M106 Summer 2020 Recitation Lectures: Game Theory

☐

6/16 Lecture:

Office Hours: Wed., Thurs. 2pm-3pm (Or by Appointment)

Today we'll cover:
-Class 1 & 2: Introduction
-Class 3 (IF TIME): Dominant Strategies

Upcoming Homework:
Game Theory Homework 1 (due Thurs.)
Game Theory Homework 2 (due Fri.)
Game Theory Homework 3 (due Mon.)

Definition of a game: (Def. 1.1 of the notes)
A **game** is a situtation where every player simultaneously makes a decision (without the other person knowing) and the combination of everyone's decision.

-USUALLY 2 PLAYER (PROBABLY ALWAYS 2 PLAYERS)

Components of a game: (Def. 2.1 of the notes)
*the **strategies**: the player's possible moves/choices
*the **outcomes**: the result of a game given the player's combinations of choices (i.e., action profile); usually given as an ordered pair of actions
*the **payoff**: The (usually numeric) result of a game

NOTE: What a "game" for the sake of this course is very specifically this definition above and it's NOT what you usually think of as a game.

EXAMPLES OF A GAME:

-The Prisoner's Dilemma
We have two prisoners, prisoner 1 and prisoner 2, locked up, and each of them are being interviewed by a detective (in isolation)
-The strategies available are either to be loyal L or betray B

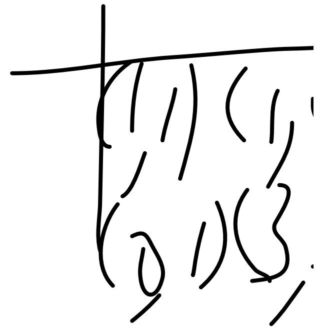
-A prisoner who is loyal doesn't rat out the other prisoner
-A prisoner who betrays the other prisoner rats out the other prisoner.
-Possible outcomes: (L, L), (B, L), (L, B), (B, B)
-If both are loyal, both only get 1 year in prison; if any prisoner betrays, they get a 1 year sentence reduction; any prisoner that is ratted out gets 3 years in prison (before determining reduction)

PAYOFF MATRICES

		2	
		L	B
1	L	(-1, -1)	(-3, 0)
	B	(0, -3)	(-2, -2)

Stag Hunt
2 players (Hunter 1 and Hunter 2) are hunting game for their next meal, and faced with the decision of either hunting the stag (S) or hunting the hare (H)
-Possible Outcomes: (S, S), (H, H), (S, H), (H, S)
-Payoff Structure: Any player that hunts the Hare has food for a day, if both players hunt the stag, then there's food for 2 days

		2	
		H	S
1	H	(1,1)	(1,0)
	S	(0,1)	(2,2)



Best Response:

The **Best response function** (BRF) for a given player is a function whose input is one of your opponents strategies, output is the best strategy you can play, given the opponents' inputted stragetgy

NOTATION:

player 1 and 2 with action A, B, C, ...

The BRF for player 1 is denoted $B_1(-)$, player 2 denoted $B_2(-)$
The BRF for player 1 given player 2 selected A is denoted $B_1(A)$
The BRF for player 2 given player 1 selected C is denoted $B_2(C)$

Let's look at the BRF's for the prisoner's dilemma:

Let's look at the BRF for player 2 B_1(-) :
If player 1 is loyal, GIVEN that player 2 is loyal, then player 1 get 1 year in prison
If player 2 betrays, GIVEN that player 2 is loyal, then player 1 gets 0 years in prison
0 years is worse than 1 year
$B_1(L) = B\ B_1(B) = B$
$B_2(L) = B\ B_2(B) = B$

1 \ 2	L	B
L	(4, 4)	(-3, 0)
B	(0, -3)	(-2, -2)

	L	B
L	(4, 4)	(-3, 0)
B	(0, -3)	(-3, -3)

6/17 Lecture

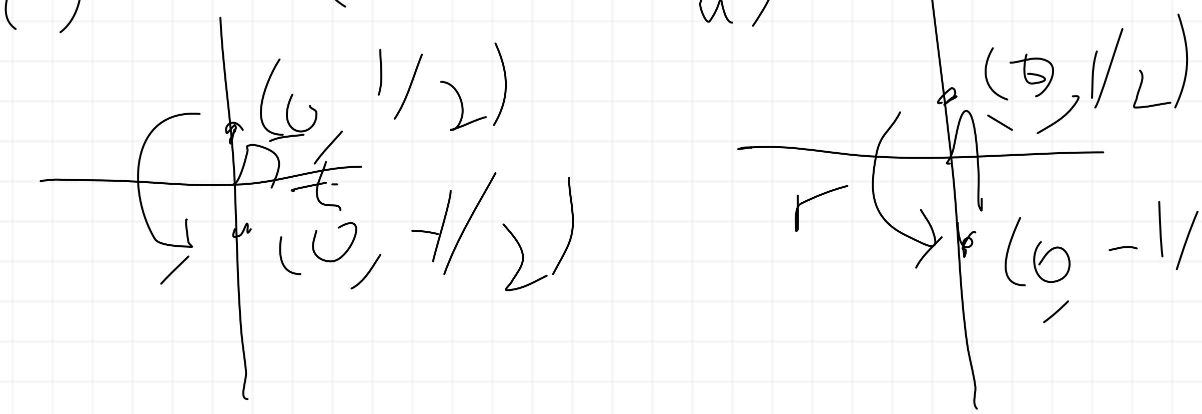
Announcements:
Office Hours: Wed., Thurs. 2pm-3pm OR by appointment
Zoom Link: iu.zoom.us/my/agoodlad

Symmetry HW 3 grades out
$t \circ r \neq r \circ t$

d) tor = 1/2 turn rotatio
(0, 1/2)

c) rot = 1/2 turn rotatio
(0, -1/2)

c) rot   d) tor



HW 3 solutions are out (IN LOCAL CANVAS)

Game theory HW 1-3 uploaded

-Questions for Homework 1?

