M622 Final Project

Alex Goodlad

December 27, 2020

Contents

1 Introduction

§ About this Draft [PLEASE READ]

The title is somewhat misleading: The paper is pretty much what it says it's not: A rough draft. This paper admittedly leaves a lot to be desired in terms of completeness. This paper is the "final draft" of an assignment for a topology class, but I plan to make it into some kind of primer on a category theoretic algebraic topology/homological algebra. Moreover, there is an appendix section that I will not create until the later draft.

We start this paper by defining categories and various categorical structures in Chapter 2, giving examples that this paper covers, and defining various properties of categories, which include products, coproducts, pullbacks, pushouts, and the cherry on top: Limits and colimits. We also define functors, and give some examples, which our paper covers. We finally pay special attention to categories of chains and cochains of morphisms

In Chapter 3, we talk about the category of topological spaces and its properties. We show how this category is small complete and small cocomplete and mention how limits generalize the various topological operations such as intersections, unions, products, and even pullbacks and pushouts.

In Chapter 4, we define CW complexes, first in our "favorite way" without the categorical concept of "pushout diagrams", then we show that this definition is equivalent to the categorical definition.

In Chapter 5, we talk about the category of abelian groups and its properties. Similar to Chapter 3 with the category of Topologies, we go through the work of defining products and coproducts, and showing that ultimately that this category is small complete. We then define $\varprojlim^1$, as well as go in depth with the category of chains and cochains of abelian groups, and define the subcategories of chain and cochain complexes.

Finally, in Chapter 6, we go over the functoriality of homologies $H_\bullet$ and cohomologies $H^\bullet$, particularly in the category of Chain Complexes of Abelian Groups, and then we give some results involving limit preservation properties of $H_\bullet$ and $H^\bullet$.

I plan in a future draft to include an Appendix chapter, which includes relevant preliminary information on Sets, Topology, Homological Algebra, and likely more.

2 Category Theory

§ Definition and Examples

Definition 2.1. A *Category* $\mathcal{C}$ is a collection of objects $C \in \mathrm{Ob}(\mathcal{C})$ (we often write $C \in \mathcal{C}$ for short) and a collection $\mathrm{Mor}(\mathcal{C})$ of morphisms $f\colon A \to B \in \mathrm{Mor}(\mathcal{C})$ between objects $A, B \in \mathrm{Ob}(\mathcal{C})$ with a composition binary operation $\circ\colon \mathrm{Mor}(\mathcal{C})^2 \to \mathrm{Mor}(\mathcal{C})$ with the following properties.

1. *(associativity)* The $\circ$ operation is associative; in other words, given $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$, there exists a composition arrow $g \circ f\colon B \to C$ such that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. *(identity)* For every object $A \in \mathcal{C}$, there exists an *identity arrow* $\mathrm{id}_A\colon A \to A$ such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$ for $f\colon A \to B$ and $g\colon A \to C$.

Example 2.2. Here are some examples that this paper will cover

1. The category **Set** has an object class containing all sets X and morphisms $f\colon X \to Y$ as functions between them.

2. The category **Top** has an object class containing all topological spaces X (see appendix) and morphisms $f\colon X \to Y$ all continuous functions between them.

3. The category **CW** has an object class containing all CW complexes X (see chapter 4) and morphisms $f\colon X \to Y$ all continuous functions that satisfy the condition $f(X^n) \subset Y^n$, where X^n and Y^n are the respective n-skeletons of x and Y.

4. The category **Grp** has an object class containing all groups G and morphisms $f\colon G \to K$ all group isomorphisms. The category **Ab** has an object class containing all abelian groups A and morphisms $f\colon A \to B$ all groups isomorphisms between abelian groups.

5. The poset category $(P, \leq)$ has an object class containing the elements of a reflexive poset $p \in P$ with a single morphism $m(p_1, p_2)\colon p_1 \to p_2$, for every $p_1, p_2 \in P$ such that $p_1 \leq p_2$.

6. Given a category $\mathcal{C}$, we define the dual category $\mathcal{C}^{op}$ to be the category with the same object class as $\mathcal{C}$, but with the source and target of each morphism reversed, i.e. a morphism $f\colon X \leftarrow Y$ for every morphism $f\colon Y \to X$ in $\mathcal{C}$.

7. Given a category $\mathcal{C}$, the category $\mathcal{C}^{(\mathbb{N}_0,\leq)}$ is the category with the object class consisting of chains of morphisms

$$\cdots C_{n+1} \xrightarrow{f_n} C_n \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} C_1 \xrightarrow{f_0} C_0,$$

and morphisms $(g_n\colon C_n \to D_n)_{n\in\mathbb{N}_0}\colon (f_n)_{n\in\mathbb{N}_0} \to (f'_n)_{n\in\mathbb{N}_0}$ latters of morphisms such that $f'_n \circ g_{n+1} = g_n \circ f_n$ commutes. We shall talk about such categories in both the § **Diagram and Chain Categories** section later in this chapter and the § **Chain Complexes** section in chapter 5.

§ Definition and Examples of Functors

Definition 2.3. We define a *covariant functor* (which we usually refer to as functors) *functor* $F\colon \mathcal{C} \to \mathcal{D}$ as a function between categories that map objects in $\mathcal{C}$ to objects in $\mathcal{D}$ and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ such that identity and composition is preserved, i.e. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$. We define a *contravariant functor* $G\colon \mathcal{C} \to \mathcal{D}$ as a function between categories that map objects in $\mathcal{C}$ to objects in $\mathcal{D}$ and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ such that $G(\mathrm{id}_A) = \mathrm{id}_{G(A)}$ and $G(f \circ g) = G(g) \circ G(f)$. Equivalently, a contravariant functor is a (covariant) functor from $\mathcal{C}$ to $\mathcal{D}^{op}$.

We define a *natural transformation*$\eta\colon F \to G$ between functors $F, G\colon \mathcal{C} \to \mathcal{D}$ as a function from $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that $\eta_C \in \mathrm{Hom}_{\mathcal{D}}(F(C), G(C))$, i.e. for every $C \in \mathcal{C}$, we have $\eta_C\colon F(C) \to G(C)$, and for every $f\colon X \to Y \in \mathcal{C}$ we have the square

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

commuting.

Example 2.4. Here are some examples of covariant and contravariant functors that our paper will cover:

1. We will look at what we call a "diagram" $F\colon J \to \mathcal{C}$, which is a functor from an arbitrary category J that we call an "index category". Most often we look at the diagrams on $(\mathbb{N}_0, \leq)$ and $(\mathbb{N}_0, \geq)$, which we call "chains" and "cochains" respectively.

2. We will look at the nth homology functor, for any $n \geq 0$, $H_n\colon \mathbf{Top} \to \mathbf{Ab}$, as well as the n-chain functor $S_n\colon \mathbf{Top} \to \mathbf{Ab}$ and n-cellular chain functor $C_n\colon \mathbf{Top} \to \mathbf{Ab}$; we moreover look at the nth cohomology contravariant functor, for any n, $H^n\colon \mathbf{Top} \to \mathbf{Ab}$, as well as the n-cochain contravariant functors $S^n\colon \mathbf{Top} \to \mathbf{Ab}$ and $C^n\colon \mathbf{Top} \to \mathbf{Ab}$ (refer to appendix)

3. In general, we can view all the nth homology functors (resp. nth cohomology functors) as one functor $H_\bullet\colon \mathbf{Top} \to \mathbf{Ch(Ab)}$ (resp. as one contravariant functor $H^\bullet\colon \mathbf{Top} \to \mathbf{CoCh(Ab)}$). The same holds for (co-)chain and cellular (co-)chain (contravariant) functors $S_\bullet, C_\bullet\colon \mathbf{Top} \to \mathbf{Ch(Ab)}$ ($S^\bullet, C^\bullet\colon \mathbf{Top} \to \mathbf{CoCh(Ab)}$). Refer to the § **Functoriality of Homologies** section for more details.

