M211 Recitation Notes Ch. 4

4.1-4.2 Expostion

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Maximum and Minimum Values

Let f be a function on domain D

absolute maximum: value c of f on D so that $f(c) \geq f(x)$ for all $x \in D$.

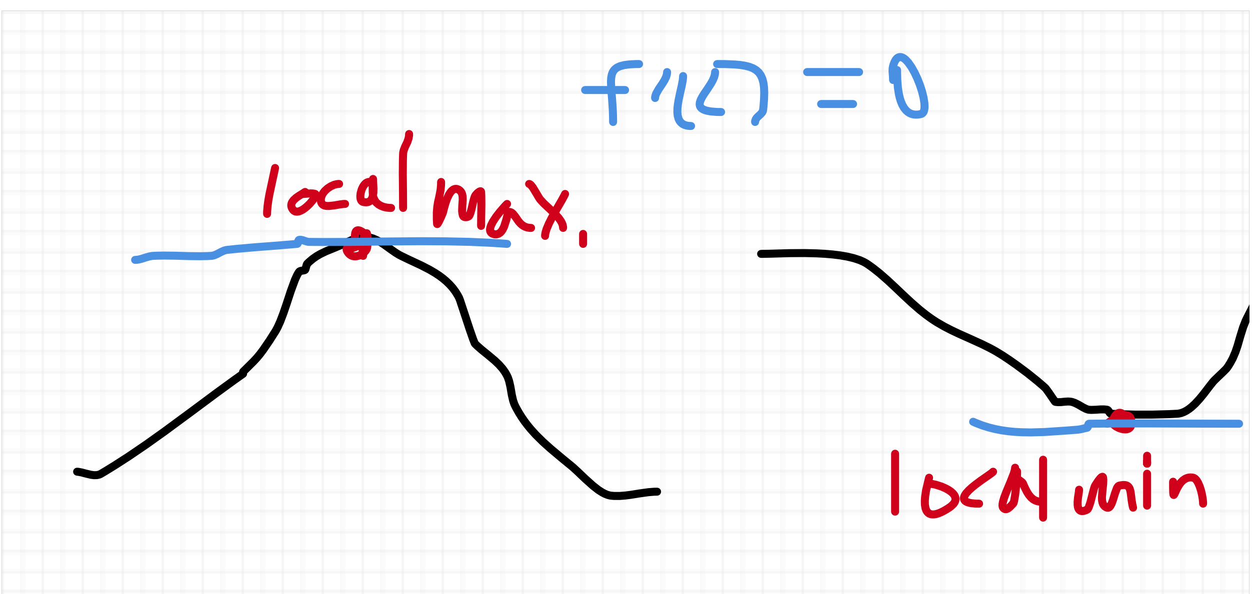
absolute minimum: value c of f on D so that $f(c) \leq f(x)$ for all $x \in D$.

local maximum: value c of f on D so that $f(c) \geq f(x)$ for all x "near" c.

local minimum: value c of f on D so that $f(c) \leq f(x)$ for all x "near" c.

By "near" c I mean that there exists some smaller domain D' of f so that c is the absolute min./max.

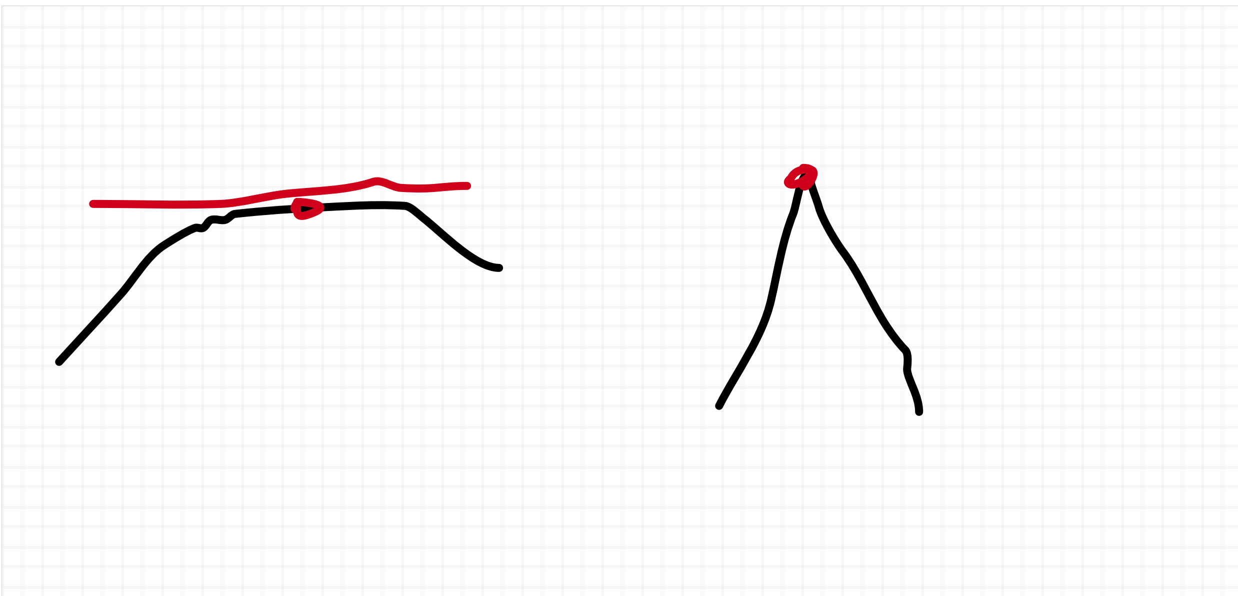
Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



NOTE: Fermat's theorem is a necessary condition for a local extreme point if the derivative exists, but not sufficient. Sometimes the derivative is 0 but it's not an absolute or even local max. or a min.

However, we do end up with a good way using the condition of Fermat's Theorem to find the absolute min. and max. values (and also the local ones later on in this course).

Definition. A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.



These sorts of points are important to finding the absolute min. and max.