Class 3: Dominant Strategies

Dominance Relations (Def. 3.3 of the Notes):
-A strategy A **weakly dominates** strategy B if strategy A is better-than OR equal to response to each of the opponents strategies, regardless of opponent's choice

-A strategy A **strictly dominates** strategy B if strategy A is *strictly* better than the response of any other strategy, regardless of opponent's choice.

Example: Prisoner's Dilemma

2

1	L	B
L	(-1, -1)	(-3, 0)
B	(0, -3)	(-2, -2)

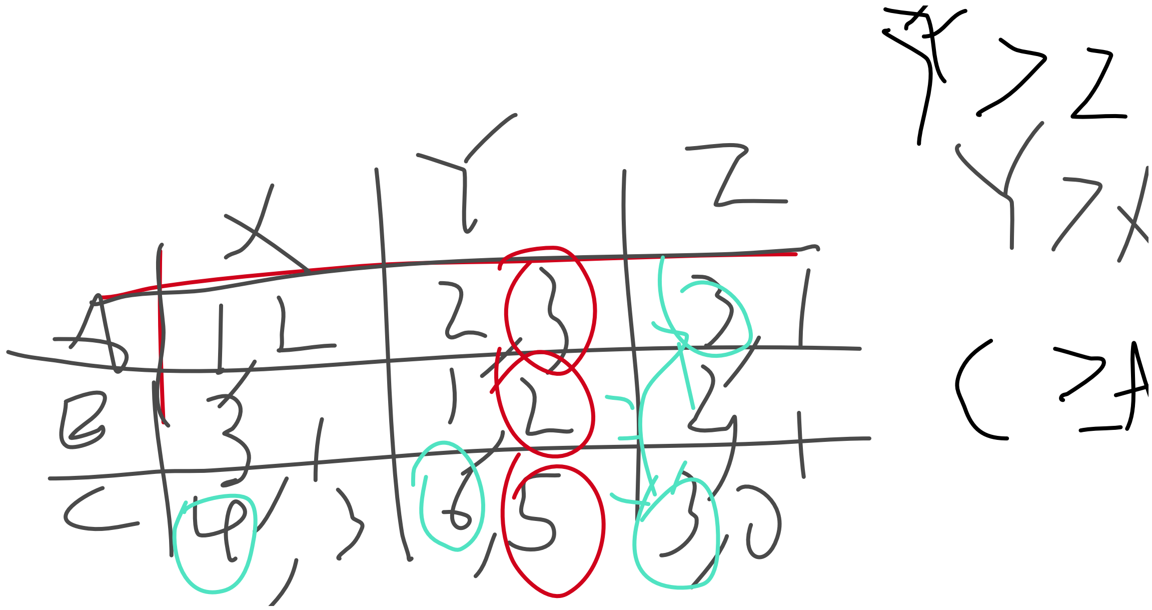
$B_1(L) = B$ $B_1(B) = B$
$B_2(L) = B$ $B_2(B) = B$

$B > L$ for player 1 AND player 2

Stag hunt

2

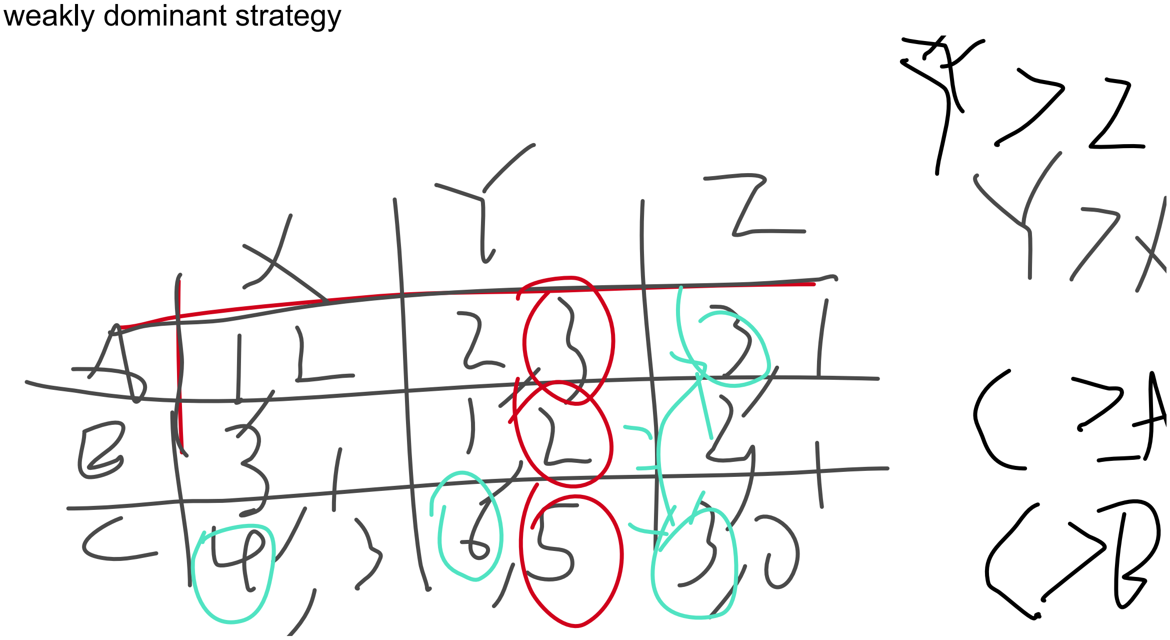
1	H	S
H	(1, 1)	(1, 0)
S	(0, 1)	(2, 2)



Dominant Strategy (Def. 3.6)
-We say that a strategy is **weakly dominant** if it weakly dominates ALL other strategies
-We say that a strategy is **strictly dominant** if it strictly dominates ALL other strategies; a strictly dominating strategy is necessarily unique

Prisoner's Dilemma: B is a strictly dominant strategy for player 1 and 2

For the game below, Y is a strictly dominant
C is a weakly dominant strategy



Class 4: Social Welfare and Pure Strategy Nash Equilibria

Social Welfare (Def. 4.1)
The net payoff for the group of all players in a given outcome is called the **social welfare** of that outcome.

