M211 Recitation Notes Ch. 5 and Ch.6

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5.3-5.5 Expostion

Question 1. What are the types of integrals?

Definite: The area under the curve between an interval

$$\int_a^b f(x)dx$$

Indefinite: Exactly the "antiderivative" (not between two intervals)

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is one antiderivative on $f(x)$, so an indefinite integral is unique up to a constant.

Question 2. Name the different versions of the fundamental theorem of calculus (FTC)

version 1. If f is continuous on $[a, b]$, then the function

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable such that $g'(x) = f(x)$.

version 2. If f is continuous on $[a, b]$ then

$$\int_a^b f(x) = F(b) - F(a)$$

where F is the antiderivative of f.

Corollary (net change theorem). The net change between two points of a differentiable function f is equal the integral of the derivative between the two points

$$\int_a^b f'(x) = f(b) - f(a)$$

Question 3. What is the substitution rule?

For indefinite integrals. Think of it as the "chain rule" in reverse

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

For two functions $g(x)$ and $f(x)$ with antiderivatives $F(x)$ and $G(x)$, we find that definite integral of functions of the form $f(G(x)) \cdot g(x)$ is

$$\int f(G(x)) \cdot g(x)dx = F(G(x)) + C,$$

because

$$\frac{d}{dx}(F(G(x)) = F'(G(x)) \cdot G'(x) = f(G(x)) \cdot g(x)$$

To apply "the substitution rule", it's helpful to find $G(x)$ and set it equal to $u = G(x)$, which then gives us

$$\int f(G(x)) \cdot g(x)dx = \int f(u)du$$

For definite integrals. We can apply the fundamental theorem of calculus and get

$$\int_a^b f(G(x)) \cdot g(x)dx = F(G(b)) - F(G(a)) = \int_{G(a)}^{G(b)} f(u)du,$$

which gives us the following u-substitution formula for definite integrals that changes the upper and lower bounds from a and b to $G(a)$ and $G(b)$, respectively:

$$\int_a^b f(G(x)) \cdot g(x)dx = \int_{G(a)}^{G(b)} f(u)du.$$

6.1, 6.5 Expostion

Question 4. Say we have f and g and we want to find the area between the two curves. How do we do that?

In the given interval $[a, b]$ be apply the formula

$$\int_a^b |f(x) - g(x)| dx$$

Note that we may not be given the explicit bounds a, b, but we might be asked to find the area "between" two curves. In this situation, we identify the function that is above and the function that is below (so identify which arrangement of functions give us $g(x) \leq f(x)$), and then solve for the bounds by setting

$$f(x) = g(x),$$

and finding the solutions $a \leq b$, which are the bounds. Then we apply formula

$$\int_a^b f(x) - g(x) dx$$

to give us the area under the curve.

Question 5. What is the "average value" of an integral and what is the mean value theorem for integrals?

$$f_{av} = \frac{1}{b-a} \int_a^b f(x) dx$$

Note that this notion of "average value" makes sense, because the fundamental theorem of calculus (if f is continuous) tells us that f_{av} is the "average rate of change" of the antiderivative $F(x)$

$$f_{av} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{F(b) - F(a)}{b-a}$$

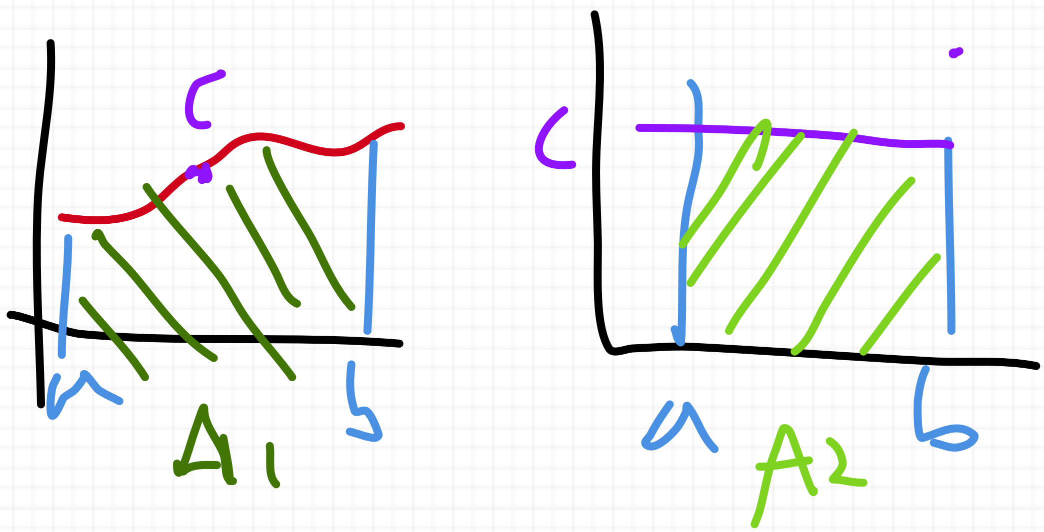
Where have we seen this before? The mean value theorem tells us that there exists a value $a < c < b$ such that

$$F'(c) = \frac{F(b) - F(a)}{b-a}$$

The mean value theorem for integrals tells us that there exists $a < c < b$ such that $f(c)$ is the average value of f

$$f(c) = f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\int_a^b f(x) = f(c)(b-a)$$



MVT for integrals tells us that there exist c s.t. $A_1 = A_2$

Homework Questions

Let's find

$$\int_1^2 \frac{1}{5+p} dp$$

using the substitution rule. What's a good "inner function"? $u = g(p) = 5 + p$ is a good inner-function. This function has derivative $g'(p) = 1$. Note that

$$\frac{1}{5+p} = g(p)^{-1} \cdot g'(p)$$

This gives us two ways to solve the problem. The first is to find the antiderivative of $\frac{1}{5+p}$ in terms of u, then plug back in p to plug in the upper bounds

$$\int \frac{1}{5+p} dp = \int g(p)^{-1} \cdot g'(p) dp = \int u^{-1} du = \ln u + C = \ln(5+p) + C$$

then we apply the FTC

$$\int_1^2 \frac{1}{5+p} dp = \ln(5+p)|_1^2 = [\ln(5+2) + C] - [\ln(5+1) + C] = \ln 7 - \ln 6$$

$$\int_1^2 \frac{1}{5+p} dp = \int_{5+1}^{5+2} u^{-1} du = \int_6^7 u^{-1} = \ln 7 - \ln 6$$

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$$\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta} d\theta$$

Note that we may have to use trig identities to find the function(s) that we apply the substitution rule to

$$\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta} d\theta = \int_0^{\pi/3} \frac{\sin\theta}{\sec^2\theta} d\theta + \int_0^{\pi/3} \frac{\sin\theta\tan^2\theta}{\sec^2\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \int_0^{\pi/3} \sin\theta\cos^2\theta d\theta + \int_0^{\pi/3} \frac{\sin\theta\sin^2\theta}{\cos^{-2}\theta\cos^2\theta} d\theta$$

$$= \int_0^{\pi/3} \sin\theta\cos^2\theta d\theta + \int_0^{\pi/3} \sin^3\theta d\theta$$

$$u = \cos\theta \;\; u' = \sin\theta$$

$$\int \sin\theta\cos^2\theta d\theta = \int u^2 du = \frac{u^3}{3} + C = \frac{\cos^3\theta}{3} + C$$

$$\int \sin^3\theta d\theta = \int \sin\theta\sin^2\theta = \int \sin\theta\left(1 - \cos^2\theta\right) d\theta = \int \sin\theta - \int \sin\theta\cos^2\theta$$

$$= \cos\theta - \frac{\cos^3\theta}{3} + C$$

$$\int_0^{\pi/3} \sin\theta\cos^2\theta d\theta + \int_0^{\pi/3} \sin^3\theta d\theta = \left[\frac{\cos^3\theta}{3} + \cos\theta - \frac{\cos^3\theta}{3} + C\right]|_0^{\pi/3}$$

$$= [\cos\theta]_0^{\pi/3} = \cos\pi/3 - 1 = 1/2 - 1 = -1/2$$