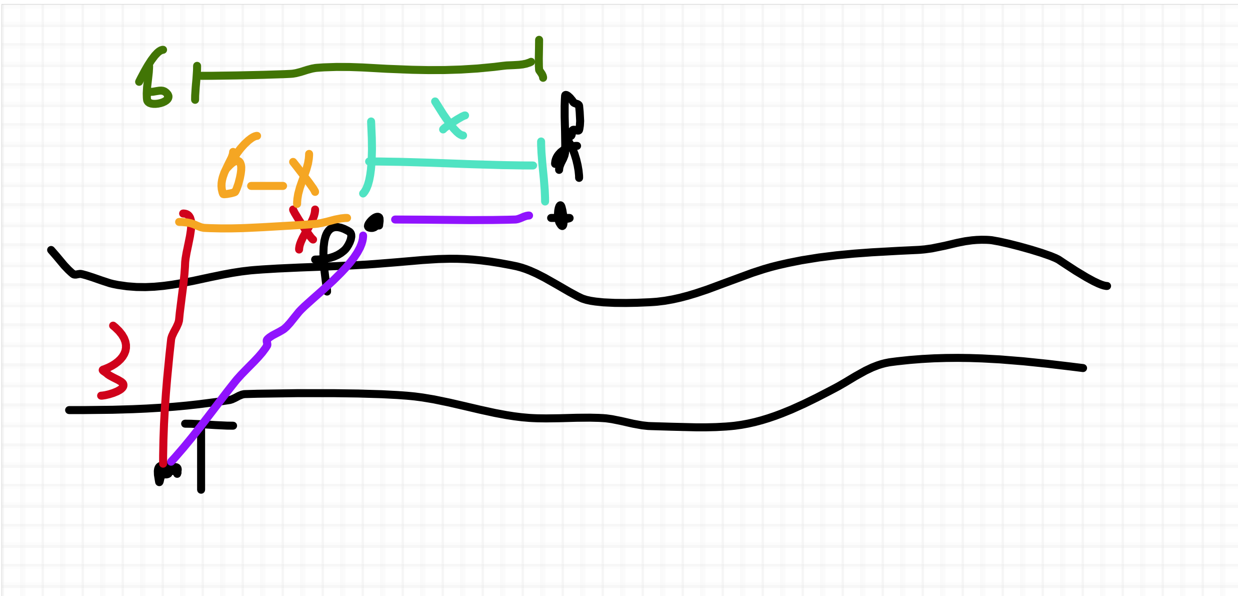
M211 Exam 3 Solutions

Problem 1.



What we're optimizing:

$c = 400,000x + 800,000h$

constraint:

$h = \sqrt{(6-x)^2 + 3^2}$

Plug in what we got for h

$c(x) = 400,000x + 800,000\sqrt{(6-x)^2 + 3^2}$,

and now we're ready to optimize:

$$0 = c'(x) = 400,000 + 800,000\frac{1}{2\sqrt{(6-x)^2 + 3^2}} \cdot 2(6-x) \cdot (-1)$$

$$0 = 400,000 + 800,000\frac{-12+2x}{2\sqrt{(6-x)^2+3^2}}$$
$$\times\sqrt{(6-x)^2+3^2} \qquad \times\sqrt{(6-x)^2+3^2}$$

$$0 = 400,000\cdot\sqrt{(6-x)^2+3^2} + 400,000\cdot(-12+2x)$$
$$0 = \sqrt{(6-x)^2+3^2} + (-12+2x)$$
$$\times\left[(-12+2x) - \sqrt{(6-x)^2+3^2}\right] \quad \times\left[(-12+2x) - \sqrt{(6-x)^2+3^2}\right]$$
$$0 = -\left[(6-x)^2+3^2\right] + (-12+2x)^2$$
$$0 = -\left[36-12x+x^2+9\right] + \left[144-48x+4x^2\right]$$
$$0 = 99-36x+3x^2$$

Using the quadratic formula, we get $x = 6 \pm \sqrt{3}$.

Problem 2.

