The Complete Guide to Solving a Single-Variable Linear Equation

Introduction

In this class, we oftentimes do algebra with a single variable. While much of this is obviously review from high school, there are extra parts to this that you may have not seen in high school. For example, we deal with "identities" and "contradictions" (as discussed in class before).

Page 50 of *Gustafson and Frisk* gives a "Five-Step Process" to solving linear equations, but this only works of course if the linear equation has a solution. What I will do is give a *more general* "Five-Step Process" that will also work for determining identities and contradictions.

The Five-Step Process

The *general* 5 step process (that includes a method for finding identities and contradictions) is as follows:

Step 1. If the equation contins fractions, multiply both sides by a number that will eliminate the denominator.

Step 2. If necesary, use the distributive property to remove all the parentheses and combine like terms.

Step 3. Use addition and subtraction to get all the variables on one side of the equation, and all the numbers on the other side.

3.1. *If the equation is an identity or contradition*, doing this will give us an equation of the form $a = 0$ or $0 = a$, where a is a constant (and no "x" appears at all), and we will be unable to move on to *Step 4*.

In the situation above where we are unable to proceed with *Step 4*, we deterine If the equation is an identity or contradiction as follows:

3.1.1. If a and 0 *are actually equal* to each other, then we have an *identity,* and we end here.

3.1.2. If a and 0 are *not equal* to each other, then we have a *contradiction*, and we end here.

3.1.3. If we are able to proceed with *Step 4*--in other words, we are left with an equation of the form $ax = b$ for $a \neq 0$--then we proceed to *Step 4*.

Step 4. Use multiplication and division properties to make the coefficient of the variable equal to 1.

Step 5. *(optional)* Check the result by plugging in the answer we got.

Some Examples

Here are some examples of how we can use this five step process to solve for some problems.

Example 1. Here, we'll solve for $2x - 4 = 16$. Note that *Step 1* and *Step 2* are not necessary, since no fractions appear, no distributive property needs to be used, and no like-terms on either side need to be combined. We go straight to *Step 3*, which in this case entails adding 4 on both sides:

$$\begin{aligned} &2x - 4 = 16 \\ &\quad +4 \qquad +4 \\ &2x \qquad = 20, \end{aligned}$$

which gives us the equation $2x = 20$. Next, we do *Step 4*, which entails dividing by 2 on both sides. We get

$$\begin{aligned} &2x = 20 \\ &\div 2 \quad \div 2 \\ &x \quad = 10. \end{aligned}$$

We can then do *Step 5* to check our solution as follows:

$$2(10) - 4 = 20 - 4 = 16,$$

which agrees with "16" on the other side of the equation

Example 2. Here, we'll solve for $5(5 - x) = 37 - 2x$. Note that there are no fractions, so we don't do anything for *Step 1*. For *Step 2*, we use distributive property as follows:

$$5(5 - x) = 5 \cdot 5 - 5 \cdot x = 25 - 5x,$$

and we get the equation $25-5x=37-2x$. Next we do *Step 3* and get

$$\begin{array}{lcl} 25-5x & = & 37-2x \\ \quad +5x & & \quad +5x \\ 25 & = & 37+3x \\ -37 & & -37 \\ -12 & = & 3x \\ \div 3 & & \div 3 \\ -4 & = & x, \end{array}$$

which gives us the solution $x=-4$. Doing *Step 5*:

$$25-5(-4)=25+20=45$$
$$37-2(-4)=37+8=45,$$

we find both sides of the equation agree at 45.

<u>Example 3.</u> Here, we'll sove for $\dfrac{x+1}{3}+\dfrac{x-1}{5}=\dfrac{2}{15}$. For *Step 1*, we multiply by common denominator 15 as follows:

$$\begin{array}{lcl} \dfrac{x+1}{3}+\dfrac{x-1}{5} & = & \dfrac{2}{15} \\ \times 15 & & \times 15 \\ 15\left(\dfrac{x+1}{3}+\dfrac{x-1}{5}\right) & = & 15\left(\dfrac{2}{15}\right), \end{array}$$

Next, we do *Step 2* and use distributive property, cancel out fractions, and combine-like terms as follows:

$$\begin{aligned} 15\left(\frac{x+1}{3}+\frac{x-1}{5}\right) &= 15\left(\frac{x+1}{3}\right)+15\left(\frac{x-1}{5}\right) \\ &= 5(x+1)+3(x-1) \\ &= 5\cdot x+5\cdot 1+3\cdot x-3\cdot 1 \\ &= 5x+5+3x-3 \\ &= 8x+2 \end{aligned}$$

$$15\left(\frac{2}{15}\right)=2,$$

to give us the equation $8x+2=2$. Next, we do *Step 3* to get

$$
\begin{aligned}
8x + 2 &= 2 \\
-2 \quad &\; -2 \\
8x \quad &= 0,
\end{aligned}
$$

then *Step 4* to get

$$
\begin{aligned}
8x &= 0 \\
\div 8 \quad &\div 8 \\
x \quad &= 0.
\end{aligned}
$$

Finally, we do *Step 5* to check our result as follows:

$$\frac{(0)+1}{3} + \frac{(0)-1}{5} = \frac{1}{3} - \frac{1}{5} = \frac{5}{5}\cdot\frac{1}{3} - \frac{3}{3}\cdot\frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}.$$

