The Complete Guide to Solving a Matrix

Introduction

In this class, you are asked to use reduced row operations, aka Gaussian elimination, to solve a system of equations. The basic idea is you start with a system of linear equations--in its general form--such as

$$\begin{aligned} A_1x + B_1y + C_1z &= D_1 \\ A_2x + B_2y + C_2z &= D_2, \\ A_3x + B_3y + C_3z &= D_3 \end{aligned}$$

and look at it from the viewpoint of the following matrix representing the coefficients:

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{bmatrix},$$

and then do a sequence of reduce row operations (that represent algebra techniques to solve for a variable), in order to get

$$\begin{bmatrix} 1 & 0 & 0 & E_1 \\ 0 & 1 & 0 & E_2 \\ 0 & 0 & 1 & E_3 \end{bmatrix},$$

which represents the solution for (x, y, z) since the above matrix represents the equation

$$\begin{aligned} x &= E_1 \\ y &= E_2. \\ z &= E_2 \end{aligned}$$

Personal Conventions (vs. Textbook Conventions)

Note that the textbook Gustafon and Frisk (used for M018), as well as many other textbooks, use the typical square bracket matrix notation below:

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{bmatrix}.$$

I, however, do it differently, with parentheses and a line between the last column and other columns (as I do below) to make more clear that we have a two-sided system of equation, with the line between the columns representing "equality".

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right).$$

We shall adapt this convention from now on.

Types of Reduced Row Operations

There are three types of reduced row operations. I shall name these operations and explain what they are.

Note as I name these reduced row operations that every operation represents some kind of algebra trick that is "fair game" to do when solving a system of equations.

1. Row Switching

We take one two rows and switch their terms; we symbolize this operation by row $i \ \leftrightarrow$ row j for distinct rows $i \ \neq j$. For example, the operation row $1 \ \leftrightarrow$ row 2 in the matriix below would look like

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right)$$

row $1 \ \leftrightarrow$ row 2

$$\left(\begin{array}{ccc|c} A_2 & B_2 & C_2 & D_2 \\ A_1 & B_1 & C_1 & D_1 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right).$$

This is equivalent to taking a system of equations rearranging the order that the equations appear, so we go from

$$A_1x + B_1y + C_1z = D_1$$
$$A_2x + B_2y + C_2z = D_2,$$

$A_3x + B_3y + C_3z = D_3$

to

$A_2x + B_2y + C_2z = D_2$
$A_1x + B_1y + C_1z = D_1.$
$A_3x + B_3y + C_3z = D_2$

2. Row Multiplication

We take a row and multiply *each entry* in it by a nonzero constant $c \neq 0$, and we symbolize this operation by $c \times$ row i, where i is a row number. For example, the operation $c \times$ row 2 would look like

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right)$$

$c \times$ row 2

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ c \cdot A_2 & c \cdot B_2 & c \cdot C_2 & c \cdot D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right)$$

This is equivalent to taking the system of equations

$A_1x + B_1y + C_1z = D_1$
$A_2x + B_2y + C_2z = D_2,$
$A_3x + B_3y + C_3z = D_3$

and multiplying one of the equation $A_2x + B_2y + C_2z = D_2$ by the nonzero constant c on both sides:
$A_2x + B_2y + C_2z = D_2$
$\times c \qquad\qquad \times c$
$cA_2x + cB_2y + cC_2z = cD_2,$
to give us the new system of equations:

$A_1x + B_1y + C_1z = D_1$
$cA_2x + cB_2y + cC_2z = cD_2.$
$A_3x + B_3y + C_3z = D_3$

3. Row Addition

We take a row and add that row by a nonzero constant multiple $c \neq 0$ of another another row; we symbolize this operation by row $i + c \cdot$ row j, for distinct rows $i \neq j$. For example, the operation row $3 + c \cdot$ row 2 would look like

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right)$$

row $3 + c \cdot$ row 2

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 + c \cdot A_2 & B_3 + c \cdot B_2 & C_3 + c \cdot C_2 & D_3 + c \cdot D_2 \end{array}\right).$$

