The Complete Guide to Solving a Two-Variable Linear System of Equations

Introduction

In this class, we solve linear systems of equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ in two variables x, y and we find the solution (x_0, y_0) such that plugging x_0 and y_0 into these equations get us the equalities:

$$a_1x_0 + b_1y_0 = c_1$$
$$a_2x_0 + b_2y_0 = c_2.$$

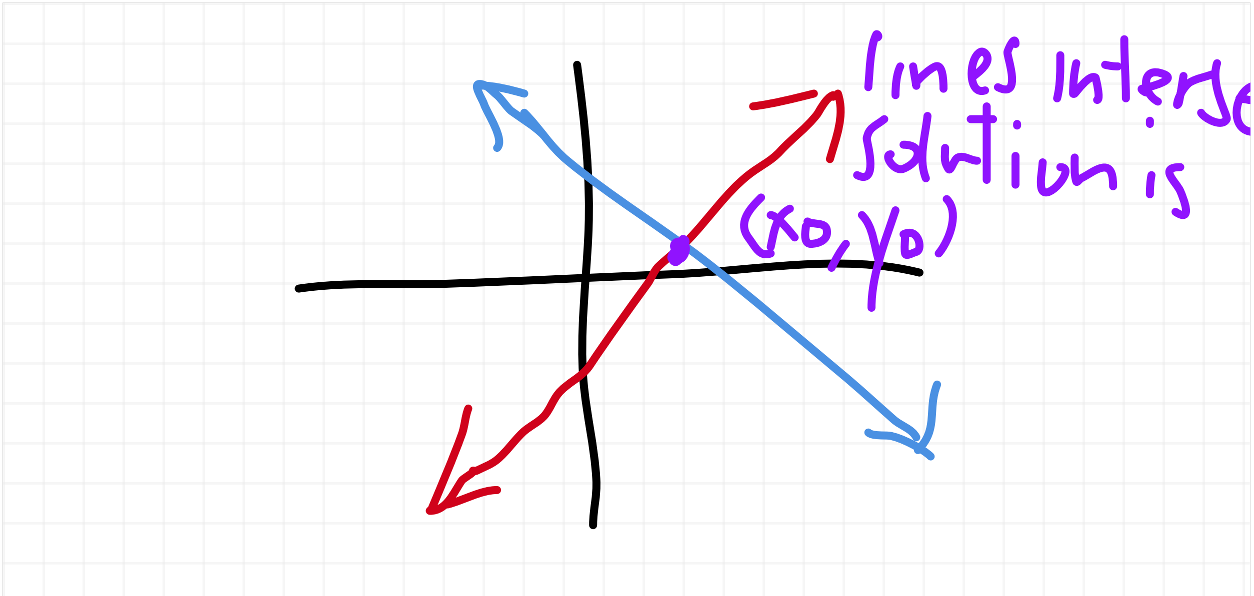
While much of this is still review for high school algebra, we also cover more fringe scenarios of these systems of equations where there isn't simply one solution. In the scenario with **dependent equations**, we have infinitely many solutions, and with **inconsistent** systems of equations, we have no solutions.

