, y_0) = (4, -3)$

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Last Time: We went over the first two methods (substitution and addition methods) of solving the two variable linear system of equations and did some example of finding equations to finish off Section 2.3.

Here's the Step-by-Step process for the substitution method (page 161):

Step 1. If necessary, solve one equation for one of its variables.

Step 2. Substitute the resulting expression for the variable abtained in *Step 1* into the other equation and solve that equation, if possible

2.1. If it's not possible to solve the resulting equation, determine instead of the resulting equation is an *identity* or *contradiction*.

Step 3. Find the value of the other variable, if possible, by substituting the value of the variable found in Step 2 into any equation containing both variables

3.1. If it's not possible to solve the resulting equation, determine instead of the resulting equation is an *identity* or *contradiction*.

Step 4. State the solution. If we determined in *2.1* or *3.1* that we have an identity, then we have a dependent equation, which is of the form $(x, ax + b)$ or $(ay + b, y)$. If we determined in *2.1* or *3.1* that we have a contradiction, then no solution exists.

Step 5. Check the solution in both of the original equations.

<u>3.2 Example 2.</u> (page 162) Solve the following system using the substitution method

$$\frac{4}{3}x + \frac{1}{2}y = -\frac{2}{3}$$

$$\frac{1}{2}x+\frac{2}{3}y = \frac{5}{3}.$$

So we do it in the following steps.

Step 1. We take $\frac{4}{3}x+\frac{1}{2}y = -\frac{2}{3}$ and solve for y

$$\frac{4}{3}x+\frac{1}{2}y = -\frac{2}{3}$$
$$\times 6 \qquad\qquad \times 6$$
$$6\left(\frac{4}{3}x+\frac{1}{2}y\right) = 6 \cdot -\frac{2}{3}$$
$$\frac{24}{3}x+\frac{6}{2}y = -\frac{12}{3}$$
$$8x+3y = -4$$
$$-8x \qquad\qquad -8x$$
$$3y = -4-8x$$
$$\div 3 \qquad \div 3$$
$$y = \frac{-4-8x}{3}$$

Step 2. We take the equation $\frac{1}{2}x+\frac{2}{3}y = \frac{5}{3}$ and substitute $y = \frac{-4-8x}{3}$, and we get

$$\frac{1}{2}x+\frac{2}{3}\left(\frac{-4-8x}{3}\right) = \frac{5}{3}$$

Step 3. Find the solution for $\frac{1}{2}x+\frac{2}{3}\left(\frac{-4-8x}{3}\right) = \frac{5}{3}$. Doing so, we get

$$\frac{1}{2}x+\frac{2}{3}\left(\frac{-4-8x}{3}\right) = \frac{5}{3}$$
$$\frac{1}{2}x+\frac{2\cdot(-4-8x)}{3\cdot 3} = \frac{5}{3}$$
$$\frac{1}{2}x+\frac{-8-16x}{9} = \frac{5}{3}$$
$$\times 18 \qquad\qquad \times 18$$

$$
\begin{aligned}
&18 \cdot \left(\frac{1}{2}x + \frac{-8-16x}{9}\right) = 18 \cdot \frac{5}{3} \\
&9x + 2(-8-16x) = 6 \cdot 5 \\
&9x - 16 - 32x = 30 \\
&-23x - 16 \quad = 30 \\
&\quad +16 \qquad\quad +16 \\
&-23x \qquad\quad = 46 \\
&\div -23 \qquad \div -23 \\
&\quad x \qquad\quad = -2
\end{aligned}
$$

Now we want to find y, and to do that, we plug in $x = -2$ into $y = \dfrac{-4-8x}{3}$. We have

$$y = \frac{-4-8(-2)}{3} = \frac{-4+16}{3} = \frac{12}{3} = 4$$

Step 4. We state the solution of $(x, y) = (-2, 4)$.

