Category Theory Notes

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Contents

1 Categories and Structure

§ Definition of a Category and Examples

The identity endofunctor of a category $\mathcal{C}$ will be denoted $\mathrm{id}_{\mathcal{C}}$. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there is a functor $F^{-1}\colon \mathcal{D} \to \mathcal{C}$, called a *quasi-inverse* of F, such that $F^{-1} \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ F^{-1} \cong \mathrm{id}_{\mathcal{D}}$.

A category is called *locally small* if for any objects X, Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set, and is called *essentially small* if in addition its isomorphism classes $\mathcal{C}$ of objects form a set. In other words, an essentially small category is a category equivalent to a small category. All categories will be considered locally small (except in the section on 2-categories), and most of them will be essentially small. So set-theoretical subtleties will not play any rule in the book.

For a category $\mathcal{C}$ the notation $X \in \mathcal{C}$ will mean that X is an object of $\mathcal{C}$, and the set of morphisms between $X, Y \in \mathcal{C}$ will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. An element $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ will be usually depicted either as $\phi\colon X \to Y$ or as $X \xrightarrow{\phi} Y$. We denote by C^{op} the category *dual* to $\mathcal{C}$, i.e., the one obtained from $\mathcal{C}$ by reversing the direction of morphisms.

§ Functors and Natural Transformation

§ Basic Category Structures

recall that in **Sets**, a function $f\colon A \to B$ is called:

injective if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$,
surjective if for all $b \in B$ there is some $a \in A$ with $f(a) = b$.

We have the following abstract characterization of these properties:

Definition 1.1. In any category $\mathcal{C}$, an arrow $f\colon A \to B$ is called a:

monomorphism if given any $g, h\colon C \to A$, $fg = fh$ implies $g = h$,

$$\begin{tikzcd} C \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & A \arrow[r, "f"] & B. \end{tikzcd}$$

epimorphism if given any $i, j\colon B \to D$, $if = jf$ implies $i = j$,

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] & D. \end{tikzcd}$$

We often write $f\colon A \rightarrowtail B$ if f is a monomorphism, and $f\colon A \twoheadrightarrow B$ if f is an epimorphism.

Definition 1.2. In any category $\mathcal{C}$, an object 0 is *initial* if for any object C, there is a unique morphism $0 \to C$, and an object 1 is *terminal* if for any object C there is a unique morphism $C \to 1$. As in the case of monos and epis, note that there is a kind of "duality" in these definitions. Precisely, a terminal object in $\mathcal{C}$ is exactly an initial object in $\mathcal{C}^{op}$.

Definition 1.3. Let $\mathcal{C}$ be a category with some objects X_1 and X_2. A *binary product* of X_1 and X_2 is an object X (often $X_1 \times X_2$) together with a pair of morphisms $\pi_1\colon X \to X_1$, $\pi_2\colon X \to X_2$ that satisfy the following universal property:

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dotted, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & D. \end{tikzcd}$$

For every object Y and every pair of morphisms $f_1\colon Y \to X_1, f_2\colon Y \to X_2$ there exists a unique morphism $f\colon Y \to X_1 \times X_2$ such that the following diagram above commutes. The unique morphism f is called the *product of morphisms* f_1 and f_2 and is denoted $\langle f_1, f_2 \rangle$. The morphisms π_1 and π_2 are called the *canonical projections* or *projection morphisms*.

Instead of two objects, we can take an arbitrary family of objects indexed by some set I. Then we obtain the general definition of a *product*. An object X is the product of a family $(X_i)_{i \in I}$ of objects if there exist morphisms $\pi_i\colon X \to X_i$ such that for every object Y and every I-indexed family of morphisms $f_i\colon Y \to X_i$, there exists a unique morphism $f\colon Y \to X$ such that the following diagrams commute for all i in I:

$$\begin{tikzcd} & Y \arrow[d, "\pi_i"] \\ Y \arrow[ur, dotted, "f"] \arrow[r, "f_i"'] & X_i. \end{tikzcd}$$

The product is denoted $\prod_{i \in I} X_i$. If $I = \{1, \ldots, n\}$ then it is denoted $X_1 \times \cdots \times X_n$ and the product of morphisms is denoted $\langle f_1, \ldots, f_n \rangle$.