This is equivalent to taking the system of equations

$$A_1x + B_1y + C_1z = D_1$$
$$A_2x + B_2y + C_2z = D_2,$$
$$A_3x + B_3y + C_3z = D_3$$

and adding each side of the equation $A_3x + B_3y + C_3z = D_3$ by the nonzero constant c times each corresponding side of the equation $A_2x + B_2y + C_2z = D_2$ in the following way

$$A_3x + B_3y + C_3z \qquad = D_3$$
$$+c(A_2x + B_2y + C_2z) \quad + cD_2$$
$$(A_3 + cA_2)x + (B_3 + cB_2)y + (C_3 + cC_2)z = D_3 + cD_2,$$

to give us the new system of equations

$$A_1x + B_1y + C_1z = D_1$$
$$A_2x + B_2y + C_2z = D_2$$
$$(A_3 + cA_2)x + (B_3 + cB_2)y + (C_3 + cC_2)z = D_3 + cD_2.$$

SOME IMPORTANT ADDITIONAL NOTATION: Note that row additon and row multiplication give rise to what you might call "row subtraction" and "row division" where we often like to write "$c^{-1} \times$ row i" as "row $i \div c$" and "row $i + (-c) \cdot$ row j" as "row $i - c \cdot$ row j"

Now that I've shown how these row operations work, let me provide you with a 2×2 matrix example to see how these reduced row operations really correspond to algebraic operations.

Example 1. Let's take equation

$$2x + y = 5$$
$$3x + 4y = 10,$$

and find the solution through looking at it as the matrix

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 3 & 4 & 10 \end{array}\right).$$

To solve this equation, we use more or less the method of cancellation (Gustafon and Frisk calls this the "addition method") to cancel first the x coefficient in the bottom equation, then the y coefficient in the top equation, in order to get the solution. The algebraic operations we do in order to do this correspond to reduce row operations in the following way:

$$2x + y = 5 \qquad \left(\begin{array}{cc|c} 2 & 1 & 5 \\ 3 & 4 & 10 \end{array}\right)$$
$$3x + 4y = 10$$

$$2x + y = 5$$
$$\div 2 \qquad \div 2 \qquad \text{row } 1 \div 2$$
$$x + \frac{1}{2}y = \frac{5}{2}$$

$$x + \frac{1}{2}y = \frac{5}{2} \qquad \left(\begin{array}{cc|c} 1 & 1/2 & 5/2 \\ 3 & 4 & 10 \end{array}\right)$$
$$3x + 4y = 10$$

$$3x + 4y \qquad = 10$$
$$+(-3)\left(x + \frac{1}{2}y\right) \quad + (-3)\left(\frac{5}{2}\right) \qquad \text{row } 2 - 3 \cdot \text{row } 3$$
$$0x + \frac{5}{2}y = \frac{5}{2}$$

$$x + \frac{1}{2}y = \frac{5}{2} \qquad \left(\begin{array}{cc|c} 1 & 1/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{array}\right)$$

$$\frac{5}{2}y = \frac{5}{2}$$

$$\frac{5}{2}y = \frac{5}{2}$$

$$\times\frac{2}{5} \quad \times\frac{2}{5} \qquad \frac{2}{5} \times \text{row } 2$$

$$y = 1$$

$$x + \frac{1}{2}y = \frac{5}{2} \qquad \left(\begin{array}{cc|c} 1 & 1/2 & 5/2 \\ 0 & 1 & 1 \end{array}\right)$$

$$y = 1$$

$$x + \frac{1}{2}y = \frac{5}{2}$$

$$-\frac{1}{2}y \quad -\frac{1}{2} \qquad \text{row } 1 - \frac{1}{2} \cdot \text{row } 2$$

$$x + 0y \;\; = \frac{4}{2}$$

$$x = 2 \qquad \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

$$y = 1$$

How do we Actually get a Solution?

%CONTINUE EDITING OUT THE OPERATIONS INTO ROW OPERATIONS
One issue that students in my experience run into is they don't know *how exactly* to use reduced row operations to get from A to B--going from the original system of equations

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right),$$

to its solution

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & E_1 \\ 0 & 1 & 0 & E_2 \\ 0 & 0 & 1 & E_3 \end{array}\right).$$

Without a sense of the bigger picture, reduced row operations can seem like an aimless series of operations that has no actual sense of direction, that just *happens* to get to the right result if you "guess" the series of operations correctly.