Step 5. Check our solution by plugging the numbers back into the equation and seeing if they work.

$$
\begin{aligned}
&\frac{4}{3}(-2) + \frac{1}{2}(4) = \frac{-8}{3} + \frac{4}{2} = \frac{-16+12}{6} = \frac{-4}{6} = -\frac{2}{3} \\
&\frac{1}{2}(-2) + \frac{2}{3}(4) = \frac{-2}{2} + \frac{8}{3} = \frac{-3+8}{3} = \frac{5}{3}.
\end{aligned}
$$

We see that the solution works.

3. addition method.

General Idea: The general idea is that we set both equations so that the constants of one variable matches, then we cancel out the first variable, then we solve for the remaining variable.

Step 1. Write both equations of the system in general form.

Step 2. Multiply the terms of one or both of the equations by constants chosen to make the coefficients of x (or y) differ only in sign.

Step 3. Add the equations to solve the resulting equation, *if possible.*

3.1. If it's not possible to solve the resulting equation, determine instead if the resulting equation is an *identity* or a *contradiction.*

Step 4. Substitute the value obtained in *Step 3* into either of the original equations and solve for the remaining variable.

Step 5. State the solution obtained in *Step 3* and *4*. If we determined in *3.1* that we have an identity, then we have a dependent equation, which is of the form $(x, ax+b)$ or $(ay+b, y)$. If we determined in *3.1* that we have a contradiction, then no solution exists.

Step 6. Check the solution in both of the original equations.

<u>3.2 Example 3</u> (page 163) Solve the system

$4x + y = 13$
$-2x + 3y = -17$

using the addition method.

Step 1. Done

Step 2. Multiply the two equations so that x has a common coefficient. To do that, we can multiply by -2 on the second equation

$-2x + 3y = -17$
$\times -2 \qquad \times -2$
$4x - 6y = 34$

Step 3. Cancel out x in the equation $4x + y = 13$ by subtracting by $4x - 6y$ on the left side and 34 on the right side (note that we can do that since $4x - 6y = 34$)

$4x + y = 13$
$-(4x - 6y) \quad -34$
$y - (-6y) = 13 - 34$
$7y = -21$
$\div 7 \quad \div 7$
$y = -3$

Step 4. We plug in $y = -3$ in the other equation $-4x + y = 13$ (you can do it for the other though; doesn't matter)

$$
\begin{aligned}
&4x + (-3) = 13 \\
&4x - 3 = 13 \\
&\quad\;\; +3 \quad\;\; +3 \\
&4x \qquad = 16 \\
&\div 4 \qquad \div 4 \\
&\;\; x \qquad = 4
\end{aligned}
$$

Step 5. We have the solution $(x, y) = (4, -3)$

Step 6. Check the solution

$$
\begin{aligned}
&4(4) + (-3) = 16 - 3 = 13 \\
&-2(4) + 3(-3) = -8 - 9 = -17
\end{aligned}
$$

and we're good.

Inconsistent and Dependent Systems of Equations

Usually, when we solve for two systems of two-variable linear equations, we get one solution and only one solution. But sometimes that happen.

3.1 Example 2. (p. 153)

$$
\begin{aligned}
&2x + 3y = 6 \\
&4x + 6y = 24
\end{aligned}
$$

We try to use the addition method. We multiply the first equation by 2 to get

$$
\begin{aligned}
&2x + 3y = 6 \\
&\times 2 \qquad\;\; \times 2 \\
&4x + 6y = 12
\end{aligned}
$$

We cancel out the x on the second equation by subtracting $4x + 6y$ on one side and 12 on the other side

$$
\begin{aligned}
&4x + 6y = 24 \\
&-(4x + 6y) \; -12 \\
&0 \qquad\quad = 12
\end{aligned}
$$

We have a *contradiction*.

Whenever we run into a contradiction when trying to use the substitution method or addition method, we have *no solution*.

It's also possible that we have infinitely many solutions.

