Spring 2023 Math 151C Chapter 2 Lecture Notes

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Concept and Definition of a Limit

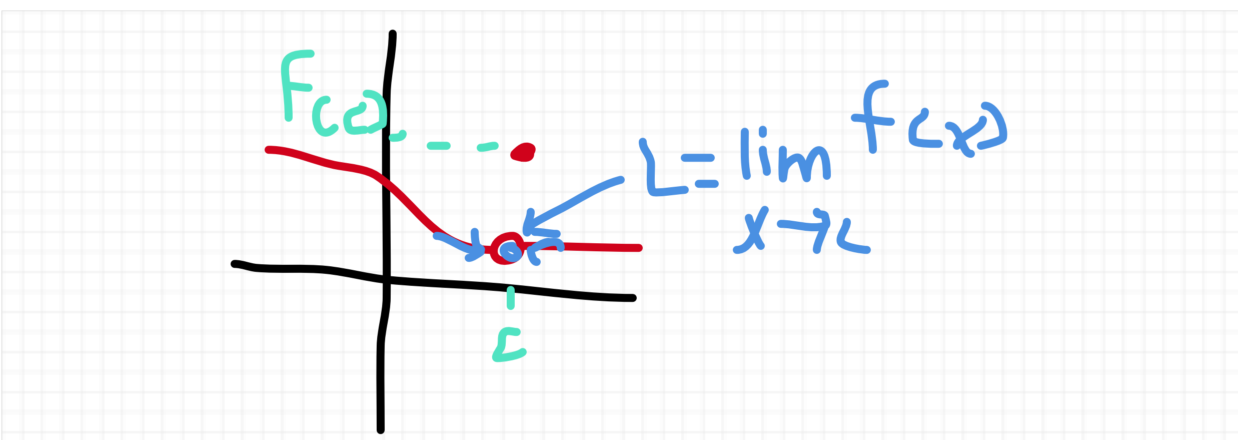
A **limit** of a function f at a value c is the value L (if it exists) the function "approaches" as the input value x gets arbitrarily close c

We write

$$\lim_{x \to c} f(x) = L$$

to talk about such a value L

One might (mistakenly) think that the limit is the actual value of the function, and as we'll learn, and hopefully as illustrated in the picture below, it might not be the case



Example 1. Let's take $f(x) = \dfrac{\sin x}{x}$. Note that it's undefined at $x = 0$, but the limit exists

$x = 1 \;\; f(x) \approx 0.841 \qquad\qquad x = -1 \;\; f(x) \approx 0.841$
$x = 0.5 \; f(x) \approx .959 \qquad\qquad x = -0.5 \;\; f(x) \approx .959$
$x = 0.001 \; f(x) \approx 0.999999983 \qquad x = -0.001 \; f(x) \approx 0.999999983$

We can see through just plugging in smaller and smaller values that

$$\lim_{x \to 0} f(x) = 1$$

The takeaway is that one can think of the limit $\lim_{x \to c} f(x)$ as a value L that the function sufficiently approximates as the input value x gets infintecimally closer to the given input value c

Given $\epsilon > 0$ arbitrarily close to 0, there exists "margin of error" $\delta > 0$ such that for any input x such $c - \delta < x < c + \delta$, we have

$$|f(x) - L| < \epsilon$$

This is as known as the **ϵ-δ definition of limits**, it's *the formal definition of limits that we won't actually cover.*

We don't need to utilize this definition in the course because in practice we usually evaluate limits not through the definition, but through various algebraic/calculus techniques that allow us to evaluate limits.

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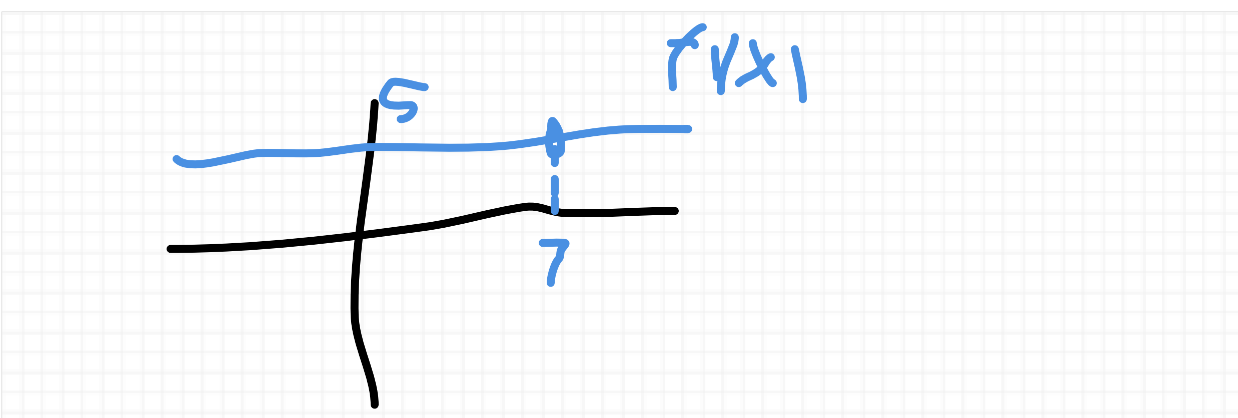
Example 1. *(from Example 1 in page 50)* $f(x) = 5,\ g(x) = 3x + 1$

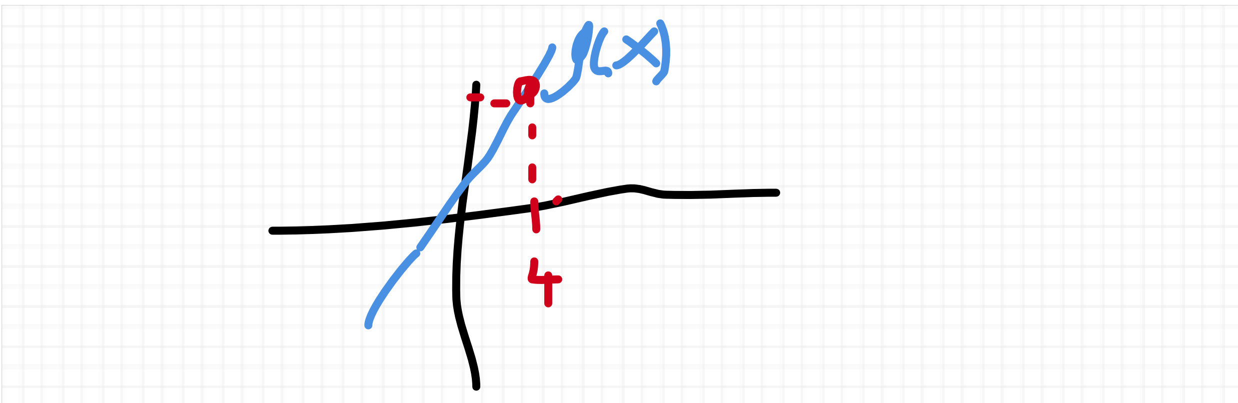
(a) $\lim_{x \to 7} f(x) = 5$



(b) $\lim_{x \to 4} g(x)$

To find the limit, we plug in 4, and see if there's any issues; there isn't any since we don't have what we'll later to be an "indeterminant form" (i.e., no division by 0)

$g(4) = 3(4) + 1 = 13$



$\lim_{x \to 4} g(x) = 13$

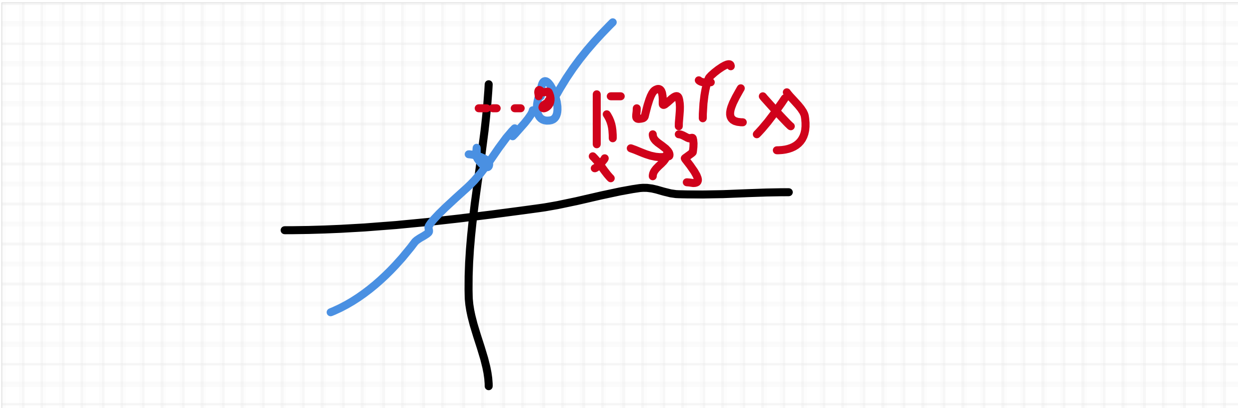
Example 2. *(from Example 3 in page 52)* $\lim_{x \to 4} x^2$

Similar to part (b) of the previous example, we have a function where the value is approached from both sides, so we can just plug in the value for the evaluation

$\lim_{x \to 4} x^2 = (4)^2 = 16$

Example 3. Define $f(x) = \dfrac{x^2 - 9}{x - 3}$ and find $\lim_{x \to 3} f(x)$. What's going on is that the function is undefined at $x = 3$, since we can't divide by zero. However, there's a bit of a wierd thing happening where the function is defined and linear at every point except $x = 3$ because

$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3}$$



$$\lim_{x \to 3} f(x) = (3) + 3 = 6$$

Basic Limit Laws

Basic Limit Laws Include the Following: Assume $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist:

(i) Identity and Constant Laws: For any constants c and k (note that k stands for any real number), we have

$$\lim_{x \to c} k = k, \quad \lim_{x \to c} x = c$$

(ii) Sum Law:

$$\lim_{x \to c} [(f + g)(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

It follows immediately from this law that

$$\lim_{x \to c} [(f - g)(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

(iii) Constant Multiple Law: For any constant k, we have

$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$$

(iv) Product Law:

$$\lim_{x \to c} (f \cdot g)(x) = \left[\lim_{x \to c} f(x)\right] \cdot \left[\lim_{x \to c} g(x)\right]$$

(v) Product Law: If $\lim_{x \to c} g(x) \neq 0$, then

$$\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

(vi) Power Law: If n is a positive integer, then

$$\lim_{x \to c} [f(x)]^n = \left(\lim_{x \to c} f(x)\right)^n$$

If q is a rational number and $\lim_{x \to c} [f(x)] \geq 0$,

$$\lim_{x \to c} [f(x)]^q = \left(\lim_{x \to c} f(x)\right)^q$$

Example 1. *(from Example 1 (a) and (b) in page 59)* Use the basic limit laws to evaluate the following:

(a) $\lim_{x \to 2} x^2$

The identity law tells us that

$$\lim_{x \to 2} x = 2$$

So then the product law tells us

$$\lim_{x \to 2} x^2 = \left[\lim_{x \to 2} x\right]\left[\lim_{x \to 2} x\right] = 4$$

(b)

$$\lim_{x \to 2} \left(x^3 + 5x + 7\right) = \lim_{x \to 2} x^3 + \lim_{x \to 2} 5x + \lim_{x \to 2} 7$$

$$\lim_{x \to 2} x^3 = \left(\lim_{x \to 2} x\right)^3 = 2^3 = 8$$

$$\lim_{x \to 2} 5x = 5\left(\lim_{x \to 2} x\right) = 5(2) = 10$$

$$\lim_{x \to 2} 7 = 7$$

$$\begin{aligned}\lim_{x \to 2} \left(x^3 + 5x + 7\right) &= 8 + 10 + 7 \\ &= 25\end{aligned}$$