What I hope most of all to accomplish through these notes is to assure you that this is *not the case* and there is an actual step-by-step process with doing reduced row operations in a way that gets you the solution, and this isn't really a matter of "guessing" the correct row operation you make and hoping it leads to the right result. The exercise of Gaussian Elimination is fundamentally an algorithm that can be simulated via a computer program! But that's another discussion for another kind of class, and I digress.

The basic idea is we want to "snake" along each column and use row operations that will give us the desired column, So if we start with the equation

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right),$$

we want to *first* do operations in a way (that I'll explain in more precise detail in a moment) that will give us a new matrix with the desired first colummn with entries (from top to bottom) of $(1, 0, 0)$, i.e., we first get the matrix:

$$\left(\begin{array}{ccc|c} 1 & B_{1,1} & C_{1,1} & D_{1,1} \\ 0 & B_{2,1} & C_{2,1} & D_{2,1} \\ 0 & B_{3,1} & C_{3,1} & D_{3,1} \end{array}\right).$$

Then we repeat this idea for each other column. So for the second column, we want do operations that gives us a second column of $(0, 1, 0)$, while preserving the first column of $(1, 0, 0)$, gives us the matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & C_{1,1} & D_{1,1} \\ 0 & 1 & C_{2,1} & D_{2,1} \\ 0 & 0 & C_{3,1} & D_{3,1} \end{array}\right),$$

and then we repeat this idea for the third and last column to give us the desired end result

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & E_1 \\ 0 & 1 & 0 & E_2 \\ 0 & 0 & 1 & E_3 \end{array}\right),$$

where every *diagonal entry* is 1 and every other entry is 0.

Getting the desired column for a given column i (for each variable representing the ith column) can be achieved in the ***following two steps:***

Step 1. Make the diagonal i, i entry 1. We can do this using the following procedure.

Check the other entries for any row $j > i$ below the diagonal i, i entry (i.e., the entry of the matrix we "want" to to be 1) that already have the desired 1 constant.

<u>GENERAL NOTE:</u> Whenever we switch rows, we *never do so with a row above* the i entry, so *never consider any row* $j < i$ whenever you switch a row.

1.1: Employ the row switching operation "row $i \leftrightarrow$ row j" if the j, i entry is 1, for some $j > i$ (if more than one $j > i$ exists, then the choice of j is unimportant). Then proceed to *Step 2* (and skip *1.2-1.4*). Otherwise, we check for the next condition below.

if no such $j > i$ exists, check to see if the i, i entry is nonzero.

1.2: Employ the row division operation "row $i \div c$" (i.e. "$c^{-1} \times$ row i") If the i, i entry is a nonzero constant $c \neq 0$. Then proceed to *Step 2* (and skip *1.3* and *1.4*). Otherwise, we check for the next condition below.

If the i, i entry is 0, then find some row below $j > i$ where the j, i entry is nonzero.

1.3: Employ the row switching operation "row $i \leftrightarrow$ row j" if the j, i entry is a nonzero constant $c \neq 0$, for some $j > i$ (if more than one j exists, then the choice of j is unimportant), now the i, i entry is c and we then divide row i by c (i.e., employ the operation "row $i \div c$"). Then proceed to *Step 2* (and skip *1.4*). Otherwise, we proceed below.

If the j, i entry turns out to be zero, *for every* $j > i$ we have some case of an inconsistent and/or dependent equation (refer to the next section "Inconsistent and Dependent Equations" about what that means).

1.4: We skip *Step 2* completely if the j, i entry turns out to be zero, *for every* $j > i$, and proceed to the next column.

Step 2. Take the ith row and for each $j \neq i$ check to see if the j, i entry c is nonzero and do the following:

2.1: If $c \neq 0$, then employ the row subtraction operation "row $i - c \cdot$ row j" (i.e., " row $i + (-c) \cdot$ row j") to cancel out the j, i column and make it zero. If c already is 0, we leave the jth row as is in this step.