Possible Solution Scenarios

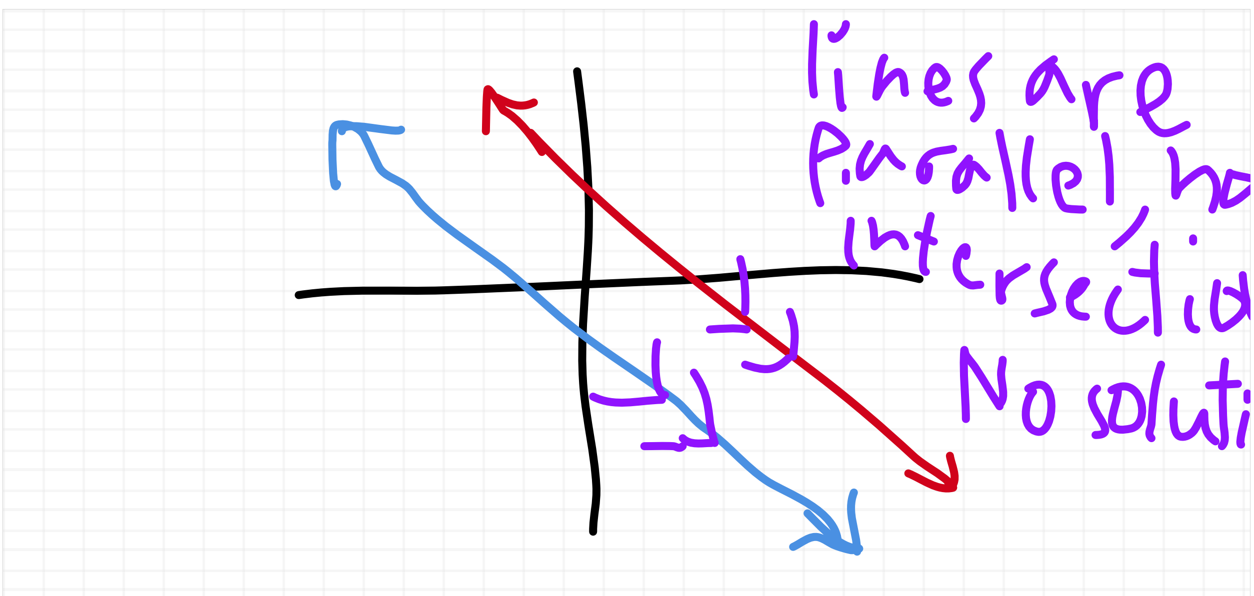
Note that a solution senario is a point on the cartesian plane that satisfies both equations of a line. Visually, this is a point of intersection of the lines, since the solution is a point that can be found one *both* lines

Just like with the solving a single-variabled equation, there's a few possible scenarios; to be exact, there are three:

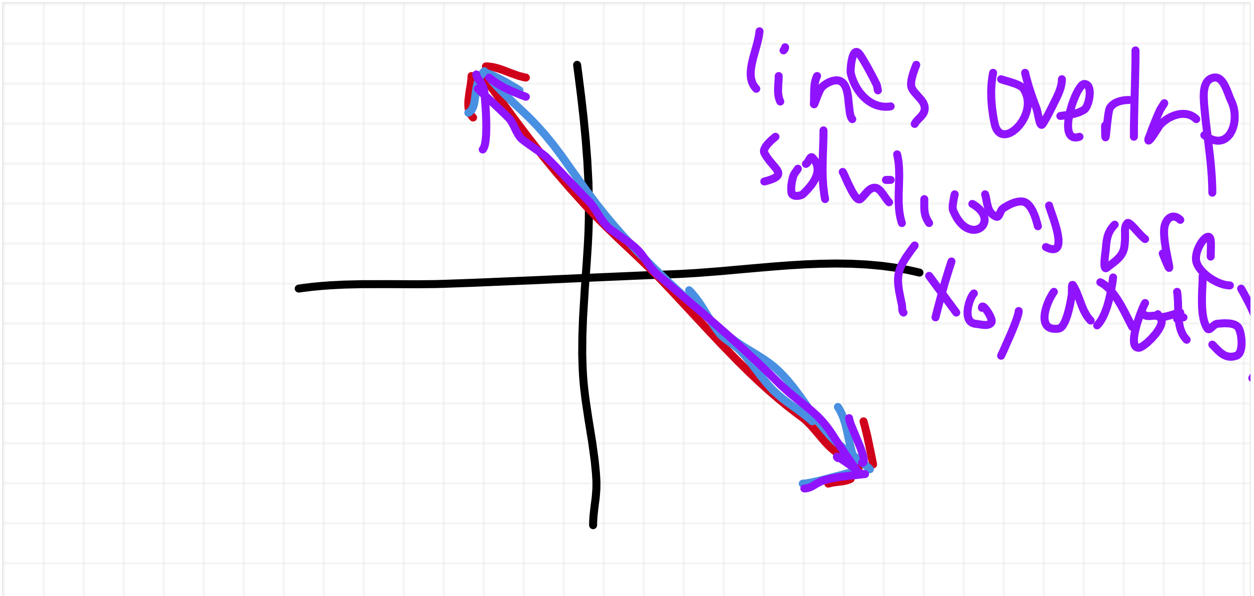
Scenario 1. The line intersects at one, *and only one*, place and there is only one solution (x_0, y_0).