Example 2. *(from Example 2 in page 59)*

(a) $\lim_{x \to -1} \frac{t+6}{2t^4}$

We want to use the quotient, but we must evaluate each limit to make sure they exist and that $\lim_{x \to -1} 2t^4 \neq 0$

$$\lim_{x \to -1} t + 6 = 5$$

$$\lim_{x \to -1} 2t^4 = 2 \neq 0$$

We find that it meets the conditions for the quotient law, so

$$\lim_{x \to -1} \frac{t+6}{2t^4} = \frac{\lim_{x \to -1}[t+6]}{\lim_{x \to -1}\left[2t^4\right]} = \frac{5}{2}$$

(b) $\lim_{t \to 3} t^{-1/4}(t+5)^{1/3}$

$$\lim_{t \to 3} t^{-1/4} = \left(\lim_{t \to 3} t\right)^{-1/4} = 3^{-1/4} = \frac{1}{3^{1/4}} \approx .760$$

$$\lim_{t \to 3} (t+5)^{1/3} = \left(\lim_{t \to 3} t + 3\right)^{1/3} = (8)^{1/3} = 2$$

Example 3. *(from 2.3 exercises 4 and 8, page 60, done as a group)*

(a)

$$\lim_{z \to 27} z^{2/3} = \left(\lim_{z \to 27} z\right)^{2/3} = (27)^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

(b)

$$\begin{aligned}\lim_{x \to 1/3} \left(3x^3 + 2x^2\right) &= \lim_{x \to 1/3} \left[3x^3\right] + \lim_{x \to 1/3} \left[2x^2\right] \\ &= 3\left(\lim_{x \to 1/3} x\right)^3 + 2\left(\lim_{x \to 1/3} x\right)^2 \\ &= 3\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 \\ &= \frac{1}{9} + \frac{2}{9} \\ &= \frac{1}{3}\end{aligned}$$

Concerning giving credit for evaluating limits on homeworks, quizzes, and exams, I'll give full credit if you evaluate the limit correctly, but if you don't, I will give partial credit for the steps you show.

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Example 3. *(from 2.3 exercise 28 on page 61)* Let's say $\lim_{x \to 6} f(x) = 4$; compute

(a) $\lim_{x \to 6} f(x)^2 = \left[\lim_{x \to 6} f(x)\right]^2 = 4^2 = 16$

(b) $\lim_{x \to 6} \frac{1}{f(x)} = \frac{1}{\lim_{x \to 6} f(x)} = \frac{1}{4} \quad \lim_{x \to 6} f(x) \neq 0$

(c) $\lim_{x \to 6} x\sqrt{f(x)} = \left(\lim_{x \to 6} x\right)\left(\lim_{x \to 6} \sqrt{f(x)}\right) = 6\left(\sqrt{\lim_{x \to 6} f(x)}\right) = 6\sqrt{4} = 6 \cdot 2 = 12$

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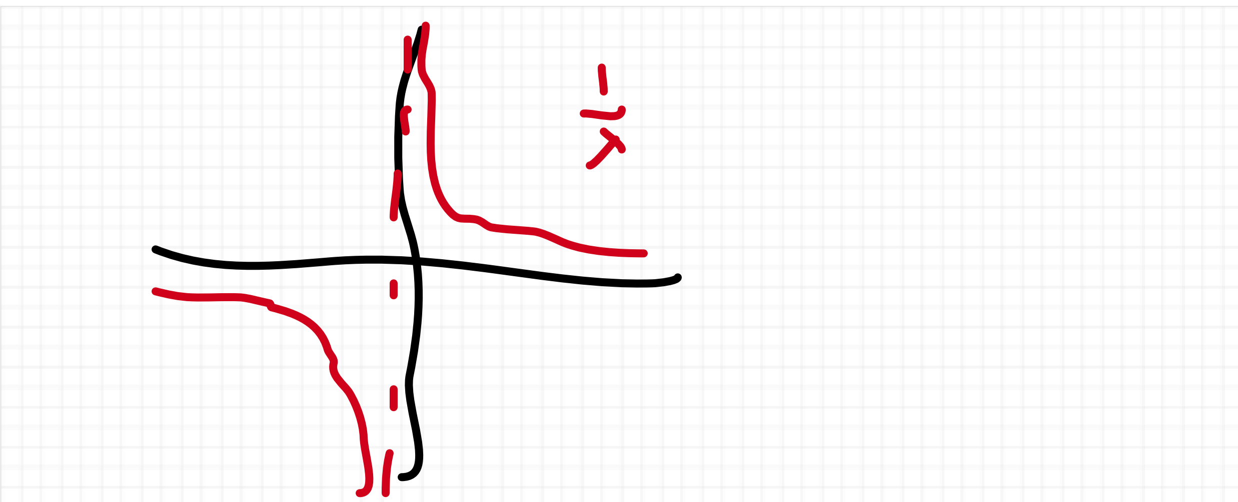
Existence and Nonexistence of a Limit

Example 1. Let's evaluate

$$\lim_{x \to 0} \frac{1}{x}$$



Clearly, the limit doesn't approach any value, and hence it doesn't exist

Example 2. *(From Example 6 on page 53)*

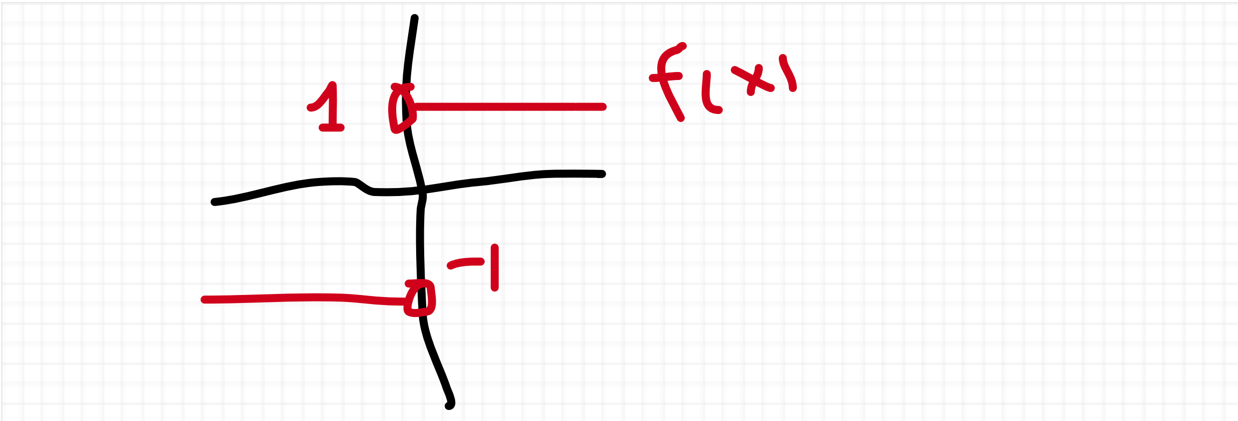
$f(x) = \dfrac{x}{|x|}$ Let's check if $\lim_{x \to 0} f(x)$ exists

If $x > 0$, then $|x| = x$

$$f(x) = \frac{x}{x} = 1$$

If $x < 0$, then $|x| = -x$

$f(x) = \frac{x}{-x} = -1$



Since it doesn't consistently approach the same value, the limit does not exist.

However, it's worth noting that from the right hand side (aka the positive side) the value of $f(x)$ as x approaches 0 converges to 1, and on the left hand side (aka the negative side) the value of $f(x)$ as x approaches 0 converges to -1. This lends itself to the notion of one-sided limits, which we'll learn about next time.

We'll also learn about continuity next time.

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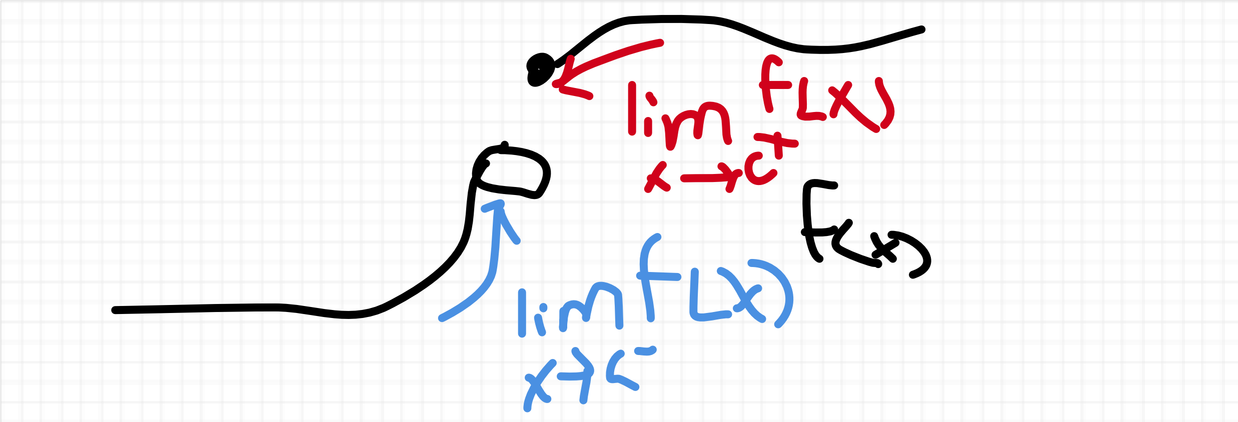
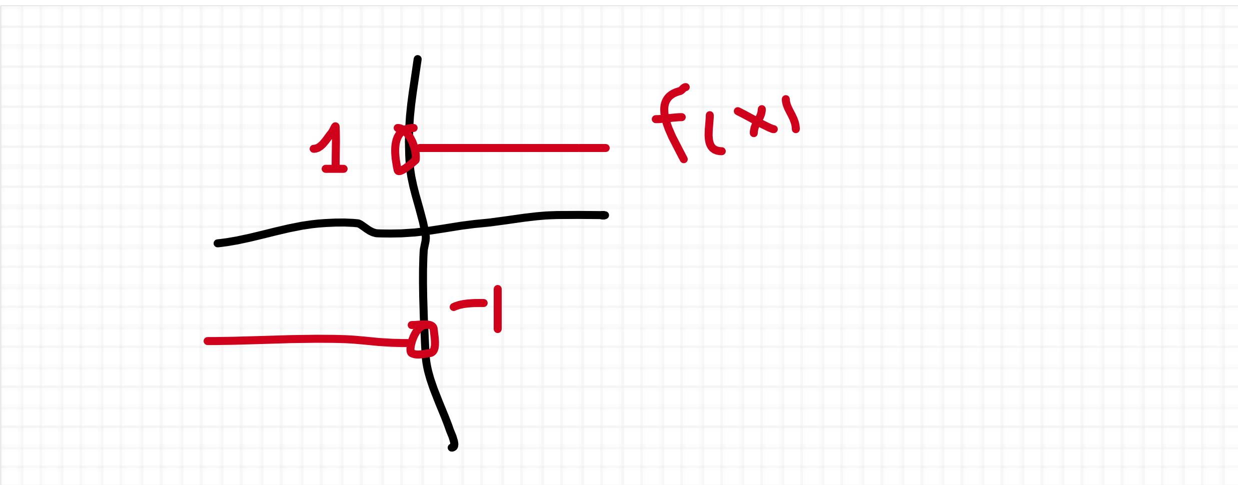
One-Sided Limit

So similar to how we look at limits, we can look at **one-sided limits** as a limit (if it exists) as x approaches some value c from one side (either the right or the left)

We write $\lim_{x \to c^+} f(x)$ to mean "the right side limit"
We write $\lim_{x \to c^-} f(x)$ to mean "the left side limit"



Going back to **Example 2** where $f(x) = \dfrac{x}{|x|}$, we the right and left limits existing