Definition 1.4. If X and Y are objects, while f and g are morphisms from X to Y, an *equilizer* of f and g consists of an object E and a morphism $\mathrm{eq}\colon E \to X$ satisfying $f \circ \mathrm{eq} = g \circ \mathrm{eq}$ and such that, given any object O and morphism $m\colon O \to X$, if $f \circ m = g \circ m$, then there exists a unique morphism $u\colon O \to E$ such that $\mathrm{eq} \circ u = m$

$$\begin{tikzcd} E \arrow[r, "\mathrm{eq}"] & X \arrow[r, shift left, bend left, "f"] \arrow[r, shift right, bend right, "g"'] & Y. \\ O \arrow[u, dotted, "u"] \arrow[ur, "m"'] & & \end{tikzcd}$$

More generally, we can define an *equilizer* of $(f_i)_{i \in I}$ for an arbitrary family of morphisms from X to Y as follows: An object E and a morphism $\mathrm{eq}\colon E \to X$ satisfying $f_i \circ \mathrm{eq} = f_j \circ \mathrm{eq}$ for every $i, j \in I$ and such that given any object O and morphism $m\colon O \to X$, if $f_i \circ f_j = g \circ m$ for every $i, j \in I$, then there exists a unique morphism $u\colon O \to E$ such that $\mathrm{eq} \circ u = m$

$$\begin{tikzcd} E \arrow[r, "\mathrm{eq}"] & X \arrow[r, shift left, bend left, "f_i"] \arrow[r, shift right, bend right, "f_j"'] & Y. \\ O \arrow[u, dotted, "u"] \arrow[ur, "m"'] & & \end{tikzcd}$$

We've seen that every subset $U \subset X$ of a set X occurs as an equilizer, and that equilizer, and that equalizers are always monomorphisms. So it's natural to regard monos as generalized subsets. That is, a mono in **Groups** can be regarded as a subgroup, a mono in **Top** as a subspace, and so on.

Definition 1.1. A *subobject* of an object X in a category $\mathcal{C}$ is a mono $m\colon \rightarrowtail X$. Given subobjects m, m' of X, a morphism $f\colon M \to M'$ is an arrow in $\mathcal{C}/X$, as in:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & X \end{tikzcd}$$

Thus we have a category, $\mathrm{Sub}_{\mathcal{C}}(X)$ of subobjects of X in $\mathcal{C}$

In this definition, since m' is monic, there is at most one f as in the diagram above, so that $\mathrm{Sub}_{\mathcal{C}}(X)$ is a preorder category. We define the relation of *inclusion* of subobjects by:

$$m \subset m' \text{ iff there exists some } f\colon m \to m'$$

Finally, we say that m and m' are *equivalent*, written $m \equiv m'$, if and ohly if they are isomorphic as subobjects, i.e., $m \subset m'$ and $m' \subset m$.

Remark: We sometimes abuse notation and language by claling M the subobject when the mono $m\colon M \rightarrowtail X$ is clear.

Note that if $M \subset M'$ then the arrow f which makes this so in,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr] & M' \arrow[d] \\ & X \end{tikzcd}$$

is also monic, so also M is a subobject of M'. In fact, we have a functor $i_*\colon \mathrm{Sub}(M') \to \mathrm{Sub}(X)$ defined by composition (since the composite of monos is monic).

In terms of generalized elements of an object X, $z\colon Z \to X$ one can define a *local membership relation*, $z \in_X M$ between these and subobjects $m\colon M \rightarrowtail X$ by:

$$z \in_X M \text{ iff there exists } f\colon Z \to M \text{ such that } z = mf.$$

Since m is monic, if z factors through it then it does so uniquely. So this is naturally a relation.