We have the following two step process to do so (in a closed interval $[a, b]$):

Step 1. Find the values of f at the critical numbers of f in (a, b), as well as the endpoints a, b

First we find the undefined points. Next we set $f'(c) = 0$ and solve for c (NOTE: multiple solutions may exist)

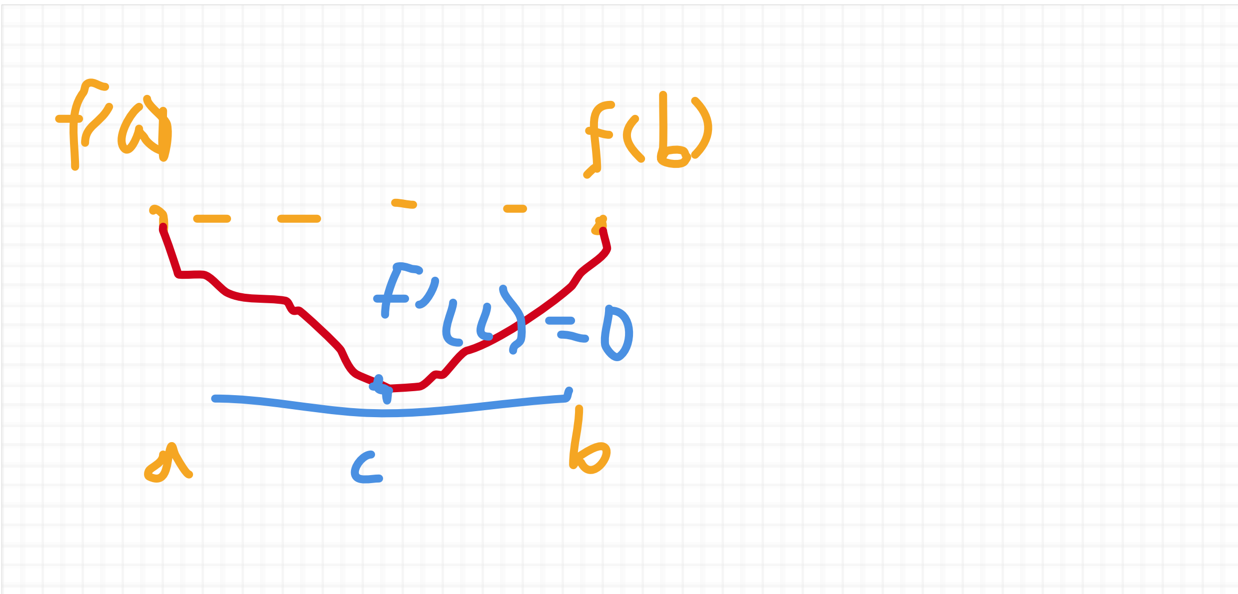
All these numbers are our candidates for the absolute min./max.

Step 2. Plug in all the candidates, the highest one is the absolute max. and the lowest one is the absolute min.

the Mean Value Theorem

The first theorem is an intuitive thoerem that tells us that between two values a and b such that $f(a) = f(b)$ we have a plateau point c (with derivative 0) *somewhere*.

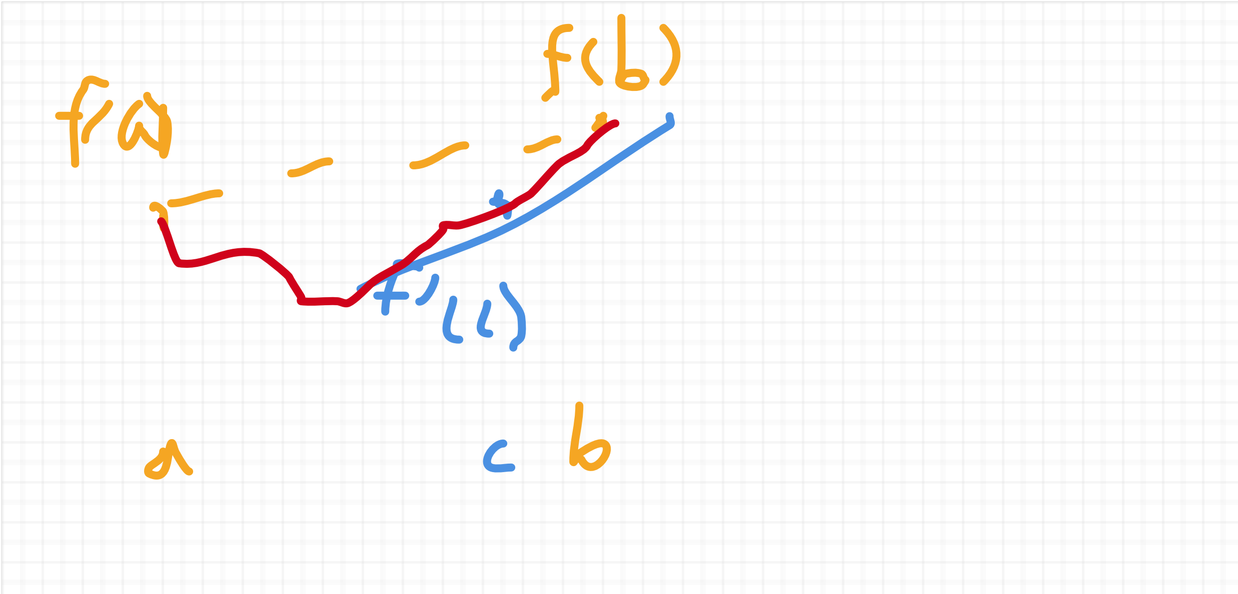
Rolle's Theorem. Let f be a function on $[a, b]$ such that $f(a) = f(b)$ and f has its derivative at (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = 0$.