<u>Example.</u>

$$2x + 3y = 6$$
$$4x + 6y = 12$$

We try to use the addition method. We multiply the first equation by 2 to get

$$\begin{aligned} 2x + 3y &= 6 \\ \times 2 \quad &\quad \times 2 \\ 4x + 6y &= 12 \end{aligned}$$

We cancel out the x on the second equation by subtracting $4x + 6y$ on one side and 12 on the other side

$$\begin{aligned} 4x + 6y &= 12 \\ -(4x + 6y) &\ -12 \\ 0 &= 0, \end{aligned}$$

We have an *identity*. In this situation, x can be arbitrary, and y can be the value that satisfies the linear equation, so we have

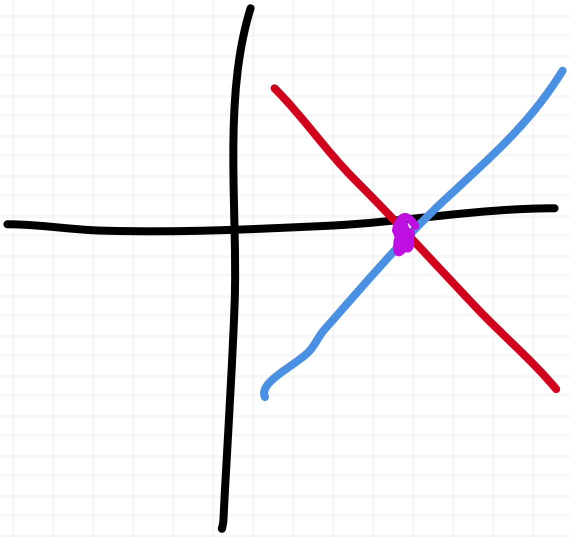
$$\begin{aligned} 4x + 6y &= 12 \\ -4x \quad &\quad -4x \end{aligned}$$

$$\begin{aligned} 6y &= -4x + 12 \\ \div 6 \ &\ \div 6 \\ y &= -\frac{2}{3}x + 2, \end{aligned}$$

and the solution is $\left(x, -\frac{2}{3}x + 2\right)$ for any x (there's infinitely many solutions of this sort).

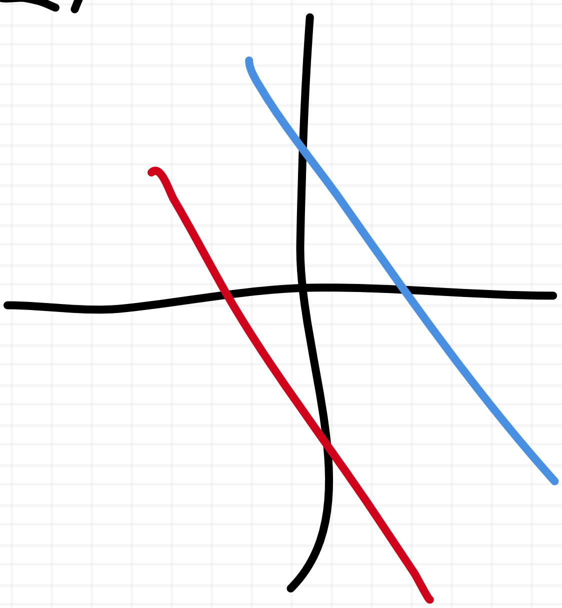
The takeaway is, we have three possibilities for a two-variabled system of two linear equations:

1.



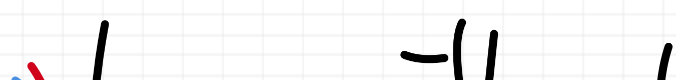
there is a single solution and the lin intersect