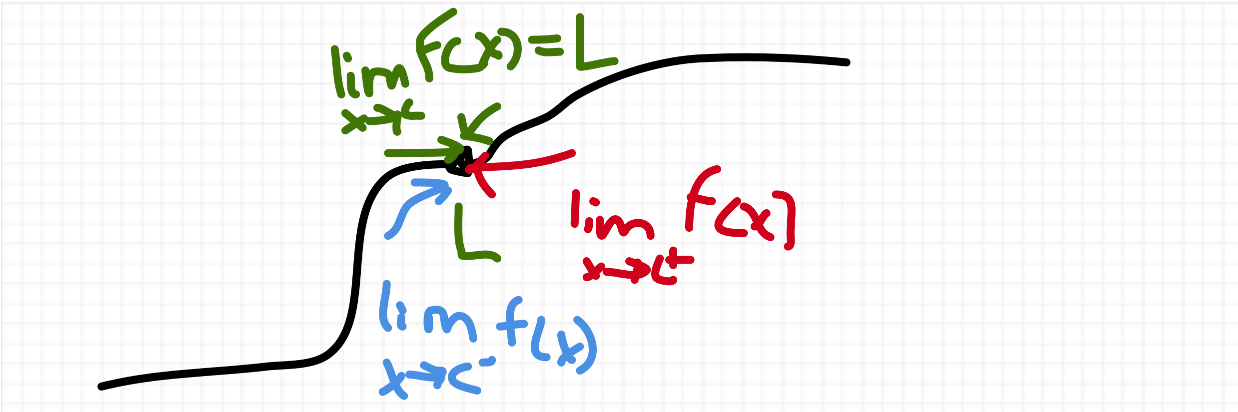


$$\lim_{x \to c^+} f(x) = 1, \ \lim_{x \to c^-} f(x) = -1$$

Theorem 1. $\lim_{x \to c} f(x)$ exists if $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$. In this situation, we have $\lim_{x \to c} f(x) = L$.

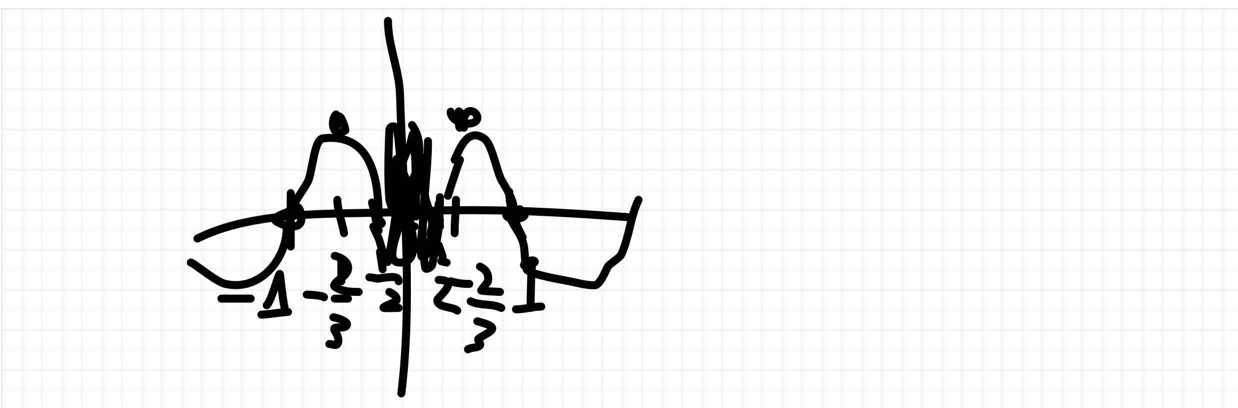
Conversely, if $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c} f(x)$$



Sometimes, we have functions that don't approach ANY value.

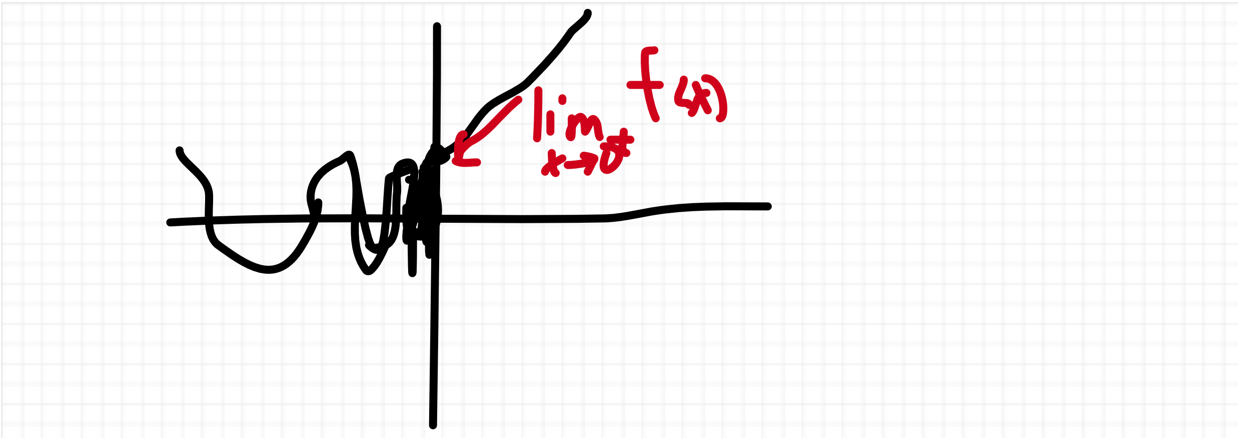
Example 4. $f(x) = \sin \dfrac{\pi}{x}$. Let's look at what happens when $x \to 0$



What happens is it oscillates between $-1 \leq y \leq 1$ and doesn't approach a consistent value from either side, so neither $\lim\limits_{x \to c^+} f(x)$ nor $\lim\limits_{x \to c^-} f(x)$ exist.

Example 5.

$$f(x) = \begin{cases} \sin \dfrac{\pi}{x} & \text{if } x < 0 \\ x + 1 & \text{if } x > 0 \end{cases}$$



$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 1$$

$\lim_{x \to 0^-} f(x)$ does not exist (DNE) because $f(x)$ on the right hand side, and of course $\lim_{x \to 0} f(x)$ DNE.

Example 6.

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

We find that

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

we have that $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1,$ therefore

$$\lim_{x \to 1} f(x) = 1$$

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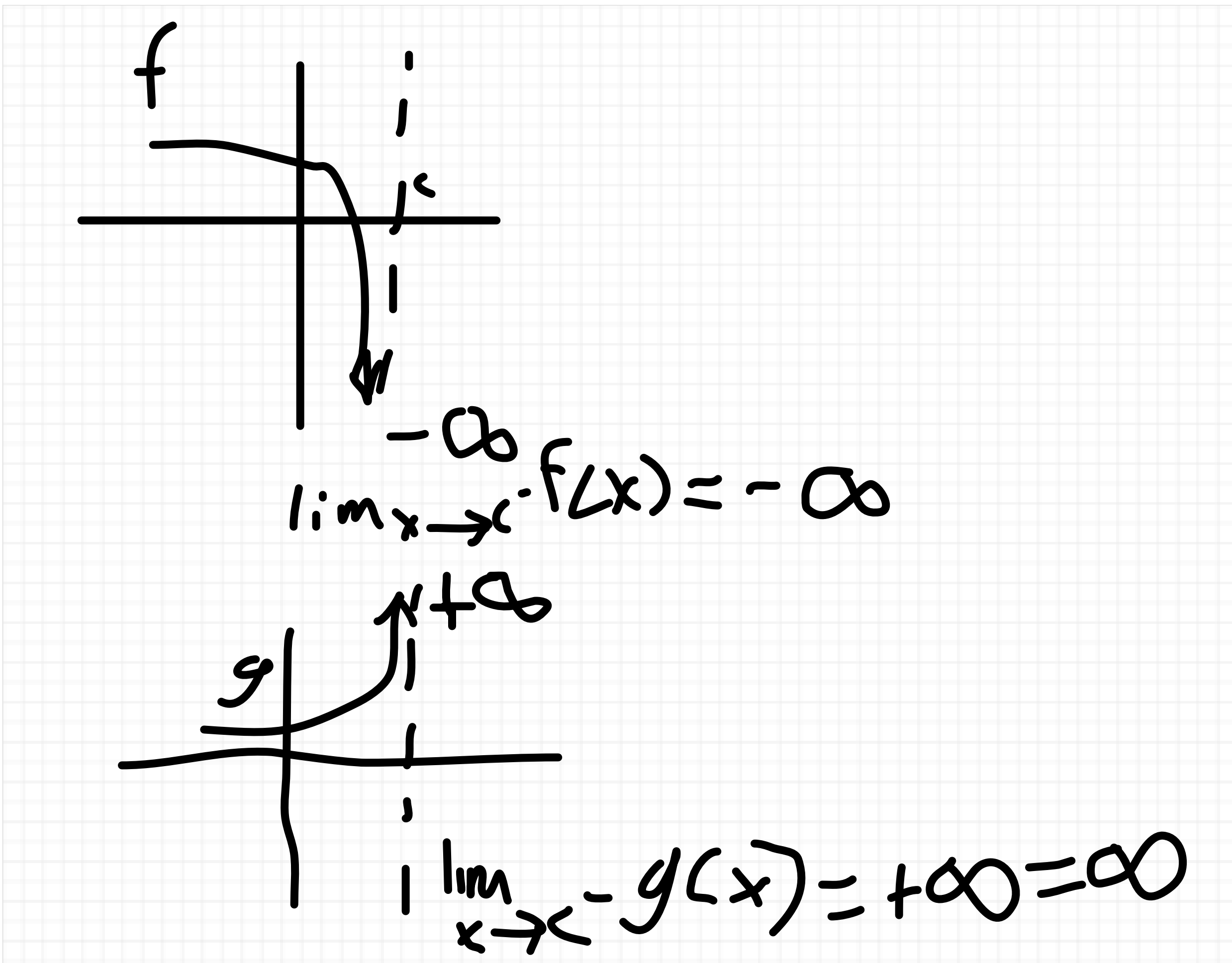
Limits at Infinity

For some functions, $f(x)$ tends to "blow up", i.e. have a vertical asymptote, and either trend infinitely upwards or infinitely downwards

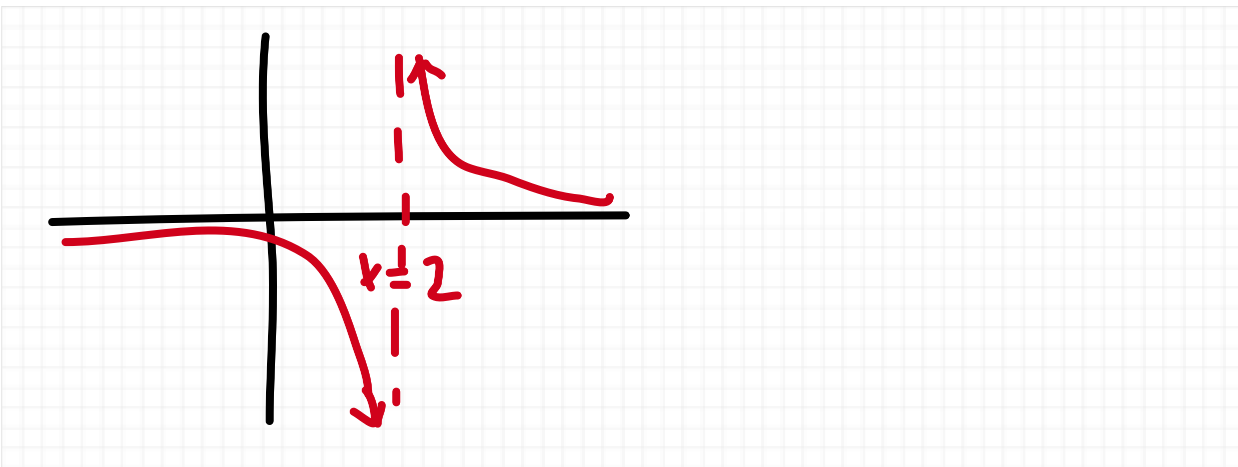


the above diagram shows it happening from the left side, but we state $\lim_{x \to c^+} g(x) = +\infty$ and $\lim_{x \to c^+} g(x) = -\infty$ if we have analogous asymptotic behavior from the right side.