Remark: It is often convenient to pass from the preorder $\mathrm{Sub}_{\mathcal{C}}(X)$ to the *poset* given by factoring out the equivalence ration "$\equiv$". Then a subobject is an equivalence class of monos under mutual inclusion.

In **Sets**, under this notion of subobject, one then has an isomorphism $\mathrm{Sub}_{\mathbf{Sets}}(X) \cong P(X)$ i.e., every subobject is represented by a unique subset. We shall use both notions of subobject, making clear when monos are intended, and when equivalence classes thereof are intended.

Next, we shall talk about pullbacks. The notion of pullback, like that of a product, is one that comes up very often in mathematics and logic. It is a generalization of both intersection and inverse image. We begin with the definition

Definition 1.2. In any category $\mathcal{C}$, a *pullback* of arrows f, g with $\mathrm{cod}(f) = \mathrm{cod}(g)$

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

consists of arrows

$$\begin{tikzcd} P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \\ A & \end{tikzcd}$$

such that $fp_1 = gp_2$, and universal with this property; i.e., given any $z_1\colon Z \to A$ and $z_2\colon Z \to B$ with $fz_1 = gz_2$ there exists a unique $u\colon Z \to P$ with $z_1 = p_1 u$ and $z_2 = p_2 u$.

$$\begin{tikzcd} Z \arrow[dr, dotted, "u"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\ & P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"'] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

One sometimes uses product-style notation $\langle z_1, z_2 \rangle$ and $A \times_C B$ for pullbacks:

$$\begin{tikzcd}
Z \arrow[dr, "\langle z_1, z_2 \rangle"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\
& A \times_C B \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"'] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Lemma 1.3. *(Two-pullbacks)* Consider the commutative diagram below in a category with pullbacks:

$$\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

1. If the two squares are pullbacks, so is the outer recatangle. Thus,

$$A \times_B (B \times_C D) \cong A \times_C D.$$

2. If the right square and the outer rectangle are pullbacks, so is the left square.

Pullbacks are clearly unique up to isomorphism since they're given by a UMP.

Corollary 1.4. the pullback of a commutative triangle is a commutative triangle. Specifically, given a commutative triangle as on the right end of the following "prism diagram":

We've already seen that the notions are product, equilizer, and pullback are not independent; the precise relation between them is this:

Proposition 1.5. A category has finite products and equilizers iff it has pullbacks and a terminal object.

§ Limit and Colimit Structures

Product, terminal object, pullback, and equilizer, are all special cases of the general notion of a *limit*, which we'll consider now. first, we need some preliminary definitions:

Definition 1.6. Let J and $\mathcal{C}$ be categories. A *diagram of type* J in $\mathcal{C}$ is a functor, $D \colon J \to \mathcal{C}$.

We'll write the objects in the "induex category" J lower case, $i, j, \ldots$ and the values in the form D_i, D_j, etc.

A *cone* to a diagram D consists of an object C of $\mathcal{C}$ and a family of arrows in $\mathcal{C}$, $c_j\colon C \to D_j$ one for each object $j \in J$, such that for each arrow $\alpha\colon i \to j$ in J, the following triangle commutes.

$$\begin{tikzcd} C \arrow[r, "c_j"'] \arrow[d, "c_i"] & D_j \\ D_i \arrow[ur, "D_\alpha"'] & \end{tikzcd}$$

A *morphism* of cones $\vartheta\colon (C, c_j) \to (C', c'_j)$ is an arrow ϑ in $\mathcal{C}$ making each triangle commute

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

i.e., such that $c_j = c'_j \circ \vartheta$ for all $j \in J$. Thus we have an apparent category: $\mathbf{Cone}(D)$ of cones to D.

We are here thinking of a diagram D as a "picture of J in $\mathcal{C}$". A cone to such diagram D is then imagined as a many-sized pyramid over the "base" D, and a morphism of cones is an arrow between the apexes of such pyramids (The reader should draw some pictures at this point!)

Definition 1.7. A *limit* for a diagram $D\colon J \to \mathcal{C}$ is a terminal object in $\mathbf{Cone}(D)$. A *finite limit* is a limit for a diagram on a finite index category J.

