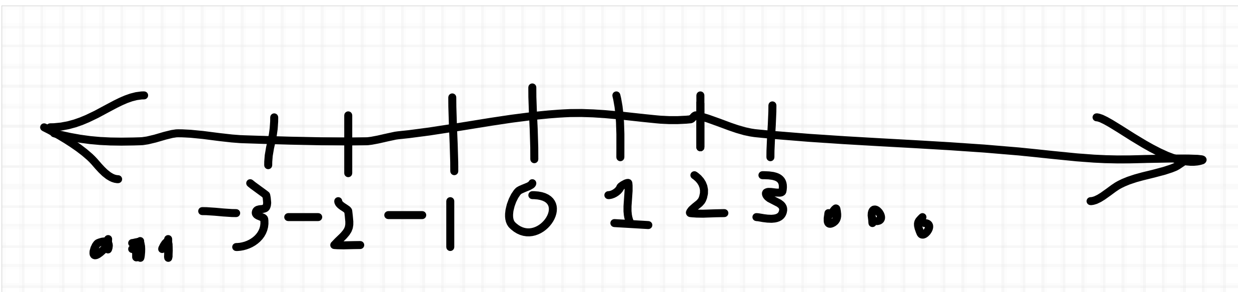
Spring 2023 Math 151C Chapter 1 Lecture Notes

1/31 (Rewrite from Chalkboard Lecture)

Sets of Real Numbers and Interval Notation

real number-a "decimal expansion" number on the "number line".



rational number-A number that can be expressed as a ratio of two integers, i.e., $\frac{a}{b}$, $b \neq 0$

Equivalently, it can be expressed as a finite decimal expansion or an eventually repeating one.

$$\frac{1}{2},\ 0.5,\ 0.1\overline{6},\ 0.12\overline{34}$$

In this course, we often talk about "sets" of numbers.

Set Notation

The set of rational numbers can be expressed as the set

$$\left\{x : \underbrace{x = \frac{a}{b} \text{ where } a \text{ and } b \neq 0 \text{ are integers}}\right\}$$

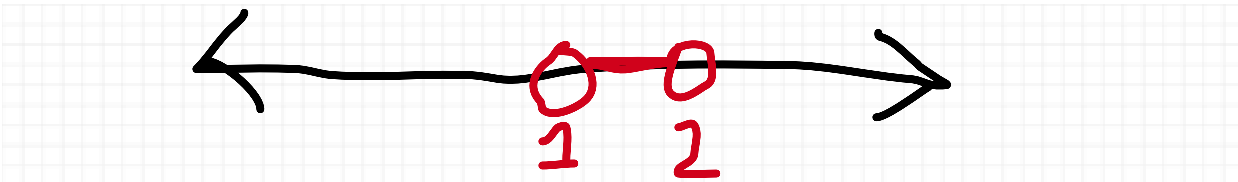
$\uparrow$ a real number $\uparrow$ description

The bracketed notation above we often like to use as notation for a set of real numbers.

Example 1.

$\{x : 1 < x < 2\}$

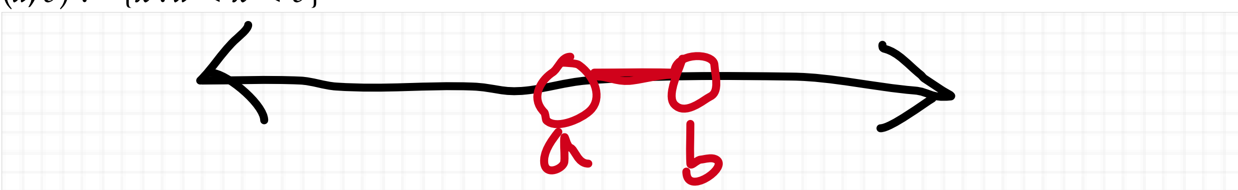
The set above we call an interval, which is a segment of the number line