This turns out to be a special case of a more general theorem known as the "mean value theorem":

Mean Value Theorem. Let f be a function on $[a, b]$ with a derivative on (a, b). Then there exists $c \in (a, b)$ where its derivative is the average rate of change of a and b, i.e., we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



4.1-4.2 Homework Questions

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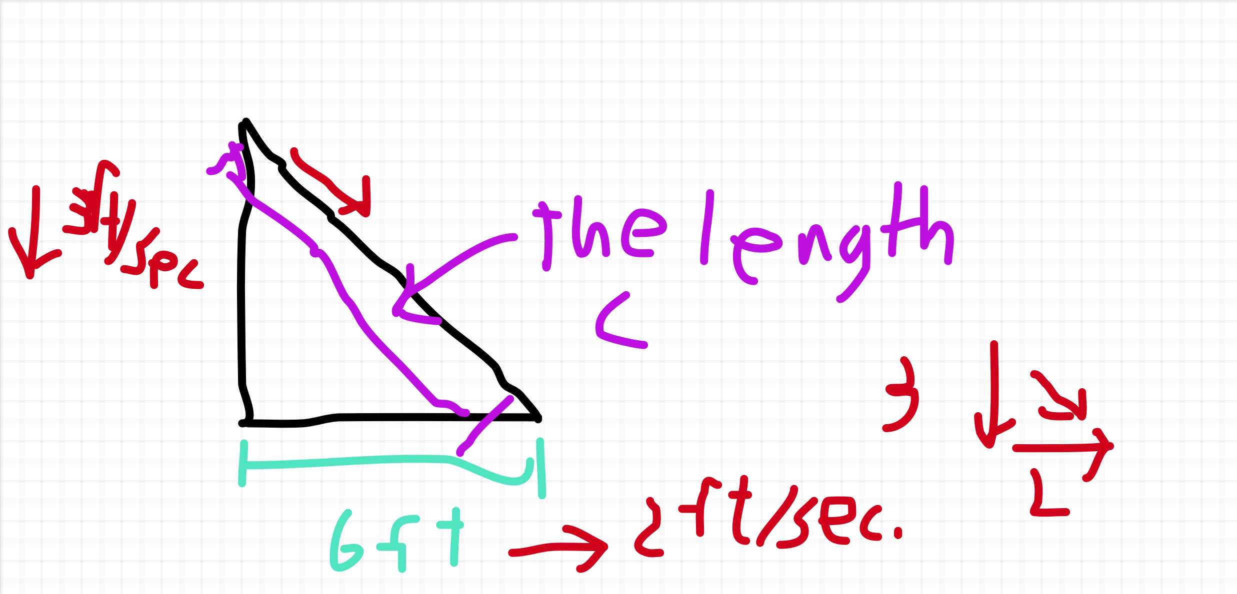
<u>Question 2 of Exam 2.</u>

Let y be the vertical location, let x be the horizontal location (away from the wall)

$$\frac{dy}{dt} = 3ft/sec. \quad \frac{dx}{dt} = 2ft/sec. \quad a = 6ft$$



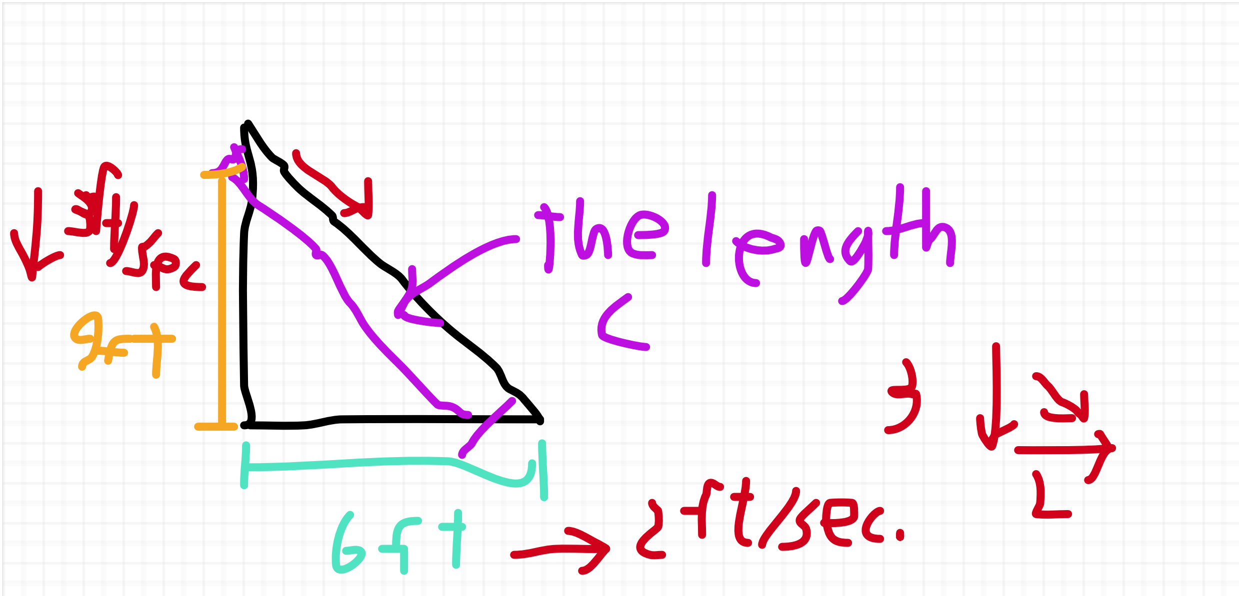
How long is the latter:

$c^2 = a^2 + b^2$

We want to find b using related rates.

We know that the derivative is related by the red triangle above similar to the other triangle, so

$$b = \frac{6}{2} \cdot 3ft = 9ft$$



$c = \sqrt{9^2 + 6^2} = \sqrt{117}$

4.3-4.4 Expostion

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First and Second Derivative Tests

When do we know if a function is Increasing/Decreasing?