Example 7. *(from Example 8 on page 46)*

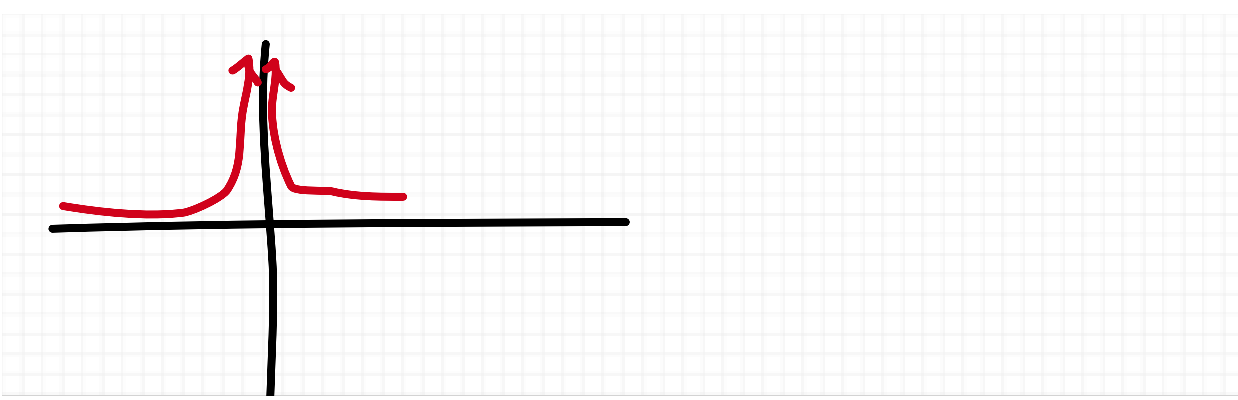
(a)

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$
$$\lim_{x \to 2^+} \frac{1}{x-2} = +\infty$$



(b)

$$\lim_{x \to 0^-} \frac{1}{x^2} = +\infty$$
$$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$$



Whenever a vertical asymptote of f at $x = c$ approaches either $+\infty$ from both sides or $-\infty$ from both sides, we say that $\lim_{x \to c} f(x) = +\infty$ or $\lim_{x \to c} f(x) = -\infty$, respecitvely, so for **Example 7 (b)**, we find that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Note that in a sense, the "limits don't exist" because they don't approach an actual real-number value, but intuitively we can talk about a side (or even both sides) of a limit approaching $\pm\infty$ when it blows up in the way we describe.

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Continuity of a Function

Definition and Examples

Assume that $f(x)$ is defined on an open interval (a, b) containing c (i.e, we have $a < c < b$). We say that f is **continuous** at the point c if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, a function that is continuous at c has its limit existing equal to the value of the function. If f is continuous at every point c inside (a, b), we say that f is **continuous on the interval $(\mathbf{a}, \mathbf{b})$**.

A value of a function that is not continuous is called a **discontinuity**.

Let's look at some of types of functions that are continuous

1. constant functions $f(x) = k$ for some real number k and the identity function x are continuous $(-\infty, +\infty)$

2. polynomial functions are also continuous on $(-\infty, +\infty)$

3. The trig functions $\sin x$ and $\cos x$ are continuous on $(-\infty, +\infty)$

4. exponential functions b^x are continous on $(-\infty, +\infty)$

5. Power functions x^a and $\log_a x$ for a real number a are continuous on $(0, +\infty)$ (logorithms are not defined on $(-\infty, 0]$ and x^a are not necessarily defined for negative values)

We have a few rules for making continuous functions from previous continuous functions

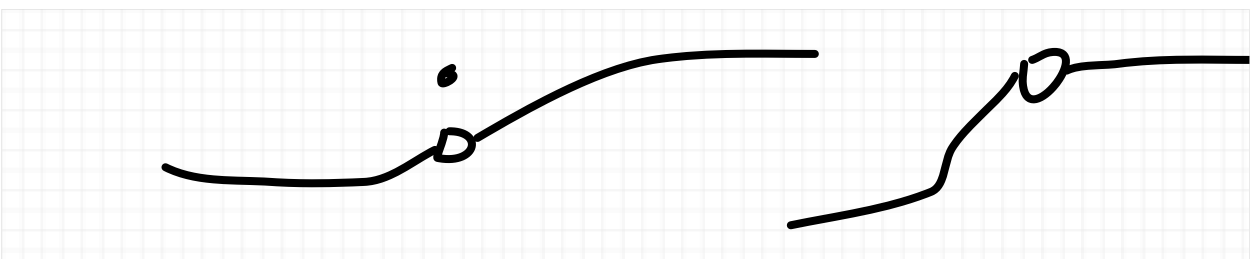
Theorem 1. If f and g are continous on $x = c$, then the following functions are also continuous at $x = c$

(i) $f + g$ *(ii)* $f \cdot g$ ($k \cdot g$ for a constant k) *(iii)* $\frac{f}{g}$ if $g(c) \neq 0$

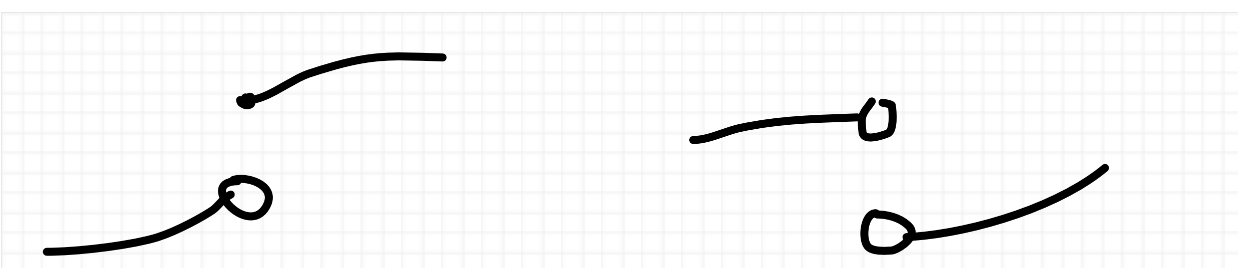
Discontinuities

Recall that a **discontinuity** is a value of a function where either the value does not exist, or the limit does not exist. Here are a few types of discontinuities

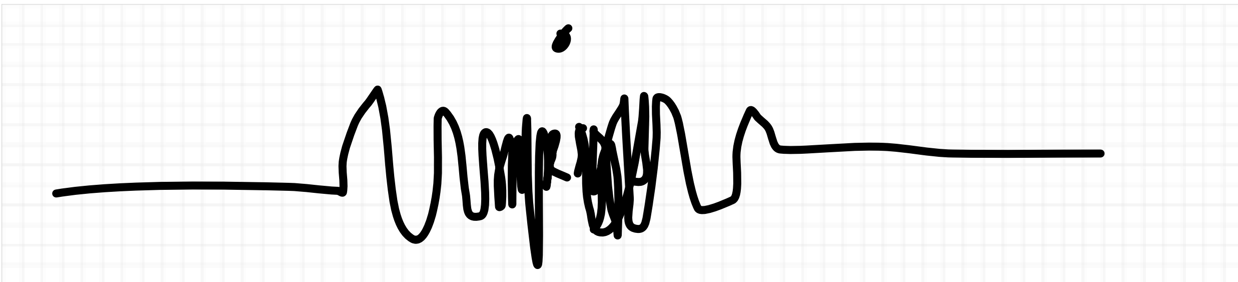
removable discontinuity. A discontinuity where the limit exists but the value is not defined at the limit (if defined at all)



jump discontinuity. A discontinuity where the one-sided limits exist but the value of each one-sided limit does not agree.



oscilating discontinuity. A discontinuity that arises from the function oscillating between many values and not approaching one from either side.



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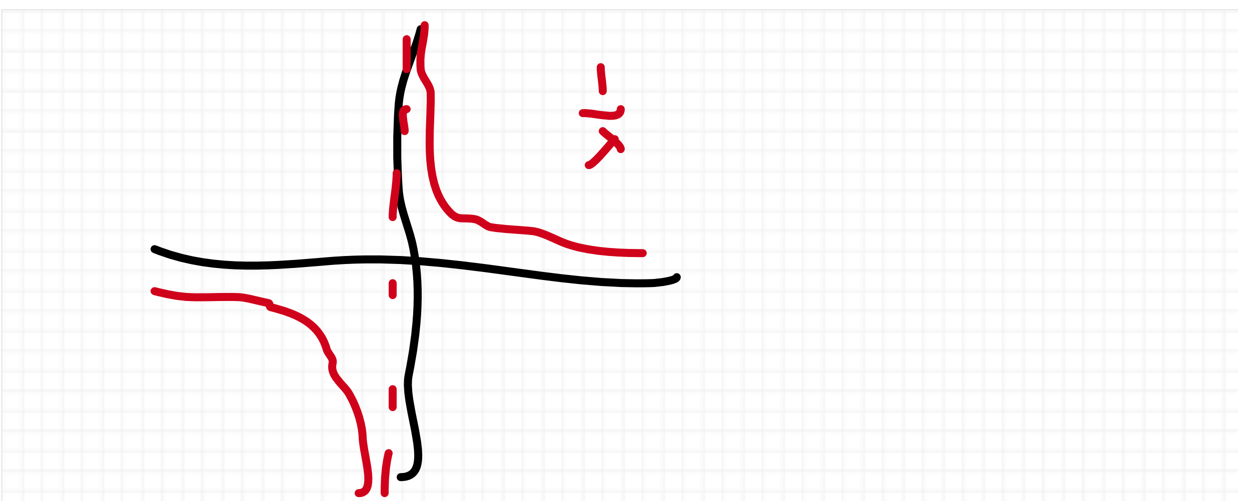
infinite discontinuities. Discontinuities that arise from vertical asympotes and the one sided limits not being a specific number.

Example 1. *(from 2.4 exercise 17 on page 70)* We shall identify the points of discontinuity of $f(x) = \frac{1}{x}$



At $x = 0$ is where we have a discontinuity, the discontinuity is an infinite discontinuity.

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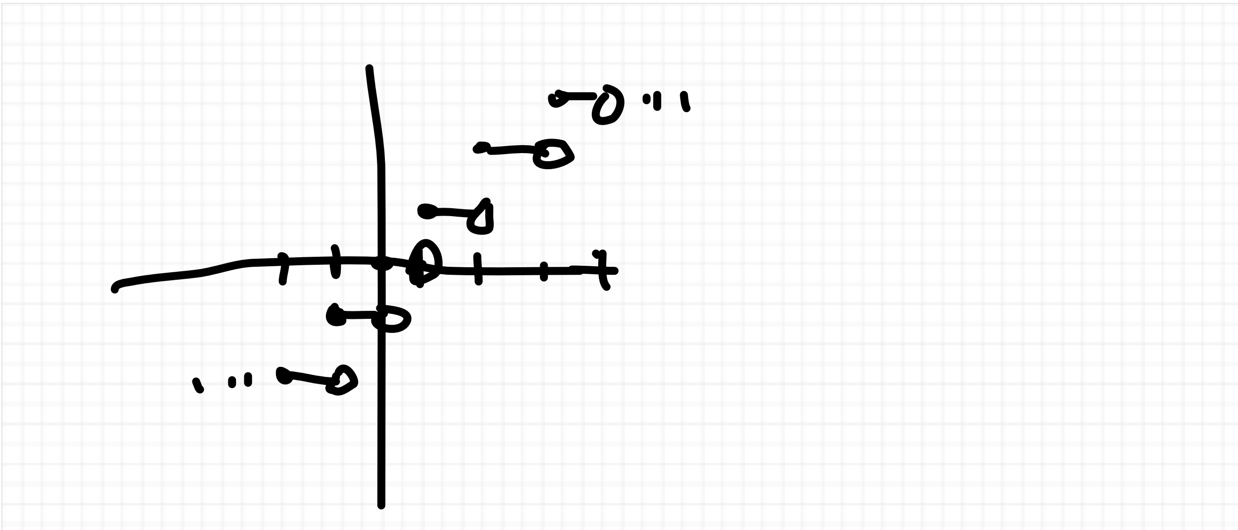
Example 2. *(from 2.4 exercise 20 on page 70)* Let's look at

$f(x) = \lfloor x \rfloor =$ the highest integer $\leq x$

For example $f(27.345) = 27$ and $f(103.\overline{3}) = 103$. Graphically, it looks as follows



We find this function has jump discontinuities for every integer value $x = n$.

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Theorem 2. *(Continuity of Composite Functions)* If g is continuous at $x = c$, and f is continuous at $x = g(c)$, then the composite function $(f \circ g)(x) = f(g(x)$ is continuous at $x = c$.

Example 3. Let's say we had $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$, Then for every $x = c$ value, we find that $(f \circ g)(x) = \sqrt{x^2 + 1}$ is continuous at any point $x = c$.

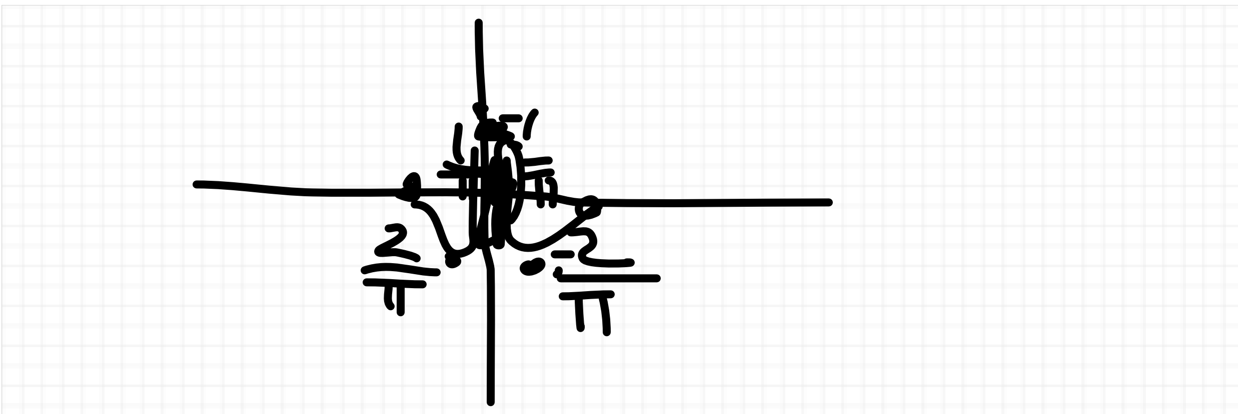
Example 4. *(from 2.4 exercise 30 on page 70)*

Determine the points of discontinuity of

$$f(x) = \begin{cases} \cos \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Continuous at $x \neq 0$ because $\frac{1}{x}$ is continuous at $x \neq 0$ and $\cos x$ is continuous, therefore, the composition $\cos \frac{1}{x}$ at $x \neq 0$ is continuous

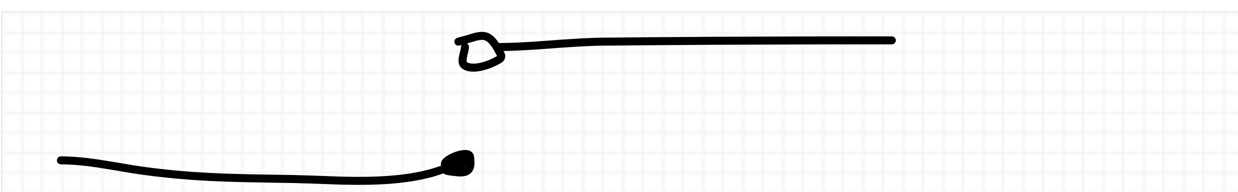
From drawing the diagram below, we find that $\cos \frac{1}{x}$ oscillates at $x = 0$, and therefore is discontinuous at $x = 0$



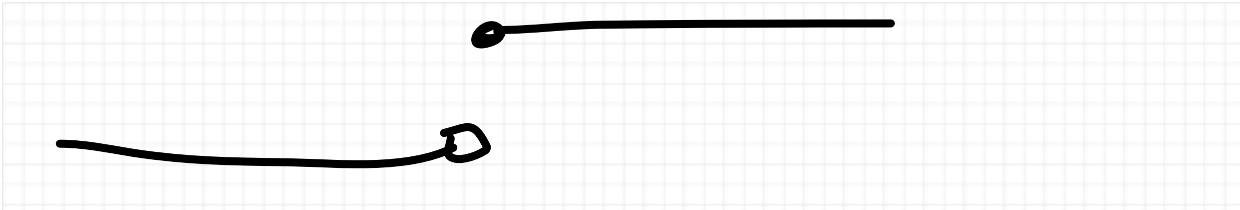
One-Sided Continuity

In connection with jump discontinuities, it is convenient to define **one-sided continuity**.

We call f **left-continuous** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$



We call f **right-continuous** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$



Theorem 1. A function is continuous precisely when it is **right-continuous** and **left-continuous**.

Proof. Follows immediately from Theorem 1. ◻

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Theorem 2. A function is left (respectively right) continuous precisely when the only discontinuities that exist are jump discontinuities with the left (respectively right) hand limits existing.

Proof. Follows immediately from Theorem 1 on the Limit and Nonexistence of Limit section, since that theorem characterizes limits existing as the left and right limit existing and agreeing. %CHANGE REFERENCE OF THEOREM 1 ◻

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Indeterminant Forms

Basic Idea and Examples

When the function is continuous, evaluating limits is usually pretty easy and involves plugging in the value, but oftentimes that isn't the case.

Given a function f, if the formula $f(c)$ yields an undefined expression in one of the forms $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \pm\infty, 0,$ or $\infty - \infty$, then we say that $f(x)$ has an **indeterminate form** (or we say is **indeterminant)** at $x = c$.

Example 1. *(from Example 1 on page 72)* Calculate $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$. Notice that we have

$x = 3$

$$f(x) = x^2 - 4x + 3 = (x-1)(x-3), \quad f(3) = 0$$
$$g(x) = x^2 + x - 12 = (x+4)(x-3), \quad g(3) = 0$$

so we end up with the indeterminant form $\frac{0}{0}$, and we need some kind of trick to get cancel a term out that causes the form to be indeterminant. In this case, we can cancel out $x-3$ on the numerator and denominator as so

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim_{x\to 3} \frac{(x-1)(x-3)}{(x+4)(x-3)} = \lim_{x\to 3} \frac{x-1}{x+4} \cdot \lim_{x\to 3} \frac{x-3}{x-3} = \lim_{x\to 3} \frac{x-1}{x+4} \cdot \lim_{x\to 3} 1$$
$$= \lim_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}.$$

Example 2. *(From Example 2 on page 73)* Evaluate $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

The issue is, we have

$$f(x) = x - 9 = \left(\sqrt{x}\right)^2 - 9 = \left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right), \quad f(9) = 0$$
$$g(x) = \sqrt{x} - 3, \ g(9) = 0$$

and hence we have the $\frac{0}{0}$ indeterminant form. Note that $\sqrt{x}-3$ factors into $x-9$ and we have

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \frac{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}{\sqrt{x}-3} = \lim_{x\to 9} \sqrt{x}+3 = 6$$

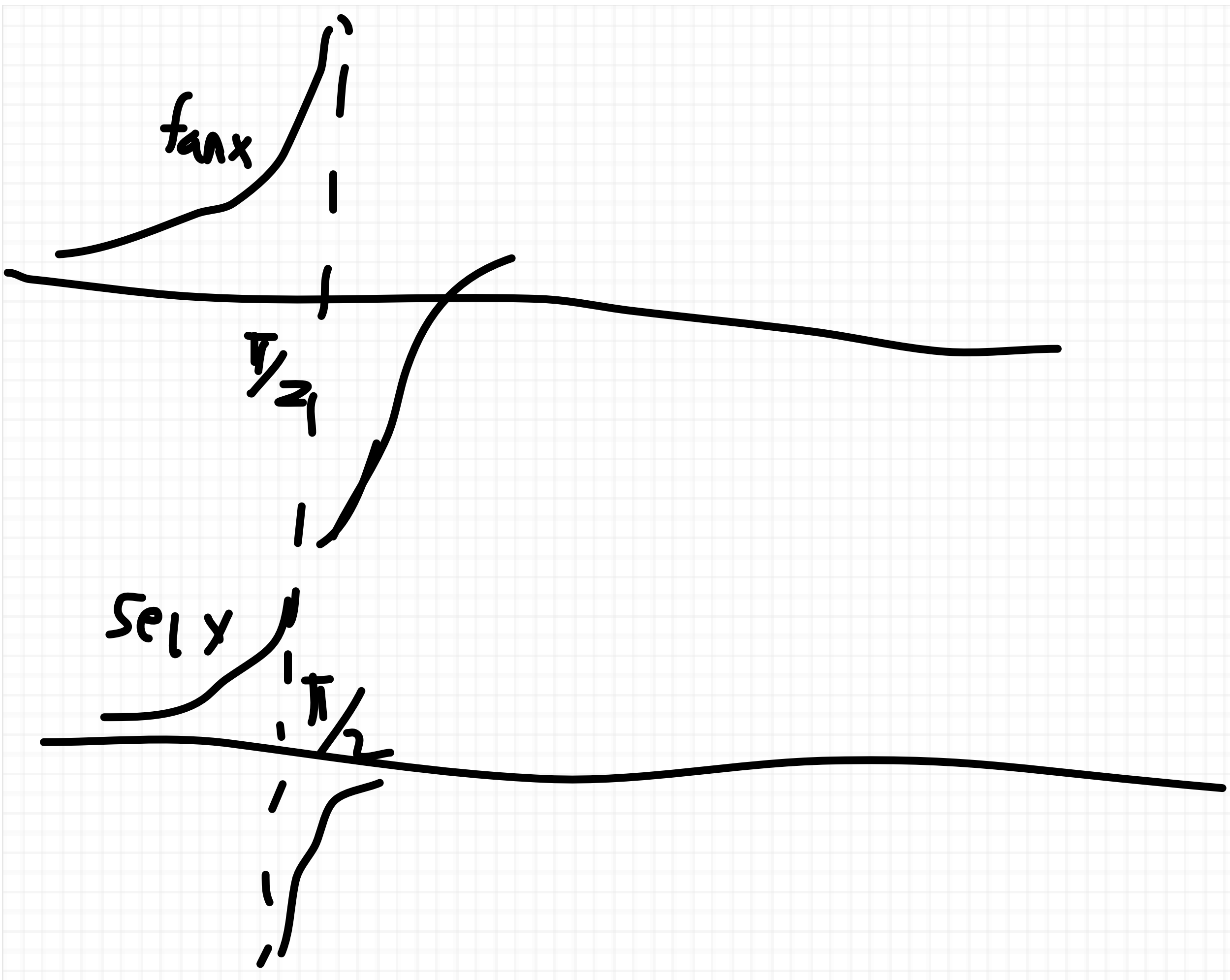
Example 3. *(From Example 4 on page 74)*

$$\lim_{x\to \pi/2} \frac{\tan x}{\sec x}$$

Note that

$$\tan x = \frac{\sin x}{\cos x} \quad \cos \pi/2 = 0$$

$\sec x = \dfrac{1}{\cos x}, \ \cos \pi/2 = 0$



As a result, we have an $\dfrac{\infty}{\infty}$ indeterminant form, because

$$\lim_{x \to \pi/2^-} \tan x = +\infty, \quad \lim_{x \to \pi/2^+} \tan x = -\infty$$
$$\lim_{x \to \pi/2^-} \sec x = +\infty, \quad \lim_{x \to \pi/2^+} \sec x = -\infty$$