§ Limits of Diagrams

First, we define some categorical concepts that turn out to be special cases of limits and colimits, which this paper will explicitly cover.

Definition 2.5.

1. Given a category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we call an object $X \times Y$ the *product* of $\mathcal{C}$ if there exists morphisms $\pi_X \colon X \times Y \to X$, $\pi_Y \colon X \times Y \to Y$ such that given morphisms $f_X \colon X \to Z$ and $f_Y \colon Y \to Z$, there exists a unique morphism $\langle f_X, f_Y \rangle \colon X \times Y \to Z$ the following diagram commutes:

$$\begin{tikzcd}
& Z \arrow[dl, "f_Y"'] \arrow[d, dashed, "\langle f_X, f_Y \rangle"] \arrow[dr, "f_X"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

More generally, we define a *product* $\prod_{i \in I} X_i$ *of a family of objects* $\{X_i\}_{i \in I} \subset \mathcal{C}$ to be an object such that given $Z \in \mathcal{C}$ and a family $\{f_i \colon X_i \to Z\}_{i \in I}$ of morphisms, we find there exists a unique morphism $\langle f_i \rangle_{i \in i} \colon Z \to \prod_{i \in I} X_i$ such that

$$\begin{tikzcd}
Y \arrow[d, dashed, "\langle f_i \rangle_{i \in I}"'] \arrow[dr, "f_j"] & \\
\prod_{i \in I} X_i \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

commutes

2. We call an object $X + Y$ the *coproduct* of $\mathcal{C}$ if there exists morphisms $p_X \colon X \to X + Y$ and $p_Y \colon Y \to X + Y$ such that given morphisms $g_X \colon W \to X$ and $g_Y \colon W \to Y$, there exists a morphism $[g_X, g_Y] \colon W \to X + Y$.

$$\begin{tikzcd}
& W & \\
X \arrow[ur, "g_X"] \arrow[r, "p_X"'] & X + Y \arrow[u, dashed, "{[g_X, g_Y]}"] & Y \arrow[ul, "g_Y"'] \arrow[l, "p_Y"]
\end{tikzcd}$$

More generally, we define *coproduct* $\coprod_{i \in I} X_i$ *of a family of objects* $\{X_i\}_{i \in I} \subset \mathcal{C}$ to be an object such that given $Z \in \mathcal{C}$ and a family $\{g_i \colon W \to X_i\}_{i \in I}$ of morphisms, we there exists a unique morphism $[f_i]_{i \in I} \colon \coprod_{i \in I} X_i \to W$ such that

$$\begin{tikzcd}
W & \\
\coprod_{i \in I} X_i \arrow[u, "{[g_i]_{i \in I}}"] & X_j \arrow[ul, "g_j"'] \arrow[l, "p_j"]
\end{tikzcd}$$

commutes.

Definition 2.6. Given an object X and Y, and morphisms $f \colon X \to Z$ and $g \colon Y \to Z$ we define the *pullback* $X \times_Z Y$ with maps $\varphi_X \colon X \times_Z Y \to X$ and $\varphi_Y \colon X \times_Z Y \to X$ such that it is a solution and every solution ϕ_X, ϕ_Y has a unique morphism $\langle \phi_X, \phi_Y \rangle \colon T \to X \times_Z Y$ that satisfies the following UMP (universal mapping property):

$$\begin{tikzcd}
T \arrow[dr, dashed, "\langle \phi_X,\phi_Y\rangle"'] \arrow[drr, "\phi_X"] \arrow[ddr, "\phi_Y"'] & & \\
& X \times_Z Y \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

With objects X and Y, and morphisms $q \colon W \to X$ and $r \colon W \to Y$, we define the *pushout* $X +_W Y$ with maps $\xi_X \colon X \to X +_W Y$ and $\xi_Y \colon Y \to X +_W Y$ such that it is a solution and every solution ζ_X, ζ_Y has a unique morphism $[\zeta_X, \zeta_Y] \colon X +_W Y \to S$ that satisfies the following UMP:

$$\begin{tikzcd}
W \arrow[r, "q"] \arrow[d, "r"'] & X \arrow[d, "\xi_X"'] \arrow[ddr, "\zeta_Y"] & \\
Y \arrow[r, "\xi_Y"] \arrow[drr, "\zeta_Y"'] & X +_W Y \arrow[dr, "{[\zeta_X,\zeta_Y]}"] & \\
& & S.
\end{tikzcd}$$

Note that we consider a *diagram* to be a functor $D \colon J \to \mathcal{C}$ from an arbitrary category J that we call an *index category* to some category $\mathcal{C}$; we often like to write D_j in place of $D(j)$.

Definition 2.7.

1. Given a diagram $D \colon J \to \mathcal{C}$, we first define a *cone* (C, ϕ_j) of D to be an object $C \in \mathcal{C}$ and morphisms $\phi_j \colon C \to D_j$ such that

$$\begin{tikzcd}
& C \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\
D_i \arrow[rr, "D(f)"'] & & D_j
\end{tikzcd}$$

commutes. We define the *limit* (L, φ_j) of D as a cone such that for any other cone (C, ϕ_j) there exist a morphism $u \colon C \to L$ such that

$$\begin{tikzcd}
& C \arrow[d, dashed, "u"] \arrow[ddl, "\phi_i"'] \arrow[ddr, "\phi_j"] & \\
& L \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\
D_i \arrow[rr, "D(f)"'] & & D_j
\end{tikzcd}$$

commutes.

2. Next, we define a *cocone* (S, ζ_j) of a diagram D to be an object $S \in \mathcal{C}$ and morphisms $\zeta_j \colon D_j \to P$ such that

$$\begin{tikzcd}
 & S & \\
D_i \arrow[ur, "\zeta_i"] \arrow[rr, "D(f)"'] & & D_j \arrow[ul, "\zeta_j"']
\end{tikzcd}$$

commutes. We define the *colimit*, (P, ξ_j) of a diagram D as a cocone such that for any other cocone (P, φ_j) there exists a morphism $v \colon P \to S$ such that

$$\begin{tikzcd}
 & S & \\
 & P \arrow[u, dashed, "v"] & \\
D_i \arrow[uur, "\zeta_i"] \arrow[ur, "\xi_i"'] \arrow[rr, "D(f)"'] & & D_j \arrow[uul, "\zeta_j"'] \arrow[ul, "\xi_j"]
\end{tikzcd}$$

commutes.

Remark 2.8. First, it's worth noting a few equivalent definitions of limits and colimits. The first is the duality of limits and colimits. Every limit (respectively colimit) object in the category $\mathcal{C}$ is a colimit (resp. limit) in the category $\mathcal{C}^{op}$. This immediately follows from the realization that every cone (resp. cocone) in $\mathcal{C}$ is a cocone (resp. cone) in the category $\mathcal{C}^{op}$.