<u>Example 4.</u> Here, we'll solve the equation $2(x-3) = \frac{3}{2}(x-4) + \frac{x}{2}$. First, we do *Step 1* and multiply both sides by 2:

$$
\begin{aligned}
2(x-3) &= \frac{3}{2}(x-4) + \frac{x}{2} \\
\times 2 \qquad &\times 2 \\
2\cdot 2(x-3) &= 2\ \cdot \left(\frac{3}{2}(x-4) + \frac{x}{2}\right)
\end{aligned}
$$

Next, we do *Step 2* and simplify both sides:

$$4(x-3) = 4\cdot x - 4\cdot 3 = 4x - 12$$

$$
\begin{aligned}
2\ \cdot \left(\frac{3}{2}(x-4) + \frac{x}{2}\right) &= 2\cdot\frac{3}{2}(x-4) + 2\cdot\frac{x}{2} \\
&= 3\cdot x - 3\cdot 4 + x \\
&= 3x - 12 + x \\
&= 4x - 12
\end{aligned}
$$

Finally, we do *Step 3*, which gives us:

$$
\begin{aligned}
4x - 12 &= 4x - 12 \\
-4x \quad &\; -4x \\
-12 &= \quad -12
\end{aligned}
$$

$$\begin{aligned} +12 \quad & +12 \\ 0 = \quad & 0, \end{aligned}$$

and we have a *identity* since $0 = 0$ does hold.

<u>Example 5.</u> Here, we'll solve the equation $2x - 6 = -2x + 4(x-2)$. There are no fractions, so no need for *Step 1.* We do *Step 2*, and get

$$\begin{aligned} -2x + 4(x-2) &= -2x + 4 \cdot x - 4 \cdot 2 \\ &= -2x + 4x - 8 \\ &= 2x - 8. \end{aligned}$$

So we do *Step 3* and get

$$\begin{aligned} 2x - 6 &= 2x - 8 \\ -2x \quad & \quad -2x \\ -6 &= -8 \\ +8 \quad & \quad +8 \\ 2 &= 0. \end{aligned}$$

Since $2 \neq 0$, we have a *contradiction*.

In this class, we often deal with "***linear formulas***" consisting of multiple "arbitrary numbers" written as letters, where we designate one arbitrary number as a "variable" and solve for it in terms of the other arbitrary numbers. Let me go over some examples of solving for linear formulas here:

<u>Example 6.</u> Here, we'll solve for m in the equation $y = mx + b$. First, note that nothing comes of *Step 1* since we have no fractions, and nothing comes of *Step 2* since no parentheses need to be distributed and no like terms on either side need to be combined. For *Step 3*, we have

$$\begin{aligned} y &= mx + b \\ -b \quad & \quad -b \\ y - b &= mx, \end{aligned}$$

and for *Step 4*, we divide by m and we have

$$\begin{aligned} y - b &= mx \\ \div m \quad & \quad \div m \end{aligned}$$

$$\frac{y-b}{m} = x,$$

and we have the solution $x = \dfrac{y-b}{m}$. We plug in and check our answer for *Step 5* and get:

$$m\left(\frac{y-b}{m}\right) + b = (y-b) + b = y,$$

as desired.

<u>Example 7.</u> Here, we'll solve for F in the equation $C = \frac{5}{9}(F-32)$. For Step 1, we do away with the fractions and multiply by 9 on both sides to get

$$\begin{aligned} &C = \frac{5}{9}(F-32) \\ &\times 9 \quad \times 9 \\ &9C = 5(F-32). \end{aligned}$$

For Step 2, we use distributive property to get

$$5(F-32) = 5F - 5 \cdot 32 = 5F - 160,$$

giving us the equation $9C = 5F - 160$. For Step 3, we get the term $5F$ factoring F by itself as follows:

$$\begin{aligned} &9C = 5F - 160 \\ &+160 \quad\quad +160 \\ &9C + 160 = 5F. \end{aligned}$$

Next, we do Step 4 and divide by 5 to get

$$\begin{aligned} &9C + 160 = 5F \\ &\div 5 \qquad\quad \div 5 \\ &\frac{9C+160}{5} = F. \end{aligned}$$

Finally, we check our solution using Step 5 to get

$$
\begin{aligned}
\frac{5}{9}\left(\left(\frac{9C+160}{5}\right)-32\right) &= \frac{5}{9}\left(\frac{9C+160}{5}-\frac{5}{5}\cdot 32\right) \\
&= \frac{5}{9}\left(\frac{9C+160}{5}-\frac{160}{5}\right) \\
&= \frac{5}{9}\left(\frac{9C}{5}\right) \\
&= C,
\end{aligned}
$$

as desired.

Final Thoughts

As we go on with this class, we'll find that we'll want to do algebra at a faster pace than thinking about it in terms of steps allows us to go, especially if there is no fractions we have to cancel out or no like-terms that we need to combine.

In all other sections, we will not go through the whole step-by-step process for every one-variable linear equation we are solving, and I will not expect you go through the steps in excruciating detail, either. Though there will be plenty of other step-by-step processes involving various concepts (including more equations in more variables) old and new that we'll be executing that I will be going over meticulously.