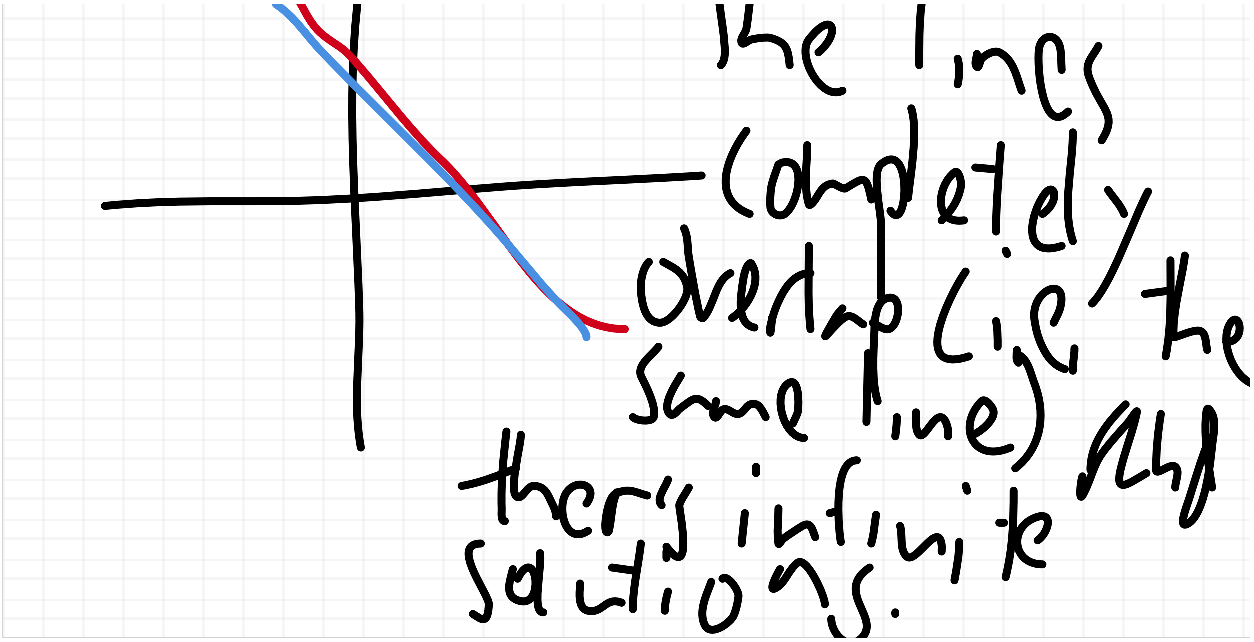
2.



the lines are parallel and distinct and no solution exists

3.





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Three Variable Systems of Equations Overview

What we did for systems of two variable equations generalize to systems of three variable equations. In particular, in the next lesson, we'll deal with systems of equations involving three equations and three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

To solve these equations, we can generalize the substtitution method and the addition method to find solutions, but we use the "matrix method" to solve such equations, which we do in the next lesson.

Questions on Homework 5

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Questions on Homework 6

<u>Question 27 (page 171).</u> Use the addition method for

$$2x + 3y = 8$$
$$3x - 2y = -1$$

Step 1. Already in general form

Step 2. Pick a variable that we want to cancel out (we'll pick y) and then we want to find a common coefficient so that we can cancel them out. So the two equations have 3 and -2 as the y coefficients, so let's make the first one 6 by multiplying by 2 on both sides

$$\begin{aligned} 2x + 3y &= 8 \\ \times 2 \qquad &\times 2 \\ 4x + 6y &= 16 \end{aligned}$$

$$\begin{aligned} 3x - 2y &= -1 \\ \times(-3) \qquad &\times(-3) \\ -9x + 6y &= 3 \end{aligned}$$

Now we have the new system of equations

$$4x + 6y = 16$$
$$-9x + 6y = 3$$

Step 3. Cancel the y out by taking second equation and subtract it by the first equation

$$\begin{aligned} 4x + 6y &= 16 \\ -(-9x + 6y) \quad &-3 \\ 4x + 9x + 6y - 6y &= 16 - 3 \\ 13x \qquad\qquad &= 13 \end{aligned}$$

and then solve for x

$$\begin{aligned} 13x &= 13 \\ \div 13 \quad &\div 13 \\ x &= 1 \end{aligned}$$

Step 4. Take $x = 1$ plug it into $4x + 6y = 16$ and solve for y as follows:

$$
\begin{aligned}
4(1) + 6y &= 16 \\
4 + 6y &= 16 \\
-4 \qquad &\ \ -4 \\
6y &= 12 \\
\div 6 \quad &\ \div 6 \\
y &= 2
\end{aligned}
$$

Step 5. We state the solution as what we got for x and y: $(x, y) = (1, 2)$.

Step 6. Check our work (if you want to) by plugging $(1, 2)$ into the original equations.