Another way to equivalently define limits is looking at the category $\mathbf{Cone}(D)$ of the diagram $D \colon I \to \mathcal{C}$ with objects consisting of all cones (X, ϕ_i) of the diagram D and morphisms $u \colon (X, \phi_i) \to (Y, \psi_i)$ consisting of morphisms $u \colon X \to Y$ in $\mathcal{C}$ such that $\phi_i = \psi_i \circ u$ for all $i \in I$. Limits are final objects (see next remark) of that category. Similarly, colimits are initial objects (see next remark) in the category $\mathbf{CoCone}(D) := \mathbf{Cone}(D)^{op}$.

Remark 2.9. We shall see that products, coproducts, pullbacks, and pushouts are special cases of limits, as well as some other diagrams. Note that we use direct graphs, with "$\cdot$" as vertices and $\to$ as a directed edge, to refer to the index category generated by the graph (i.e., one with objects as vertices and morphisms as paths).

1. First, we start with the empty set index category $\emptyset$. We find the limit of any diagram on this category is the *final object* and the colimit of this diagram is the *initial object*, since *every object* itself is a cone (resp. a cocone), so the limit (resp. a colimit) is the final (resp. initial) object.

2. Next, we look at diagrams of the two-vertex index category $\{\cdot\ \cdot\}$ with no directed edges. We find the limit of any diagram $\{A, B\}$ on this category is the *product*, since any object X that has morphims $f_A \colon X \to A$ and $f_B \colon X \to B$ has morphisms $\langle f_A, f_B \rangle \colon X \to A \times B$ has morphisms that commute with the cone structure. Similiarly, the colimit of any diagram $\{\cdot\ \cdot\}$ is the *coproduct*. More generally, we find the limit (resp. colimit) diagram of the n-vertex index category $\{\cdot \underbrace{\cdots}_{n \text{ times}} \cdot\}$ with no directed edges is the *n-product* (resp. *n-coproduct*).

3. Let's look at some diagrams of index cagegory with arrows. First, let's look at the index category $\{\cdot \to \cdot \leftarrow \cdot\}$. We find that the limit of diagrams of that index category is the *pullback*. The *pushout* ends up being the colimit of the diagram of the index category $\{\cdot \leftarrow \cdot \to \cdot\}$.

4. In the next chapter (**Definition 3.7**), we talk about the "direct limit" and "inverse limit", and it's worth mentioning the index category structure in this instance here. We find the *direct limit* is a colimit of a diagram of the index category $\{\cdot \to \cdot \to \cdot \to \cdots\}$ and the *inverse limit* is a limit of a diagram of the index category $\{\cdots \leftarrow \cdot \leftarrow \cdot \leftarrow \cdot\}$.

5. Lastly, it's worth talking about diagrams with an index category of the form $\{\cdot \rightrightarrows \cdot\}$ where there is multiple directed edges between two vertices. The limit of a diagram on that index category is what we call an *equilizer* and the colimit of such a diagram is what we call a *coequilizer*. This paper does not explicitly utilize equilizers and coequilizers since we can define limits in the categories we cover without explicit mention of them. However, it still is worth mentioning them to talk about the underlying category theoretic principles that go into figuring out the construction of limits (resp. colimits) in a given category.

In this paper, we talk about limits and colimits in the categories **Top** and **Ab**, as well as **Ch(Ab)** **CoCh(Ab)** (see **Definition 5.8**), and whether or not they exist. We call a diagram $D\colon J \to \mathcal{C}$ a *small diagram* if J is a "small category", by which we mean that both the object and morphism class are "small sets"–i.e., can be expressed as a real set in the ZFC-axiom treatment of set theory. We consider a category *small complete* (resp. *small cocomplete*) if every small diagram has a limit (resp. colimit).

When showing whether categories are small complete or small cocomplete, you will notice a patten where small limits (resp. colimits) are often formulated as subojects of the small products (resp. quotient objects of small colimits). When we notice that subobjects (resp. quotient objects) go hand-in-hand with equilizing (resp. coequilizing) properties of categories, we find this is no accident when we reflect on this important equivalent condition of limits (resp. colimits) which we shall state for intuition (but not prove).

Theorem 2.10. A category $\mathcal{C}$ is small complete (resp. small cocomplete) if and only if $\mathcal{C}$ has small products and equilizers (resp. small coproducts and coequilizers).

§ Diagram and Chain Categories

Definition 2.11.

1. Given a category $\mathcal{C}$ and an index category I, we define the *category of diagrams* $\mathcal{C}^I$ with object class the diagrams from I to $\mathcal{C}$ with morphisms as natural transformations between them. More specifically, we define the *category of chains of* $\mathcal{C}$ as the diagram category $\mathcal{C}^{(\mathbb{N}_0,\leq)}$, i.e., the objects are sequences of morphisms $(f_n\colon C_n \to C_{n+1})_{n\in\mathbb{N}_0}$ and the morphisms $\eta\colon (f_n)_{n\in\mathbb{N}_0} \to (f'_n)_{n\in\mathbb{N}_0}$ are

"latters of morphisms" $\eta_n\colon C_n \to C'_n$ such that

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "\eta_0"] & C_1 \arrow[r, "f_1"] \arrow[d, "\eta_1"] & \cdots \arrow[r, "f_{n-1}"] & C_n \arrow[r, "f_n"] \arrow[d, "\eta_n"'] & \cdots \\
C'_0 \arrow[r, "f'_0"'] & C'_1 \arrow[r, "f'_1"'] & \cdots \arrow[r, "f'_{n-1}"] & C'_n \arrow[r, "f'_n"] & \cdots
\end{tikzcd}$$

commutes. We can define the *category of cochains of* $\mathcal{C}$ as the diagram category $\mathcal{C}^{(\mathbb{N}_0,\geq)}$, with the sequence of morphisms $(g_n\colon D_{n+1} \to D_n)_{n\in\mathbb{N}_0}$ in the reverse direction with morphisms $\gamma\colon (f_n)_{n\in\mathbb{N}_0} \to (f'_n)_{n\in\mathbb{N}_0}$ similarly defined as the latters morphisms $\gamma_n\colon D_n \to D'_n$ so that the diagrams

$$\begin{tikzcd}
D_0 \arrow[d, "\gamma_0"] & D_1 \arrow[l, "g_0"] \arrow[d, "\gamma_1"] & \cdots \arrow[l, "g_2"] & D_n \arrow[l, "g_{n-1}"] \arrow[d, "\gamma_n"'] & \cdots \arrow[l, "g_n"] \\
D'_0 & D'_1 \arrow[l, "g'_0"] & \cdots \arrow[l, "g'_2"] & D'_n \arrow[l, "g'_{n-1}"] & \cdots \arrow[l, "g'_n"]
\end{tikzcd}$$

commutes.

Remark 2.12. Note that in homological algebra, it is mainstream convention to reverse the terms "chain" and "cochain" and when we refer to chain and cochain "complexes". In other words, when we refer to "chain complexes", we are referring to *cochains* of exact morphisms (i.e., morphisms going in the *decreasing direction*), and when we refer to a "cochain complex", we are actually referring to a *chain* of exact morphisms (i.e., morphisms going in the *increasing direction*). We maintain this convention in **Definition 5.8** when we define the categories of chain and cochain complexes, despite the somewhat conflicting terminology.

