M211 Recitation Notes Ch. 3

3.1-3.2 Exposition

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Constant Rule 1:

$$\frac{d}{dx}(c) = 0$$

Constant Rule 2:

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Sum/Difference Rule:

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$
$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

Power Rule:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Derivative of the Natural Exponential Function:

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Fun fact: e^x is the unique function (up to a constant multiple) such that $f'(x) = f(x)$. (we'll learn that later during implicit differentiation)

Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

3.1-3.2 Homework Questions

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Some Webwork due on Monday.

$x^2 + 6$ point at $(0, -2)$

some a where a tangent line would hit the point $(0, -2)$

To do that, let's find the derivative $f'(x)$ of $f(x) = x^2 + 6$

We use the sum rule then the power rule to get

$$f'(x) = \frac{d}{dx}\left(x^2 + 6\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(6) = 2x + 0 = 2x$$

We need to find the tangent line (in general) of a point a. Note that the line we're looking for contains the point $(x_1, y_1) = (a, f(a)) = \left(a, a^2 + 6\right)$ and it has slope $m = 2a$. So using the point-slope formula for a point (x_1, y_1) and slope m, which is

$$y - y_1 = m(x - x_1)$$

the equation of the line is

$$y - \left(a^2 + 6\right) = 2a(x - a)$$

Now we plug in the point $(x, y) = (0, -2)$ that the line contains and solve for a as follows:

$$\begin{aligned} -2 - \left(a^2 + 6\right) &= 2a(0 - a) \\ -2 - a^2 - 6 &= 2a(-a) \\ -8 - a^2 &= -2a^2 \end{aligned}$$

$$
\begin{array}{ll}
+a^2 & +a^2 \\
-8 & = -a^2 \\
\div -1 & \div -1 \\
8 & = a^2
\end{array}
$$

$$a^2 = 8 \Longrightarrow a = \pm\sqrt{8} = \pm 2\sqrt{2}$$

NOTE: In general, you might have to factor out a polynomial to solve this kind of problem.

<u>WW12Sec3.2 Problem 2.</u>

Differentiate $R(t) = \left(3t + e^t\right)\left(2 - \sqrt{t}\right)$.

Use the product rule as follows $f(t) = 3t + e^t$, $g(t) = 2 - \sqrt{t}$:

$$\frac{d}{dt}R(t) = \left[\frac{d}{dt}\left(3t + e^t\right)\right]\left(2 - \sqrt{t}\right) + \left(3t + e^t\right)\left[\frac{d}{dt}\left(2 - \sqrt{t}\right)\right]$$

Next, we want to find:

$$\frac{d}{dt}\left(3t + e^t\right) = \frac{d}{dt}(3t) + \frac{d}{dt}\left(e^t\right) = 3 + e^t$$

$$\frac{d}{dt}(3t) = 3\frac{d}{dt}(t) = 3 \cdot 1 = 3, \quad \frac{d}{dt}\left(e^t\right) = e^t$$

$$\frac{d}{dt}\left(2 - \sqrt{t}\right) = \frac{d}{dt}(2) - \frac{d}{dt}\left(\sqrt{t}\right) = 0 - \frac{1}{2}t^{-1/2} = -\frac{1}{2}t^{-1/2}$$

$$\frac{d}{dt}(2) = 0, \ \frac{d}{dt}\left(\sqrt{t}\right) = \frac{d}{dt}\left(t^{1/2}\right) = \frac{1}{2}t^{1/2-1} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}},$$

and then we plug it all in and multiply everything together

$$\frac{d}{dt}R(t) = \left(3 + e^t\right)\left(2 - \sqrt{t}\right) + \left(3t + e^t\right)\left(-\frac{1}{2}t^{-1/2}\right).$$

3.3-3.4 Exposition

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$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1, \ \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

These limits are important and lead us to knowing the derivatives of Trigonometric functions

$$\frac{d}{dx}\sin x = \cos x \quad \frac{d}{dx}\cos x = -\sin x \quad \frac{d}{dx}\tan x = \sec^2 x \quad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \quad \frac{d}{dx}\csc x = -\csc x \cot x$$

The Chain Rule:

$$F(x) = (f \circ g)(x) = f(g(x)$$

$$\begin{aligned} F'(x) &= (f' \circ g)(x) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

Another way to write it in Lebiniz notation If we have $y(u(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example.

$$\frac{d}{dx}\left(e^{ax}\right) = e^{ax} \cdot a$$

The chain rule can help us figure out a lot of other derivatives:

Example.

Note that $b^x = e^{\ln b \cdot x}$

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{\ln b \cdot x}\right) = e^{\ln b \cdot x} \cdot \ln b = b^x \cdot \ln b = \ln b \cdot b^x.$$

chain rule is applied with outter function e^x and inner function $u = \ln b \cdot x$

$$\frac{d}{du}e^u = e^u$$
$$\frac{du}{dx} = \ln b$$

3.3-3.4 Homework Questions

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WW15Sec3.4 Problem 15.

(a)

$s(t) = A\cos(\omega t + d)$

$v(t) = s'(t)$

outer function:
$y = A\cos(u)$
$y' = -A\sin(u)$

inner function:
$u = \omega t + d$
$u' = \omega$

$s'(t) = y' \cdot u' = -A\sin(u)\cdot\omega = -A\sin(\omega t + d)\cdot\omega$

(b) What is the smallest positive value of t for which the velocity is 0? Assume that w and d are positive. (i.e. $w > 0, 0 < d \leq \pi$

$0 = -A\sin(\omega t + d)\cdot\omega$ when $\omega t + d = a = 0, \pi, 2\pi, 3\pi, \ldots$

$$\begin{aligned}
\omega t + d &= a \\
-d \quad & \quad -d \\
\omega t \quad &= a - d \\
\div\,\omega \quad & \quad \div\,\omega \\
t \quad &= \frac{\pi - d}{\omega}
\end{aligned}$$

WW15Sec3.4 Problem 10.