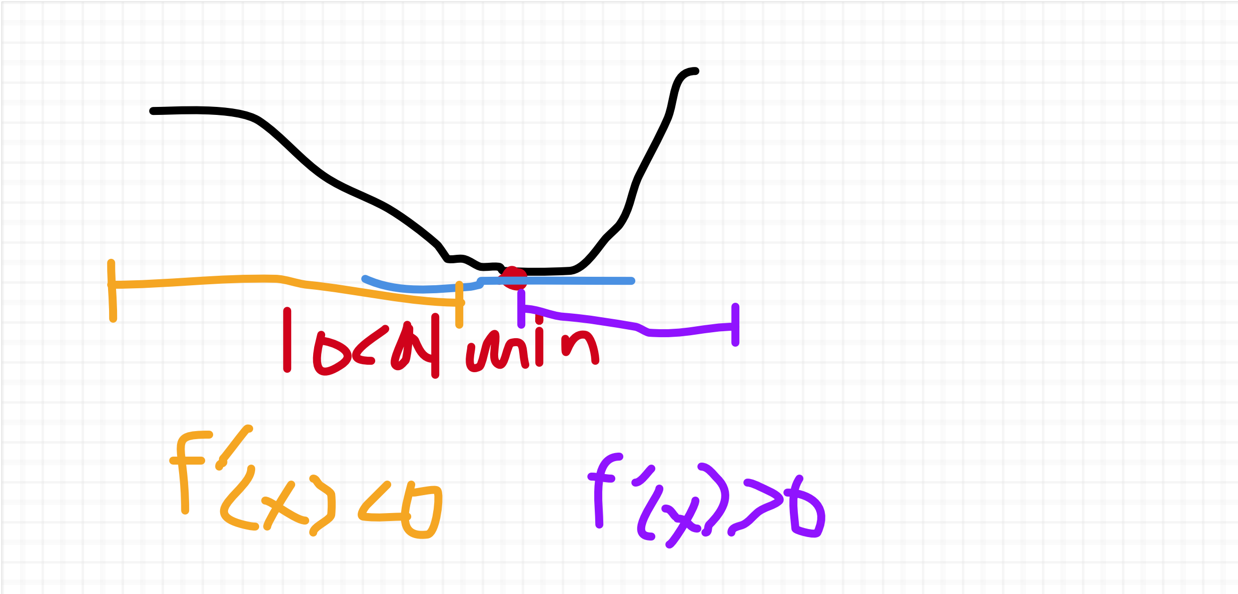
If $f'(x) > 0$, then the function is increasing, and if $f'(x) < 0$, then the function is decreasing.

How do we find local extreme values?

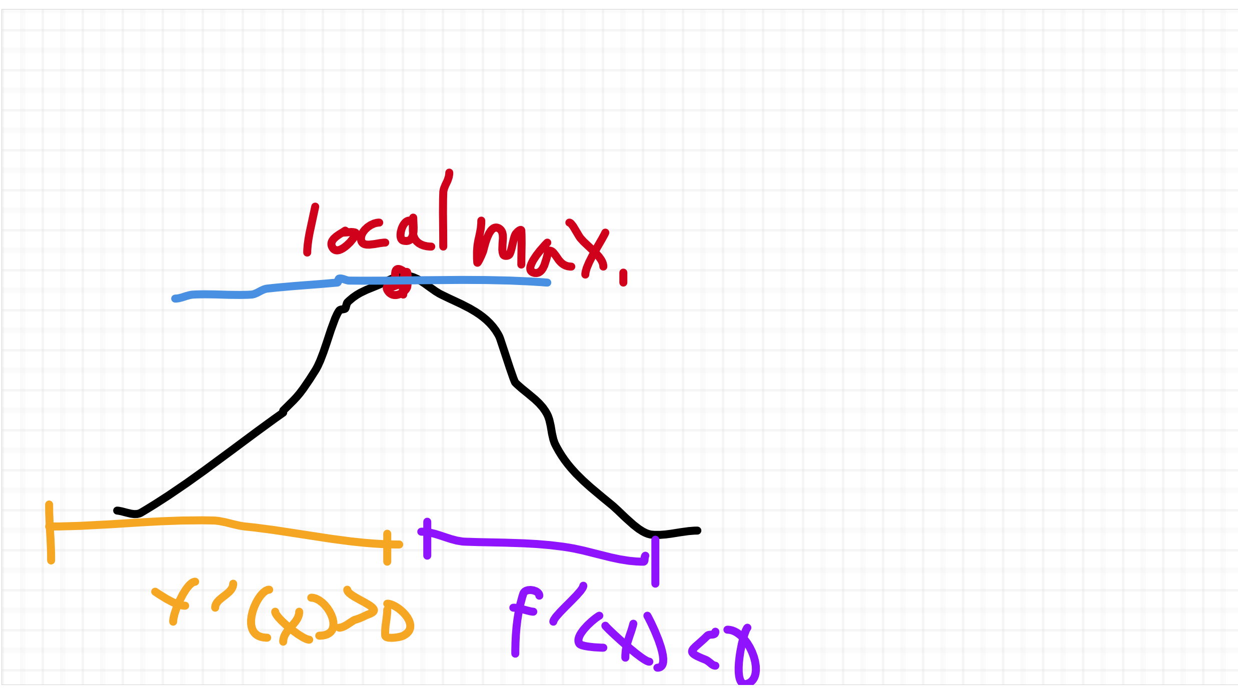
We do it through the first derivative test:

Given a criticial point c:

- If $f'(x)$ changes from negative at c to positive, then $f'(x)$ has a local min.



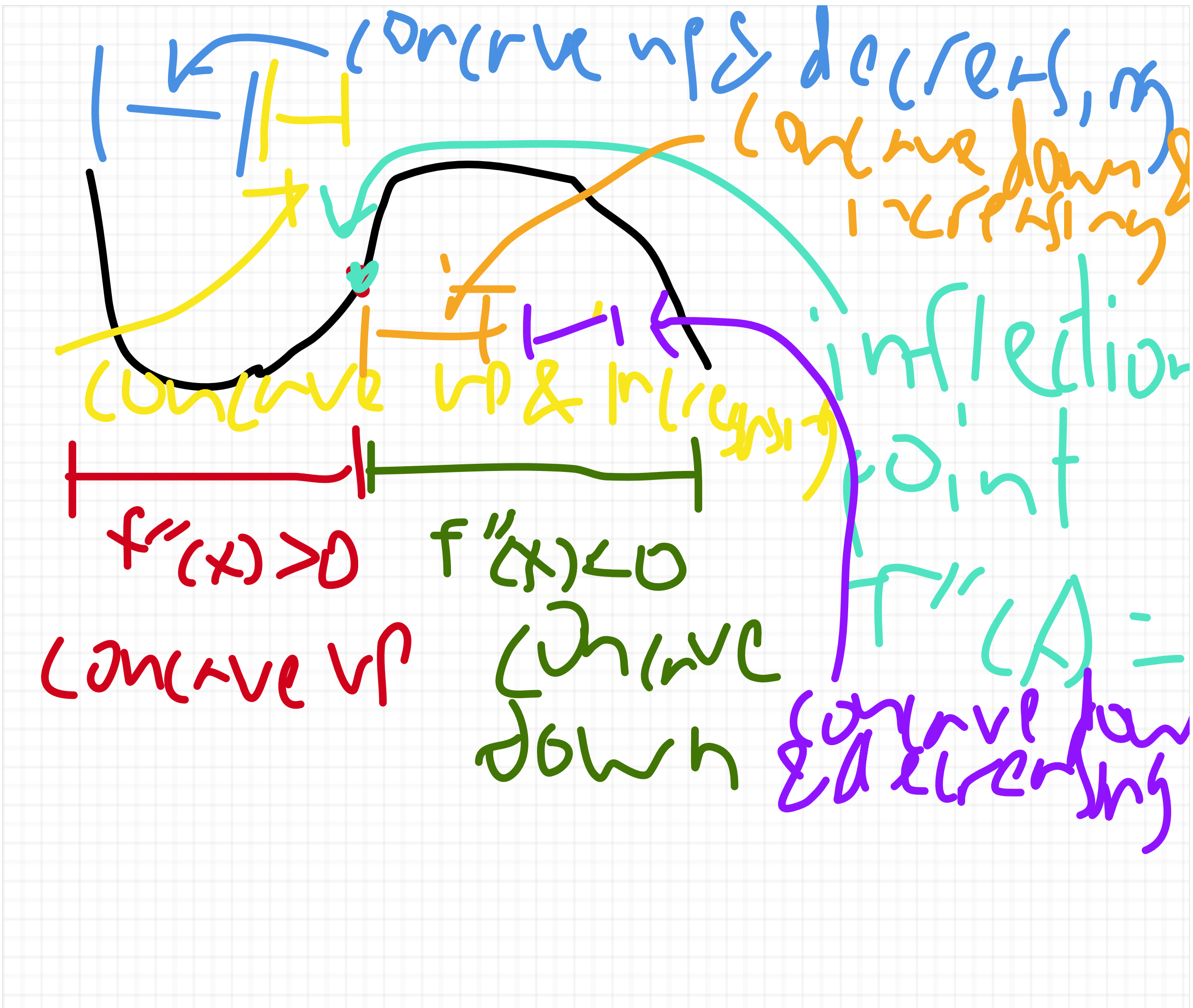
- If $f'(x)$ changes from positive at c to negative, then $f'(x)$ has a local max.



What do we do to determine concavity?

We use the second derivative.

- If $f''(x) > 0$ on an interval then f is concave up at that interval
- If $f''(x) < 0$ on an interval then f is concave down at that interval



L'Hopital's Rule

If either:

1. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

2. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

4.3-4.4 Homework Questions

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<u>Written HW 9 4.3 Question 40.</u> Given the function $g(x) = 200 + 8x^3 + x^4$

(a) When is g increasing and decreasing

We find the first derivative function $g'(x)$:

$$g'(x) = 24x^2 + 4x^3$$

We want to find the intervals where $g'(x) > 0$ and $g'(x) > 0$ to determine whether it's increasing and decreasing. How do we go about finding these intervals?

We find the critical points and evaluate $g'(x)$ in each of the regions between the critical points

$$0 = g'(x) = 24x^2 + 4x^3$$

$$0 = 4x^2(6 + x) \implies x = 0, -6$$

Now we find

$$x < -6 \implies \begin{array}{ccc} 4x^2 & (6+x) & \text{decreasing} \\ +\uparrow & -\uparrow & f'(x) < 0 \end{array}$$

$$-6 < x < 0 \implies \begin{array}{ccc} 4x^2 & (6+x) & \text{increasing} \\ +\uparrow & +\uparrow & f'(x) > 0 \end{array}$$

$$x > 0 \implies \begin{array}{ccc} 4x^2 & (6+x) & \text{increasing} \\ +\uparrow & +\uparrow & f'(x) > 0 \end{array}$$

(b) We want to find if the critical points $-6, 0$ are local mins. and max.'s

We see that using the first derivative test (refer to part (a)), we see that -6 is a local min. and 0 is neither, since f' doesn't change from the left and the right.

(c) We want to find the regions of concavity by finding when the $g''(x) > 0$ and $g''(x) < 0$ and we do that through finding the points of inflection and evaluating the regions between the critical points

$$0 = g''(x) = 48x + 12x^2$$

$$0 = 12x(4 + x) \implies x = -4, 0$$

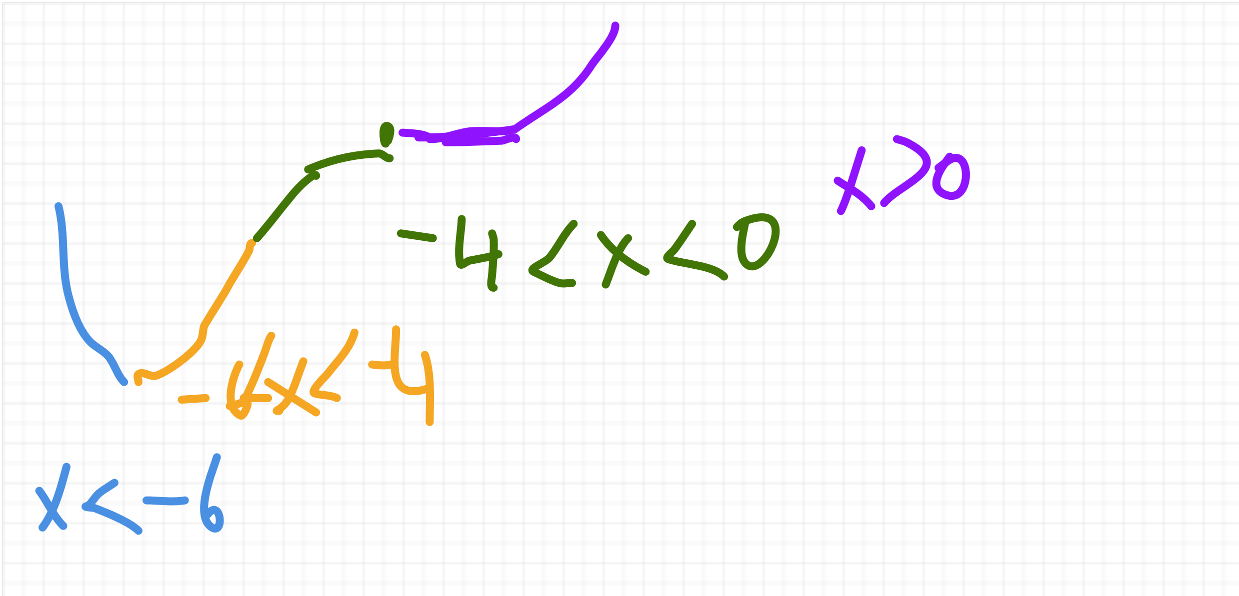
Now we find

$$x < -4 \implies \begin{array}{cc} 12x & (4+x) \\ -\uparrow & -\uparrow \end{array} \quad \begin{array}{l} \text{concave up} \\ g''(x) > 0 \end{array}$$

$$-4 < x < 0 \implies \begin{array}{cc} 12x & (4+x) \\ -\uparrow & +\uparrow \end{array} \quad \begin{array}{l} \text{concave down} \\ g''(x) < 0 \end{array}$$

$$x > 0 \implies \begin{array}{cc} 12x & (4+x) \\ +\uparrow & +\uparrow \end{array} \quad \begin{array}{l} \text{concave up} \\ g''(x) > 0 \end{array}$$

(d) Now we can draw the region



$$\lim_{x \to \infty} \left(\frac{5x-3}{5x+5} \right)^{3x+1}$$

WW24Sec4.4: Problem 14. We want to find the limit of $\lim_{x \to \infty} (\ln f(x))$ if we have an exponent

1^{∞}

$$\begin{aligned}
\lim_{x\to\infty} \ln\left(\frac{5x-3}{5x+5}\right)^{3x+1} &= \lim_{x\to\infty} (3x+1)\ln\left(\frac{5x-3}{5x+5}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(\frac{5x-3}{5x+5}\right)}{1/(3x+1)} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(\frac{5x-3}{5x+5}\right)}{\frac{d}{dx}1/(3x+1)} \\
&= \lim_{x\to\infty} \frac{\frac{5x+5}{5x-3}\cdot\frac{5(5x+5)-5(5x-3)}{(5x+5)^2}}{-(3x+1)^{-2}\cdot 3} \\
&= \lim_{x\to\infty} \frac{\frac{1}{5x-3}\cdot\frac{25+15}{5x+5}(3x+2)^2}{-3} \\
&= \lim_{x\to\infty} \frac{40(3x+2)^2}{-3(5x-3)(5x+5)} \\
&= \lim_{x\to\infty} \frac{40\cdot\left[9x^2+12x+4\right]}{-3\cdot\left[25x^2+10x-15\right]} \\
&= \lim_{x\to\infty} \frac{40\cdot 9}{-3\cdot 25} = -\frac{120}{25} = -\frac{24}{5}\,.
\end{aligned}$$

NOTE: To calculate $\lim_{x\to\infty} \frac{40\cdot\left[9x^2+12x+4\right]}{-3\cdot\left[25x^2+10x-15\right]}$, we can use L'Hopital's rule twice, but we can also just remember that the limit of two polynomials is just the ratio of the two lead coefficients, (which are in this case 9 and 25).

$$\lim_{x\to\infty}\left(\frac{5x-3}{5x+5}\right)^{3x+1} = exp\left(\lim_{x\to\infty}\ln\left(\frac{5x-3}{5x+5}\right)^{3x+1}\right) = e^{-24/5}.$$

4.5-4.7 Expostion

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Guidelines to Sketch a Curve

1. Find the domain.

2. x and y intercepts

3. Look for "symmetries". One possibility is even or odd functions, another possibility is periodic.

NOTE: Step 3 is not a super-essential step, but it's a good shortcut so good to check. But step 4 and onwards is usually more important and what we emphasize in this class.

4. Check for asymptotes: Horizontal asymptotes and vertical asymptotes. (slant asymptotes are mentioned but not discussed in class)

5. Critical points/intervals of increasing and decreasing: We find the critical points and the intervals that are increasing (positive derivative) and decreasing (negative derivative). We also determine whether the critical points are local mins and maxes.

6. Concavity and points of inflection. We find the points of inflection and intervals where the function is concave up (second derivative positive) or concave down (second derivative negative), and the points of inflection are where the concavity changes (second derivative is zero).

7. Sketch the curve by marking the intercepts, critical points, inflection points, and asymptotes and using the information about when it's increasing and the concavity to sketch the way the curve progresses.

Guidelines to Optimization

Usually you're given something you want to optimize, based on some variables (it could be more than one) and at least one constraint. We do this in the following steps.

1. Interpret the problem. We try to find what quantity Q we are optimizing what the variables $x_1, \ldots, x_n$ are, and what the constraints are. Drawing a picture often helps.

NOTE: Usually in these optimization problems, you're given two variables.

2. Figure out how to express the constraints an some kind of equation where we can manipulate the equations and solve for the other variables in terms of a primary variable x.

3. We plug what we've determined for the other variables in terms of x into the equation for Q, giving us $Q(x)$ as a function of x.

4. We use the usual methods of single variable optimization (i.e. finding the critical points) to

find the extreme values.

4.5-4.7 Homework Questions

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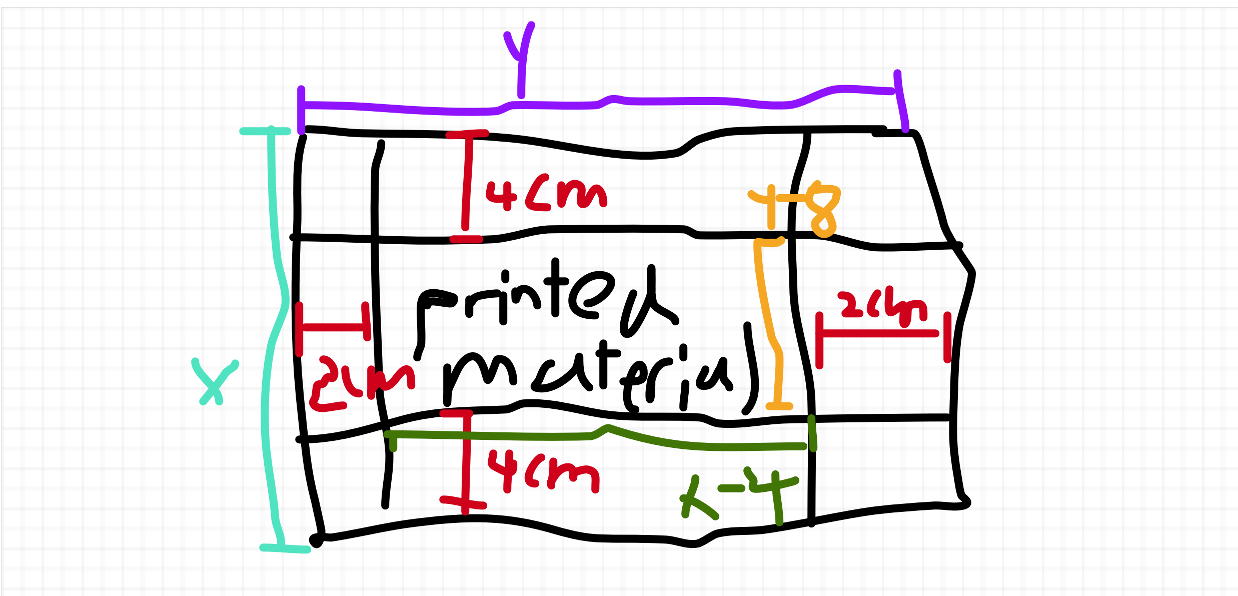
WW27Sec4.7a: Problem 6.

The top and bottom margins of a poster are 4 cm and the side margins are each 2 cm. If the area of printed material on the poster is fixed at 380 square centimeters, find the dimensions of the poster with the smallest area.



We want to find the smallest area

$A = xy$

Constraint:

area of printed material $= 380cm^2 = (x-8)(y-4)$

Next, we want to make A into a function of x (y works as well): We do this by solving for y in the equation

$380 = (x-8)(y-4)$

$$\div (x-8) \quad \div (x-8)$$
$$\frac{380}{x-8} = y - 4$$
$$+4 \qquad\qquad +4$$
$$\frac{380}{x-8} + 4 = y$$

Now we plug in y and get

$$A(x) = x \cdot \left[\frac{380}{x-8} + 4\right] = x \cdot \left[380(x-8)^{-1} + 4\right]$$

Next, we find the derivative A'

$$A'(x) = \left[\frac{380}{x-8} + 4\right] + x \cdot \left[-380(x-8)^{-2}\right]$$
$$= \frac{380}{x-8} + 4 - \frac{380x}{(x-8)^2}$$

Solve for x when $A'(x) = 0$ (find the critical points)

$$0 = A'(x) = \frac{380}{x-8} + 4 - \frac{380x}{(x-8)^2}$$
$$\times (x-8)^2 \qquad \times (x-8)^2$$
$$0 = 380(x-8) + 4(x-8)^2 - 380x$$
$$0 = 380x - 380 \cdot 8 + 4\left[x^2 - 16x + 64\right] - 380x$$
$$0 = -3040 + 4x^2 - 64x + 256$$
$$\div 4 \qquad \div 4$$
$$0 = -760 + x^2 - 16x + 64$$
$$0 = x^2 - 16x - 696$$

We have two solutions:

$$x = 8 \pm 2\sqrt{190}$$

Which solution?

$$x = 8 + 2\sqrt{190}$$

If we really wanted to check whether minimum or maximum, we could test for that, but it's one solution, and we're running out of time, so we'll skip that

$$y = \frac{380}{8 + 2\sqrt{190}}$$

Exam 3 Review

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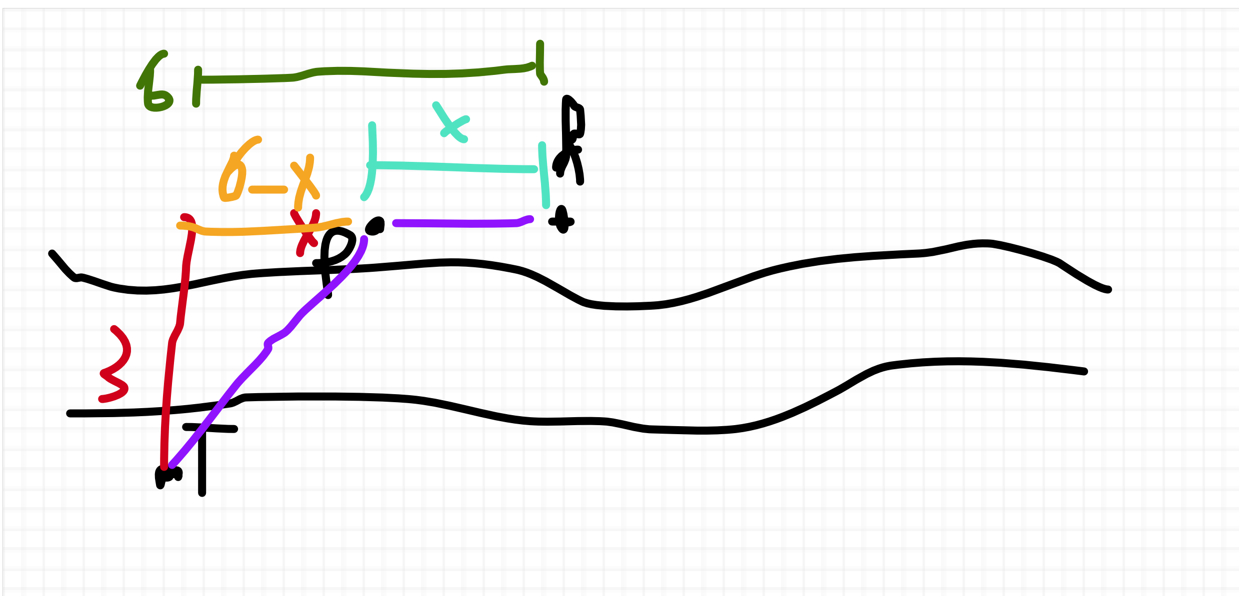
NOTE: This review is woefully incomplete and contains a few errors. To make up for this, I wrote up some *complete* exam 3 solutions, which I'll [link here]().

1.