Prisoner's Dilemma (see below payoff matrix)

Social welfare of (L, L) is $-1 + \ -1 = -2$
Social welfare of (L, B) is $-3 + 0 = -3$
Social welfare of (B, L) is $0 + \ -3 = -3$
Social welfare of (B, B) is $-2 + \ -2 = -4$

Example: Prisoner's Dilem

1 \ 2	L	B
L	(-1, -1)	(-3, 0)
B	(0, -3)	(-2, -2)

PURE Nash Equilibrium (Def. 4.4)
A **Nash Equilibrium (NE)** a pair of strategies which are a best response to each other, i.e., the outcome (A, X) is an NE if $B_1(X) = A$ and $B_2(A) = X$.
Equivalently, an NE is an outcome where no player can unilaterally improve their payoff by changing strategies

for the Prisoner's Dilemma:
$B_1(B) = B,\ B_2(B) = B$
(B, B) as a NE

for the Stag Hunt:
$B_1(H) = H,\ B_1(S) = S$

$B_2(H) = H,\ B_2(S) = S$

Stag hunt

1 \ 2	H	S
H	(1, 1)	(1, 0)
S	(0, 1)	(2, 2)

(H, H) and (S, S) are NE

There is a shortcut of computing a NE
The shortcut is a 3 step process as follows:
1.circle the maxium of first numbers on each column
2. circle the maximum of second in each row
3. Check for any outcome with two circles, and that's our NE

Example 4.7

	X	Y
A	1,1	2,2
B	3,3	1,1

~~NE: (A,Y) and (B,X)~~

(B,X), (A,Y)

NOTE: circle both in case of a tie

Ex. 4.12

(A,X) (B,X)

(B,Y)

	X	Y
A	2,3	1,2
B	2,1	1,1

6/17 Lecture

Announcements:

- +1 point for everyone on Symmetry HW 3
- Exam 1 Grades ETA: Sometime today, no later than tonight, solutions will be uploaded in the LOCAL canvas once grades are uploaded.

-Homework 4 and next week lecture uploaded (Exam 2 Thurs. June 25).

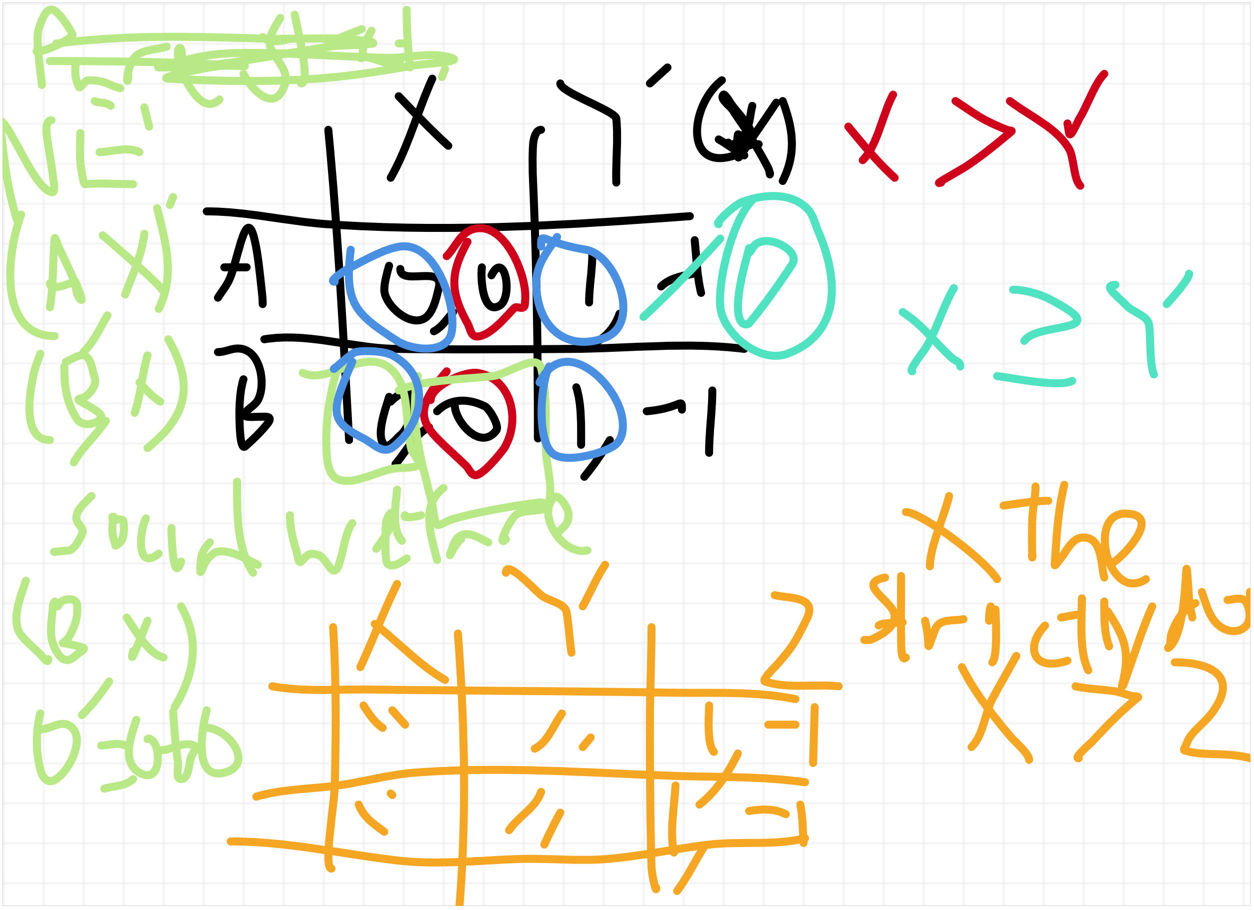
Questions for Homework 1-2:

Previously:

Class 3: Weak and Strict Dominance:

$X > Y$ if X strictly dominates Y

$X \geq Y$ if X weakly dominates Y



Class 4: Social Welfare and Pure Strategy NE

Class 6: Types of Games

Cooridination Game: (Def. 6.1) A **coordination game** is a game with multiple pure strategy NE.
Non-Example: Prisoner's Dilemma
Example: Stag Hunt, (*)

Many variations of coordination games

***Pure Coordination:** the strategy of both players directly align
Example: Stag Hunt

***Bias Coordination:** a coordination where both of the goals are parially but the benifit is unequal
Example: Battle of the Sexes
Better Name: Battle of the Couple

Person 2

		B	D
P1	B	4,2	0,0
	D	0,0	2,4

*__Anti-Coordination__: Where the goals are in direct conflict

Example: Cold War

	W	P
W	[illegible]	10,-10
P	-10,10	0,0

-Pure Coordination
-Bias Coordination
-Anti-Coordination

Zero-Sum Game (Def. 6.7): A **zero sum game** is a game where the social welfare of every outcome is zero.

Example: Rock-Paper Scissors

Fixed Sum Game (Def. 6.10): A **fixed sum game** is a game where the social welfare of every outcome is fixed

Exam. 6.11

	H	T
H	3, 1	1, 3
T	1, 3	3, 1

In Example 6.11, the social welfare (i.e. the payoff sum) of every outcome is 4, so the game is fixed sum.

Mixed Motive Game (Def. 6.12): a **mixed motive game** is a game which are neither coordination games nor zero sum games.
Equivalently, there exists ZERO OR ONE pure NE AND the social welfare of some outcome is nonzero
Prisoner's Dilemma

Matching Coins

		2	
1		H	T
	H	1,-1	-1,1
	T	-1,1	1,-1

Sol, welfar is zero for all outcomes

so zero-sum

Chicken

2 NE
(St,Sw)
(Sw,St

		Car 2	
Car 1		Sw	St
	Sw	0,0	-1,1
	St	1,-1	-99,-99

Chicken is an anti-coordination game because it's a coordination game where both of the objectives completely conflict.

Two main types games:

Coordination Games

Zero Sum Games

"trivial" game

	A	B
A	0,0	0,0
B	0,0	0,0

all outcomes are NE

all outcomes add to zero (*)

Another example:

Example 6.11 is also an example of a fixed motive game since it has fixed sum 4 (not 0) and has no NE

6/19 Lecture

Announcements:

Thursday's Office Hours: About to change
 Juneteenth
What to do with Game Theory Homework 5: I'll grade it early if you turn it in early
Exam 1 grades have been released (I'll upload a file with corrections shortly after class)

Questions for Homework 2-3 OR Past Material:

Previously:

Class 6: Types of games
*Coordination Games
  *Pure Coordination
  *Bias Coordination
  *Anti-Coordination

*Zero-Sum/fixed-sum games
*Mixed Motive Games

Class 7: Dominated Strategies, Rational Outcome, and IESDS

Dominated Strategies (Def. 7.1): A **strictly dominated strategy** is a strategy which is strictly dominated by another strategy

	X	Y
A	1,4	2,4
B	-2,0	3,9
C	-1,0	0,2

$A > C$, $Y \geq X$ but not $Y > X$

So C is a *strictly dominated* by A
Note that A *strictly dominates* C

Rational Person (Def. 7.3): A **rational person** is a person who always acts to maximize their own self interest in a rational mathematical way.

Personal Definition: A **rational outcome** is a outcome which a rational person will always choose

NOTE:
1. If a rational outcome exists, then it's ALWAYS the unique NE.
2. If there's a single NE, that doesn't mean it's a rational outcome.

Theorem 7.4:
A rational player will
*always play a strictly dominant strategy, if they have one.
*never play a strictly dominated strategy.

	X	Y
A	0,1	4,2
B	-2,	1,0

Y > X

A > B

(Y, A) is the rational outcome

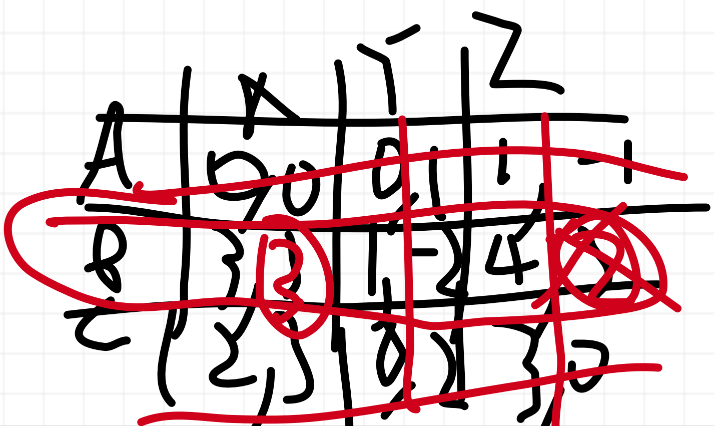
Example 7.6

	X	Y	Z
A	0,-1	2,1	1,2
B	1,6	4,1	3,4
C	1,-2	0,1	0,2

$B > A,\ B > C,\ Z > Y,\ Z > X$

(B, Z) is the rational outcome

Example 78



$B > A$ and $B > C$

(B, Y) is the only rational outcome

IESDS (Def. 7.9): The **Iterative Elimination of Strictly Dominated Strategies (IESDS)** algorithm:

1. Identify strictly dominated strategy (by either player)
2. Eliminate the Strictly dominated strategy and draw a new table
3. Repeat back to step 1. and do that repitition until there are no more strictly dominated strategies

NOTE:

1. If you do IESDS and it ends at a 1x1 table, then you have a rational outcome (also NE)
2. IESDS doesn't always end at a rational outcome

Prisoner's Dilemma

	L	B
L	-1, -1	-3, 0
B	0, -3	-2, -2

L ~~B~~

B > L f(

P2

	B
L	~~-3, 0~~
B	-2, -2

B > L fo

P1

	B
B	-2, -2

(B, B) is a rational outcome for the prisoner's dilemma

Ex. 7.10

	X	Y	Z
A	1,2	2,1	5,0
B	2,4	4,5	1,6
C	4,0	5,3	3,2

C > B

Y > Z

	X	Y	Z
A	1,2	2,1	5,0
C	4,0	5,3	3,2

Y > Z

C > A

	X	Z
A	1,2	5,0
C	4,0	3,2

	X	Y
C	4,0	3,2

rational outcome is (C, Y)

6/22 Lecture

Announcements:

-Homework 1 grades are out
-Homework 2 grades later this afternoon
-Office Hours OFFICIALLY: Mon., Wed.: 2pm-3pm OR by appointment
-Class 9 Not too important

Questions for Homework 3-4 (and past material):

Plan for this Week:

TODAY: IESDS Review, Probability and Expected Value (Class 10) (HW 3 due)
Tomorrow (6/23): Mixed Strategy NE (Class 11-13) (HW 4 due)
Wed. (6/24): Nash's Theorem, Review (Office Hours 2pm-3pm) (HW 5 due)
Thurs. (6/25): Exam
Fri. (6/26): Start Graph Theory

Previously:

Class 7-8
*rational person, rational outcome
*IESDS algorithm

Class 10: Probability and Expected Value

Sample Spaces and Events (Def. 10.1):
*The **sample space** the set of all possible outcomes. We usually this set by S
*An **event** is any subset of the sample space

Ex. 10.2: Fipping a coin; the sample space for a coin flip $\{H, T\}$
$\{H\}, \{T\}, \{H, T\}$ are events

Ex. 10.3: Roling 6 sided die; the sample space for rolling a six sided die $\{1, 2, 3, 4, 6\}$
$\{n\}, 1 \leq n \leq 6, \{2, 3, 6\}$
$\{even\}\ \{2, 4, 6\}$

Ex. 10.4: Flipping a coin twice; $\{HH,\ HT,\ TH,\ TT\}$

-Probability of an event (Rehash Thm. 10.7 of the notes)
-A **probability space** is a sample space S with an assigned number to *each outcome*
-The **probability of an event** E, denoted $P(E)$, is the SUM of the assigned numbers of outcomes, which follow the following rules:
1. for each outcome w $0\ \leq P(\{w\})\ \leq 1$
2. All outcomes must sum up to 1; in other words $P(S) = 1$

Ex. 10.8: A fair coin, when one flips a coin, each outcome $\{H,\ T\}$ is *equally likely, so* we assign the number $1/2$ to both H and T.
$P(\{H\}) = 1/2$ $P(\{T\}) = 1/2$, $P(S) = 1$

Ex. 10.10: Rolling a fair 6-sided die. each outcome of $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ is *equally likely*, so we assign the number $1/6$ to all the outcomes (numbers 1-6).
$P(\{n\}) = 1/6,$
$P(\{1,\ 6\}) = 1/6 + 1/6 = 1/3,$
$P(\{even\}) = P(\{2,\ 4,\ 6\}) = 1/6 + 1/6 + 1/6 = 1/2.$

Expected Value:

Motivation: There are more NE's than just pure strategy (what we talked about before); there are mixed strategy NE, and we formulate payoff in terms of <u>expected payoff</u>.

The **payoff** of an event is an number to that event

(think of gambling)

Ex. 10.12:
Flip a fair coin $S = \{H,\ T\}$
We assign payoff values for the event $\{H\}$ and the event $\{T\}$ in the following way:
If $\{H\}$, we get the payoff +\$2, and if it lands T, then we $-\$1$

Expected Value/Payoff (Def. 10.14): An **expected value** is the weighted average of the payoffs of the events, weighted by the probability that they occur.

6/23 Lecture

Announcements:

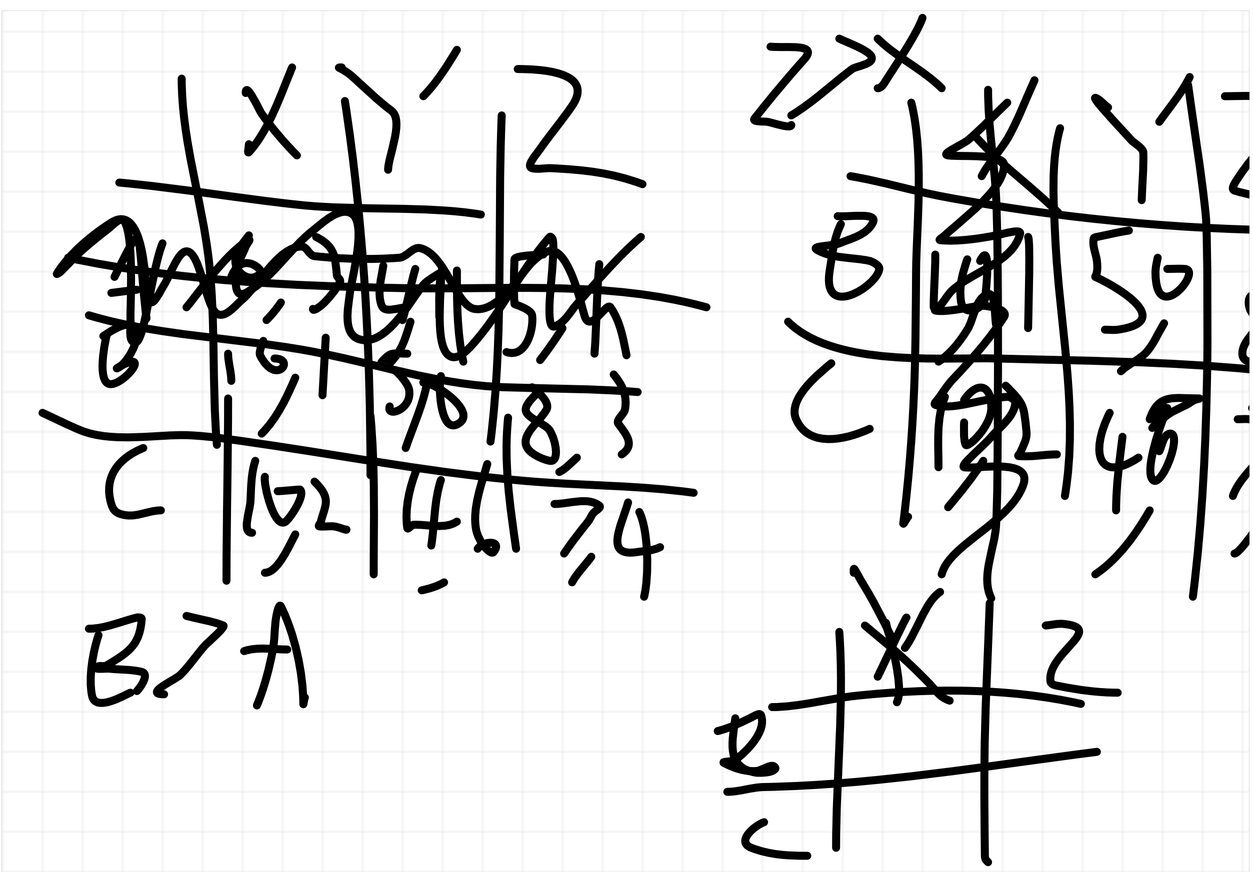
-Homework 2 grades, Homework 1-2 solutions are out
-Homework 3 grades this afternoon, Homework 3 solution will be out
-Homework 4 and early HW 5 grades this afternoon (HW 4 solution uploaded then)

Questions for Homework 4-5 (and past material):

Homework 4
Q3 part b) E={exactly one is tails}

$\{HHT,\ HTH,\ THH\}$
$1/8$
$P(\{\textit{exactly one tails}\}) = P(\{HHT,\ HTH,\ THH\}) = 1/8+1/8+1/8 = 3/8$

Homework 5 is due on WEDNESDAY June 24 11:59pm
Based on Classes 11-13



Previously:

-We talked about **sample spaces**, i.e., a set of outcomes, and **events**, i.e. subsets of those outcomes.
-We defined **probability space**, i.e. weights on the outcomes summing up to one, measures the liklihood that they would occur.
-We talked about finding the **probability of an event** through <u>summin</u>g up all the oucomes in the event

Expected Value:

Motivation: There are more NE's than just pure strategy (what we talked about before); there are mixed strategy NE, and we formulate payoff in terms of <u>expected payoff</u>.

The **payoff** of an event is an number to that event

(think of gambling)

Ex. 10.12:
Flip a fair coin $S = \{H, T\}$
We assign payoff values for the event $\{H\}$ and the event $\{T\}$ in the following way:
If $\{H\}$, we get the payoff +$2, and if it lands T, then we $-\$1$

Expected Value/Payoff (Def. 10.14): An **expected value** is the weighted average of the payoffs of the events, weighted by the probability that they occur.

For a sample space $S = \{w_1, w_2, \ldots, w_n\}$ and a payoff assignment scheme $\{x_1, x_2 \ldots x_n\}$

outcome	payoff
w_1	x_1
w_2	x_2
$\vdots$	$\vdots$
W_n	X_n

The expected value is computed as follows:
$expected\ value = x_1 \cdot P(w_1) + x_2 \cdot P(w_2) + \ \cdots \ + x_2 \cdot P(w_n)$

Ex. 10.12:
Flip a fair coin $S = \{H,\ T\}$
We assign payoff values for the event $\{H\}$ and the event $\{T\}$ in the following way:
If $\{H\}$, we get the payoff +$2, and if it lands T, then we $-\$1$