Find an equation of the tangent line to the curve
$y = \sin(7x) + \cos(4x)$
at $(\pi/6, y(\pi/6))$.

First, let's find the derivative:
$$\frac{dy}{dx} = \frac{d}{dx}\sin(7x) + \frac{d}{dx}\cos(4x) = 7\cos(7x) - 4\sin(4x).$$

Next, we find the linear equation using the point slope formula for the point (x_0, y_0) and slope m

$$y - y_0 = m(x - x_0)$$

So note that

$$(x_0, y_0) = (\pi/6,\ \sin(7\pi/6) + \cos(4\pi/6))$$

$$m = 7\cos(7\pi/6) - 4\sin(4\pi/6)$$

So plugging everything in, we get

$$y - (\sin(7\pi/6) + \cos(4\pi/6)) = (7\cos(7\pi/6) - 4\sin(4\pi/6))(x - \pi/6)$$

$$y = (7\cos(7\pi/6) - 4\sin(4\pi/6))(x - \pi/6) + \sin(7\pi/6) + \cos(4\pi/6)$$

3.5-3.8 Exposition

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Derivatives of Logorithmic functions

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}(\ln |x|) = \frac{1}{x}$$
$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}(\log_b |x|)\frac{1}{x \ln b}$$

Opens the door to logorithmic differentiation. If we have a differentiable function $g(x)$, using the chain rule, we can set $u = g(x)$ and get

$$\frac{d}{dx}(\ln g(x)) = \frac{d}{dx}(\ln u) = \frac{d}{du}\ln(u)\cdot\frac{d}{dx}u = \frac{1}{u}\cdot g'(x) = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$$

When you have a function $g(x)$ that you want to use logorithmic differentiation on, what are the steps to do so?

Step 1. Take the natural log of $g(x)$ and then set $\ln y = \ln g(x)$
Step 2. Differentiate $\ln y$ with respect to x, setting us up for implicit differentiation with respect to y
Step 3. Solve for y'

Example 8 (page 222).

differentiate $y = \ln x^{\sqrt{x}}$ use logorithmic diffentiation, using each of the steps as follows

Step 1. We set up the problem of

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln x^{\sqrt{x}}$$

Step 2. Implicitly differntiate

$$\frac{d}{dx}\ln y = \frac{y'}{y}$$

$$\frac{d}{dx}\ln x^{\sqrt{x}} = \frac{d}{dx}\sqrt{x}\ln x = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}$$

So we end up with

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}$$

Step 3. We solve for y' as follows

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}$$
$\times y \qquad \times y$

$$y' = y\left(\frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}\right) \quad y = x^{\sqrt{x}}$$

$$y' = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}\right)$$

Exponential growth and decay

What is the law of natural growth and decay?

$$\frac{dy}{dt} = ky \quad k > 0$$

This law can be equivalently formulated by the solution of the differential equation.

What is the solution to the above differential equation?

The solution is all functions of the form

$$y(t) = y(0)e^{kt}$$

3.5-3.8 Homework Questions

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WW18Sec3.8: Problem 8. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 190 degrees Fahrenheit when freshly poured, and 1 minutes later has cooled to 171 degrees in a room at 68 degrees, determine when the coffee reaches a temperature of 146 degrees.
The coffee will reach a temperature of 146 degrees in t' minutes?
equation editor
Equation Editor

minutes.

Newtown's law is just another way of formulation the law of natural growth/decay (in this case decay)

$$\frac{dT}{dt} = k(T - T_s)$$

If we set set $y(t) = T(t) - T_s$, then we can work with the original set up

$$\frac{dy}{dt} = ky \Longleftrightarrow y(t) = y(0)e^{kt}$$

$T_s = 68$

$T(0) = 190$
$T(1) = 171$

$y(0) = 190 - 68 = 122$
$y(1) = 171 - 68 = 103$

Using that we want to solve for k

$103 = y(1) = y(0)e^{k \cdot 1} = 122e^k$

$103 = 122e^k$
$\div 122 \quad \div 122$
$\frac{103}{122} = e^k$

$$\ln\left(\frac{103}{122}\right) = \ln e^k = k$$

$T(t') = y(t') + 68 = 122exp(\ln(103/122)t') + 68 = 122(103/122)^{t'} + 68$

solve for t' setting $T(t') = 146$

$146 = 122(103/122)^{t'} + 68$
$-68 \qquad\qquad\qquad\qquad -68$
$78 = 122(103/122)^{t'}$

$$\div 122 \qquad \div 122$$

$$\frac{78}{122} = \left(\frac{103}{122}\right)^{t'}$$

$$\ln\frac{78}{122} = \ln\left(\frac{103}{122}\right)^{t'} = t'\ln\left(\frac{103}{122}\right)$$

$$\ln\frac{78}{122} = t'\ln\left(\frac{103}{122}\right)$$

$$\div\ln\left(\frac{103}{122}\right) \quad \div\ln\left(\frac{103}{122}\right)$$

$$\frac{\ln\frac{78}{122}}{\ln\left(\frac{103}{122}\right)} = t'$$

3.9-3.10 Exposition

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3.9-3.10 Homework Questions

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