We often denote a limit in the form, $p_i\colon \varprojlim_j D_j \to D_i$

Spelling out the definition, the limit of a diagram D has the following UMP. Given any cone (C, c_j) to D, there is a unique arrow $u\colon C \to \varprojlim_j D_j$ such that for all j, we have $p_j u = c_j$.

Proposition 1.8. A category has all finite limits iff it has finite products and equilizers (resp. pullbacks and a terminal object by the last proposition).

We can also do a similar thing with arbitrary sized indeces.

Corollary 1.9. A category has all limits of some cardinality iff it has all equalizers and products of that cardinality, where $\mathcal{C}$ has limits (resp. products) of cardinality κ iff $\mathcal{C}$ has a limit for every diagram $D\colon J \to \mathcal{C}$ where $\mathrm{Card}(J) \leq \kappa$ (resp. $\mathcal{C}$ has all products of κ many objects).

The notions of cones and limits of course dualize to give those of *cocones* and *colimits*. One then has the following dual theorem.

Theorem 1.10. A category $\mathcal{C}$ has finite colimits iff it has finite coproducts and coequilizers (resp. iff it has pushouts and initial object). $\mathcal{C}$ has all colimits of size κ iff it has coequilizers and coproducts of size κ.

Definition 1.11. A functor $F: \mathcal{C} \to \mathcal{D}$ is said to *preserve limits of type J* if, whenever $p_j: \to D_j$ is a limit for a diagram $D: J \to \mathcal{C}$, the cone $Fp_j: FL \to FD_j$ is then a limit of the diagram $FD: J \to \mathcal{D}$. Briefly:

$$F(\varprojlim D_j) \cong \varprojlim F(D_j).$$

A functor that preserves all limits is said to be *continuous*.

For example, let $\mathcal{C}$ be locally small and recall the representable functor,

$$\mathrm{Hom}_{\mathcal{C}}(C, -): \mathcal{C} \to \mathbf{Sets}$$

for any object $C \in \mathcal{C}$, taking $f: X \to Y$ to

$$f_*: \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$$

where $f_*(g) = f \circ g$, for $g: C \to X$.

Proposition 1.12. Representable functors preserve all limits.

Definition 1.13. A functor of the form $F: \mathcal{C}^{op} \to \mathcal{D}$ is called a *contravariant functor* on $\mathcal{C}$. Explicitly, such a functor takes $f: A \to B$ to $F(f): F(B) \to F(A)$ and $F(g \circ f) = F(f) \circ F(g)$.

A typical example of a contravariant functor is a representable functor of the form,

$$\mathrm{Hom}_{\mathcal{C}}: \mathcal{C}^{op} \to \mathbf{Sets}$$

for any $C \in \mathcal{C}$ (where $\mathcal{C}$ is any locally small category). Such a contravariant functor takes $f: X \to Y$ to,

$$f^*: \mathrm{Hom}(Y, C) \to \mathrm{Hom}(X, C)$$

by $f^*(g) = g \circ f$, for alll $g: X \to C$.

The dual version of the foregoing proposition is then this:

Corollary 1.14. Contravariant representable functors map all colimits to limits.

For example, given a coproduct $X + Y$ in any locally small category $\mathcal{C}$, there is a canonical isomorphism

$$\mathrm{Hom}(X + Y, C) \cong \mathrm{Hom}(X, C) \times \mathrm{Hom}(Y, C).$$

2 Specific Category Theory Structure

§ Exponentials

Definition 2.1. Let the category $\mathcal{C}$ have binary products. an *exponential* of objects B and C of $\mathcal{C}$ consists of an object C^B and an arrow $\epsilon\colon C^B \times B \to C$ such that, for any object Z an arrow $f\colon Z \times B \to C$, there's a unique arrow $\widetilde{f}\colon Z \to C^B$ such that $\epsilon \circ (\widetilde{f} \times 1_B) = f$, all as in the diagram:

$$
\begin{tikzcd}
C^B \times B \\
Z \arrow[u, "\widetilde{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
Z \times B \arrow[u, "\widetilde{f} \times 1_B"] \arrow[ur, "f"'] &
\end{tikzcd}
,
$$