Outcome	payoff
H	\$2
T	-\$1

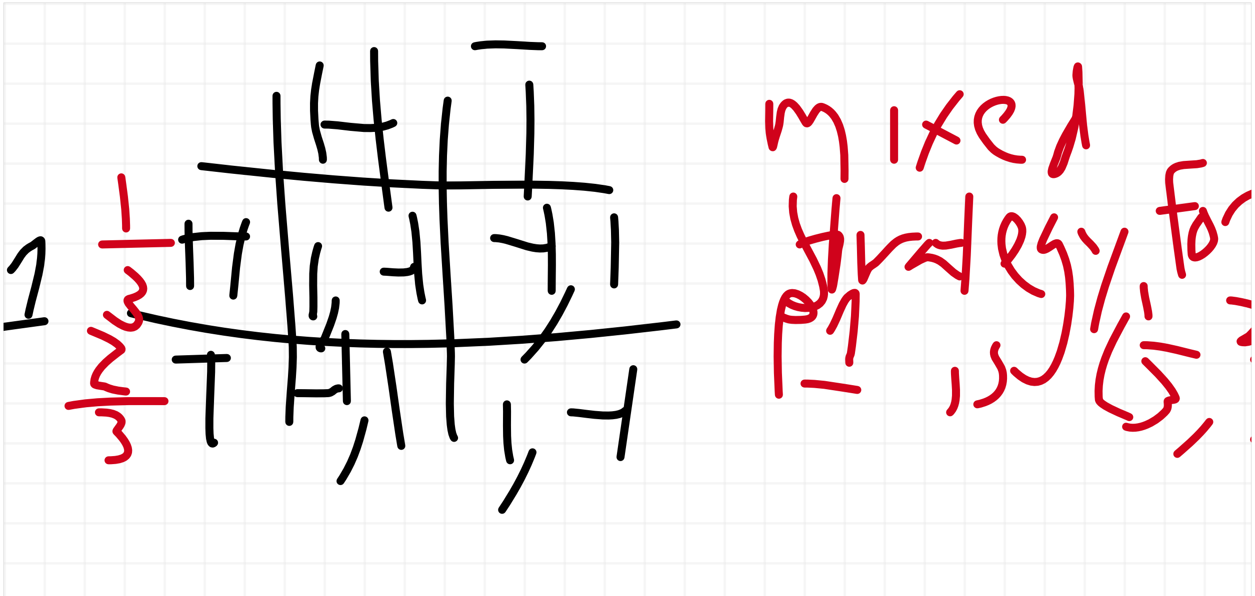
$expected\ value = 2P(H) + (-1)P(T) = 2 \cdot 1/2 + (-1) \cdot 1/2 = 1 - 1/2 = 1/2$

Class 11: Mixed Strategies and Expected Payoff

Mixed Strategy (Def. 11.1): The available named strategies are **pure strategies**
A **mixed strategies** is an assignement of probability to each pure strategy

Ex. 11.2 (Matching Coins)

	H	T
H	1, 1	-1, 1
T	-1, 1	1, -1



Expected payoff (Def. 11.3): X is a pure strategy for player 2, and player 1 has a mixed strategy $(p,\ 1-p)$, then we let $E_1(X)$ denote the expected value of the payoff for player 1 if player 2 picks X

$E_1(H) = 1/3 \cdot 1 + 2/3 \cdot -1 = 1/3 - 2/3 = -1/3$
$E_1(T) = 1/3 \cdot (-1) + 2/3 \cdot 1 = -1/3 + 2/3 = 1/3$

Class 12-13: Indifference Strategies and Mixed NE

Class 12 deals with "indifference strategies" which are the main pillar of <u>mixed strategy NE</u>.

Indifference Strategies (Def. 12.4): A mixed strategy is an **indifference strategy** when the payoff of your opponent's strategies are equal.
$E_1(H) = E_1(T)$

is $(1/3,\ 2/3)$ an indifference strategy? No.
$E_1(H)\ \neq E_1(T)$

Next Time: Define formally mixed strategy NE and Nash's Theorem.

6/24 Lecture

Announcements:

-Homework 3 grades are out
-Another chance to turn in Homeworks 3, 4 (5 points off), by 9pm tonight
-Post solutions to Homework 3, 4 will be posted 9pm
-Exam 2 TOMORROW: Classes 1-13 (probably won't include class 9)
-Office hours THIS AFTERNOON 2pm-3pm (by appointment)
-Homework 5 is due tonight 11:59

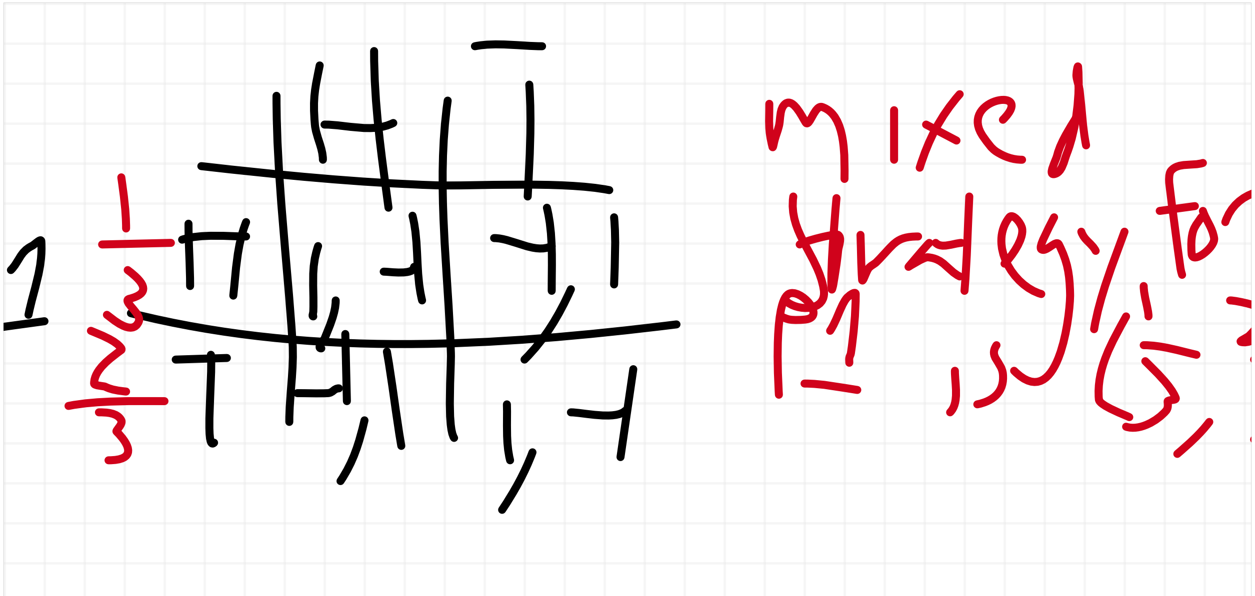
Questions for Homework 3-5 (and past material):

Previously:

-We talked about **mixed strategies**, which is a probability assignement of all the strategies (eg (1/3, 2/3), (1/3, 1/2, 1/6))
-We talked about **expected payoff**, i.e. **expected value** of a given player, give a mixed strategy of the other player
-We talked about **indifference strategies**, which is a mixed strategy, for a player that yields the same expected payoff for the opponent.