Scenario 2. The lines are parallel, never intersect, and there is no solution. As stated before, we call such equations "inconsistent".



Scenario 3. The lines completely overlap (i.e., they are the same line) and there is infinitely many solutions. As stated before, we call this such equations "dependent equations" because
all the solutions are of the form $(x_0, ax_0 + b)$ or $(ay_0 + b, y_0)$ where one coordinate is "dependent" of another.



Graphing Method

We do the following steps:

Step 1. On a single set of coordinate axes, graph each equation.

Step 2. Find these coordinates of the point (or points) where the graphs intersect. These coordinates give the solution of the system.

NOTE: Make sure the graph is drawn *very precisely*. Otherwise, you will get the wrong solution

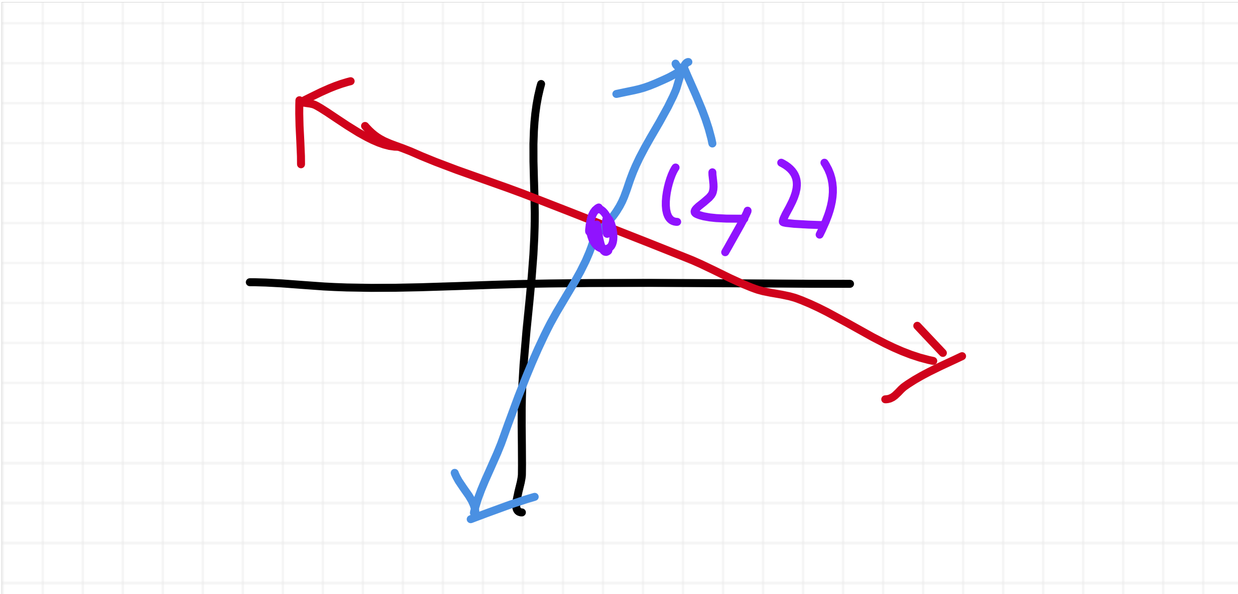
Step 3. If the graphs have no point in common, the system has no solution.

Step 4. If the graphs of the equations coincide, the system has infinitely many solutions.

Example 1. We find the equation to

$x + 2y = 6$
$5x - y = 8$

by graphing the line. Note that the equations are in general form and we know that $x + 2y = 6$ has slope $-\dfrac{1}{2}$ and y-intercept $(0, 3)$, and $5x - y = 81$ has slope 5 and y-intercept $(0, -8)$, and so we get the following graph