Here's some terminology:

- $\epsilon\colon C^B \times B \to C$ is called an *evaluation*.
- $\widehat{f}\colon Z \to C^B$ is called the (exponential) *transpose* of f.
- Given an arrow $g\colon Z \to C^B$ we write

$$\overline{g} := \epsilon(g \times 1_B)\colon Z \times B \to C$$

and also call $\overline{g}$ the *transpose* of g. By the uniqueness clause of the definition, then, $\widetilde{\overline{g}} = g$, and for any $f\colon Z \times B \to C$, $\overline{\widetilde{f}} = f$. Briefly, transposition of transposition is the identity. Thus transposition provides the desired isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Z \times B, C) \cong \mathrm{Hom}_{\mathcal{C}}(Z, C^B),$$

where $f \mapsto \widetilde{f}$ and $g \mapsto \overline{g}$.

Definition 2.2. A category is called *cartesian closed* if it has all finite products and exponentials.

Proposition 2.3. In any cartesian closed category $\mathcal{C}$, exponentiation by a fixed object A, is a functor $-^A\colon \mathcal{C} \to \mathcal{C}$.

§ Categories of Diagrams

We are going to focus on special functor categories of the form $\mathrm{Sets}^{\mathcal{C}}$ where the category $\mathcal{C}$ is locally small. Thus the objects are set-valued functors, $P, Q\colon \mathcal{C} \to \mathrm{Sets}$ and the arrows are natural transformations $\alpha, \beta\colon P \to Q$.

Remember that, for each object $C \in \mathcal{C}$ be can evaluate any commutative diagram in $\mathrm{Sets}^{\mathcal{C}^{op}}$

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[dr, "\beta\alpha"'] & Q \arrow[d, "\beta"] \\ & R \end{tikzcd}$$

at any object C to get a commutative diagram on Sets

$$\begin{tikzcd} PC \arrow[r, "\alpha_C"] \arrow[dr, "(\beta\alpha)_C"'] & QC \arrow[d, "\beta_C"] \\ & RC. \end{tikzcd}$$

Thus for each object C there is an evaluation functor, $\mathrm{ev}_C\colon \mathrm{Sets}^{\mathcal{C}^{op}} \to \mathrm{Sets}$.

One way of thinking about thinking about such functor categories is considering the case where $\mathcal{C}$ is the category Γ pictured:

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r] & 0 \end{tikzcd}$$

then a set-valued functor $G\colon \Gamma \to \mathrm{Sets}$ is just a graph, and a natural transformation, $\alpha\colon G \to H$ is a graph homomorphism. Thus, for this case, $\mathrm{Sets}^{\Gamma} = \mathrm{Graphs}$. This suggests regarding an arbitrary category of the form $\mathrm{Sets}^{\mathcal{C}}$ as a generalized "category of structured sets" and their "homomorphisms"; indeed, this is a very useful way of thinking of such functors and their natural transformations.

Among the objects of $\mathrm{Sets}^{\mathcal{C}}$ are certain very special ones, namely the (co-variant) representable functors, $\mathrm{Hom}_{\mathcal{C}}(C, -)\colon \mathcal{C} \to \mathrm{Sets}$. Observe that for each $h\colon C \to D$ in $\mathcal{C}$, we have a natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(h, -)\colon \mathrm{Hom}_{\mathcal{C}}(D, -) \to \mathrm{Hom}_{\mathcal{C}}(C, -)$$

(note the direction!) where the component at X is defined by

$$(f\colon D \to X) \mapsto (f \circ h\colon C \to X).$$

Thus we have a *contravariant functor* $k\colon \mathcal{C}^{op} \to \mathrm{Sets}^{\mathcal{C}}$ defined by $k(C) = \mathrm{Hom}_{\mathcal{C}}(C, -)$. Note that this functor k is the exponential transpose of the bifunctor,

$$\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Sets}.$$

If we instead transpose $\mathrm{Hom}_{\mathcal{C}}$ with respect to its other argument, we get a *covariant* functor $y\colon \mathcal{C} \to \mathrm{Sets}^{\mathcal{C}^{op}}$ from $\mathcal{C}$ to a category of *contravariant* set-valued functors, sometimes called "presheaves". More formally