Note that there is a canonical way to think about a limit of a diagram L that is helpful to think about in this paper, which is a chain of limits in the original category. This result follows by design of the definitions.

Proposition 2.13. Any colimit $P \in \mathcal{C}^{(\mathbb{N}_0,\leq)}$ of any diagram $D\colon J \to \mathcal{C}^{(\mathbb{N}_0,\leq)}$ is such that $P(n)$ is the colimit of the diagram $D(n)\colon J \to \mathcal{C}$ defined by

$$(f\colon i \to j) \mapsto (D(f)\colon D_i(n) \to D_j(n))$$

Similarly, any limit $L \in \mathcal{C}^{(\mathbb{N}_0,\geq)}$ of any diagram $E\colon J \to \mathcal{C}^{(\mathbb{N}_0,\geq)}$ is such that $L(n)$ is the limit of the diagram $E(n)\colon J \to \mathcal{C}$.

Remark 2.14. Note that the previous proposition can be generalized to arbitrary diagram categories $\mathcal{C}^I$, but we state this proposition in this special case, since the paper only really deals with this special case, so the intuition of the special case far outweighs the generality of the general case.

3 Topological Limits

§ Products, Disjoint Unions, Pullbacks, and Pushouts

Definition 3.1.

1. Given two topologies X and Y , the *cartesian product* $X \times Y$ is defined as the set $X \times Y$ (see appendix) with the topology generated by all subsets of $X \times Y$ of the form $U \times V$ as a basis, where $U \subset X$ and $V \subset Y$ are open in their respective topology. We shall define the projection maps $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ by

$$\pi_X(x, y) = x,\ \ \pi_Y(x, y) = y.$$

2. Given two topologies X and Y, the *disjoint union* $X \sqcup Y$ is defined as the disjoint union set $X \sqcup Y$ (see appendix) with the topology generated by all subsets of the form $U \times \{X\}$ and $V \times \{Y\}$ as a basis, where $U \subset X$ and $V \subset Y$ are open in their respective topology. We shall define the injective maps $p_X : X \to X \sqcup Y$, $p_Y : Y \to X \sqcup Y$ by

$$p_X(x) = (x, X),\ \ p_Y(y) = (y, Y).$$

Definition 3.2.

1. Given a family of topologies $\{X_i\}_{i\in I}$, the *cartesian product* $\prod_{i\in I} X_i$ of $\{X_i\}_{i\in I}$ is defined as the set $\prod_{i\in I} X_i$ (see appendix). We shall define the projection maps $\pi_j : \prod_{i\in I} X_i \to X_j$, for each $j \in I$, by $\pi_j((x_i)_{i\in I}) = x_j$. The topology of this product is the topology generated by sets of the form $\pi_j^{-1}(U_j)$ as a subbasis, where $U_j \subset X_j$ is open.

2. Given a family of topologies $\{X_i\}_{i\in I}$, the *disjoint union* $\bigsqcup_{i\in I} X_i$ of $\{X_i\}_{i\in I}$ is defined as the set $\bigsqcup_{i\in I} X_i$ (see appendix). We shall define the injective maps $p_j : X_j \to \bigsqcup_{i\in I} X_i$, for each $j \in I$, by $p_j(x_j) = (x_j, j)$. The topology of this disjoint union is the topology generated by sets of the form $p_j(U_j)$ as a basis, where $U_j \subset X_j$ is open.

Proposition 3.3. Let $\{X_i\}_{i\in I}$ be a family of topologies. $\prod_{i\in I} X_i$ are products and $\bigsqcup_{i\in I} X_i$ are coproducts in the category **Top**.

Proof. Given a family $\{f_i\colon Y \to X_i\}_{i\in I}$ of continuous maps, we find $(f_i)_{i\in I}\colon Y \to \prod_{i\in I} X_i$ defined by

$$(f_i)_{i\in I}(y) = (f_i(y))_{i\in I},$$

is a map such that

$$\begin{tikzcd} Y \arrow[d, "(f_i)_{i\in I}"'] \arrow[dr, "f_j"] & \\ \prod_{i\in I} X_i \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

commutes.

Moreover, given a family $\{g_i\colon X_i \to Z\}_{i\in I}$ of continuous maps, we find $\sqcup_{i\in I} g_i\colon \coprod_{i\in I} X_i \to Z$ defined by

$$\sqcup_{i\in I} g_i(x, j) = g_j(x),$$

is map such that

$$\begin{tikzcd} Z & \\ \coprod_{i\in I} X_i \arrow[u, "\sqcup_{i\in I} g_i"] & X_j \arrow[l, "p_j"] \arrow[ul, "g_j"'] \end{tikzcd}$$

commutes. □

Definition 3.4.

1. Given two topological maps $f\colon X \to Z$ and $g\colon Y \to Z$, the *pullback* $X \times_Z Y$ of f, g is defined as the set

$$X \times_Z Y = \{(x, y) \in X \times Y\colon f(x) = g(y)\},$$

with the subspace topology of the product topology $X \times Y$, along with the maps φ_X, φ_Y defined to be the projections of $X \times Y$ restricted to $X \times_Z Y$

$$\varphi_X := \pi_X|(X \times_Z Y),\ \varphi_Y := \pi_X|(X \times_Z Y).$$

2. Given two topological maps $q\colon W \to X$ and $r\colon W \to Y$, the *pushout* $X +_W Y$ is defined as the set

$$X +_W Y := (X \sqcup Y)\, / q(w) \sim r(w)$$

endowed with the quotient topology, along with the maps ξ_X, ξ_Y defined to be the composition of injections defined by

$$\xi_X := [-]_\sim \circ p_X,\ \xi_Y := [-]_\sim \circ p_Y,$$

where $[-]_\sim\colon X \sqcup Y \to X +_W Y$ is the quotient map of the equivalence relation $q(w) \sim r(w)$.

Lemma 3.5. Let $\equiv$ be an equivalence relation on X. For any map $g\colon X \to Y$ such that the image $g[[x]]$ of every equivalence class is a singleton, there exists a map $\overline{g}\colon X/\equiv \to Y$ such that

$$\begin{tikzcd} X \arrow[d, "{[-]_\equiv}"'] \arrow[dr, "g"] & \\ X/\equiv \arrow[r, "\overline{g}"'] & Y \end{tikzcd}$$

commutes.

Proof. Note that $\overline{g}\colon [x] \mapsto g(x)$ is well defined in this situation. $\square$

Proposition 3.6. The notions of "pullbacks" and "pushouts" defined in **Definition 2.4** are pullbacks and pushouts, respectively, in the category **Top**.

Proof. First, let

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

be a pullback diagram. We find $X \times_Z Y$ and φ_X, φ_Y is a solution such that for any other solution $\phi_X\colon T \to X, \phi_Y\colon T \to Y$, we find $(\phi_X, \phi_Y)\colon T \to X \times_Z Y$ defined by $t \mapsto (\phi_X(t), \phi_Y(t))$ is a map such that

$$\begin{tikzcd} T \arrow[dr, "{(\phi_X,\phi_Y)}"] \arrow[drr, bend left, "\phi_X"] \arrow[ddr, bend right, "\phi_Y"'] & & \\ & X \times_Z Y \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

commutes.