NOTE: ERRORS IN NOTATION IN THE LAST LECTURE

Expected payoff (Def. 11.3): X is a pure strategy for player 2, and player 1 has a mixed strategy $(p,\ 1-p)$, then we let $E_2(X)$ denote the expected value of the payoff for player 1 if player 2 picks X



$E_2(H) = 1/3 \cdot 1 + 2/3 \cdot -1 = 1/3 - 2/3 = -1/3$
$E_2(T) = 1/3 \cdot (-1) + 2/3 \cdot 1 = -1/3 + 2/3 = 1/3$

Class 12 deals with "indifference strategies" which are the main pillar of <u>mixed strategy NE</u>.

Indifference Strategies (Def. 12.4): A mixed strategy is an **indifference strategy** when the payoff of your opponent's strategies are equal.
$E_2(H) = E_2(T)$

is $(1/3,\ 2/3)$ an indifference strategy? No.
$E_2(H) \neq E_2(T)$

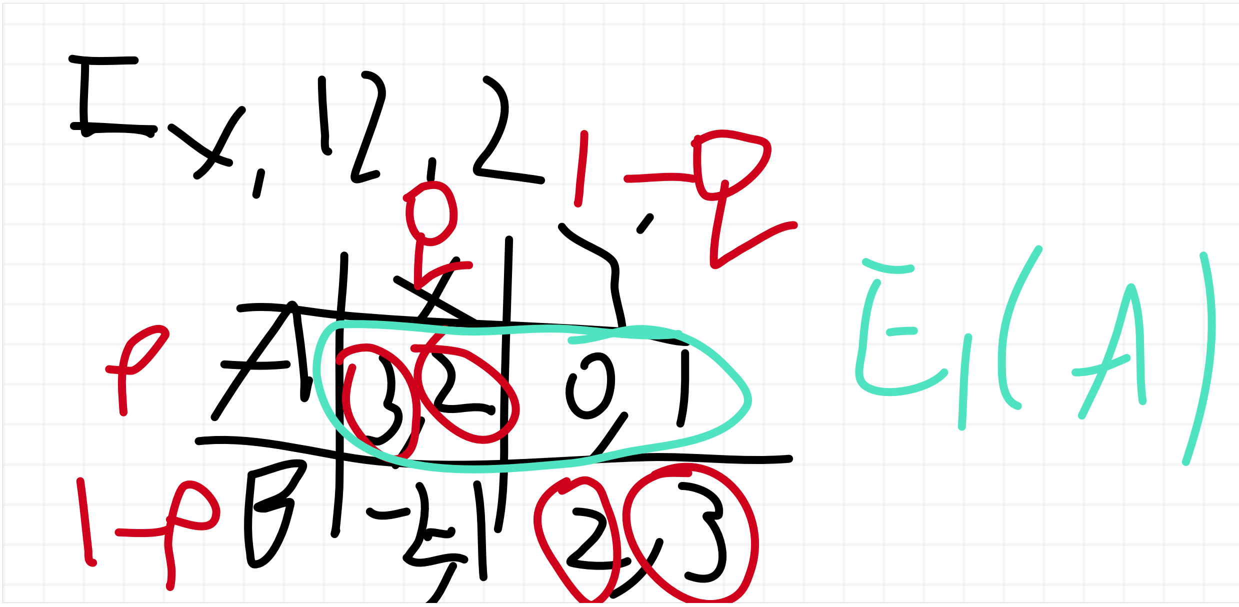
This Time: Define formally mixed strategy NE and Nash's Theorem.

Class 12-13 (Cont.): Mixed Strategy NE and Nash's Theorem

Pure and Mixed NE (Def. 12.4):
*call the old style NE **pure strategy NE**
*A **mixed strategy NE** is a pair of indifference strategies, player 1 and player 2 choose indifference strategies, i.e. $E_2(X) = E_2(Y)$ $E_1(A) = E_1(B)$

NOTE: There MAY NOT be a mixed strategy NE (just like how there may not be a pure strategy NE)

Nash's Theorem (Theorem 13.6): <u>Every game</u> has at least one NE, either pure, mixed, or both



Pure NE: (A, X), (B, Y)

Compute mixed Strategy NE:
First step is: Know the expected payoff indifference strategy condition for mixed NE
$E_2(X) = E_2(Y)$, $E_1(A) = E_1(B)$

Second step is: Compute (in terms p and q) what the expected payoffs
$E_2(X)$, $E_2(Y)$, $E_1(A)$, $E_1(B)$ are:

$E_2(X) = 2p + 1(1-p)$
$E_2(Y) = 1p + 3(1-p)$
$E_1(A) = 3q + \ 0(1-q)$
$E_1(B) = (-2)q + 2(1-q)$

Midstep: Simplify the computation

$E_2(X) = 3p - 1$
$E_2(Y) = 3 - 2p$
$E_1(A) = 3q$
$E_1(B) = 2 - 4q$

Final Step: Solve for p and q:
Plug in the expected into these equations:

$E_2(X) = E_2(Y)$, $E_1(A) = E_1(B)$

$3p - 1 = 3 - 2p$
$5p = 4$
$p = 4/5$
$(4/5,\ 1/5)$

$3q = 2 - 4q$
$7q = 2$
$q = 2/7$

Review: