A Guide to the Language of Sets

This guide is intended to be a general reference to the language of sets, and what the various symbols and operations mean. The point of this guide is twofold:

1. Provide in the first two sections provide a brief reference to all the linguistic concepts of sets that the reader (who may naturally forget something) can quickly look at to easily recall something they've learned before.

2. Provide in the third (and last) section a guide to "Sets to English" (and conversely "English to Sets") translation that is often asked of the reader on homeworks, and the reader will find is implicitly asked to think like when solving a probability and counting problem.

Let's begin.

1 What are Sets and How do we Talk About Them?

A **set** is a collection of objects, which are referred to as **elements**.

These collections of objects gives us a whole new mathematical language. In this language, we like to use capital letters $A, B, C, \ldots$ to refer to sets.

We write "$a \in A$" when a **is an element of** a set A

We write "$A = B$" and say that A and B are **equal** when they have the same elements, i.e. $a \in A \implies a \in B$ and $b \in B \implies b \in A$.

We write "$A \subset B$" as say that A **is a subset of** B when every element of A is also an element of B, i.e. $a \in A \implies a \in B$.

Note that $A \subset B$ *does not necessarily* mean $A = B$ but $A \subset B$ *and* $B \subset A$ does mean $A = B$.

There are many ways to talk about and look at sets. Here are three of these ways.

1.1 Sets as Bracketed Lists

One way is to write a set as a bracketed list, for example, the set A of whole numbers between 1 and 5 (including 1 and 5) can be written as $A = \{1, 2, 3, 4, 5\}$.

Note that the same set can be written in multiple different ways, since equality is only based on wich elements there are, not on how many times they're mentioned or the order they're mentioned, so

$\{1,2,3,4,5\} = \{5,4,3,2,1\} = \{1,1,2,3,3,4,5\}$, and so on...

Infinite sets can be also given in a list, as long as there's a clear pattern that signifies the elements in it, so the set of X of positive even numbers can be written as $X = \{2,4,6, \ldots\}$ and the set Y of all integers (positive and negative) can be written as $Y = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

1.2 Sets as a Description of Objects ("Set-Builder Notation")

Perhaps the most descriptive way to talk about sets is write sets as a description of objects (what the book calls set-builder notation), so we can alternatively write the set X of even number and the set Y of integers as follows:

$$X = \left\{x | x \text{ is a positive even number}\right\},$$
$$Y = \left\{x | x \text{ is an integer}\right\}.$$

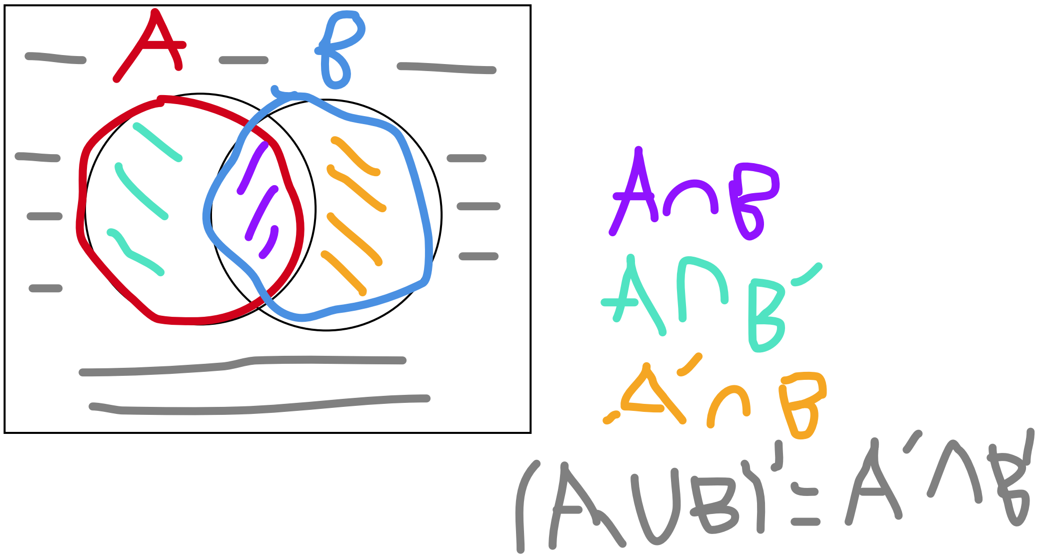
1.3 Sets as a Diagram

Here is what to think about when drawing or looking at Venn-Diagrams. There are three possible ways that a set can interact.

When Some of the Set Overlaps:

Here, some of the set (but not all of it) overlaps with the other set. There is an intersection, in addition to the set $A \cap B'$ consiting of elements in A that are not in B and the set $A' \cap B$ consisting of elements in B that are not in A.

We can also express the region outside of both sets with the symbol $(A \cup B)'$, or equivalently $A' \cap B'$ (note that $(A \cup B)' = A' \cap B'$ is a property often referred to as one of DeMorgan's Laws, the other "DeMorgan Law" being the property $(A \cap B)' = A' \cup B'$.



When the Set Completely Overlaps:

When the set B completely overlaps with A, this is precisely the situation that $B \subset A$. We find in this situation that $A \cap B = B$ and $A' \cap B = \varnothing$ (since you cannot be in B without also being in A).

$A \cap B = B$

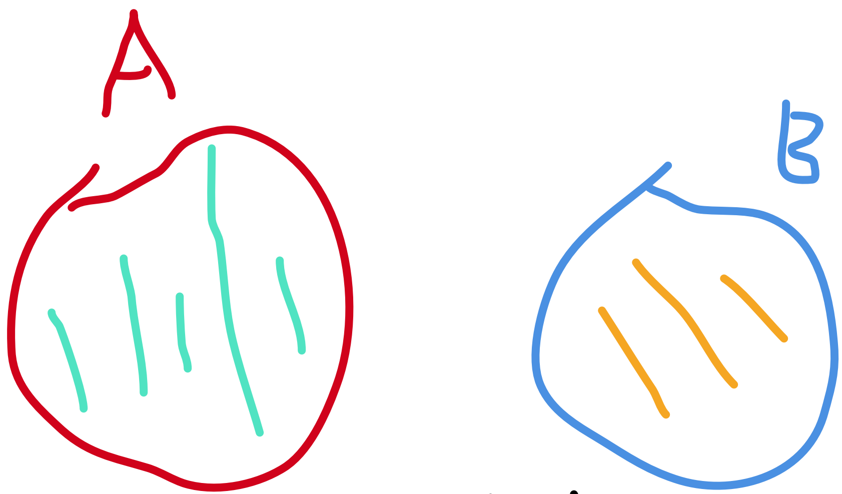
$A \cap B'$

$A' \cap B = \varnothing$

$B \subset A$

When The Set Doesn't at all Overlap:

When the set A doesn't overlap with B at all, this is precisely the situation when A and B are **disjoint**, or have no elements in common (and $A \cap B = \varnothing$ as a result). In this situation, we moreover have the regions $A \cap B' = A$ and $A' \cap B = B$.



A and B disjoint.

$A \cap B = \emptyset$

$A \cap B' = A$

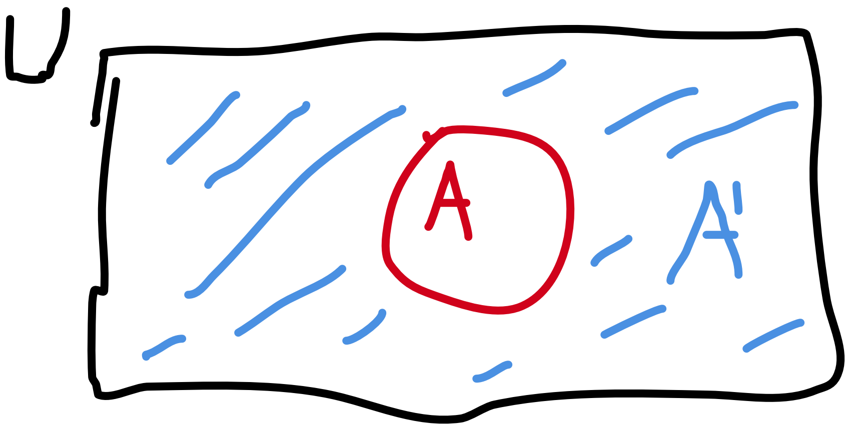
$A' \cap B = B$

2 Set Operations

2.1 Complement

To define the complement, i.e. negation $(-)'$, we designate a set U as our **universal set,** which serves as the set with <u>all the elements</u> in the context of the given scenario. Then for any set A, we define