Interval Notation

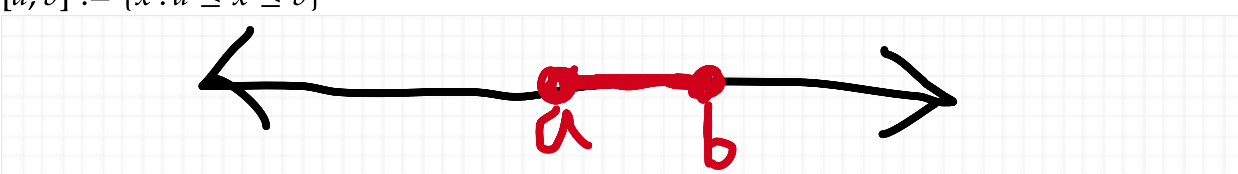
Open Interval

$(a, b) := \{x : a < x < b\}$



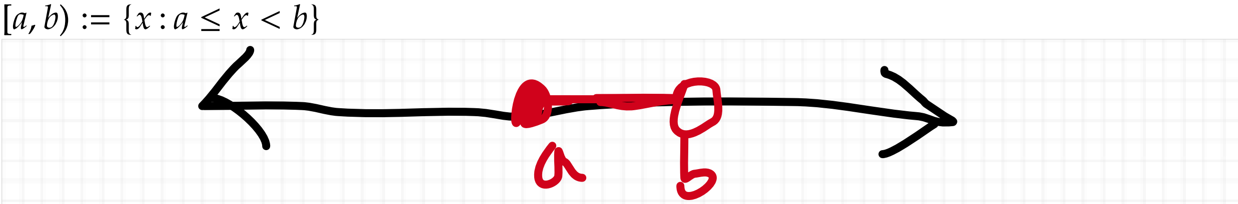
Closed Interval

$[a, b] := \{x : a \leq x \leq b\}$

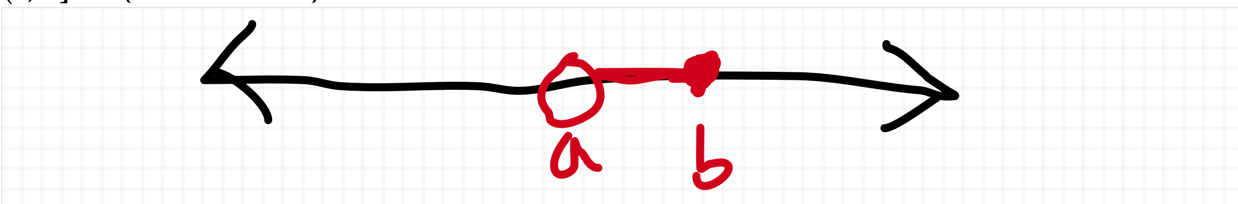


Half-Open Interval

$[a, b) := \{x : a \leq x < b\}$

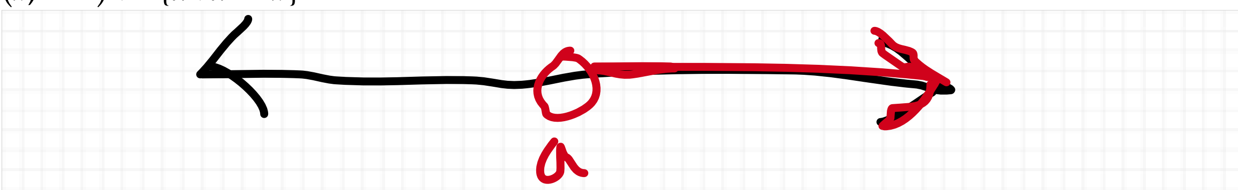


$(a, b] := \{x : a < x \leq b\}$

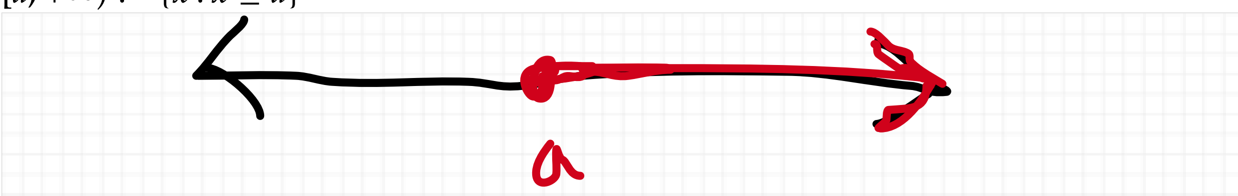


Unbounded Interval

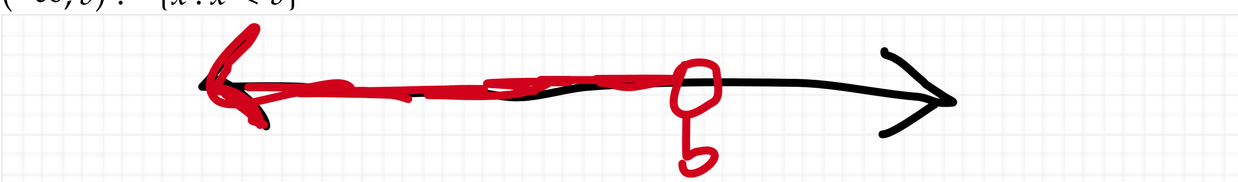
$(a, +\infty) := \{x : x > a\}$