What we're optimizing:

$$c = 400,000x + 800,000h$$

constraint:

$$h = \sqrt{(6-x)^2 + 3^2}$$

Plug in what we got for h

$$c(x) = 400,000x + 800,000\sqrt{(6-x)^2+3^2},$$

and now we're ready to optimize:

$$0 = c'(x) = 400,000 + 800,000\frac{1}{2\sqrt{(6-x)^2+3^2}} \cdot 2(6-x)\cdot(-1)$$

$$0 = 400,000 + 800,000\frac{-12+2x}{2\sqrt{(6-x)^2+3^2}}$$

$$\cdot\sqrt{(6-x)^2+3^2} \qquad \cdot\sqrt{(6-x)^2+3^2}$$

$$0 = 400,000\cdot\sqrt{(6-x)^2+3^2} + 400,000\cdot(-12+2x)$$

$$0 = \sqrt{(6-x)^2+3^2} + (-12+2x)$$

$$\cdot(-12+2x) - \sqrt{(6-x)^2+3^2} \qquad \cdot(-12+2x) - \sqrt{(6-x)^2+3^2}$$

$$0 = -\left[(6-x)^2+3^2\right] + (-12+2x)^2$$

$$0 = -\left[36-12x+x^2+9\right] + \left[144-48x+4x^2\right]$$

$$0 = 99-36x+3x^2$$

2a. continuous on $[a, b]$
differentiable on (a, b)

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

2b. $F(x) = 4x^3 - 8x + 5$ on $[0, 2]$

$$F'(x) = 12x^2 - 8$$

find

$$\frac{F(2)-F(0)}{2-0} = \frac{21-5}{2} = 8$$

Set that equal to

$8 = F'(c) = 12c^2 - 8$

and solve for c

$8 = 12c^2 - 8$
$+8 \qquad +8$
$16 = 12c^2$
$\div 12 \quad \div 12$
$\frac{16}{12} = c^2$

$$\sqrt{\frac{16}{12}} = c = \frac{2}{\sqrt{3}}$$

3. $y = (1-7x)^{1/x}$ This a 1^∞ problem, so we use the $\ln$ trick

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} 1/x \cdot \ln(1-7x)$$
$$= \lim_{x \to 0} \frac{\frac{d}{dx}\ln(1-7x)}{\frac{d}{dx}x}$$
$$= \lim_{x \to 0} \frac{\frac{1}{1-7x} \cdot 7}{1}$$
$$= \lim_{x \to 0} \frac{\frac{1}{1-7(0)} \cdot 7}{1}$$
$$= 7$$

$$\lim_{x \to 0} y = \lim_{x \to 0} e^{\ln y} = e^{\lim_{x \to 0} \ln y} = e^7.$$

4.
$$F(x) = \frac{x^2+3x}{x^2-9},\ F' = \frac{-3}{(x-3)^2},\ F'' = \frac{-3}{(x-3)^2}$$

A. domain: all reals except ± 3

B. Two vertical asymptotes (at $x = \pm 3$)

C.

$$\lim_{x\to\infty}\frac{x^2+3x}{x^2-9}=1,\ \lim_{x\to-\infty}\frac{x^2+3x}{x^2-9}=1$$

At $y = 1$, we have both a right and left horizontal asymptote.

D. Find the critical points of F'', by finding the zero values of the derivatives and undefined points. There are no zero values, since

$$0=\frac{-3}{(x-3)^2}$$

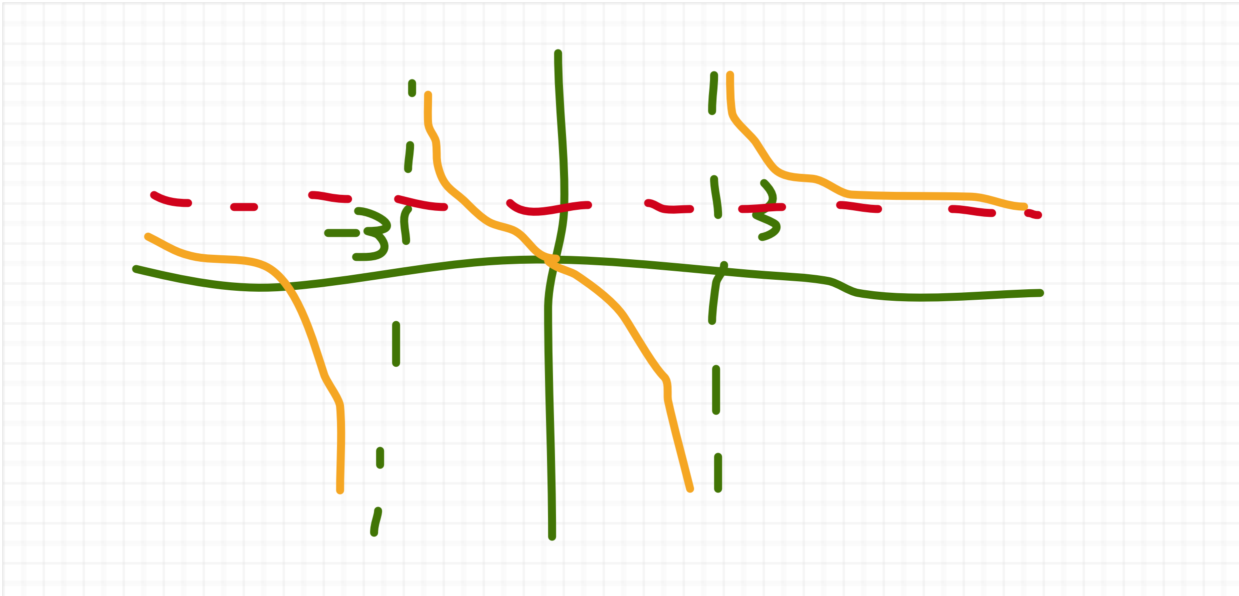
is a contradiction, but we have $x = \pm 3$ as asymptotes. At $x < -3$, we find both values in the denomenator are negative, so the function is decreasing. At $-3 < x < 3$, it's decreasing as well. And same with $x > 3$.

E.

$$F''=\frac{6}{(x-3)^3}$$

As before, no zero values, but undefined points at $x = \pm 3$. At $x < -3$, we have all the values
negative, so it's concave down. And then for $-3 < x < 3$, we find that the values are negative, and still concave down. It becomes concave up at $x > 3$, since the F'' becomes positive then.

F.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