We can evaluate this limit doing a similar trick to the $\dfrac{0}{0}$ indeterminant form of simplifying the fraction

$$\lim_{x \to \pi/2} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x \to \pi/2} \frac{\frac{1}{\cos x} \cdot \sin x}{\frac{1}{\cos x}} = \lim_{x \to \pi/2} \sin x = 1.$$

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Example 4. *(From Example 5 on page 74)* Calculate

$$\lim_{x \to 1}\left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$$

This is a $\infty - \infty$ indeterminant form since the discontinuities on both $\frac{1}{x-1}$ and $\frac{2}{x^2-1}$ are infinite. To evaluate this limit, we'll combine the partial fractions and simplify as follows

$$\begin{aligned}
\lim_{x \to 1}\left(\frac{1}{x-1} - \frac{2}{x^2-1}\right) &= \lim_{x \to 1}\left(\frac{1}{x-1} - \frac{2}{(x-1)(x+1)}\right) = \lim_{x \to 1}\left(\frac{1}{x-1} \cdot \frac{x+1}{x+1} - \frac{2}{(x-1)(x+1)}\right) \\
&= \lim_{x \to 1}\left(\frac{(x+1)-2}{(x-1)(x+1)}\right) = \lim_{x \to 1}\left(\frac{x+(-1)}{(x-1)(x+1)}\right) \\
&= \lim_{x \to 1}\left(\frac{x-1}{(x-1)(x+1)}\right) = \lim_{x \to 1}\left(\frac{1}{x+1}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Example 5. *(From 2.5 exercise 26 on page 76)*

$$\sqrt{x^2+x} = \sqrt{x}\sqrt{x+1}$$

$$\begin{aligned}
\lim_{x \to 0^+}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2+x}}\right) &= \lim_{x \to 0^+}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}\sqrt{x+1}}\right) = \lim_{x \to 0^+}\left(\frac{1}{\sqrt{x}}\frac{\sqrt{x+1}}{\sqrt{x+1}} - \frac{1}{\sqrt{x}\sqrt{x+1}}\right) \\
&= \lim_{x \to 0^+} \frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}} = \lim_{x \to 0^+} \frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\
&= \lim_{x \to 0^+} \frac{(x+1)-1}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x \to 0^+} \frac{x}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)} \\
&= \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x+1}\left(\sqrt{x+1}+1\right)} \\
&= \lim_{x \to 0^+} \frac{\sqrt{0}}{\sqrt{0+1}\left(\sqrt{0+1}+1\right)} \\
&= 0.
\end{aligned}
$$

The Squeeze Theorem

Theorem 1. *(Squeeze Theorem)* Let f, g, h be functions such that $f(x) \leq g(x) \leq h(x)$ on some interval (a, b) such that $a < c < b$. Suppose that $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then

$$\lim_{x \to c} g(x) = L = \lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Proof. Intuitive. ◻

Example 1. *(from Example on page 78)* We shall evaluate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

Note that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ and $-|x| \leq x \leq |x|$, so

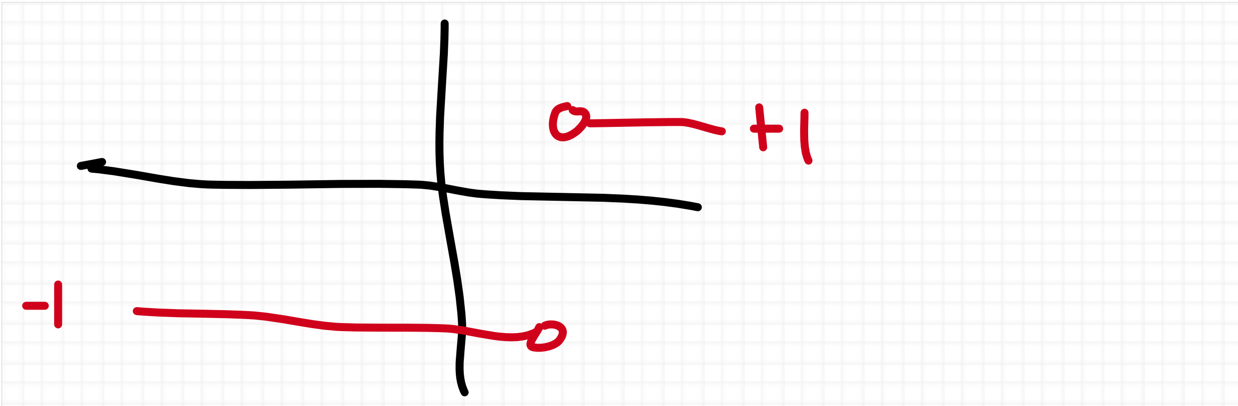
$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$. Note additionally that

$$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0.$$

Then by the squeeze theorem, we find that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Example 2. *(from exercise 4 on page 81)* We shall evaluate $\lim_{x \to 3} \left(x^2 - 9\right) \frac{x-3}{|x-3|}$



Note from the graph above that

$$-1 \le \frac{x-3}{|x-3|} \le 1,$$

hence

$$-\left(x^2 - 9\right) \le \left(x^2 - 9\right) \frac{x-3}{|x-3|} \le x^2 - 9.$$

Since

$$\lim_{x \to 3} -\left(x^2 - 9\right) = \lim_{x \to 3} x^2 - 9 = 0,$$

we find using the Squeeze Theorem, we have

$$\lim_{x \to 3} \left(x^2 - 9\right) \frac{x-3}{|x-3|} = 0.$$

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A few general tips with finding limits using the squeeze theorem: Whatever function you have, if there are "complicated terms" that result in determinant forms (such as

$\frac{x-3}{|x-3|}$, $\sin\left(\frac{1}{x}\right)$, $2^{\cos\frac{1}{x}}$ with discontinuities) that happens to be bounded find a bound for them, and try to squeeze the function between two rational/polynomial functions/power functions/algebraic functions/nonoscilating trig functions

Example 3. *(from exercise 8 on page 81)*

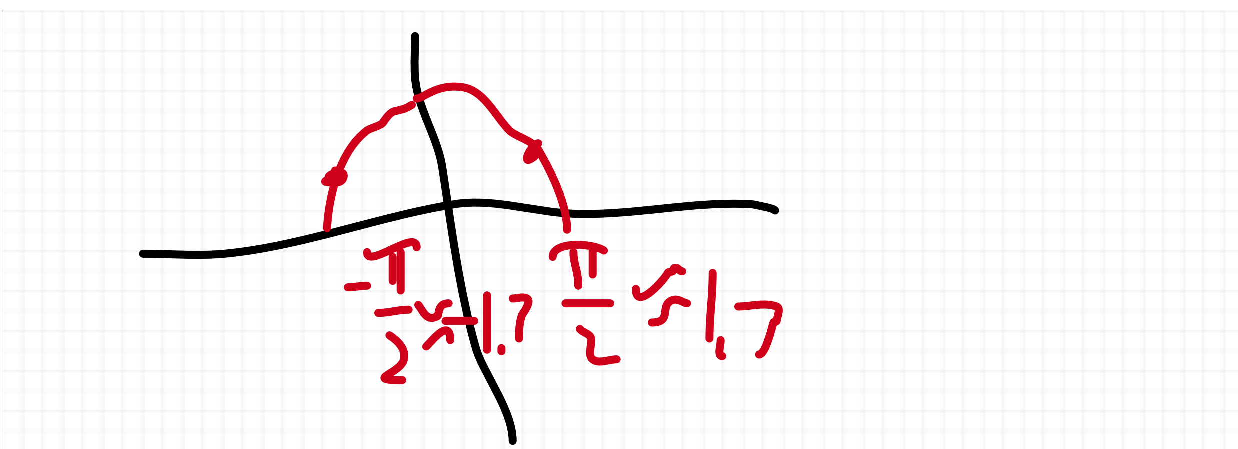
$$\lim_{x \to 0} \tan x \cos\left(\sin \frac{1}{x}\right)$$

$\tan x = \frac{\sin x}{\cos x} = 0$ as $x \to 0$

$\lim_{x \to 0} \tan x = 0$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-1 \leq \cos\left(\sin \frac{1}{x}\right) \leq 1$$



We shall use the squeeze thoerem on $-\tan x$ and $\tan x$ as so: Observe that

$$-|\tan x| \leq \tan x \cos\left(\sin \frac{1}{x}\right) \leq |\tan x|$$

$$\lim_{x \to 0} -|\tan x| = \lim_{x \to 0} |\tan x| = 0$$

then we can use the squeeze theorem and conclude that

$$\lim_{x \to 0} \tan x \cos\left(\sin \frac{1}{x}\right) = 0.$$

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Some Important Trig. Limits

Theorem 1.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \ \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

Proof. Note by derivations done in page 78-79 in the book (see Theorem 3) that

$$\cos \theta \le \frac{\sin \theta}{\theta} \le 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Since $-\frac{\pi}{2} < 0 < \frac{\pi}{2}$, we can use the squeeze theorem to get $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ since

$$\lim_{\theta \to 0} \cos \theta = \lim_{\theta \to 0} 1 = 1.$$

For evaluating $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$, it follows from the pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ and the limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ we previously derived that

$$\begin{aligned}\lim_{\theta \to 0}\left[\frac{1 - \cos \theta}{\theta}\right] &= \lim_{\theta \to 0}\left[\frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta}\right] = \lim_{\theta \to 0}\left[\frac{1 - \cos^2\theta}{\theta(1 + \cos \theta)}\right] = \lim_{\theta \to 0}\left[\frac{\sin^2\theta}{\theta(1 + \cos \theta)}\right] \\ &= \lim_{\theta \to 0}\left[\frac{\sin \theta}{\theta}\right] \cdot \lim_{\theta \to 0}\left[\frac{\sin \theta}{1 + \cos \theta}\right] = 1 \cdot \frac{0}{2} = 0.\end{aligned}$$

□

Change of Variables

Theorem 1. *(Change of variables)* If h is a continuous on $x = c$, and $\lim_{y \to h(c)} f(y)$ exists, then $\lim_{x \to c}(f \circ h)(x)$ and

$$\lim_{x \to c}(f \circ h)(x) = \lim_{y \to h(c)} f(y)$$

Proof. Follows immediately from Theorem 1 of the Continuity of a Function section.
%CHANGE REFERENCE OF THEOREM 1 □

Example 1. Evaluate $\lim_{h \to 0} \dfrac{\sin 4h}{h}$. Let's define $\theta = 4h$. Note that $\theta(0) = 4(0) = 0$

$$\lim_{h \to 0} \frac{\sin 4h}{h} = \lim_{h \to 0} \frac{4 \sin 4h}{4h} = \lim_{\theta \to 0} \frac{4 \sin \theta}{\theta} = 4 \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 4 \cdot 1 = 4.$$

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Example 2. *(from Example 3 on page 80)* Evaluate $\lim_{x \to 0} \dfrac{\tan 3x}{\tan 2x}$. When evaluating limits of functions involving $\tan x$, it helps to put it in terms of $\sin x$ and $\cos x$ using the $\tan x = \dfrac{\sin x}{\cos x}$ identity. Set $\theta = 2x,\ \phi = 3x$