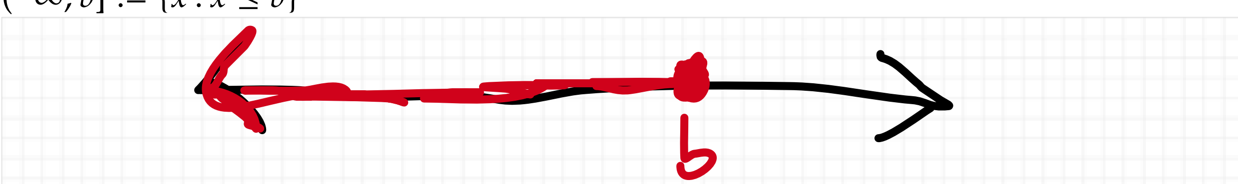
$[a, +\infty) := \{x : x \geq a\}$



$(-\infty, b) := \{x : x < b\}$



$(-\infty, b] := \{x : x \leq b\}$



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Functions and Their Graphs

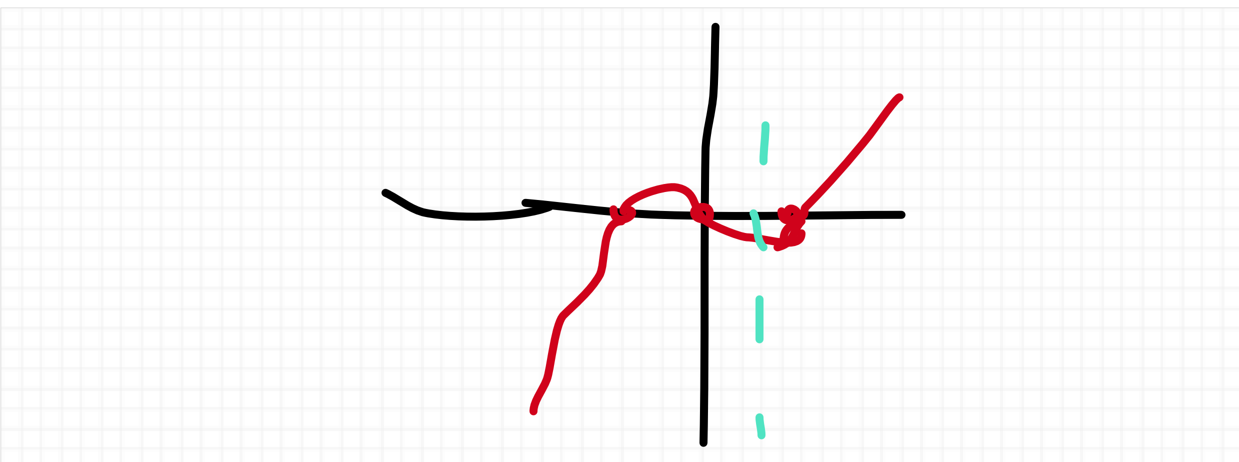
A **function** f is a mapping from a input set D (called the domain) and the output set Y (called the range), and more specifically it's "rule" that takes an input x in D and assigns it to an *unique* output value $f(x)$ in Y