Do this two-step process *for every column* $i = 1, 2, 3$ (or $i = 1, 2$ in a two-variabled equation, and this can of course be generalized for a higher number of variables) and we will certainly get a solution, provided that we run into no problems at ***1.3***--a scenario we'll deal with during the *next section*.

To make this more clear, allow me to illustrate this process with the following three-variable example:

<u>Example 2.</u> Let's take the equation

$2x + 2y + 2z = 8$
$x + 3z = 4$
$3y + 6z = 12$

and find the solution through looking at it as the matrix

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 8 \\ 1 & 0 & 3 & 4 \\ 0 & 3 & 6 & 12 \end{array}\right).$$

First, we try to make column 1 (representing the x variable) the desired column (from top to bottom) $(1, 0, 0)$ as follows:

Employ *step 1* to make the $1, 1$ entry 2 into 1 by noticing that the $2, 1$ entry already has a 1 and we can switch the second row with the first row to get

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 8 \\ 1 & 0 & 3 & 4 \\ 0 & 3 & 6 & 12 \end{array}\right)$$

row $1 \leftrightarrow$ row 2

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 2 & 2 & 2 & 8 \\ 0 & 3 & 6 & 12 \end{array}\right).$$

Next, we employ *step 2* to make the $2,1$ and $3,1$ entries 0 as follows: We cancel out the $2,1$ entry to make that zero by subtract the second row by 2 times the first row to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 2 & 2 & 2 & 8 \\ 0 & 3 & 6 & 12 \end{array}\right)$$

row $2\ -2 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 2-2(1) & 2-2(0) & 2-2(3) & 8-2(4) \\ 0 & 3 & 6 & 12 \end{array}\right),$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & 6 & 12 \end{array}\right),$$

and then leave the third row alone since the $3,1$ entry is already 0.

Now we try to make column 2 (representing the y variable) the desired column $(0,1,0)$ as follows:

Employ *step 1* to make the $2,2$ entry 2 into 1 by noticing that there is no below entry that is 1 but that the $2,2$ entry is nonzero, so we can divide the second row by 2 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & 6 & 12 \end{array}\right)$$

row $2 \div 2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0\div 2 & 2\div 2 & -4\div 2 & 0\div 2 \\ 0 & 3 & 6 & 12 \end{array}\right),$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 6 & 12 \end{array}\right).$$

Next, we employ *step 2* to make the $1, 2$ and $3, 2$ entries 0 as follows: We notice that the $1, 2$ entry is already 0 and leave the first row alone, then we go to the third row and cancel the $2, 3$ entry by subtracting the third row by 3 times second row to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 6 & 12 \end{array}\right)$$

row $3 - 3 \cdot$ row 2

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 3-3(1) & 6-3(-2) & 12-3(0) \end{array}\right),$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12 & 12 \end{array}\right).$$

Finally, we try to make column 2 (representing the z variable) the desired column $(0, 0, 1)$ as follows:

Employ *step 1* to make the $3, 3$ entry 12 into 1 by dividing the third row by 12 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12 & 12 \end{array}\right)$$

row $3 \div 12$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12\div 12 & 12\div 12 \end{array}\right),$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Next, we employ *step 2* to make the $1, 3$ entry and the $2, 3$ entry 0's by noticing that the $1, 3$ and $2, 3$ entries are 3 and -2 respectively, so we in order to cancel out those entries, we subtract the first row by -3 times the third row, followed by adding the second row by 2 times the third row, and we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

row $1 - 3 \cdot$ row 3

$$\left(\begin{array}{ccc|c} 1-3(0) & 0-3(0) & 3-3(1) & 4-3(1) \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right),$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

row $2 + 2 \cdot$ row 3

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0+2(0) & 1+2(0) & -2+2(0) & 0+2(1) \\ 0 & 0 & 1 & 1 \end{array}\right),$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

We have now reduced the matrix to the desired reduced form, and see that the solution for (x, y, z) is $(1, 2, 1)$.

Inconsistent and Dependent Systems of Equations

Of course, as we've learned in the context two linear equations with two variables, systems of linear equations do not necessarily have a single unique solution. Sometimes they have *infinitely many solutions*, in which case we have a **Dependent System of Equation**, and other times they have *no solution*, in which case we have an **Inconsistent System of**

Equations.

Consistent Equations are equations with unique solutions, i.e., when we apply the row reduction procedure discussed above, we get a matrix of the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & E_1 \\ 0 & 1 & 0 & E_2 \\ 0 & 0 & 1 & E_3 \end{array}\right) \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & E_1 \\ 0 & 1 & 0 & E_2 \\ 0 & 0 & 1 & E_3 \end{bmatrix}.$$

In general, the final matrix result we get when applying the same procedure above is called the **Reduced Row Echelon Form** (abbreviated **RREF**), which is always matrix of the form

$$\left(\begin{array}{ccc|c} \epsilon_1 & a_{1,2} & a_{1,3} & E_1 \\ 0 & \epsilon_2 & a_{2,3} & E_2 \\ 0 & 0 & \epsilon_3 & E_3 \end{array}\right) \quad \text{or} \quad \begin{bmatrix} \epsilon_1 & a_{1,2} & a_{1,3} & E_1 \\ 0 & \epsilon_2 & a_{2,3} & E_2 \\ 0 & 0 & \epsilon_3 & E_3 \end{bmatrix},$$

where $\epsilon_i = 0$ or 1 and $a_{i,j}$ are entries that are necessarily zero if $\epsilon_i = 1$ and necessarily nonzero if $\epsilon_i = 0$.

Dependent Systems of Equation

A dependent system of equation always has an RREF where some row has $\epsilon_i = 1$ and some $a_{i,j}$ is nonzero, and any row i with E_i nonzero means that some *different* entry in that row must be nonzero.

<u>Example 3.</u> We shall solve

$$\begin{aligned} &2x + 6y + 2z = 12 \\ &3x + 9y + 3z = 18. \\ &4x + 8y + 2z = 4 \end{aligned}$$

The equation is yields the following matrix

$$\left(\begin{array}{ccc|c} 2 & 6 & 2 & 12 \\ 3 & 9 & 3 & 18 \\ 4 & 8 & 2 & 4 \end{array}\right),$$

and we apply the following reduced row operations to get its RREF:

$$\left(\begin{array}{ccc|c} 2 & 6 & 2 & 12 \\ 3 & 9 & 3 & 18 \\ 4 & 8 & 2 & 4 \end{array}\right)$$

row $2 \div 2$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 3 & 9 & 3 & 18 \\ 4 & 8 & 2 & 4 \end{array}\right)$$

row $2 - 3 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 0 & 0 & 0 \\ 4 & 8 & 2 & 4 \end{array}\right)$$

row $3 - 4 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -2 & -20 \end{array}\right)$$

row 2 $\leftrightarrow$ row 3

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & -4 & -2 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

row $2 \div - 4$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & \frac{1}{2} & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

row 1 $-$ row 2

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -9 \\ 0 & 1 & \frac{1}{2} & 5 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and we are finished with solving this equation since the $3, 3$ entry is zero, and we are unable to continue. This RREF matrix fits the criteria of a dependent equation since the first and second row has a 1 diagonal along with a nonzero entry in a later column before the last. This nonzero entry signifies a free variable, which we can see from converting the RREF back into equation form. We have the equation

$$\begin{aligned} x - \frac{1}{2}z &= -9 \\ y + \frac{1}{2}z &= 5, \end{aligned}$$

and we have the dependent solution (with z left free)

$$\begin{aligned} x &= \frac{1}{2}z - 9 \\ y &= -\frac{1}{2}z + 5. \end{aligned}$$

Inconsistent Systems of Equations

An inconsistent system of equations always has an RREF where E_i is nonzero and *every other entry in that row is* 0.