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan 3x}{\tan 2x} &= \lim_{x \to 0} \frac{\frac{\sin 3x}{\cos 3x}}{\frac{\sin 2x}{\cos 2x}} = \lim_{x \to 0} \frac{\frac{1}{x} \cdot \frac{\sin 3x}{\cos 3x}}{\frac{1}{x} \cdot \frac{\sin 2x}{\cos 2x}} = \lim_{x \to 0} \frac{\frac{1}{\cos 3x} \frac{\sin 3x}{x}}{\frac{1}{\cos 2x} \frac{\sin 2x}{x}} \\
&= \lim_{x \to 0} \frac{\frac{1}{\cos 3x}}{\frac{1}{\cos 2x}} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{x}}{\lim_{x \to 0} \frac{\sin 2x}{x}} = \lim_{x \to 0} \frac{\frac{1}{\cos 3x}}{\frac{1}{\cos 2x}} \cdot \frac{\lim_{x \to 0} \frac{3 \sin 3x}{3x}}{\lim_{x \to 0} \frac{2 \sin 2x}{2x}} \\
&= \lim_{x \to 0} \frac{\frac{1}{\cos 3x}}{\frac{1}{\cos 2x}} \cdot \frac{\lim_{x \to 0} \frac{3 \sin 3x}{3x}}{\lim_{x \to 0} \frac{2 \sin 2x}{2x}} = \lim_{x \to 0} \frac{\frac{1}{\cos 3x}}{\frac{1}{\cos 2x}} \cdot \frac{\lim_{\phi \to 0} \frac{3 \sin \phi}{\phi}}{\lim_{\theta \to 0} \frac{2 \sin \theta}{\theta}}
\end{aligned}$$

$$\lim_{x\to 0}\frac{\frac{1}{\cos 3x}}{\frac{1}{\cos 2x}} = \frac{\frac{1}{1}}{\frac{1}{1}} = 1$$

$\lim_{\phi\to 0}\frac{3\sin\phi}{\phi} = 3\cdot\lim_{\phi\to 0}\frac{\sin\phi}{\phi} = 3\cdot 1 = 3$ by Theorem 2 (the property that $\lim_{x\to 0}\frac{\sin x}{x} = 1$)

$$\lim_{\theta\to 0}\frac{2\sin\theta}{\theta} = 2\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 2\cdot 1 = 2$$

We then conclude that

$$\lim_{x\to 0}\frac{\tan 3x}{\tan 2x} = 1\cdot\frac{3}{2} = \frac{3}{2}.$$

Exam 1 Review

Example 3. *(from exercise 26 on page 70)* Determine the points of discontinuity of

$$h(z) = \frac{1-2z}{z^2-z-6}$$

With rational functions, discontinuities are found where the function is undefined, and rational functions are undefined when the denominator is zero.

$$z^2 - z - 6 = (z-3)(z+2)$$

$$h(z) = \frac{1-2z}{(z-3)(z+2)}$$

We find that the points of discontinuity are $z = -2, 3$.

In general for rational functions, the points of discontinuity are removable if the numerator and denominator cancel out, we infinite discontinuities if the numerator and denominator do not cancel out.

In this case, $z = -2, 3$ don't cancel out in the denominator, so they're vertical asymptotes, i.e. infinite discontinuities

Example 4. *(from exercise 24 on page 70)*

$$k(z) = \frac{x-2}{|2-x|}$$

The discontinuity is found at $x = 2$ and is a jump discontinuity.

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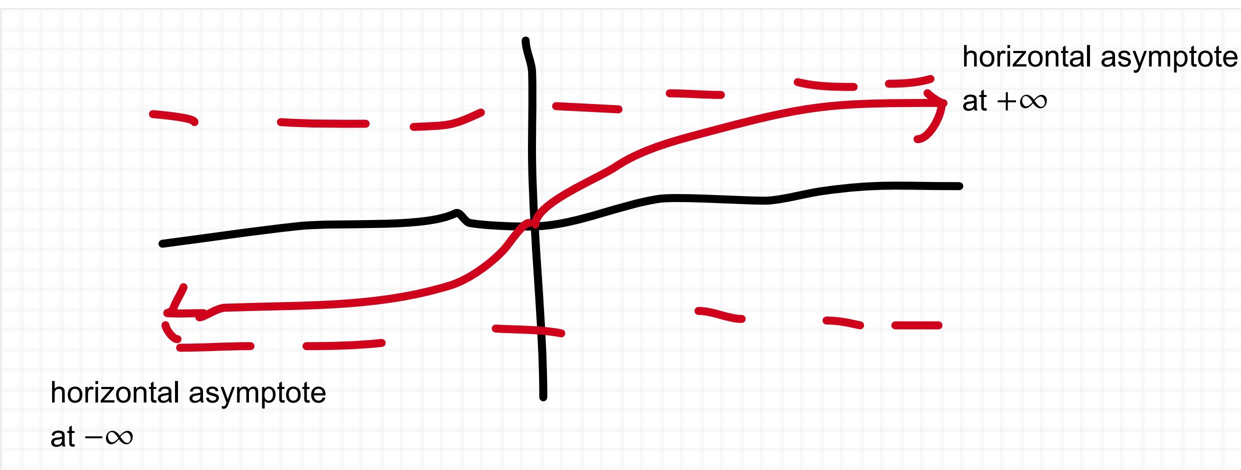
Limits to Infinity

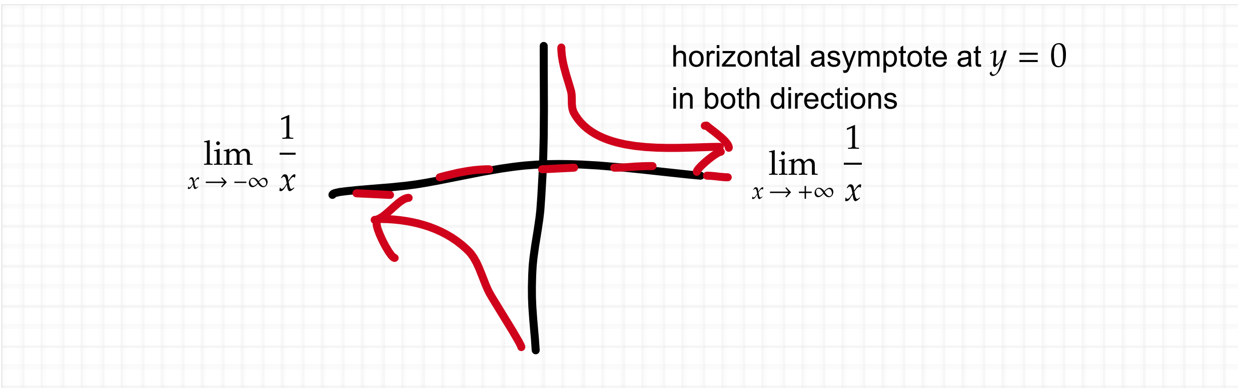
Earlier we talked about limits at infinity to mean whenever $\lim_{x \to c^+} f(x) = \pm\infty$, $\lim_{x \to c^-} f(x) = \pm\infty$. We haven't yet talked about the concept of $x \to \pm\infty$. You might recall from high school the notion of a **horizontal asymptote**, which is horizontal line that the function approaches and doesn't surpass as x gets large, either in the positive or negative direction



Thinking about how obtain a horizontal asymptote allows us to then conceptualize the idea of limits to infinity.

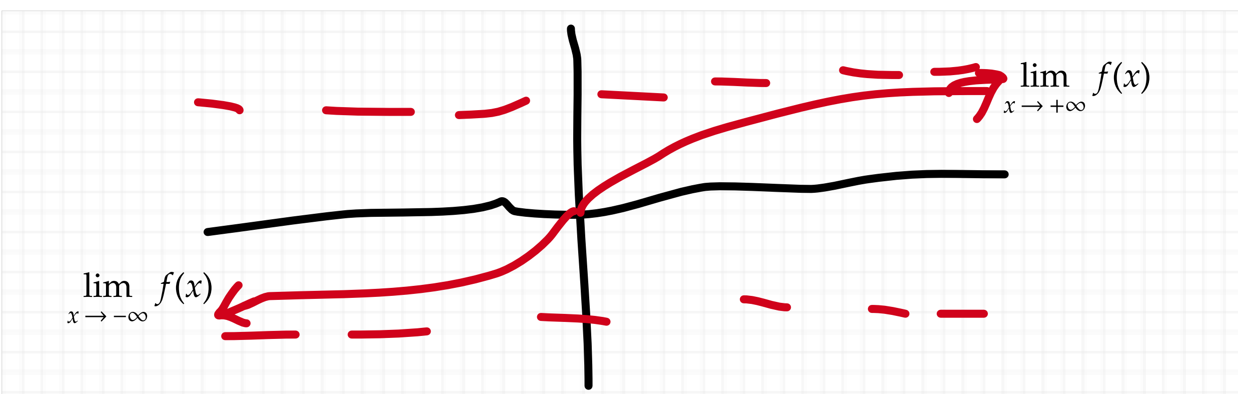
We say that $x \to \pm\infty$ when we're looking x as it gets infinitely large (in the positive direction for $+\infty$ and the negative direction for $-\infty$). And we say that $\lim_{x \to +\infty} f(x)$ (resp. $\lim_{x \to -\infty} f(x)$) exist whenever there's a value that $f(x)$ approaches as x gets infinitely larger.

Example 1. To evaluate $\lim_{x \to +\infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$, we look at the horizontal asymptote in both of those directions and conclude that $\lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$



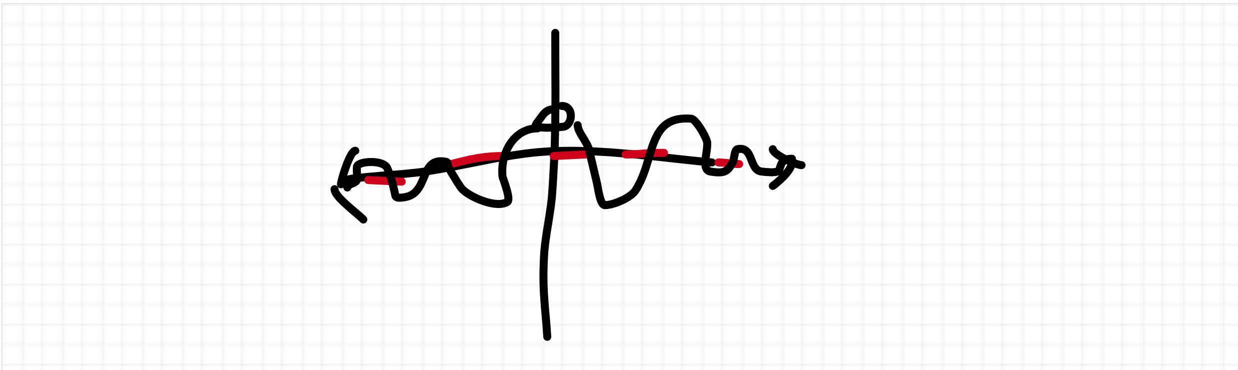
More generally, involving horizontal asymptotes, we find the following theorem holds:

Theorem 1. If there is a horizontal asymptote $y = b$ the right (resp. left) side of the graph of f, we find that the limit as $\lim_{x \to +\infty} f(x)$ (resp. $\lim_{x \to -\infty} f(x)$ exists) and $\lim_{x \to +\infty} f(x) = b$ (resp. $\lim_{x \to -\infty} f(x) = b$ exists).