Example 1. Let's look at the function

$$f(x) = x^3 - 2x = x\left(x^2 - 2\right) = x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right).$$

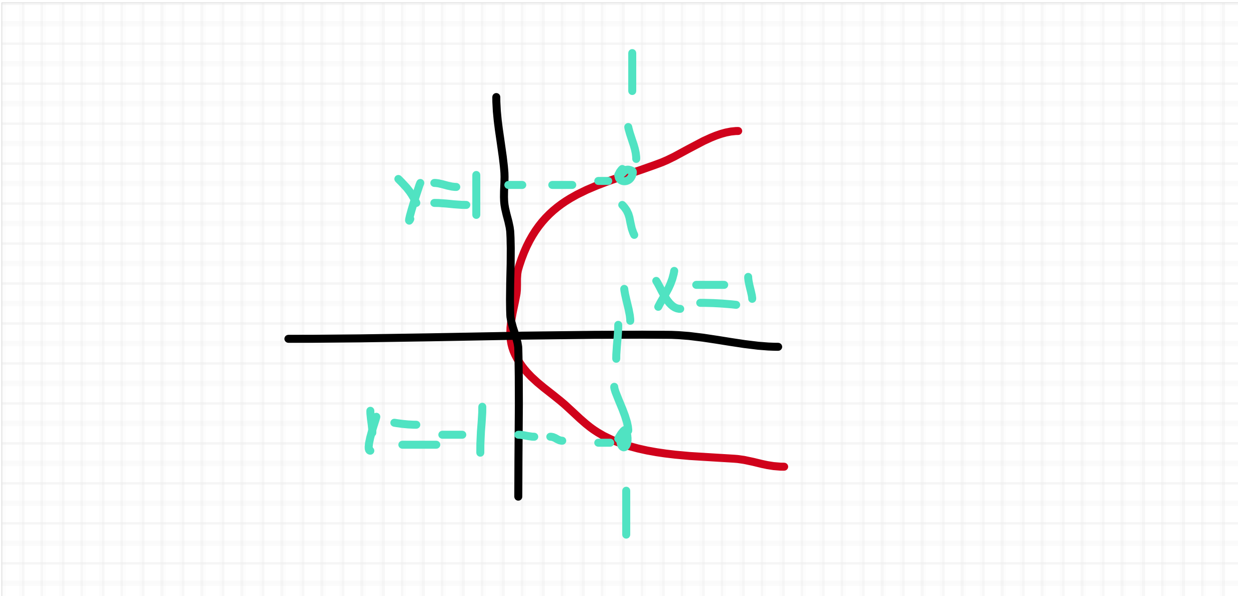
This is a function from the set of real numbers (we call this set $\mathbb{R}$ or $(-\infty, +\infty)$) to the real numbers $\mathbb{R}$.

With functions, we often like to graph them using the cartesian plane (the two dimensional xy-plane). The graph of f is illustrated as follows.



We find the graph above is a function since every input value has a unique output value.

The curve with the equation $x = y^2$ is not a function because the output value is not unique.



A way to see whether a curve is a function is through the **vertical line test**, where if a vertical line through the curve always intersects with the curve, then it's function. Otherwise it's not.