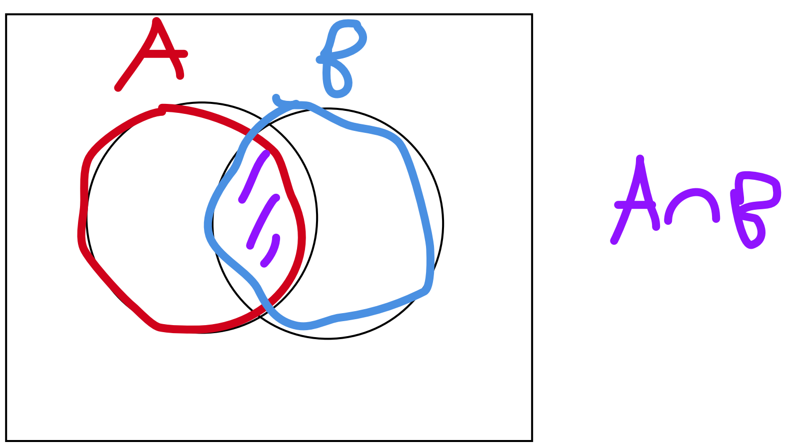
$$A' := \left\{x | x \in U \text{ and } x \notin A\right\}$$



Then, for any given two sets A and B, we define

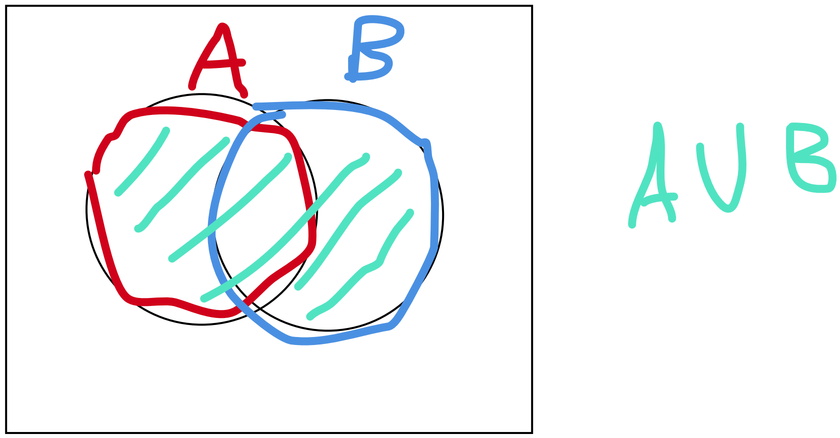
2.2 Intersection

$A \cap B := \left\{x | x \in A \text{ and } x \in B\right\}$



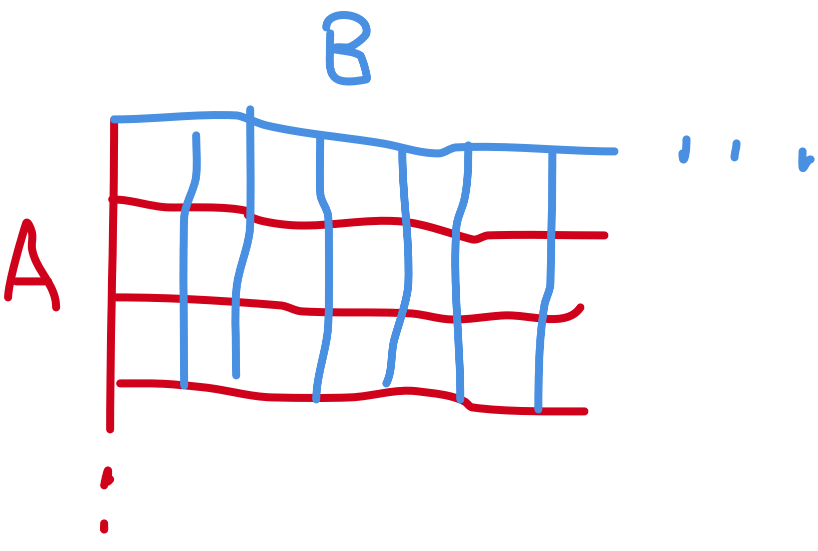
2.3 Union

$A \cup B := \left\{x | x \in A \text{ or } x \in B\right\}$



2.4 Cartesian Product

$A \times B := \{x | x = (a, b),\ \text{where } a \in A \text{ and } b \in B\}$



2.5 Order of Operations of Sets

Whenever we apply set operations to determine what a given set is, we do it in the following order of operations:

1. Work inside innermost parenthesis ( ).

2. Complements $'$ *before intersections* $\cap$ *and unions* $\cup$ *and products* $\times$

3. Products $\times$ *before intersections* $\cup$ *and unions* $\cap$.

3. find intersections $\cap$ *and unions* $\cup$ *from <u>left to right.</u>*

NOTE: Usually in this M018 class, we don't apply products in the order of operations, but I'm writing this order for completion.

3 Translating the Language of Sets to English (and Vice Versa)

3.1 Sets as "Noun Phrases"

Note that the paradigm of sets that we're talking is a mathematical language with operations $\cup$, $\cap$, $'$, $\times$ (just like the language of arithmetic with $+, -, \cdot$), and a useful one for making sense of finding probabilities.

And this language of sets has an "english translation", and it works as follows

First, we look a given "atomic set", i.e. sets in terms of set builder notation, which uses an english phrase to describe all its elements, and it translates in english to that english phrase

$\left\{x|x \text{ is a whole number}\right\}$ translates to the english phrase "the whole numbers"

$\left\{x|x \text{ is an IU student}\right\}$ translates to the english phrase "the IU students"

$\left\{x|x \text{ is a fruit}\right\}$ translates to the english phrase "the fruits"

Note that sometimes sets are given as a list, such as the set $\{1, 2, 3\}$, so then we have to find a phrase that describes the set in set-builder notation and that's how we translate that set, so since

$\{1, 2, 3\} = \left\{x|x \text{ is a whole number between 1 and 3}\right\}$,

we find that:

$\{1, 2, 3\}$ translates to "the whole numbers between 1 and 3

intersection, union, and complement translate to the words "and", "or", "not" respectively that modifies/combines atomic phrases.

$\left\{x|x \text{ is a fruit}\right\} \cup \left\{x|x \text{ is a vegetable}\right\}$ translates to "fruits or vegetables"

$\left\{x|x \text{ is a fruit}\right\} \cap \left\{x|x \text{ is a vegetable}\right\}$ translates to "fruits and vegetables", but not necessarily the collection that consists of "all fruits and vegetables" (that's what the union $\cup$ describes). What ends up being described is objects that are both fruits and vegetables

$U = \left\{x|x \text{ an IU student}\right\}$
$S = \left\{x|x \text{ is a Kelly Student}\right\}$
S' translates to "Non-kelly students".

$U = \left\{x|x \text{ is an integer (positive or negative)}\right\}$
$S = \left\{x|x \text{ is a nonnegative integer}\right\} = \{0, 1, 2, 3, \dots\}$

S' translates to "the nonnegative integers"
The negation symbol cancels out with "nonnegative" and replaces it with "negative"
$S' = \left\{x|x \text{ is a negative integer}\right\}$ translates to "the negative integers"

In general, applying $'$ to a set S translates to the negation phrase of the translation of S

In conclusion, we can think about sets as "noun phrases" in the english language.

3.2 Sentences in the Language of Sets

Sentences involve objects and relations: the relations are $\in$, $\subset$, $=$.

In the elementhood relation $\in$ we link it to an object and a set. When we have $x \in S$ that translates in general to "x is [insert translation of S here]"

<u>Example.</u>

$x \in \left\{x|x \text{ is a fruit}\right\} \cap \left\{x|x \text{ is a vegetable}\right\}$ translates to "x is a fruit and a vegetable".

"$A \subset B$" that translates to "all [insert translation of A] are [insert translation of B]"

<u>Example:</u>

$A = \left\{x | x \text{ is an integer}\right\},\ B = \left\{x | x \text{ is a rational number}\right\}$

"$A \subset B$" translates to "all integers are rational numbers".

"$A = B$" that translates to "all [insert translation of A] are *precisely* the [insert transation of B]"

Example:

$A = \left\{x | x \text{ fractions of integers}\right\},\ B = \left\{x | x \text{ is a rational number}\right\}$

"$A = B$" translates to "all fractions of integers are precisely the rational numbers"