Definition 2.4. The *Yoneda embedding* is the functor $y\colon \mathcal{C} \to \mathrm{Sets}^{\mathcal{C}^{op}}$ taking $C \in \mathcal{C}$ to the contravariant representable functor,

$$yC = \mathrm{Hom}_{\mathcal{C}}(-, C)\colon \mathcal{C}^{op} \to \mathrm{Sets}$$

and taking $f\colon C \to D$ to the natural transformation

$$yf = \mathrm{Hom}_{\mathcal{C}}(-, f)\colon \mathrm{Hom}_{\mathcal{C}}(-, C) \to \mathrm{Hom}_{\mathcal{C}}(-, D).$$

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an *embedding* if it is full, faithful and injective on objects. We will soon show that y really *is* an embedding; this is a corollary of the Yoneda Lemma.

One should think of the Yoneda embedding y as a "representation" of $\mathcal{C}$ in a category of set-valued functors and natural transformations on *some* index category. Compared to the Caley representation, this one has the virtue of being *full*. Indeed, recall that the Cayley representation of a group G was an injective group homomorphism

$$G \rightarrowtail \mathrm{Aut}(|G|) \subset |G|^{|G|},$$

where each $g \in G$ is represented as an automorphism $\widetilde{g}$ of the set $|G|$ of elements (i.e. a "permutation"), by letting it "act on the left", $\widehat{g}(x) = g \cdot x$, and the group multiplication is represented by composition of permutations, $\widetilde{g \cdot h} = \widetilde{g} \circ \widetilde{h}$. We also showed a generalization of this representation to arbitrary categories. Thus for any monoid M, there is an analogous representation,

$$M \rightarrowtail \mathrm{End}(|M|) \subset |M|^{|M|}$$

by left action, representing the elements of M as endomorphisms of $|M|$.

The representation given by the Yoneda embedding is "better" than these in that it cuts down the arrows in the codomain category to just those in the image of the representation functor $y\colon \mathcal{C} \to \mathrm{Sets}^{\mathcal{C}^{op}}$ (since y is full). Indeed, there may be many automorphisms $\alpha\colon G \to G$ of a group G that are not left actions by an element, but if we require α to commute with all *right* actions $\alpha(x \cdot g) = \alpha(x) \cdot g$, then α must itself be a left action. This is what the Yoneda embedding does in general; it adds enough "structure" to the objects yA in the image of the representation that every "homomorphism" $\zeta\colon yA \to yB$ can be represented as $\zeta = yh$ for some $h\colon A \to B$.

We shall now prove this result, which is called the "Yoneda lemma".

Lemma 2.5. *(Yoneda)* Let $\mathcal{C}$ be locally small. For any object $C \in \mathcal{C}$, and functor $F \in \mathrm{Set}^{\mathcal{C}^{op}}$, there's an isomorphism,

$$\mathrm{Hom}(yC, F) \cong FC,$$

which, moreover, is natural in both F and C.

Here:

1. the Hom is $\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}$

2. naturality in F means that, given any $\eta\colon F \to G$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(yC,F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(yC,\eta)"'] & FC \arrow[d, "\eta_C"] \\ \mathrm{Hom}(yC,G) \arrow[r, "\cong"'] & GC \end{tikzcd}$$

3. naturality in C means that, given any $h\colon C \to D$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(yC,F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(yh,F)"'] & FC \arrow[d, "Fh"] \\ \mathrm{Hom}(yD,F) \arrow[r, "\cong"'] & FD \end{tikzcd}$$

Theorem 2.6. The Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Sets}^{C^{op}}$ is full and faithful.