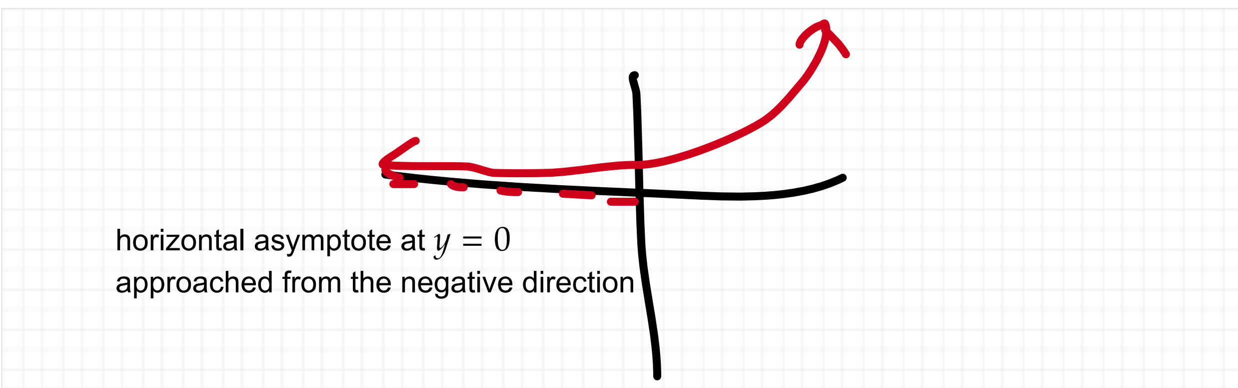


However, although horizontal asymptotes give us an intuitive picture of what limits to infinity might look like, and existence of horizontal asymptotes imply existence of a limit to infity, the converse does not hold. In other words, just because the limit exists as $x \to \pm\infty$ doesn't mean that limit is a horizontal asymptote. The following is an example where a limit exists as $x \to \pm\infty$ but isn't a horizontal asymptote:

Example 2. $\lim_{x \to +\infty} \frac{1}{x} \sin x = 0$, $\lim_{x \to -\infty} \frac{1}{x} \sin x = 0$. We can find this limit using the squeeze theorem (which still holds for infinite limits) with $-\frac{1}{|x|} \leq \frac{1}{x} \sin x \leq \frac{1}{|x|}$.



Example 3. *(from Example 1 in page 82-93)* Let's look at the $\lim_{x \to +\infty} 2^x$, $\lim_{x \to -\infty} 2^{-x}$



$\lim_{x \to +\infty} 2^x = +\infty$, $\lim_{x \to -\infty} 2^x = 0$

IMPORTANT NOTE: The limit laws that we learned in the "Basic Limit Laws" of the notes (i.e. section 2.2 of the book) also hold for limits at infinity, with the additional addendums to each of the properties

addendum (i) If $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} g(x) = +\infty$, we have $\lim_{x \to \pm\infty} f(x)g(x) = +\infty$.

addendum (ii) If $\lim_{x \to \pm\infty} f(x) = \pm\infty$, then $\lim_{x \to +\infty} \frac{1}{f(x)} = 0$.

%ADD THESE CONCEPTS OF LIMIT PROPERTIES TO "LIMITS AT INFINITY" FOR THE

NEXT TIME

Same goes for other properties and techniques we've learned involving limits such as the *Squeeze Theorem* and *Change of Variables*. This allows us to evaluate the limit for the following limits:

Note by the identity rule, we have $\lim\limits_{x \to +\infty} x = +\infty$ and $\lim\limits_{x \to -\infty} x = -\infty$,

and more generally we have the following theorem involving limits of integer power functions

Theorem 2. for any integer n, we have

$$\lim_{x \to +\infty} x^n = \begin{cases} +\infty & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n < 0 \end{cases}, \quad \lim_{x \to -\infty} x^n = \begin{cases} +\infty & \text{if } n > 0 \text{ and even} \\ -\infty & \text{if } n > 0 \text{ and odd} \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Proof. The case where $n = 0$, note that $x^0 = 1$, and hence this case is trivial. For $n > 0$, apply repeatedly the multiplication property $\lim\limits_{x \to +\infty} x = +\infty$ and $\lim\limits_{x \to -\infty} x = -\infty$, apply the substitution $y = -x$, and note by repeated use of *addendum (i)* that

$$\lim_{x \to -\infty} x^n = (-1)^n \lim_{x \to -\infty} (-x)^n = (-1)^n \lim_{y \to +\infty} y^n = (-1)^n(+\infty).$$

For the case where $n < 0$, note that we've found $\lim\limits_{x \to \pm\infty} \dfrac{1}{x} = 0$, and we can repeatedly use the multiplication property $-n$ times to get

$$\lim_{x \to \pm\infty} x^n = \left(\lim_{x \to \pm\infty} \frac{1}{x} \right)^{-n} = 0.$$

We can also prove this case using *addendum (ii)* on the fact that $x^n = \dfrac{1}{x^{-n}}$ and $\lim\limits_{x \to \pm\infty} x^{-n} = \pm\infty$, which we proved from the previous case. $\square$

We can then use *Theorem 2* above to find the limit in the following example:

Example 4. *(from Example 2 in page 84)* $\lim\limits_{x \to +\infty} \dfrac{20x^2 - 3x}{3x^5 - 4x^2 + 5}$

Note by *Theorem 2*, we have

$$\lim_{x \to +\infty} \frac{1}{x^n} = \lim_{x \to +\infty} x^{-n} = 0,$$

for $n > 0$. As a result, we find that

$$\lim_{x \to +\infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} = \lim_{x \to +\infty} \frac{\frac{1}{x^5}}{\frac{1}{x^5}} \cdot \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} = \lim_{x \to +\infty} \frac{\frac{20x^2}{x^5} - \frac{3x}{x^5}}{\frac{3x^5}{x^5} - \frac{4x^2}{x^5} + \frac{5}{x^5}} = \lim_{x \to +\infty} \frac{\frac{20}{x^3} - \frac{3}{x^4}}{3 - \frac{4}{x^3} + \frac{5}{x^5}}$$

$$= \frac{\lim_{x \to +\infty} \frac{20}{x^3} - \frac{3}{x^4}}{\lim_{x \to +\infty} 3 - \frac{4}{x^3} + \frac{5}{x^5}} = \frac{0 - 0}{3 - 0 + 0} = 0.$$

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Example 5. $\lim\limits_{x \to -\infty} \dfrac{50x^3 + x}{x^2 + 3x - 4}$

$$\lim_{x \to -\infty} \frac{50x^3 + x}{x^2 + 3x - 4} = \lim_{x \to -\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{50x^3 + x}{x^2 + 3x - 4} = \lim_{x \to -\infty} \frac{\frac{50x^3}{x^2} + \frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}} = \frac{\lim_{x \to -\infty} 50x - \frac{1}{x}}{\lim_{x \to -\infty} 1 + \frac{3}{x} - \frac{4}{x^2}}.$$

Since

$$\lim_{x \to -\infty} 50x - \frac{1}{x} = -\infty, \quad \lim_{x \to -\infty} 1 + \frac{3}{x} - \frac{4}{x^2} = 1,$$

we conclude by *addendum (ii)* that

$$\lim_{x \to -\infty} \frac{50x^3 + x}{x^2 + 3x - 4} = -\infty$$

Example 6. $\lim_{x \to -\infty} \frac{11x^4 + 3x^3}{23x^4 + 7x - 2}$
%DERIVE THIS EXAMPLE

From a proof that is a generalization of what we did in *Example 4-6*, we have the following theorem:

Theorem 3. *(Limit to infinity for a Rational Function)* The asymptotic behavior of a rational function depends only on the leading terms of its numerator and deonminator. If $a_n, b_m \neq 0$, then

$$\lim_{x \to \pm\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x \to \pm\infty} x^{n-m}.$$

3/2

To better understand *Theorem 3*, observe that

$$\lim_{x \to +\infty} x^{n-m} = \begin{cases} +\infty & \text{if } n > m, \\ 1 & \text{if } n = m, \\ 0 & m > n, \end{cases}$$

and

$$\lim_{x \to -\infty} x^{n-m} = \begin{cases} -\infty & \text{if } n > m \text{ and } n - m \text{ is odd,} \\ +\infty & \text{if } n > m \text{ and } n - m \text{ is even,} \\ 1 & \text{if } n = m, \\ 0 & m > n. \end{cases}$$

It follows that

$$\lim_{x \to +\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \begin{cases} +\infty & \text{if } n > m, \\ \frac{a_n}{b_m} & \text{if } n = m, \\ 0 & m > n, \end{cases}$$

and

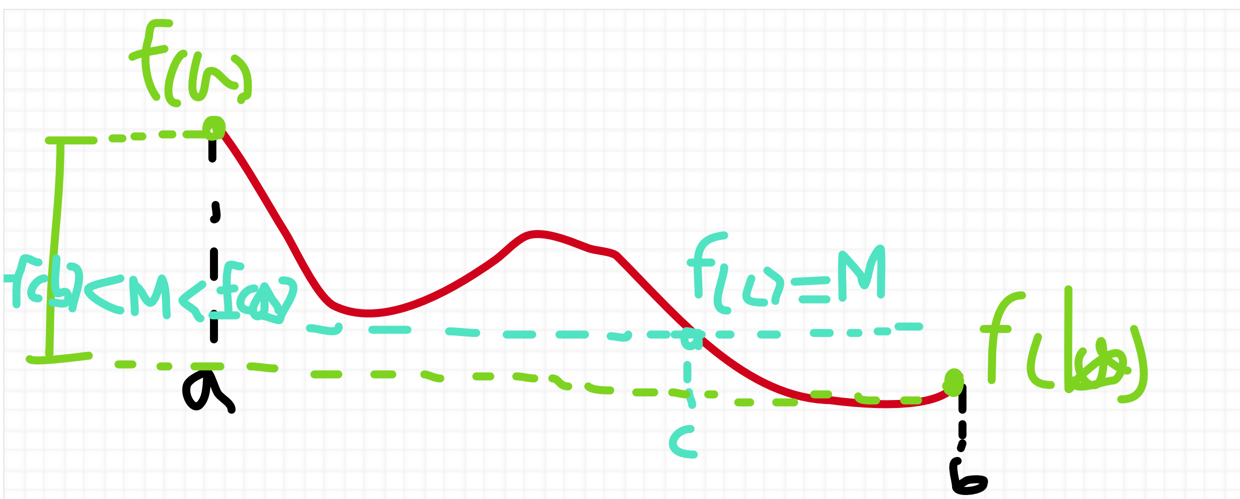
$$\lim_{x \to -\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \begin{cases} -\infty & \text{if } n > m \text{ and } n - m \text{ is odd,} \\ +\infty & \text{if } n > m \text{ and } n - m \text{ is even,} \\ \dfrac{a_n}{b_m} & \text{if } n = m, \\ 0 & m > n. \end{cases}$$

Intermediate Value Theorem

IVT is the intermediate value theorem for short.

Theorem 1. *(Intermediate Value Theorem)* If f is continuous on a closed interval $[a, b]$, then for every value M strictly between $f(a)$ and $f(b)$ (i.e. M is in the interval $(\min(f(a), f(b)),\ \max(f(a), f(b)))$, then the equation $f(x) = M$ has a solution $a < c < b$.

In other words, there exists some number c such that $a < c < b$ that we can plug into the equation $f(x) = M$ and the equation holds true and we have $f(c) = M$.



Proof. Continuous are functions that you can "draw without lifting the pencil", so you can't take on the value $f(b)$ without taking on $f(c) = M$ for some M. □

Example 1. We want to show that $\sin x = .3$ has at least one solution in the interval $\left(0, \dfrac{\pi}{2}\right)$.

To show this, note that $\sin 0 = 0,\ \sin \dfrac{\pi}{2} = 1$. Then the intermediate value theorem tells us that there is some c such that $\sin c = M$, for any $\sin 0 = 0 < M < 1 = \sin\left(\frac{\pi}{2}\right)$. We conclude that $c \in (0, 1)$ such that $\sin c = .3$ is the solution.