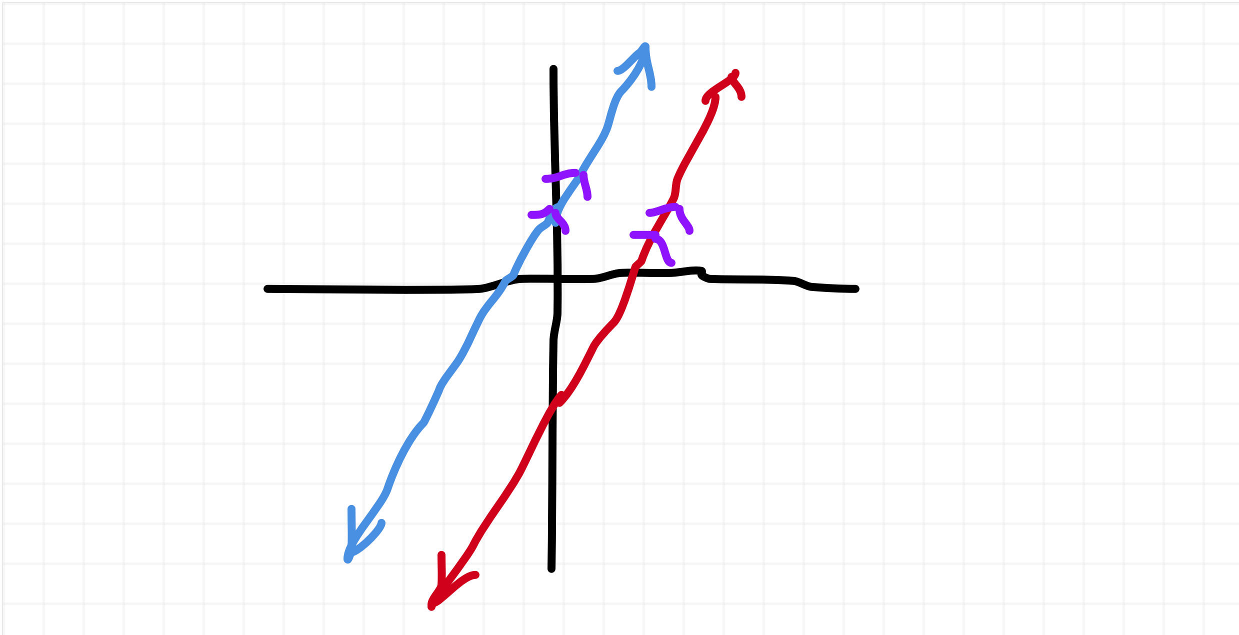


and we find $(2, 2)$ is the solution.

<u>Example 2.</u> We find the equation to

$$3x - 2y = 6$$
$$-9x + 6y = 9$$

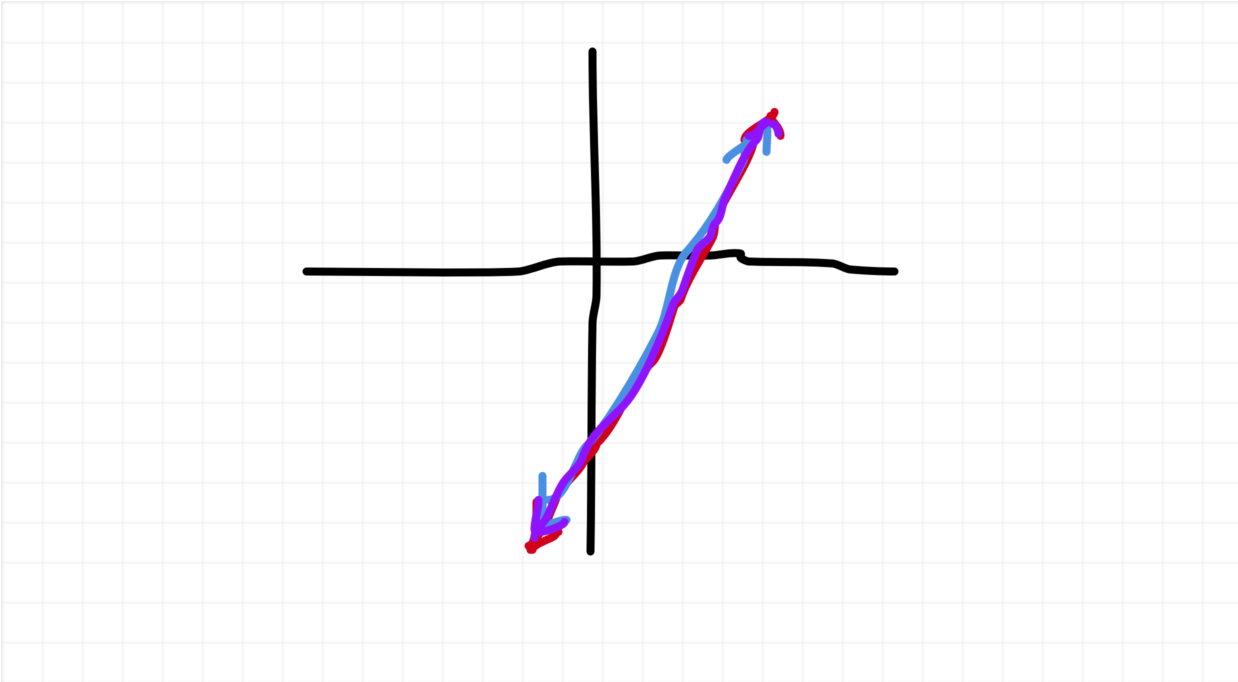
by graphing the line. Note that the equations are in general form and we know that $3x - 2y = 6$ has slope $\frac{3}{2}$ and y-intercept $(0, -3)$, and $-9x + 6y = 9$ has slope $\frac{3}{2}$ and y-intercept $\left(0, \frac{3}{2}\right)$, and so we get the following graph