Remarks: Note the following

• If $\mathcal{C}$ is small, then $\mathrm{Sets}^{\mathcal{C}^{op}}$ is locally small, and so $\mathrm{Hom}(yC,P)$ in $\mathrm{Sets}^{\mathcal{C}^{op}}$ is a set.

• If $\mathcal{C}$ is locally small, then $\mathrm{Sets}^{\mathcal{C}^{op}}$ need not be locally small. In this case, the Yoneda Lemma tells us that $\mathrm{Hom}(yC,P)$ is always a set.

• If $\mathcal{C}$ is not locally small, then $y\colon \mathcal{C} \to \mathrm{Sets}^{\mathcal{C}^{op}}$ won't even be defined, so the Yoneda lemma does not apply.

Next, we shall discuss some applications of the Yoneda Embedding.

§ Adjoints

§ Monads and Algebras

§ Comonads and Coalgebras

§ Abelian Categories and Exact Sequences

Definition 2.7. An *additive category* is a category $\mathcal{C}$ satisfying the following axioms:

(A1) Every set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is equippened with a structure of an abelian group (written additively) such that the composition of morphisms biadditive with respect to this structure.
(A2) There exists a zero object $0 \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(0,0) = 0$
(A3) (Existence of direct sums) For any objects $X_1, X_2 \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ and morphisms such that $p_1 i_1 = \mathrm{id}_{X_1}$, $p_2 i_2 = \mathrm{id}_{X_2}$, and $i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$.

In (A3), the object Y is unique up to a unique isomorphism, is denoted $X_1 \oplus X_2$, and is called the *direct sum* of X_1 and X_2. Thus, every additive category is equipped with a bifunctor $\oplus\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Definition 2.8. Let K be a field. An additive category $\mathcal{C}$ is said to be K-linear (or defined over K) if for any objects $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a structure of a vector space over K, such that composition of morphisms is K-linear.

Definition 2.9. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between two additive categories, the functor F is called *additive* if the associated maps

$$(*) \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ X, Y \in \mathcal{C},$$

are homomorphisms of abelian groups. If $\mathcal{C}$ and $\mathcal{D}$ are K-linear categories then F is called K*-linear* if the homomorphisms are K-linear.

Proposition 2.10. For any additive functor $F\colon \mathcal{C} \to \mathcal{D}$ there exists a natural isomorphism $F(x) \oplus F(Y) \xrightarrow{\sim} F(X \oplus Y)$.

Definition 2.11.

1. Let $\mathcal{C}$ be an additive category and $f\colon X \to Y$ a morphism in $\mathcal{C}$. The *kernel* $\mathrm{Ker}(f)$ of f (if exists) is an object K together with a morphism $k\colon K \to X$ such that $fk = 0$ and if $k'\colon K' \to X$ is such that $fk' = 0$ then there exists a unique morphism $l\colon K' \to K$ such taht $kl = k'$. If $\mathrm{Ker}(f)$ exists then it is unqieu up to unique isomorphism.

2. Dually, the *cokernel* $\mathrm{Coker}(f)$ of a morphism $f\colon X \to Y$ in $\mathcal{C}$ (if exists) is an object C together with a morphism $c\colon Y \to C$ such that $cf = 0$ and if $c'\colon Y \to C'$ is such that $c'f = 0$ then there exists a unque morphism $l\colon C \to C'$ such that $lc = c'$. If $\mathrm{Coker}(f)$ exists then it is unique up to unique isomorphism.

Definition 2.12. An *abelian category* is an additive category $\mathcal{C}$ in which for every morphism $\varphi\colon X \to Y$ there exists a sequence

$$(*) \quad K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

with the following properties:

1. $ji = \varphi$,
2. $(K, k) = \mathrm{Ker}(\varphi)$, $(C, c) = \mathrm{Coker}(\varphi)$,
3. $(I, i) = \mathrm{Coker}(k)$, $(I, j) = \mathrm{Ker}(c)$

A sequence (*) is called a "canonical decomposition of φ". The object I is called the image of φ and is denoted by $\mathrm{Im}(\varphi)$.