2a. continuous on $[a,b]$
differentiable on (a,b)

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

2b. $F(x) = 4x^3 - 8x + 5$ on $[0,2]$

$$F'(x) = 12x^2 - 8$$

find

$$\frac{F(2)-F(0)}{2-0} = \frac{21-5}{2} = 8$$

Set that equal to

$$8 = F'(c) = 12c^2 - 8$$

and solve for c

$$8 = 12c^2 - 8$$

$+8 \qquad +8$

$16 = 12c^2$

$\div 12 \quad \div 12$

$\frac{16}{12} = c^2$

$\sqrt{\frac{16}{12}} = c = \frac{2}{\sqrt{3}}$

Problem 3. $y = (1-7x)^{1/x}$ This a 1^∞ problem, so we use the $\ln$ trick

$$\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} 1/x \cdot \ln(1-7x) \\
&= \lim_{x\to 0} \frac{\frac{d}{dx}\ln(1-7x)}{\frac{d}{dx}x} \\
&= \lim_{x\to 0} \frac{\frac{1}{1-7x}\cdot 7}{1} \\
&= \lim_{x\to 0} \frac{\frac{1}{1-7(0)}\cdot 7}{1} \\
&= 7
\end{aligned}$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0}\ln y} = e^7.$$

Problem 4.

$$F(x) = \frac{x^2+3x}{x^2-9},\ F' = \frac{-3}{(x-3)^2},\ F'' = \frac{-3}{(x-3)^2}$$

A. domain: all reals except ± 3.

B. One vertical asymptotes (at $x = 3$)

C.

$$\lim_{x\to\infty} \frac{x^2+3x}{x^2-9} = 1,\ \lim_{x\to-\infty} \frac{x^2+3x}{x^2-9} = 1$$

At $y = 1$, we have both a right and left horizontal asymptote.

D. Find the critical points of F'', by finding the zero values of the derivatives and undefined points. There are no zero values, since

$$0 = \frac{-3}{(x-3)^2}$$

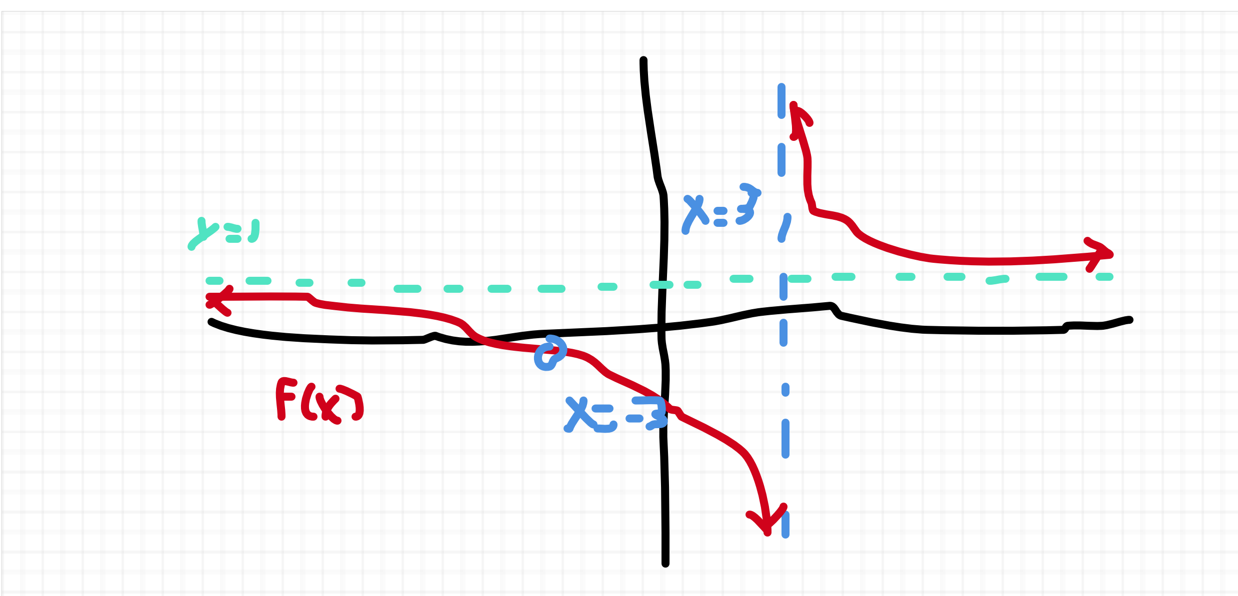
is a contradiction, but we have $x = 3$ as asymptotes. At $x < 3$, we find both values in the denomenator are negative, so the function is decreasing. And same with $x > 3$.