Types of Functions

Even and Odd Functions

Here is some types of functions that we'll deal with in this class.

f is an **even function** if $f(-x) = f(x)$ for all input values x
f is an **odd function** if $f(-x) = -f(x)$ for all input values x

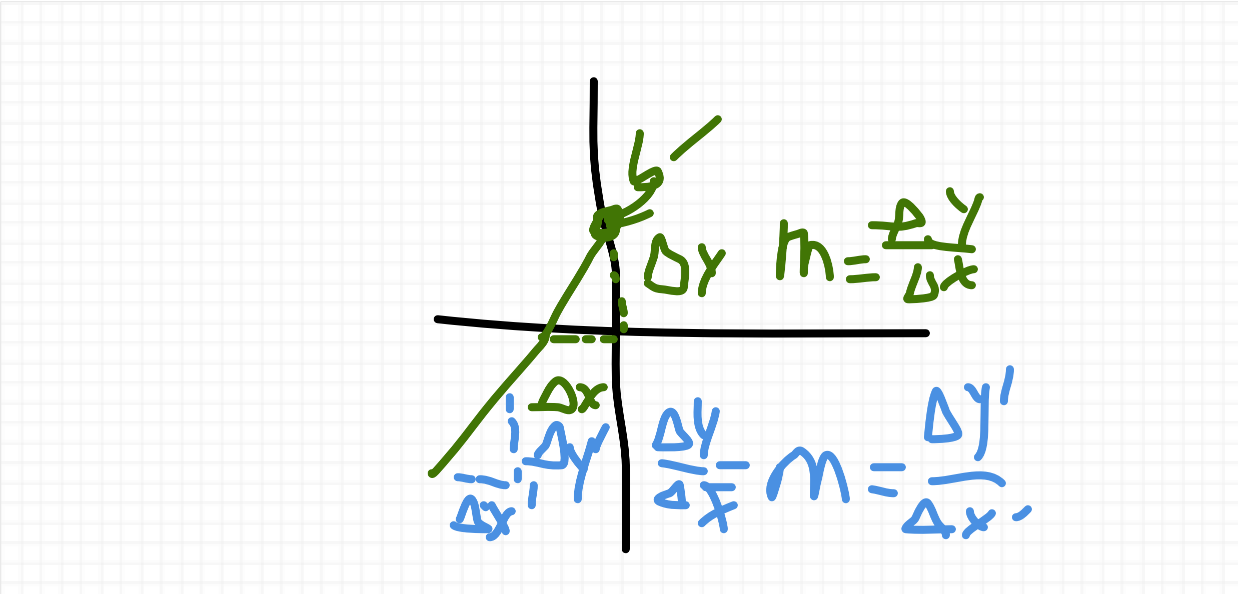
Examples of even functions include x^2, x^4, x^{2n}, and examples of and odd function include x, x^3, x^{2n+1}

Linear Functions

The next type of function is a **linear function**, which could be arranged in following possible equations.

Slope-intercept form

$y = mx + b$ where m refers to the slope and b refers to the y intercept



NOTE: A line has the crucial property where the ratio of the rise and run is always the same.

Point-slope form

Given two points $(x_0, y_0), (x_1, y_1)$, or the slope m

$$y - y_0 = m(x - x_0)$$
$$m = \frac{y - y_0}{x - x_0}$$

$y - y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}(x - x_0)$ (noting that $m = \dfrac{y_1 - y_0}{x_1 - x_0}$)

Note that usually the point-slope form is easier to find the equation of a line given two points (or given a point and a slope).

Polynomials, Rational Functions, and Algebraic Functions

Note first that polynomial functions include **quadratic functions**, which are of the form $y = ax^2 + bx + c$

They also include **power functions**, which are functions that takes the input value to some exponential power a (a real number)

$f(x) = x^a$

Polynomials are linear combinations of powers of the input. It's any function that can be expressed in the form

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$, we call $a_0, \ldots, a_n \neq 0$ the coefficients, and the value n of the highest nonzero coefficient a_n (the leading coefficient) the degree of the polynomial.