In the future, for brevity we will write K instead of (K, k) and C instead of (C, c).

Remark: In an abelian category, the canonical decomposition of a morphism is unique up to unique isomorphism.

Example 2.14. The category of abelian groups is an abelian category. The category of modules over a ring is an abelian category. the category of modules over a ring is an abelian category. The category Vec of vector spaces over a field K and its subcategory Vec_0 of finite dimensional vector spaces are K-linear abelian categories. More generally, the category of modules over an associative K-algebra and the category of comodules over a coassociative K-coalgebra (see coalgebra section) are K-linear abelian categories.

Definition 2.15. Let $\mathcal{C}$ be an abelian category. A morphism $f\colon X \to Y$ is said to be a *monomorphism* if $\mathrm{Ker}(f) = 0$. It is said to be an *epimorphism* if $\mathrm{Coker}(f) = 0$.

It is easy to see that a morphism is both a monomorphism and epimorphism if and only if it is an isomorphism.

Definition 2.16. A *subobject* of an object Y is an object X together with a monomorphism $i\colon X \to Y$. A *quotient object* of Y is an object Z with an epimorphism $p\colon Y \to Z$. A *subquotient object* of Y is a quotient object of a sobobject of Y.

We shall write $X \subset Y$ whenever X is a subobject of Y for some monomorphism $i\colon X \to Y$, and we shall write Y/X, for any $X \subset Y$, to describe the cokernel of the monomorphism $i\colon X \to Y$ (note again that the cokernel is unique up isomorphism)

Definition 2.17. An abelian category $\mathcal{C}$ is said to be *indecomposable* if it is not equivalent to a direct sum of two nonzero categories.

The following theorem is psychologically useful, as it allows one to think of morphisms, kernels, cokernels, subobjects, quotient objects, etc. in an abelian category in terms of usual linear algebra.

Theorem 2.18. *(Mitchell)* Every abelian category is equivalent, as an additive category, to a full subcategory of the category of left modules over an associative unital ring A.

Remark:

1. If the category is K-linear, the ring in Theorem 2.18 can be chosen to be a K-algebra in such a way that the corresponding equivalence is K-linear.

2. A major drawback of Theorem 2.18 is that the ring A is not unique, and in many important cases there are no manageable choices of A.

Definition 2.19. A sequence of morphisms

$$\cdots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \cdots$$

in an abelian category is called *exact in degree* i if $\mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i)$. It is called *exact* if it is exact in every degree. An exact sequence

$$0 \to X \to Y \to Z \to 0$$

is called a *short exact sequence.*

In a short exact sequence X is a subobject of Y and $Z \cong Y/X$ is the corresponding quotient.

Definition 2.20. Let

$$S\colon 0 \to Z \to Y \to 0 \text{ and } S'\colon 0 \to X \to Z' \to Y \to 0$$

be short exact sequence. A *morphism* from S to S' is a morphis $f\colon Z \to Z'$ such that it restricts to the identity morphism $X \to X$. And induces the identity morphism $Y \to Y$. The set of exact sequences $0 \to X \to Z \to Y \to 0$ up to isomorphism is denoted $\mathrm{Ext}^1(Y, X)$ and is called the set of *extensions* of Y by X.

One can define an operation of addition on $\mathrm{Ext}^1(Y, X)$. Namely, let S and S' be the short exact sequence as above. Let X_{antidiag} denote the antidiagonal copy of X in $X \oplus X$ (i.e., the image of $(\mathrm{id}_X, -\mathrm{id}_X)\colon X \to X \oplus X$), and similarly Y_{antidiag} denote the antidiagonal copy of Y in $Y \oplus T$. Define $S + S'$ to be the exact sequence

We shall assume for the remainder of this text that $\mathcal{C}$ is an abelian category.

Definition 2.21. A nonzero object X is $\mathcal{C}$ is called *simple*

§ Monoidal Categories

3 n-Categories and Their Structure

§ 2-Categories and Bi-Categories

§ Strict n-Category Definition