Next, let

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "r"'] & X \\ Y & \end{tikzcd}$$

be a pushout diagram. We find $X +_W Y$ and ξ_X, ξ_Y is a solution such that for any other solution $\zeta_X\colon X \to S, \zeta_Y\colon Y \to S$, we find $\zeta_X \sqcup \zeta_Y\colon X \sqcup Y \to S$ defined by $(x, X) \mapsto \zeta_X(x)$ and $(y, Y) \mapsto \zeta_Y(y)$ is a map such that

$$\begin{tikzcd} X \arrow[dr, "p_X"] \arrow[ddr, "\zeta_X"'] & & Y \arrow[dl, "p_Y"'] \arrow[ddl, "\zeta_Y"] \\ & X \sqcup Y \arrow[d, "\zeta_X \sqcup \zeta_Y"'] & \\ & S & \end{tikzcd}$$

commutes. Since $\zeta_X(q(w)) = \zeta_Y(r(w))$, we find by **Lemma 2.5** that there exists $\overline{\zeta_X \sqcup \zeta_Y}\colon X \sqcup Y \to S$ exists such that

$$\begin{tikzcd} X \sqcup Y \arrow[d, "{[-]_\sim}"'] \arrow[dr, "\zeta_X \sqcup \zeta_Y"] & \\ X +_W Y \arrow[r, "\overline{\zeta_X \sqcup \zeta_Y}"'] & S \end{tikzcd}$$

commutes, and we can verify that

$$
\begin{tikzcd}
W \arrow[r, "q"] \arrow[d, "r"'] & X \arrow[d, "\xi_X"'] \arrow[ddr, "\zeta_Y"] & \\
Y \arrow[r, "\xi_Y"] \arrow[drr, "\zeta_Y"'] & X +_W Y \arrow[dr, "\overline{\zeta_X \sqcup \zeta_Y}"] & \\
& & S
\end{tikzcd}
$$

commutes. $\square$

§ Topological Limits and Colimits

Definition 3.7.

1. Let $(h_n \colon X_n \twoheadrightarrow X_{n-1})_{n \geq 1}$ be a sequence of epimorphisms (i.e., surjective open maps) between the topologies

$$\cdots \xrightarrow{h_{n+1}} X_n \xrightarrow{h_n} X_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_3} X_2 \xrightarrow{h_2} X_1 \xrightarrow{h_1} X_0.$$

We define the *inverse limit* of this sequence $\varprojlim_n X_n$ as follows:

$$\varprojlim_n X_n := \left\{ (x_n)_{n \geq 0} \in \prod_{n \geq 0} X_n \colon h_n(x_n) = x_{n-1} \right\}.$$

2. Let $(i_n \colon X_{n-1} \rightarrowtail X_n)_{n \geq 1}$ be a sequence of monomorphisms (i.e., open injective maps) between the topologies

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X_n \xrightarrow{i_{n+1}} \cdots.$$

We define the *direct limit* of this sequence $\varinjlim_n X_n$ as follows:

$$\varinjlim_n X_n := \left(\bigsqcup_{n \geq 0} X_n \right) / x \sim i_n(x).$$

Definition 3.8. Let $X \colon I \to \mathbf{Top}$ be a small diagram.

1. We define the *limit of the diagram X* as

$$\varprojlim_I X_i := \left\{ (x_i)_{i \in I} \prod_{i \in I} X_i \colon x_j = X(f)(x_j), \text{ where } f \colon i \to j \right\}$$

2. We define the *colimit of the diagram X* as

$$\varinjlim_I X_i := \left(\bigsqcup_{i \in I} X_i \right) / \sim_X$$

where

$$x \sim_X y \iff y = X(f)(x) \text{ for some } f\colon i \to j.$$

Theorem 3.9. Let $X\colon I \to \mathbf{Top}$ be a small diagram.

1. The cone $(\varprojlim_I X_i, \varphi_i)$ with maps $\varphi_i := \pi_i|\varprojlim_I X_i$ (recall that π_i is the ith projection map of $\prod_{i\in I} X_i$) is a limit in **Top** of the diagram X.

2. The cocone $(\varinjlim_I X_i, \xi_i)$, with maps $\xi_i := [-]_{\sim_X} \circ p_i$, where $[-]_{\sim_X}\colon \bigsqcup_{i\in I} X_i \to \varinjlim_I X_i$ is the quotient map of $\sim_X$ (and recall that p_i is the ith injection map of $\bigsqcup_{i\in I} X_i$), is a colimit in **Top** of the diagram X.

3. **Top** is small complete and small cocomplete.

Proof.

1. Set $L := \varprojlim_I X_j$ as defined in the definition, and let (C, ϕ_i) be a cone of X. We find $(\phi_i)_{i\in I}C\colon \to L$ is a map such that for any morphism $f\colon i \to j$ in the index category I, we find

$$\begin{tikzcd}
& C \arrow[d, "(\phi_i)_{i\in I}"] \arrow[ddl, "\phi_i"'] \arrow[ddr, "\phi_j"] & \\
& L \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\
X_i \arrow[rr, "X(f)"'] & & X_j
\end{tikzcd}$$

commutes.

2. Set $P := \varinjlim_I X_j$ as defined in the definition, and let (E, ϕ_i) be a cocone of X. We shall show that (P, ξ_i) is a cocone such that for any other cocone (S, ζ_i), we find $\sqcup_{i\in I}\zeta_i\colon \bigsqcup_{i\in I} X_j \to S$ defined by $(x, i) \mapsto \zeta_i(x)$ is a map such that

$$\begin{tikzcd}
& S & \\
& \bigsqcup_{i\in I} X_i \arrow[u, "\sqcup_{i\in I}\zeta_i"] & \\
X_i \arrow[ur, "p_i"'] \arrow[uur, "\zeta_i"] & & X_j \arrow[ul, "p_j"] \arrow[uul, "\zeta_j"']
\end{tikzcd}$$

commutes. Since $\zeta_i(x) = \zeta_j(X(f)(x))$, we find by **Lemma 2.5** that there exists $\overline{\sqcup_{i\in I}\zeta_i}\colon P \to S$ exists such that

$$\begin{tikzcd}
\bigsqcup_{i\in I} X_i \arrow[d, "{[-]_\sim}"'] \arrow[dr, "\sqcup_{i\in I}\zeta_i"] & \\
P \arrow[r, "\overline{\sqcup_{i\in I}\zeta_i}"'] & S
\end{tikzcd}$$

commutes, and we find that

$$\begin{tikzcd}
& S & \\
& P \arrow[u, "\overline{\sqcup_{i\in I}\zeta_i}"] & \\
X_i \arrow[uur, "\zeta_i"] \arrow[ur, "\xi_i"'] \arrow[rr, "X(f)"'] & & X_j \arrow[uul, "\zeta_j"'] \arrow[ul, "\xi_j"]
\end{tikzcd}$$

commutes.

3. We have proven with 1. and 2. that given a small diagram $X\colon I \to \mathbf{Top}$, we have $\varprojlim_I X_i$ and $\varinjlim_I X_i$ as defined in **Definition 2.8** are limits and colimits respectively. $\square$

Proposition 3.10.