Rational functions are those that can be expressed as a (fraction) ratio of two polynomials $P(x), Q(x)$

$$f(x) = \frac{P(x)}{Q(x)}$$

Algebraic Functions are produced by taking sums, products, and quotients of roots of polynomials and rational functions

$$f(x) = \sqrt{1 + 3x^2 - x^4}, \;\; g(x) = \left(\sqrt{x} - 2\right)^{-2}, \; h(x) = x^{1/3}$$

Exponential and Logorithmic Functions

Exponential function: A function that takes the input value x and outputs a^x, i.e. some real number a taken to the power x of the input value.

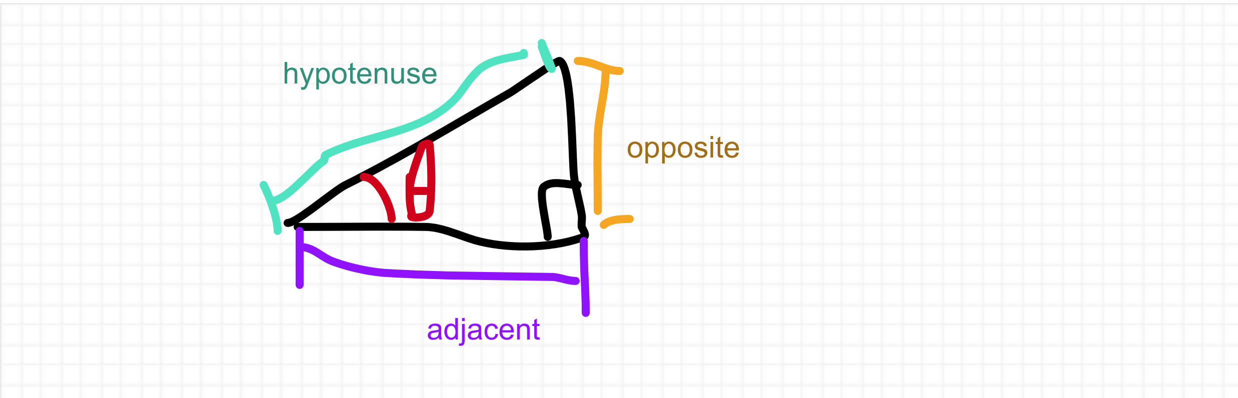
$$f(x) = 2^x, \quad h(x) = \left(-\frac{1}{3}\right)^{1/3}, \; g(x) = e^x$$

logorithmic function is the inverse of an exponential function.

$\log_a(x)$ is the value y such that $x = a^y$
$\ln(x) = \log_e(x)$

Trigonometric functions

Trigonometric functions are functions that input the angle θ of a right triangle and output the ratio of two sides of a triangle



$$\sin\theta := \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\theta := \frac{\text{ajacent}}{\text{hypotenuse}} \quad \tan\theta := \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta := \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}$$

$$\sec\theta := \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta}$$

$$\cot\theta := \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Constructing New Functions

We can construct new functions both by **adding, multiplying and dividing functions**:

$$(f+g)(x) := f(x)+g(x), \; (f-g)(x) := f(x)-g(x)$$
$$(f\cdot g)(x) := f(x)\cdot g(x), \; \left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)} \quad g(x) \neq 0$$

We can moreover find the **composition** of two functions f, g to get

$$(f\circ g)(x) = f(g(x))$$

Example 1. Let's take $f(x) = \sqrt{x}$, $g(x) = 1 - x$, and find $f \circ g$ and $g \circ f$

$(f \circ g)(x) = f(g(x)) = f(1 - x) = \sqrt{1 - x}$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = 1 - \sqrt{x}$

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Piecewise Functions

So piecewise are functions that are defined on a case-by-case basis

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$