Example 3. We find the equation to

$$3x - 2y = 6$$
$$-9x + 6y = -18$$

by graphing the line. Note that the equations are in general form and we know that $3x - 2y = 6$ has slope $\frac{3}{2}$ and y-intercept $(0, -3)$, and $-9x + 6y = 9$ has slope $\frac{3}{2}$ and y-intercept $(0, -3)$, and so we get the following graph



and we find the solution consists of all coordinates of the form $\left(x_0,\ \frac{3}{2}x_0 - 3\right)$ for any x_0.

Substitution Method

Step 1. If necessary, solve one equation for one of its variables.

Step 2. Substitute the resulting expression for the variable abtained in *Step 1* into the other equation and solve that equation, if possible

2.1. If it's not possible to solve the resulting equation, determine instead of the resulting equation is an *identity* or *contradiction*.

Step 3. Find the value of the other variable, if possible, by substituting the value of the variable found in Step 2 into any equation containing both variables

3.1. If it's not possible to solve the resulting equation, determine instead of the resulting equation is an *identity* or *contradiction*.

Step 4. State the solution. If we determined in *2.1* or *3.1* that we have an identity, then we have a dependent equation, which is of the form $(x, ax+b)$ or $(ay+b, y)$. If we determined in *2.1* or *3.1* that we have a contradiction, then no solution exists.

Step 5. Check the solution in both of the original equations.

Example 4. Let us solve the equation

$$5x + \frac{1}{2}y = 8$$
$$2x - \frac{1}{3}y = 0.$$

For step 1, take the equation $2x - \frac{1}{3}y = 0$ and solve for y. We get

$$\begin{aligned} 2x - \frac{1}{3}y &= 0 \\ \times 3 \qquad &\times 3 \\ 6x - y &= 0 \\ 6x &= y. \end{aligned}$$

For step 2, we plug in $y = 6x$ into the equation $5x + \frac{1}{2}y = 8$, and we get

$$\begin{aligned} 5x + \frac{1}{2}(6x) &= 8 \\ 5x + 3x &= 8 \\ 8x &= 8 \\ \div 8 \qquad &\div 8 \\ x &= 1. \end{aligned}$$

For step 3, we plug in $x = 1$ into $y = 6x$ and we get

$$y = 6(1) = 6.$$

And for step 4, we state $(1, 6)$ as the solution. For step 5, we will check the solution by plugging it in into the equation

$$5(1) + \frac{1}{2}(6) = 5 + 3 = 8$$
$$2(1) - \frac{1}{3}(6) = 2 - 2 = 0,$$

and we see that the solution works.

Example 5. Let us solve the equation

$$4x + 12y = 4$$
$$\frac{1}{3}x + y = 1.$$

For step 1, we take the equation $4x + 12y = 4$ and solve for x. We get

$$\begin{aligned} 4x + 12y &= 4 \\ -12y \quad & -12y \\ 4x \quad &= 4 - 12y \\ \div 4 \quad & \div 4 \\ x \quad &= 1 - 3y \end{aligned}$$

For step 2, we plug in $x = 1 - 3y$ into the equation $\frac{1}{3}x + y = 1$ and we get

$$\frac{1}{3}(1 - 3y) + y = 1$$
$$\frac{1}{3} - y + y = 1$$
$$\frac{1}{3} = 1$$
$$-\frac{1}{3} \quad -\frac{1}{3}$$
$$0 = -\frac{1}{3},$$

and we get a contradiction. We conclude that no solution exists

Example 6. Let us solve the equation

$$4x + 12y = 4$$
$$8x + 24y = 8.$$

For step 1, we take the equation $4x + 12y = 4$ and solve for x. We get

$$\begin{aligned} 4x + 12y &= 4 \\ -12y \quad & -12y \\ 4x \quad &= 4 - 12y \\ \div 4 \quad & \div 4 \end{aligned}$$

$x \qquad = 1 - 3y$

For step 2, we plug in $x = 1 - 3y$ into the equation $8x + 24y = 8$ and we get

$8(1 - 3y) + 24y = 8$
$8 - 24y + 24y = 1$
$8 = 8$
$-8 \ \ -8$
$0 = 0,$

and we get an identity. We find that both lines completely overlap, using the general form formula that the slope is $-\dfrac{1}{3}$ and the y-intercept is $\left(0, \dfrac{1}{3}\right)$ and the solution consists of all coordinates of the form $\left(x_0, -\dfrac{1}{3}x_0 + \dfrac{1}{3}\right)$.

Addition Method

Step 1. Write both equations of the system in general form.

Step 2. Multiply the terms of one or both of the equations by constants chosen to make the coefficients of x (or y) differ only in sign.

Step 3. Add the equations to solve the resulting equation, *if possible.*

3.1. If it's not possible to solve the resulting equation, determine instead if the resulting equation is an *identity* or a *contradiction.*

Step 4. Substitute the value obtained in *Step 3* into either of the original equations and solve for the remaining variable.

Step 5. State the solution obtained in *Step 3* and *4*. If we determined in *3.1* that we have an identity, then we have a dependent equation, which is of the form $(x, ax + b)$ or $(ay + b, y)$. If we determined in *3.1* that we have a contradiction, then no solution exists.

Step 6. Check the solution in both of the original equations.

<u>Example 7.</u> Let us solve the equation

$x - 3y = -4$

$2x + y = 13$

We can skip step 1 because both equations are in general form.

For step 2, we multiply the first equation by 2, which gives us

$$\begin{aligned} &x - 3y = -4 \\ &\times 2 \qquad \times 2 \\ &2x - 6y = -8, \end{aligned}$$

and we have the new system of equations

$$\begin{aligned} &2x - 6y = -8 \\ &2x + y = 13. \end{aligned}$$

For step 3, we subtract the second equation by the first equation as follows:

$$\begin{aligned} &2x + y = 12 \\ &-(2x - 6y) \ -(-8) \\ &2x - 2x + y - (-6y) = 13 - (-8) \\ &\qquad\quad 7y \qquad\quad = 21, \end{aligned}$$

and then solve for y:

$$\begin{aligned} &7y = 21 \\ &\div 7 \ \ \div 7 \\ &\ \ y = 3. \end{aligned}$$

For step 4, we plug in $y = 3$ in the equation $2x + y = 13$ and we have

$$\begin{aligned} &2x + (3) = 13 \\ &2x + 3 = 13 \\ &\quad -3 \quad -3 \\ &2x \qquad = 10 \\ &\div 2 \qquad \div 2 \\ &\ \ x \qquad = 5. \end{aligned}$$

For step 5, we state the solution is $(5, 3)$, and we check this solution for step 6 as follows:

$$\begin{aligned} &(5) - 3(3) = 5 - 9 = -4 \\ &2(5) + (3) = 10 + 3 = 13. \end{aligned}$$

Example 8. Let us solve the equation

$$2x - 4y = 2$$
$$-6x + 12y = 9.$$

We can skip step 1 since both equations are already in general form. For step 2, we divide the second equation by -3 and we get

$$\begin{array}{ll} -6x + 12y & = 9 \\ \div(-3) & \div(-3) \\ 2x - 4y & = -3, \end{array}$$

and we get the new system of equation

$$2x - 4y = 2$$
$$2x - 4y = -3.$$

For step 3, we subtract the first row by the second row, and we get

$$\begin{array}{ll} 2x - 4y = 2 & \\ -(2x - 4y) & -(-3) \\ 0 & = 5, \end{array}$$

and we get a contradiction. We conclude that no solution exists.

Example 9. Let us solve the equation

$$-4x - 12y = 12$$
$$x + 3y = -3.$$

We can skip step 1 since both equations are already in general form. For step 2, we divide the first equation by -4 and we get

$$\begin{array}{ll} -4x - 12y & = 12 \\ \div(-4) & \div(-4) \\ x + 3y & = -3, \end{array}$$

and we get the new system of equation

$$x + 3y = -3$$

$x + 3y = -3.$

For step 3, we subtract the first row by the second row, and we get

$$\begin{aligned} &x + 3y = -3 \\ &-(x + 3y) \ -(-3) \\ &0 \qquad\quad = 0, \end{aligned}$$

and we get an identity. We find the equation is a dependent equation, and we know from the general form formulas that the slope of the line $x + 3y = -3$ is $-\frac{1}{3}$ and the y-intercept is $(0, -1)$, and we find the solution consists of all coordinates in the form $\left(x_0, -\frac{1}{3}x_0 - 1\right)$.

Segway into Matrix Method

The final method we do is the matrix method, which we cover in lesson 4 (and in the matrix guide). Here, we give a sneak peak of this method. The basic idea is to look at the system of general form equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

and convert the coefficients into a matrix with columns corresponding to each variable, and constant, giving us

$$\left(\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right)$$

We can look at what we did in the addition method in Example 7 in terms of matrix row reduction operations as follows:

$x - 3y = -4$ $\qquad$ $\left(\begin{array}{cc|c} 1 & -3 & -4 \\ 2 & 1 & 13 \end{array}\right)$

$2x + y = 13$

$$\begin{aligned} &x - 3y = -4 \\ &\times 2 \qquad \times 2 \\ &2x - 6y = -8 \end{aligned}$$

$2 \times row\ 1$

$$2x - 6y = -8 \qquad \left(\begin{array}{cc|c} 2 & -6 & -8 \\ 2 & 1 & 13 \end{array}\right)$$

$$2x + y = 13$$

$$\begin{aligned} &2x + y = 12 \\ &-(2x - 6y) \;\; -(-8) \qquad \qquad row\ 2 - row\ 1 \\ &\quad 7y = 21 \end{aligned}$$

$$2x - 6y = -8 \qquad \left(\begin{array}{cc|c} 2 & -6 & -8 \\ 0 & 7 & 21 \end{array}\right)$$

$$7y = 21$$

$$\begin{aligned} &7y = 21 \\ &\div 7 \quad \div 7 \qquad \qquad 1/7 \times row\ 2 \\ &\ y = 3 \end{aligned}$$

$$2x - 6y = -8 \qquad \left(\begin{array}{cc|c} 2 & -6 & -8 \\ 0 & 1 & 3 \end{array}\right)$$

$$y = 3$$

$$\begin{aligned} &2x - 6y = -8 \\ &\quad +6y \quad + 6(3) \qquad \qquad row\ 1 + 6 \cdot row\ 2 \\ &2x \qquad = 10 \end{aligned}$$

$$2x = 10 \qquad \left(\begin{array}{cc|c} 2 & 0 & 10 \\ 0 & 1 & 3 \end{array}\right)$$

$$y = 3$$

$$\begin{aligned} &2x = 10 \\ &\div 2 \quad \div 2 \qquad \qquad 1/2 \times row\ 1 \\ &\ x = 5 \end{aligned}$$

$$x = 5 \qquad \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \end{array}\right).$$

$$y = 3$$

We see that algebra operations correspond to the "row reductions", so we can do algebra from the perspective of matrix row reductions. In the next lesson, we'll give a more specific step-by-step process to solve systems of equation using a method called "Gaussian Elimination", and we'll use this method to solve systems of equations with three variables.