E.

$$F'' = \frac{6}{(x-3)^3}$$

As before, no zero values, but undefined points at $x = 3$. At $x < 3$, we have all the values negative, so it's concave down. It becomes concave up at $x > 3$, since the F'' becomes positive then.

F.



Problem 5.

A.

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$$

$$F(x) = .1x^2 + \sin x - 1$$
$$F'(x) = .2x + \cos x,$$

$$x_{n+1} = x_n + \frac{.1x^2 + \sin x - 1}{.2x + \cos x}.$$

B. $x_1 = \pi/2$

$$x_2 = x_1 + \frac{.1x_1^2 + \sin x_1 - 1}{.2x_1 + \cos x_1} = \pi/2 + \frac{.1(\pi/2)^2}{.2(\pi/2)} = \pi/4.$$

C. Either a stationary point (a point with derivative zero) or a poor initial estimate that doesn't create a converging sequence.

Problem 6.

$$x_0 := 1000$$

$$\begin{aligned} F(1002) &\approx F(1000) + F'(1000)\Delta x \\ &= \sqrt[3]{1000} + \frac{1}{3}(1000)^{-2/3} \cdot 2 \\ &= 10 + \frac{2}{3 \cdot 100} \\ &= 10 + \frac{1}{150} \\ &= \frac{1501}{150}. \end{aligned}$$

Problem 7.

$$S = 4r^2, \;\; r = 3, \; dr = .5$$

$$dS = \frac{d}{dr}\left(4r^2\right)dr = 8rdr = .8(3)(.5) = .4(3) = 1.2.$$

Problem 8.

A.

$$F(x) = x^3 + 6x^2 - 15x$$
$$F'(x) = 3x^2 + 12x - 15$$

$$0 = 3x^2 + 12x - 15$$
$$0 = 3\left(x^2 + 4x - 5\right)$$
$$0 = x^2 + 4x - 5$$
$$0 = (x+5)(x-1)$$

$$x = -5, 1$$

x has a critical point of 1 in the interval $[0, 5]$

B.

$$F(0) = (0)^3 + 6(0)^2 - 15(0) = 0,$$
$$F(1) = (1)^3 + 6(1)^2 - 15(1) = -9,$$
$$F(5) = (5)^3 + 6(5)^2 - 15(5) = 125 + 150 - 75 = 200.$$

The absolute minimum is -9 at the point $x = 1$ and the absolute maximum is 200 at the point $x = 5$

The extreme value theorem tells us there must be a zero on the interval.

Problem 9. Choice C.: G has a local maximum at $x = 3$.

Problem 10.

$$\lim_{t \to 0} \frac{\cos(2t) - \cos(3t)}{t^2} = \lim_{t \to 0} \frac{\frac{d}{dt}[\cos(2t) - \cos(3t)]}{\frac{d}{dt}\left[t^2\right]}$$
$$= \lim_{t \to 0} \frac{2\sin(2t) - 3\sin(3t)}{2t}$$

$$
\begin{aligned}
&= \lim_{t \to 0} \frac{\frac{d}{dt}[2\sin(2t) - 3\sin(3t)]}{\frac{d}{dt}[2t]} \\
&= \lim_{t \to 0} \frac{4\cos(2t) - 9\cos(3t)}{2} \\
&= \frac{4\cos(2(0)) - 9\cos(3(0))}{2} \\
&= \frac{-5}{2}.
\end{aligned}
$$

Problem 11.

A. There is one local maxima.

B. There is one local minima.

C. There is one inflection point.

Problem 12.

$$y = \sqrt{x}$$

$$
\begin{aligned}
d(x) &= \sqrt{(x-5)^2 + \left(\sqrt{x}\right)^2} \\
&= \sqrt{(x-5)^2 + x}
\end{aligned}
$$

$$
\begin{aligned}
d'(x) &= \frac{1}{2\sqrt{(x-5)^2 + x}} \cdot [2(x-5) + 1] \\
&= \frac{2(x-5) + 1}{2\sqrt{(x-5)^2 + x}}
\end{aligned}
$$

$$0 = \frac{2(x-5) + 1}{2\sqrt{(x-5)^2 + x}}$$

$\times 2\sqrt{(x-5)^2 + x} \quad \times 2\sqrt{(x-5)^2 + x}$

$0 = 2x - 11$

$+11 \quad +11$

$11 = 2x$

$\div 2 \quad \div 2$

$11/2 = x$