Example 4. We shall (try to) solve

$$\begin{aligned} 2x + 4y + 2z &= 5 \\ x + 2y + z &= 2. \\ 2x + 4y + 5z &= 3 \end{aligned}$$

The equation is yields the following matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & 5 & 3 \end{array}\right),$$

and we apply the following reduced row operations to get its RREF:

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & 5 & 3 \end{array}\right)$$

row $1 \leftrightarrow$ row 2

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 5 \\ 2 & 4 & 5 & 3 \end{array}\right)$$

row $2 - 2 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 5 & 3 \end{array}\right)$$

row $3 - 2 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

row $1 -$ row 3

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

which is an inconsistent equation since the second row has all zeros before the last column but a nonzero entry on the last column, i.e., we get system of equations

$$\begin{aligned} x + 2y &= 3 \\ 0 &= 1 \\ z &= -1, \end{aligned}$$

which contains the contradiction $0 = 1$.

Systems of Equations With Matrices That Aren't $\mathrm{n} \times (\mathrm{n} + 1)$

Usually we do systems of equations that contain 2 equations and 2 unknowns, or systems of equations that contain 3 equations and 3 unknowns. The matrices that result from this are 2×3 and $3 \ \times 4$, respectively.

Sometimes, we have situations where we have systems of equations that are 2 equations and 3 unknowns, or 3 equations and 2 unknowns. The reduced row procedure above generalizes to that situation. with the additional nuance that we keep on going through columns ***until we've exhausted all the columns or all the rows*** (whichever comes first). I'll demonstrate this by doing an example involving a $2 \ \times 4$ matrix and 3×3 matrix.

<u>Example 5.</u> We shall solve

$$2x + 2y + 6z = 10$$
$$3x + y - z = 9.$$

The equation is yields the following matrix

$$\left(\begin{array}{ccc|c} 2 & 2 & 6 & 10 \\ 3 & 1 & -1 & 9 \end{array}\right),$$

and we apply the following reduced row operations to get its RREF:

$$\left(\begin{array}{ccc|c} 2 & 2 & 6 & 10 \\ 3 & 1 & -1 & 9 \end{array}\right)$$

row $3 \div 2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 3 & 1 & -1 & 9 \end{array}\right)$$

row $2 - 3 \cdot$ row 1

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & -2 & -10 & -6 \end{array}\right)$$

row 2 $\div -2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 1 & 5 & 3 \end{array}\right)$$

row $1-$ row 2

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 5 & 3 \end{array}\right),$$

and we have no below rows after the second row, so we are done with our row reductions. We have the final equation

$$\begin{aligned} x - 2z &= 2 \\ y + 5z &= 3, \end{aligned}$$

which gives us the solution

$$\begin{aligned} x &= 2z + 2 \\ y &= -5z + 3, \end{aligned}$$

with z a free variable. In general, a system of linear equations with more variables than equations is a dependent system of equations--and has infinitely many solutions--provided that the equation is not inconsistent.

<u>Example 6.</u> We shall solve

$$\begin{aligned} 2x + 6y &= 20 \\ 4x + y &= 7 \\ 2x + 7y &= 25. \end{aligned}$$

The equation is yields the following matrix

$$\left(\begin{array}{cc|c} 2 & 6 & 20 \\ 4 & 1 & 7 \\ 2 & 7 & 25 \end{array}\right),$$

and we apply the following reduced row operations to get its RREF:

$$\left(\begin{array}{cc|c} 2 & 6 & 20 \\ 4 & 1 & 7 \\ 2 & 7 & 25 \end{array}\right)$$

row $1 \div 2$

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 4 & 1 & 7 \\ 2 & 7 & 23 \end{array}\right)$$

row $2 - 4 \cdot$ row 1

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -11 & -33 \\ 2 & 7 & 25 \end{array}\right)$$

row $3 - 2 \cdot$ row 1

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -11 & -33 \\ 0 & 1 & 3 \end{array}\right)$$

row $2 \leftrightarrow$ row 3

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 1 & 3 \\ 0 & -11 & -33 \end{array}\right)$$

row $1 - 3 \cdot$ row 2

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -11 & -33 \end{array}\right)$$

row $3 + 11 \cdot$ row 2

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right),$$

and we have the single solution $(x, y) = (1, 3)$ for the equation.

Final Note

We can further generalize this for $n \times m$ matrices where either n or m is 4 or greater, but such examples are outside the scope of this class.