Algebraic Topology Notes

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1 Singular Homology Review

§ The Eilenberg-Steenrod Axioms

§ CW Complexes and Simplicial Complexes

Definition 1.1. A relative CW complex is a map $A \to X$ together with a filtration $X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$ (called the *skiletal filtration*) such that

• $X^{-1} = A$.
• Each X^n is formed from X^{n-1} by attaching n-cells.
• X has the topology of the union: A set closed in X if and only if the restriction to each X^n is closed.

A CW complex is a relative CW complex with $X^{-1} = \emptyset$.

Definition 1.2. Let X be a CW complex and let A be a subset of X. A is a *subcomplex* of X means that it is a union of closed cells in X with the property that if an n-cell of A intersects the relative interior of a lower dimensional closed cell of X, then it contains the entire closed lower dimensional cell. Equivalently, A is a CW complex with its set of cells a subset of the cells of X.

Definition 1.3. Let X and Y be CW complexes. A cellular map $f\colon X \to Y$ is a continuous map that preserves the skeletal filtration: $f(X^n) \subset Y^n$.

Definition 1.4. An *abstract simplicial complex* consists of a set V and a set S of non-empty finite subsets of V, containing all singleton subsets and satisfying the condition that if $\sigma \in S$ then all non-empty subsets of σ are in S.

• The elements of V are called vertices.
• The elements of S are called simplices.
• $\sigma \in S$ is an *n-simplex* means that it is an $n+1$ element subset of V.
• An ordered abstract simplicial complex is an abstract simplicial complex (V, S) together with a total ordering of V.
• A map of abstract simplical complexes $(V, S) \to (V', S')$ is a map of sets $V \to V'$ such that S loands in S' under the induced map of power sets.

Example 1.5. The standard n-Simplex is defined as follows:

$\Delta[n] : V_{\Delta[n]} = \{0, \ldots, n\}$, $S_{\Delta[n]}$ = all finite subsets.
Face maps $\partial_i \colon \Delta[n-1] \to \Delta[n]$

$$0, \ldots, n-1 \mapsto 0, \ldots, \widehat{i}, \ldots, n$$

$\Delta[n] \subset \mathbb{R}^{n+1}$, standard basis $e_0, \ldots, e_n$, Barycentric coordinate $t_0 e_0 + \cdots + t_n e_n$, $t_0, \ldots, t_n \geq 0$, $t_0 + \cdots + t_n = 1$.

Definition 1.6. Let $X = (V_X, S_X)$ be an abstract simplicial complex. Let $\mathbb{R}(V_X)$ denote the real vector space with basis V_X. For $\sigma = \{v_0, \ldots, v_n\}$ an n-simplex of X, let $|\sigma|$ be the subspace of $\mathbb{R}(\sigma) \subset \mathbb{R}(V_X)$ (with its topology) of elements of the form $t_0 v_0 + \cdots + t_n v_n$ with $t_i \geq 0$ for all i, and $t_0 + \cdots + t_n = 1$. We call $|\sigma|$ the geometric n-simplex of X.

Let $|X|$ be the subset of $\mathbb{R}(V_X)$ consisting of the union of the geometric n-simplices, with the union topology: A subset of $|X|$ is closed if and only if its restriction to each geometric n-simplex is closed.

§ Chain Complexes, Tensor, and Hom

Chain Complexes:
$\mathbb{Z}$-graded (homological grading)

$$\cdots \xrightarrow{d} A_{n+1} \xrightarrow{d} A_n \xrightarrow{d} A_{n-1} \xrightarrow{d} \cdots \quad d \circ d = 0$$

(cohomological grading)

$$\cdots \xrightarrow{d} B^{n-1} \xrightarrow{d} B^n \overset{B}{\longrightarrow}^{n} \xrightarrow{d} B^{n+1} \xrightarrow{d} \cdots \quad d \circ d = 0$$

Switch: Set $B^n = A_{-n}$ or $A_n = B^{-n}$.

Homology of Chain Complex:

$$H_n(A_*) = \frac{\operatorname{Ker}(d \colon A_n \to A_{n-1})}{\operatorname{Img}(d \colon A_{n+1} \to A_n)} \text{ or } H^n(B^*) = \frac{\operatorname{Ker}(d \colon B^n \to B^{n+1}}{\operatorname{Img}(d \colon B^{n-1} \to B^n}$$

If $B^* = A_{-*}$, then $H^*(B^*) = H_{-*}(A_*)$

Map of Chain Complexes

A map of a chain complexes $A_* \to A'_*$ consists of maps of abelian groups $A_n \to A'_n$ for all n that commute with d.

Tensor Product:

Context: R is a commutative ring, M, N are R-modules
Construction:

$$M \otimes_R N = \frac{F(M \times N)}{R(M, N)},$$

where $F(M\times N)$ is the free R-module on the set $M\times N$ and $R(M,N)$ is the submodule of $F(M\times N)$ generated by
• $(a_1+a_2,b)-(a_1,b)-(a_2,b)$ for all $a_1,a_2\in M$, $b\in N$
• $(a,b_1+b_2)-(a,b_1)-(a,b_2)$ for all $a\in M$, $b_1,b_2\in N$
• $r(a,b)-(ra,b)$ for all $a\in M$, $b\in N$, $r\in R$
• $r(a,b)-(a,rb)$ for all $a\in M$, $b\in N$, $r\in R$

Canonical map of sets $M\times N\to M\otimes_R N$. Image of (a,b) in $M\otimes_R N$ denoted $a\otimes b$.

Bilinear Map

Let P be an R-module. A bilinear map $\phi\colon M\times N\to P$ is a map of sets such that

• $\phi(a_1+a_2,b)=\phi(a_1,b)+\phi(a_1,b)+\phi(a_2,b)$ for all $a_1,a_2\in M$, $b\in N$
• $\phi(a,b_1+b_2)=\phi(a,b_1)+\phi(a,b_2)$ for all $a\in M, b_1,b_2\in N$
• $\phi(ra,b)=r\phi(a,b)=\phi(a,rb)$ for all $a\in M$, $b\in N$, $r\in R$.

Proposition 1.7. Let $\phi\colon M\times N\to P$ be a bilinear map. Then there is a unique R-module map $\widetilde{\phi}\colon M\otimes_R N\to P$ such that ϕ is the composite $M\times N\to M\otimes_R N\to P$.

Some Basic Properties:

• $M\otimes_R N\cong N\otimes_R M$
• $R\otimes_R M\cong M$
• $(\bigoplus M_\alpha)\otimes_R N\cong\bigoplus(M_\alpha\otimes_R N)$
• $(M\otimes_R N)\otimes_R P\cong M\otimes_R(N\otimes_R P)$
• $\otimes_R$ is a functor R-Mod $\times$ R-Mod $\to$ R-Mod
• For fixed N, the functor $(-)\otimes N$ is right exact.

An additive functor $F(A)\oplus F(B)\cong F(A\oplus B)$ preserve split exact sequences.

The Functor Hom:

Context: R is a commutative ring, M,N are R-modules
$\mathrm{Hom}_R(M,N)$ has the canonical structure of an R-module:

• $(f+g)(m):=f(m)+g(m)$
• $(rf)(m)=rf(m)=f(rm)$

Properties of Hom

• $\mathrm{Hom}_R(R,M)\cong M$
• $\mathrm{Hom}_R\left(\bigoplus M_\alpha,N\right)\cong\prod\mathrm{Hom}_R(M_\alpha,N)$
• $\mathrm{Hom}_R\left(M,\prod N_\alpha\right)\cong\prod\mathrm{Hom}_R(M,N_\alpha)$
• Hom_R is a functor $R\text{-Mod}^op\times R-\mathrm{Mod}\to R-\mathrm{Mod}$.
For fixed N, the functor $\mathrm{Hom}_R(-,N)$ is left exact
For fixed M, the functor $\mathrm{Hom}_R(M,-)$ is left exact

The Tensor-Hom Adjunction
$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$
Let $F(-) = (-) \otimes_R N$, $G = \mathrm{Hom}_R(N, -)$,

$$\mathrm{Hom}_R(F(M), P) \cong \mathrm{Hom}_R(M, G(P))$$

Tensor and Hom on Chain Complexes

Let C_* be a chain complex of R-modules

Construction:
$C_* \otimes_R M$ is the chain complex with abelian groups (or R-modules)

$$(C_* \otimes_R M)_n = C_n \otimes_R M$$

and differential $d(c \otimes m) := d(c) \otimes m$

$\mathrm{Hom}_R(C_*, M)$ is the chain complex with abelian groups (or R-modules)

$$\mathrm{Hom}_R(C_*, M)^n = \mathrm{Hom}(C_n, M)$$

and differential $df = -(-1)^n f \circ d$ in dimension n (to dimension $n+1$)

We can check and see these differentials have our desired properties.

§ Chain Complexes Associated to Spaces: Singular Theory

§ Chain Complexes Associated to Spaces: Simplicial and Cellular Theory

2 Homological Algebra

§ Derived Functors, Tor, and Ext

Definition 2.1.

1. A functor F between module categories is a *right exact functor*. Then for every short exact sequence

$$0 \to A \to B \to C \to 0$$

we get an exact sequence

$$F(A) \to F(B) \to F(C) \to 0.$$

2. A contravariant functor G between module categories. We say that F is a *left exact functor* if for every SES $0 \to A \to B \to C \to 0$, we get an exact sequence

$$0 \to F(C) \to F(B) \to F(A).$$

Proposition 2.2. Let F be a right exact functor from a module category to a module category. F is a *left derived functor* if there exist functors $L^n F, n = 0, 1, 2, \dots$ and natural transformations $L^n F(C) \to L^{n-1} F(A)$ associated to a SES satisfying

• $L^0 F = F$ (or at least naturally isomorphic)

• For $0 \to A \to B \to C \to 0$ SES, the sequence

$$\cdots \to L^n F(A) \to L^n F(B) \to L^n F(C) \to L^{n-1} F(A) \to \cdots \to L^1 F(C) \to L^0 F(A) \to L^0 F(B) \to L^0 F(C) \to 0$$

is long exact.

• If M is free then $L^n F(M) = 0$ for $n > 0$.

Moreover, the system of functors and natural transformation satisfying the four conditions is unique up to unique compatible isomorphism.

Moreover, the system is natural in maps (= natural transformations) of right exact functors F.

Definition 2.3. Define $\mathrm{Tor}_n^R(-, M) := L^n((-) \otimes_R M)$ or $\mathrm{Tor}_n^R(M, -) := L^n(M \otimes_R (-))$

Bifunctor: R-Mod $\times$ R-Mod $\to$ R-Mod

Note:

For $R = \mathbb{Z}$, $\mathrm{Tor}_n^{\mathbb{Z}}(M,N) = 0$, for $n > 1$ and we write $\mathrm{Tor}(M,N)$ for $\mathrm{Tor}_1^{\mathbb{Z}}(M,N)$

More generally, if R is a PID, then $\mathrm{Tor}_n^R(M,N) = 0$ for $n > 1$ and if R is a field, then $\mathrm{Tor}_n^R(M,N) = 0$ for $n > 0$.

Let's compute $\mathrm{Tor}_n^R(R/a, -)$,

$$0 \to {}_aR \xrightarrow{\subset} R \xrightarrow{\cdot a} R \xrightarrow{R} /a \to 0$$

Notation ${}_aM = \{m \in M | am = 0\}$. Assume ${}_aR = 0$.

$$\begin{aligned} \cdots \to &\mathrm{Tor}_2^R(R,M) \to \mathrm{Tor}_2^R(R,M) \to \mathrm{Tor}_2^R(R/a,M) \to \\ &\mathrm{Tor}_1^R(R,M) \to \mathrm{Tor}_1^R(R,M) \to \mathrm{Tor}_1^R(R/a,M) \to R \otimes_R M \to R \otimes_R M \to R/a \otimes_R M \to 0 \\ &R/a \otimes_R M \cong M/a, \quad \mathrm{Tor}_1^R(R/a,M) \cong {}_aM, \quad \mathrm{Tor}_n^R(R/a,M) = 0 \text{ for } n > 1 \end{aligned}$$

Let F be a contravariant left exact functor from a module category to a module category. For a SES $0 \to A \to B \to C \to 0$, we get an exact sequence

$$0 \to F(C) \to F(B) \to F(A)$$

Can we continue it to the right?

Proposition 2.4. We say that F is a *contravariant right derived functor* if there exist functors $R^nF, n = 0, 1, 2, \ldots$ and natural transformations $R^{n-1}F(A) \to R^nF(C)$ associated to a SES satisfying

- $R^0F = F$ (or naturally isomorphic)

- For $0 \to A \to B \to C \to 0$ SES, the sequence

$$0 \to R^0F(C) \to R^0F(B) \to R^0F(a) \to R^1F(C) \to \cdots \to R^{n-1}F(A) \to R^nF(C) \to R^nF(B) \to R^nF(A) \to \cdots$$

is long exact

- If M is free then $R^nF(M) = 0$ for $n > 0$.

Moreover, the system of functors and natural transformations satisfying the above properties is unique up to unique isomorphism. Moreover, the system is natural in maps (= natural transformations) of contravariant left exact functors F.

Definition 2.5. Define $\mathrm{Ext}_R^n(-,N) := R^n(\mathrm{Hom}_R(-,N))$ Bifunctor: R-Mod ^{op} $\times$ R-Mod $\to$ R-Mod.

Note:

For $R=\mathbb{Z}$, $\mathrm{Ext}^n_{\mathbb{Z}}(M,N)=0$ for $n>1$ and we write $\mathrm{Ext}(M,N)$ for $\mathrm{Ext}^1_{\mathbb{Z}}(M,N)$

More generally, if R is a PID, then $\mathrm{Ext}^n_R(M,N)=0$ for $n>1$ and if R is a field, then $\mathrm{Ext}^n_R(M,N)=0$ for $n>0$.

Let's compute $\mathrm{Ext}^n_R(R/a,-)$ in case when ${}_aR=0$,

$$0\to R\xrightarrow{\times a} R\to R/a\to 0.$$

We have

$$\begin{aligned}0\to&\mathrm{Hom}_R(R/a,M)\to\mathrm{Hom}_R(R,M)\to\mathrm{Hom}_R(R,M)\to\\ &\mathrm{Ext}^1_R(R/a,M)\to\mathrm{Ext}^1_R(R,M)\to\mathrm{Ext}^1_R(R,M)\to\mathrm{Ext}^2_R(R/a,M)\to\mathrm{Ext}^2_R(R,M)\to\mathrm{Ext}^2_R(R,M)\to\cdots\\ &\mathrm{Hom}_R(R/a,M)\cong {}_aM,\quad \mathrm{Ext}^1_R(R/a,M)\cong M/a,\quad \mathrm{Ext}^n_R(R/a,M)=0 \text{ for } n>1\end{aligned}$$

§ Projective and Injective Modules

We left off with the questions: What about the derived functros of $\mathrm{Hom}_R(M,-)$?

Let F be a contravariant left-exact functor.F takes SES $0\to A\to B\to C\to 0$ to exact sequence

$$0\to F(A)\to F(B)\to F(C)$$

Natural continuation to the right? Not necessarily exact on free modultes: $\mathrm{Ext}(\mathbb{Z}/p,\mathbb{Z})\cong\mathbb{Z}/p$. $\mathrm{Hom}(\mathbb{Z}/p,-)$ applied to $0\to\mathbb{Z}\xrightarrow{p}\mathbb{Z}\to\mathbb{Z}/p\to 0$

$$0\to\mathrm{Hom}(\mathbb{Z}/p,\mathbb{Z})\to\mathrm{Hom}(\mathbb{Z}/p,\mathbb{Z})\to\mathrm{Hom}(\mathbb{Z}/p,\mathbb{Z}/p)\to\mathrm{Ext}(\mathbb{Z}/p,\mathbb{Z})\to\mathrm{Ext}(\mathbb{Z}/p,\mathbb{Z})\to\mathrm{Ext}(\mathbb{Z}/p,\mathbb{Z}/p)\to 0.$$

What plays the rules of free modules?

SES $0\to A\to B\to C\to 0$, if C is free, then sequence splits; more generally, given P free and $f\colon P\to C$, then f lifts a map $P\to B$.

Definition 2.6. A module F is *projective* if given any epimorphism $\alpha\colon A\to B$ and any map $\beta\colon P\to B$, there exists a map $\gamma\colon P\to A$ such that $\beta=\alpha\circ\gamma$

"The problem
$$\begin{tikzcd} & F \arrow[d, "\beta"] \arrow[dl, dotted] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$$
has a solution."

Observations:

• A direct summand of a projective module is projective, i.e. $P\oplus Q$ projective $\implies$ P and Q projective.

• A module P is projective if and only if it is the direct summand of a free module, i.e., if and only if there exists Q such that $P \oplus Q$ is free.
• A projective module is finitely generated if and only if it is the direct summand of a free module.
• A module P is projective if and only if $\operatorname{Hom}_R(P, -)$ is exact.

Definition 2.6. A module Q is *injective* if given any monomorphism $\alpha\colon A \to B$ and any map $f\colon A \to Q$, there exists a map $g\colon B \to I$ such that $f = g \circ \alpha$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, dotted, "g"] \\ & Q & \end{tikzcd}$$

Equivalently: A module Q is injective if and only if $\operatorname{Hom}_R(-, Q)$ is exact.

Examples:

• In $\mathbb{Z}$-modules, $\mathbb{Q}$ is injective
• In $\mathbb{Z}$-modules, $\mathbb{Q}/\mathbb{Z}$, any injective, any "divisible" abelian group is injective (if and only if)
• In R-modules, $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q})$ and $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ are injective
• A product of injective R-modules is injective

Theorem 2.7. A R-module M is injective if and only if for any ideal I of R, any map of R-modules $I \to M$ extends to a map of R-modules $R \to M$.

Definition 2.8. A *projective resolution* of M consists of a chain complex P, and a map $\epsilon\colon P_0 \to M$ such that

• P_n is projective for all n and $P_n = 0$ for $n < 0$

• $H_n(P_*) = 0$ for $n > 0$

• The map ϵ induces an isomorphism $H_0(P_*) \to M$

Definition 2.9. An *injective resolution* of M consists of a chain complex Q^* and a map $\eta\colon M \to Q^0$ such that

• Q^n is injective for all n and $Q^n = 0$ for $n < 0$ (cohomological degree)

• $H^n(Q_*) = 0$ for $n > 0$

• The map η induces a morphism $M \cong H^0(Q^*)$.

§ The Fundamantal Lemma of Homological Algebra

Theorem 2.10. Let P_* be a complex of projectives that is zero below (homological) degree zero, let C_* be an almost acyclic complex, and let $\phi\colon H_0(P_*) \to H_0(C_*)$ be a homomorphism. Then there exists a map of complex $\Phi\colon P_* \to C_*$ that induces ϕ on H_0; moreover, ϕ is unique up to chain homotopy.

Theorem 2.11. Let Q^* be a complex of injectives that is zero below (cohomological) degree zero, let C^* be an almost acyclic complex, and let $\phi\colon H^0(C^*) \to H^0(Q^*)$ be a homomorphism. Then there exists a map of complexes $\phi\colon C^* \to Q^*$ that induces ϕ on H^0; moreover, ϕ is unique up to chain homotopy.

Review of Chain Homotopies:

Given a chain complex C_*, define $C_* \otimes I = C_* \otimes \Delta_*(\Delta[1])$ to be the chain complex

$$(C_* \otimes I)_n = C_n \oplus C_n \oplus C_{n-1} \qquad \text{write } (x, y, z) \text{ as } x \otimes [0] + y \otimes [1] + z \otimes [I]$$

$$\begin{aligned} &d(x \otimes [0]) = (dx) \otimes [0], \quad d(y \otimes [1]) = (dy) \otimes [1], \\ &d(z \otimes [I])(dx) \otimes [I] - (-1)^{|z|} z \otimes [0] + (-1)^{|z|} z \otimes [0] + (-1)^{|z|} z \otimes [1]. \end{aligned}$$

A map of chain complexes $\Phi_*\colon C_* \otimes I \to D_*$ consists of maps of modules

$$\begin{aligned} f_n\colon C_n \to D_n &\quad f_n(x) := \phi_n(x \otimes [1]) \\ g_n\colon C_n \to D_n &\quad g_n(x) := \phi_n(x \otimes [0]) \\ h_n\colon C_n \to D_{n+1} &\quad h_n(z) := \phi_{n+1}(z \otimes [I]) \end{aligned}$$

such that f_* and g_* are chain maps and h_* satisfies

$$d(h_n(z)) = h_{n-1}(dz) + (-1)^n(f_n(z) - g_n(z)).$$

(if and only if) Let $S_n := (-1)^n h_n$. Then the last equation is equivalent to

$$d \circ S_n + S_{n-1} \circ d = f_n - g_n.$$

So a map $C_* \otimes I \to D$ contains exactly the data of two chain maps $f, g\colon C_* \to D_*$ and a chain homotopy between them.

$C_* \otimes I$ comes with maps $C_* \oplus C_* \to C_* \otimes I$ and $C_* \otimes I \to C_*$.

The Fundamental Lemma of Homological Algebra after Quillen

Theorem 2.12. given a commutative (solid arrow) diagram of chain complexes

$$\begin{tikzcd} A_* \arrow[r] \arrow[d] & X_* \arrow[d] \\ B_* \arrow[r] \arrow[ur, dotted] & Y_* \end{tikzcd}$$

such that

• $A_* \to B_*$ is a monomorphism with cokernel projective and bounded below, and
• $X_* \to Y_*$ an epimorphism and quasi-isomorphism; Then there exists a map $B_* \to X_*$ making the diagram commute.

Bounded below: C_* is *bounded below* means there exists N such that $C_n = 0$ for $n < N$.

Application to the Original Fundamental Lemma

Let $A_* = 0$, $B_* = P_*$ projective in all degrees and zero in negative degrees.
Let $X_* = C_*$ be almost acyclic. WLOG X_* is zero in negative degrees..
Let $Y_* = H_0(C_*)$ concentrated in degree zero, with zero differentials.

$$\begin{tikzcd}
0 \arrow[rr] \arrow[d] & & C_* \arrow[d] \\
P_* \arrow[r] \arrow[urr, dotted, "\phi_*"] & H_0(P_*) \arrow[r, "\phi"'] & H_0(C_*)
\end{tikzcd}
\qquad
\begin{tikzcd}
P_* \oplus P_* \arrow[rr] \arrow[d] & & C_* \arrow[d] \\
P_* \otimes I \arrow[r] \arrow[urr, dotted, "\phi_*"] & H_0(P_*) \arrow[r, "\phi"'] & H_0(C_*)
\end{tikzcd}$$

NOTE: There is a “reverse arrow” way to write this by refersing the arrows and replacing “projective” with “injective”.

§ Existence of Derived Functors

§ Universal Coefficients Theorems

3 Topological Products

§ Tensor Product and Hom of Chain Complexes

§ The Kunneth Theorem

§ The Eilenberg-Zilber Theorem

§ The Alexander-Whitney and Shuffle Maps

§ Products and Pairings

§ Acyclic Models and the Eilenberg-Zilber Theorem

4 Fibre Bundles

§ Group Actions

§ Fiber Bundles

§ Examples of Fiber Bundles

§ Principal Bundles

§ Maps of Fiber Bundles

§ Reduction of Structure Group

5 Homology with Local Coefficients

§ Homology with Twisted Coefficients

§ Examples and Basic Properties of Homology with Twisted Coefficients

§ Homology with Local Coefficients

§ Functoriality of Homology with Local Coefficients