1. The notion of "inverse limits" in **Definition 2.7-1** is a limit–with appropriate cone structure- of the diagram $X\colon (\mathbb{N}, \geq) \to \mathbf{Top}$ defined by $n \geq n-1 \mapsto h_n\colon X_n \to X_{n-1}$ for every $n \geq 1$. The notion of "direct limit" in **Definition 2.7-2** is the colimit–with appropriate cocone structure–of the diagram $X\colon (\mathbb{N}, \leq) \to \mathbf{Top}$ defined by $n-1 \leq n \mapsto i_n\colon X_{n-1} \to X_n$ for every $n \geq 1$.

2. Given a chain of injections

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X_n \xrightarrow{i_{n+1}} \cdots,$$

we find that there exists $X'_n \subset X$, where $X := \varinjlim_n X_n$, such that $X'_n \cong X_n$ and $X'_n \subset X'_{n+1}$, for every $n \geq 0$, and

$$X = \bigcup_n X'_n = \varinjlim_n X'_n.$$

In other words, X is a colimit of the diagram of inclusions

$$X'_0 \hookrightarrow X'_1 \hookrightarrow X'_2 \hookrightarrow \cdots \hookrightarrow X'_{n-1} \hookrightarrow X'_n \hookrightarrow \cdots.$$

Proof.

1. Applying induction on n, we find the inverse limits and direct limits are a special case of limits and colimits, respectively as they are defined in **Definition 2.8**, i.e., $\varprojlim_n X_n$ and $\varprojlim_{(\mathbb{N},\geq)} X_n$ are the same subset of $\prod_{i\in I} X_i$, and $\varinjlim_n X_n$ and $\varinjlim_{(\mathbb{N},\leq)} X_n$ are the same quotient relation of $\bigsqcup_{i\in I} X_i$.

2. We shall show that $\xi_n\colon X_n \rightarrowtail X$ is a monomorphism for every $n \in \mathbb{N}$. Suppose $f, g\colon X \to Y$ and $f \circ \xi_n = g \circ \xi_n$. Note that since each i_n is a monomorphism, we find the diagram

$$\begin{tikzcd}
& & & Y & & \\
& & & X \arrow[u, shift right, "f"] \arrow[u, shift left, "g"'] & & \\
X_1 \arrow[r, "i_1"'] \arrow[urrr, "\xi_1"'] & X_2 \arrow[r, "i_2"'] \arrow[urr, "\xi_2"'] & \cdots \arrow[r, "i_{n-1}"'] & X_n \arrow[r, "i_n"'] \arrow[u, "\xi_n"'] & X_{n+1} \arrow[r, "i_{n+1}"'] \arrow[ul, "\xi_{n+1}"'] & \cdots
\end{tikzcd}$$

commutes, hence $(Y, f \circ \xi_n)$ is a cocone and it follows that $f = g$ by uniqueness of colimits. Take $X'_n := \xi_n[X_n]$, observe that the codomain restriction $\xi'_n \colon X_n \xrightarrow{\sim} X'_n$ are isomorphisms, and that the diagram

$$\begin{tikzcd}
 & & & & X & & \\
 & & & & \bigcup_n X'_n \arrow[u, hook] & & \\
X'_1 \arrow[r, "i_1"'] \arrow[urrrr, hook] & X'_2 \arrow[r, "i_2"'] \arrow[urrr, hook] & \cdots \arrow[r, "i_{n-1}"'] & & X'_n \arrow[r, "i_n"'] \arrow[u, hook] & X'_{n+1} \arrow[r, "i_{n+1}"'] \arrow[ul, hook] & \cdots \\
X_1 \arrow[r, "i_1"'] \arrow[u, "\xi'_1"'] & X_2 \arrow[r, "i_2"'] \arrow[u, "\xi'_2"'] & \cdots \arrow[r, "i_{n-1}"'] & & X_n \arrow[r, "i_n"'] \arrow[u, "\xi'_n"'] & X_{n+1} \arrow[r, "i_{n+1}"'] \arrow[u, "\xi'_{n+1}"'] & \cdots
\end{tikzcd}$$

commutes. The conclusion that

$$X = \bigcup_n X'_n = \varinjlim_n X'_n,$$

immediately follows. $\square$

4 CW Complexes

§ My Favorite Definition and its Properties

Definition 4.1. A *CW-Complex* is a pair $(X, \{e_i^n\})$ consisting of Hausdorff spaces X and a collection $\{e_i^n\}$ n-cells indexed by $n = 0, 1, 2, \ldots$ and $i \in I_n$, satisfying the following properties:

1. $X = \bigcup_{n,i} e_i^n$ endowed with the union topology (see appendix), with every point in X is contained in a *unique* e_i^n.

2. For each e_i^n, there exists a map $\chi_i^n \colon D^n \to X$ that we call the *attaching map* with the following properties

(a) The restriction $\chi_i^n|(\text{int } D^n)$ is a homeomorphism to e_i^n
(b) For each e_i^n, $\chi_i^n[S^{n-1}] = \bigcup_{k=1}^N e_{i_k}^{n_k}$ where $n_k < n$ for each $1 \leq k \leq N$.

Definition 3.2. Given a CW-Complex $(X, \{e_i^n\})$, we shall call $X^n := \bigcup_{1 \leq k \leq n} \bigcup_{i \in I_k} e_i^k$ *n-skeleton* of X. In general, a topological space X' is an n-skeleton if it is a CW complex that contains only cells up to dimension n.

Definition 4.3.

1. A *cellcular map* $f \colon X \to Y$ is a continuous map that preserves the skeletel filtration: $f(X^n) \subset Y^n$.

2. The category **CW** of CW-complexes consists of all CW complex with its morphisms consisting of skeletal filtrations.

Proposition 4.4. Every CW-Complex is the direct limit of its n-skeletons, i.e., $X = \varinjlim_n X^n$. Moreover, every X^n $(n \geq 0)$ is the pushout of the diagram

$$\begin{tikzcd} \bigsqcup_{i \in I_n} S^{n-1} \arrow[r, "\bigsqcup_{i \in I_n} \chi_i^n"] \arrow[d, hook] & X^{n-1}, \\ \bigsqcup_{i \in I_n} D^n & \end{tikzcd}$$

with $X^{-1} := \emptyset$.

Proof. First, note that $X = \varinjlim_n X^n$ since $X = \bigcup_n X^n$, which we proved previously is the colimit of an increasing sequence of subspaces.

To show that X^{n+1} is a pushout of the above diagram, for $n \geq -1$, we shall show that there exists exists a map $h \colon X^{n+1} \to \bigsqcup_{i \in I_{n+1}} D^{n+1} +_{\bigsqcup_{i \in I_{n+1}} S^n} X^n$ such that

$$\begin{tikzcd}
\bigsqcup_{i \in I_{n+1}} S^n \arrow[r, "\sqcup_{i \in I_{n+1}} \chi_i^{n+1}"] \arrow[d, hook] & X^n \arrow[d, hook] \arrow[ddr, "\xi_{X^n}"] & \\
\bigsqcup_{i \in I_{n+1}} D^{n+1} \arrow[r, "\sqcup_{i \in I_{n+1}} \chi_i^{n+1}"] \arrow[drr, "\xi_{\bigsqcup_{i \in I_{n+1}} D^{n+1}}"'] & X^{n+1} \arrow[dr, "h"] & \\
& & \bigsqcup_{i \in I_{n+1}} D^{n+1} +_{\bigsqcup_{i \in I_{n+1}} S^n} X^n,
\end{tikzcd}$$

commutes, and by uniqueness of limits, it will then follows that h is a homeomorphism and $X^{n+1} \cong \bigsqcup_{i \in I_{n+1}} D^{n+1}$.

Define $h_0 \colon X^{n+1} \to \bigsqcup_{i \in I_{n+1}} D^{n+1} \sqcup X^n$ as follows:

$$h_0(x) = \begin{cases} x & \text{if } x \in X^n \\ (\sqcup_{i \in I_{n+1}} \chi | D^n)^{-1} & \text{otherwise,} \end{cases}$$

which note is continuous by the *Pasting Lemma*. We then define $h = [h_0]_{n+1}$, where $[-]_{n+1} \colon \bigsqcup_{i \in I_{n+1}} D^{n+1} \sqcup X^n \to \bigsqcup_{i \in I_{n+1}} D^{n+1} +_{\bigsqcup_{i \in I_{n+1}} S^n} X^n$ is the quotient map. It follows by topological construction of the pullback that the diagram

$$\begin{tikzcd}
\bigsqcup_{i \in I_{n+1}} S^n \arrow[r, "\sqcup_{i \in I_{n+1}} \chi_i^{n+1}"] \arrow[d, hook] & X^n \arrow[d, hook] \arrow[ddr, "\xi_{X^n}"] & \\
\bigsqcup_{i \in I_{n+1}} D^{n+1} \arrow[r, "\sqcup_{i \in I_{n+1}} \chi_i^{n+1}"] \arrow[drr, "\xi_{\bigsqcup_{i \in I_{n+1}} D^{n+1}}"'] & X^{n+1} \arrow[dr, "h"] \arrow[r, "h_0"] & \bigsqcup_{i \in I_{n+1}} D^{n+1} \sqcup X^n \arrow[d, "{[-]_{n+1}}"] \\
& & \bigsqcup_{i \in I_{n+1}} D^{n+1} +_{\bigsqcup_{i \in I_{n+1}} S^n} X^n,
\end{tikzcd}$$

commutes. $\square$

§ Equivalence with the Pushout Definition

Theorem 4.5. X is a CW-Complex–with appropriate n-cell structure $\{e_i^n\}$ unique up to equivalence of n-skeletons–if and only if it the direct limit of n-skeletons $X^n \subset X$ such that X^n is the pushout of the diagram

$$\begin{tikzcd}
\bigsqcup_{i\in I_n} S^{n-1} \arrow[r] \arrow[d, hook] & X^{n-1} . \\
\bigsqcup_{i\in I_n} D^n &
\end{tikzcd}$$

Lemma 4.6. X is an n-skeleton (with appropriate k-cell structure $(0 \le k \le n)$ $\{e_i^k\}$ unique up to equivalence of k-skeletons) if and only if X^n is the pushout of every diagram for every $1 \le k < n$. Moreover, every k-cell structure $\{e_i^k\}$ of X^n of a CW complex X expands to a k-cell structure of X^{n+1}.

Proof. We shall prove this by induction on $n \geq 0$. For $n = 0$, note that

$$X^n \cong \bigsqcup_{i\in I_0} D^0 +_{\emptyset} \emptyset \cong \bigsqcup_{i\in I_0} D^0,$$

hence X^n is a disjoint union of points, which is a CW-complex, and our base case is satisfied.

For the inductive step, we know by hypothesis that

$$X^{n+1} \cong \bigsqcup_{i\in I_{n+1}} D^{n+1} +_{\bigsqcup_{i\in I_{n+1}} S^n} X^n,$$

and that a solution exists that makes the diagram

$$\begin{tikzcd}
\bigsqcup_{i\in I_{n+1}} S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \arrow[ddr, "\xi_{X^n}"] & \\
\bigsqcup_{i\in I_{n+1}} D^{n+1} \arrow[r] \arrow[drr, "\xi_{\bigsqcup_{i\in I_{n+1}} D^{n+1}}"'] & X^{n+1} \arrow[dr, dashed, "\sim"] & \\
& & \bigsqcup_{i\in I_{n+1}} D^{n+1} +_{\bigsqcup_{i\in I_{n+1}} S^n} X^n,
\end{tikzcd}$$

commute. it follows that ξ_{X^n} is a monomorphism, and we can then embed X^n into $\bigsqcup_{i\in I_{n+1}} D^{n+1} +_{\bigsqcup_{i\in I_{n+1}} S^n} X^n$ with the codomain restriction $h\colon X^n \xrightarrow{\sim} \xi_{X^n}[X^n]$ on ξ_{X^n}, and now we have the following commutative diagram:

$$\begin{tikzcd}
\bigsqcup_{i\in I_{n+1}} S^n \arrow[r] \arrow[d, hook] & X^n \arrow[r, "h"] \arrow[d] \arrow[ddr, "\xi_{X^n}"] & \xi_{X^n}[X^n] \arrow[dd, hook] \\
\bigsqcup_{i\in I_{n+1}} D^{n+1} \arrow[r] \arrow[drr, "\xi_{\bigsqcup_{i\in I_{n+1}} D^{n+1}}"'] & X^{n+1} \arrow[dr, dashed, "\sim"] & \\
& & \bigsqcup_{i\in I_{n+1}} D^{n+1} +_{\bigsqcup_{i\in I_{n+1}} S^n} X^n,
\end{tikzcd}$$

We can now suppose without loss of generality that $X^{n+1} = \bigsqcup_{i\in I_{n+1}} D^{n+1} +_{\bigsqcup_{i\in I_{n+1}} S^n} X^n$. By inductive hypothesis, there exists CW k-cell structure $\{e_i^k\}$ for $0 \le k \le n$ of the n-skeleton $X^n \subset$

X^{n+1}, and it remains to show $e_i^{n+1} := \xi_{\bigsqcup_{i \in I_{n+1}} D^{n+1}}[\text{int } D^{n+1} \times \{i\}]$ are the appropriate $n+1$-cells and $\chi_i^{n+1} = \xi_{\bigsqcup_{i \in I_{n+1}} D^{n+1}}|(D^{n+1} \times \{i\})$ are attaching maps. Property 2(a) and 2(b) of **Definition 4.1** follow from the fact that $\xi_{\bigsqcup_{i \in I_{n+1}} D^{n+1}} := p_{\bigsqcup_{i \in I_{n+1}} D^{n+1}} \circ [-]_{n+1}$, where $[-]_{n+1} \colon \bigsqcup_{i \in I_{n+1}} D^{n+1} \sqcup X^n \to \bigsqcup_{i \in I_{n+1}} D^{n+1} +_{\bigsqcup_{i \in I_{n+1}} S^n} X^n$ is the quotient map, which by construction is a one-to-one map when restricted to interior of $\bigsqcup_{i \in I_{n+1}} D^{n+1}$ and maps $\bigsqcup_{i \in I_{n+1}} S^n$ to X^n; so $\chi_i^{n+1}|\text{int } D^{n+1}$ is a homeomorphism to e_i^{n+1} and the image $\xi_i^{n+1}[S^n]$ is a closed subset of X^n, and hence a union of cells of dimension less than $n+1$.

Note that in the inductive step of this proof, we took a k-cell structure of X^n and expanded it to a k-cell structure of X^{n+1}, so we conclude for $n > 0$ that we can expand every k-cell structure of an n-skeleton to a k-cell structure of an $n+1$-skeleton. $\square$

Proof of Theorem 4.5. Note that in **Lemma 4.6**, we proved that the hypothesis implies that every X^n is a CW complex such that we can repeatedly take a k-cell structure $\{e_i^k\}$ of X^n and expand it into a k-cell structure of X^{n+1}. Continually extending an k-cell structure of X^n to a k-cell structure of X^{n+1} for every n gives us a k-cell structure $\{e_i^k\}$ of X, and our conclusion is reached. $\square$

5 Algebraic Limits and Sequences

§ Cartesian Products, Direct Sums, Limits, and Colimits

Definition 5.1.

1. Given two abelian groups A and B, the *cartesian product* $A \times B$ is defined as the set $A \times B$ with binary operation $+$ defined for $(a_1, b_1), (a_2, b_2) \in A \times B$ by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. We shall define the projection maps $\pi_A \colon A \times B \to A$, $\pi_B \colon A \times B \to B$ by

$$\pi_A(a, b) = a, \ \pi_B(a, b) = b.$$

2. Given two abelian groups A and B such that $A \cap B = \{0\}$ contained in some abelian group G, the *direct sum* $A \oplus B$ is defined as subgroup $\langle x \colon x \in A \text{ or } x \in B \rangle$ of G with every $a \in A$ and $b \in B$ as a generator. We shall define the injective maps $p_A \colon A \to A \oplus B$, $p_B \colon B \to A \oplus B$ by the inclusion maps

$$p_A(a) = a, \ p_B(b) = b,$$

of A and B.

Definition 5.2.

1. Given a family of abelian groups $\{A_i\}_{i \in I}$, the *cartesian product* $\prod_{i \in I} A_i$ is defined as the set $\prod_{i \in I} A_i$ with binary operation $+$ defined for $(a_i)_{i \in I}, (a_i')_{i \in I} \in \prod_{i \in I} A_i$ by

$$(a_i)_{i \in I} + (a_i')_{i \in I} = (a_i + a_i')_{i \in I}.$$

We shall define the projection maps $\pi_j \colon \prod_{i \in I} A_i \to A_j$, for each $j \in I$, by $\pi_j((a_i)_{i \in I}) = a_j$.

2. Given a family of abelian groups $\{A_i\}_{i \in I}$, the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the following subgroup of $\prod_{i \in I} A_i$:

$$\bigoplus_{i \in I} A_i := \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \colon a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

We shall define the injective maps $p_j \colon A_j \to \bigoplus_{i \in I} A_i$, for each $j \in I$, by $p_j(a) = (a_i)_{i \in I}$ with $a_i = a$ for $i = j$ and $a_i = 0$ otherwise.

Proposition 5.3. Let $\{A_i\}_{i\in I}$ be a family of abelian groups. $\prod_{i\in I} A_i$ are products and $\bigoplus_{i\in I} A_i$ are coproducts in the category **Ab**.

Proof. More or less analogous to the proof of **Proposition 3.3**, though the intricacies of the direct sum coproduct is a little different. I will provide details in a later draft. $\square$

Definition 5.4.

1. Let $(\psi_n\colon A_n \twoheadrightarrow A_{n-1})_{n\geq 1}$ be a sequence of epimorphisms (i.e., surjective open maps) between the abelian groups $A_0, A_1, \ldots, A_n, \ldots$

$$\cdots \xrightarrow{\psi_{n+1}} A_n \xrightarrow{\psi_n} A_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_3} A_2 \xrightarrow{\psi_2} A_1 \xrightarrow{\psi_1} A_0.$$

We define the *inverse limit* of this sequence $\varprojlim_n A_n$ as follows:

$$\varprojlim_n A_n := \left\{ (a_n)_{n\geq 0} \in \prod_{n\geq 0} A_n \colon \psi_n(a_n) = a_{n-1} \right\}.$$

2. Let $(\alpha_n\colon A_{n-1} \rightarrowtail A_n)_{n\geq 1}$ be a sequence of monomorphisms (i.e., open injective maps) between the abelian groups $A_0, A_1, \ldots, A_n, \ldots$

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} A_{n-1} \xrightarrow{\alpha_n} A_n \xrightarrow{\alpha_{n+1}} \cdots.$$

We define the *direct limit* of this sequence $\varinjlim_n A_n$ as follows:

$$\varinjlim_n A_n := \left(\bigoplus_{n\geq 0} A_n \right) / x \sim \alpha_n(x).$$

Definition 5.5. Let $A\colon I \to \mathbf{Ab}$ be a small diagram.

1. We define the *limit of the diagram* A as

$$\varprojlim_I A_i := \left\{ (a_i)_{i\in I} \prod_{i\in I} A_i \colon a_j = A(f)(a_j), \text{ where } f\colon i \to j \right\}.$$

2. We define the *colimit of the diagram* A as

$$\varinjlim_I A_i := \left(\bigoplus_{i\in I} A_i \right) / \sim_A$$

where

$$x \sim_A y \iff y = A(f)(x) \text{ for some } f\colon i \to j.$$

Theorem 5.6. Let $A\colon I \to \mathbf{Ab}$ be a small diagram.

1. The cone $(\varprojlim_I A_i, \varphi_i)$ with maps $\varphi_i := \pi_i|\varprojlim_I A_i$ (recall that π_i is the ith projection map of $\prod_{i\in I} A_i$) is a limit in **Ab** of the diagram A.

2. The cocone $(\varinjlim_I A_i, \xi_i)$, with maps $\xi_i := [-]_{\sim_A} \circ p_i$, where $[-]_{\sim_A} \colon \bigsqcup_{i\in I} A_i \to \varinjlim_I A_i$ is the quotient map of $\sim_A$ (and recall that p_i is the ith injection map of $\bigsqcup_{i\in I} A_i$), is a colimit in **Ab** of the diagram A.

3. **Ab** is small complete and small cocomplete.

Proof. More or less analogous to the proof of **Theorem 3.9**, though the intricacies of this proof involve a little bit of additional work in showing that the subset condition is a well-defined subgroup and the equivalence relation condition is a creates a well-defined quotient group. I will provide the details in a later draft. $\square$

Proposition 5.7.

1. The notion of "inverse limits" in **Definition 2.7-1** is a limit–with appropriate cone structure- of the diagram $A\colon (\mathbb{N}, \geq) \to$ **Top** defined by $n \geq n-1 \mapsto \psi_n \colon A_n \to A_{n-1}$ for every $n \geq 1$. The notion of "direct limit" in **Definition 2.7-2** is the colimit–with appropriate cocone structure–of the diagram $A\colon (\mathbb{N}, \leq) \to$ **Ab** defined by $n-1 \leq n \mapsto \alpha_n \colon A_{n-1} \to A_n$ for every $n \geq 1$.

2. Given a chain of injections

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} A_{n-1} \xrightarrow{\alpha_n} \alpha_n \xrightarrow{\alpha_{n+1}} \cdots,$$

we find that there exists $A'_n \subset A$, where $A := \varinjlim_n A_n$, such that $A'_n \cong A_n$ and $A'_n \subset A'_{n+1}$, for every $n \geq 0$, and

$$A = \bigcup_n A'_n = \varinjlim_n A'_n.$$

In other words, A is a colimit of the diagram of inclusions

$$A'_0 \hookrightarrow A'_1 \hookrightarrow A'_2 \hookrightarrow \cdots \hookrightarrow A'_{n-1} \hookrightarrow A'_n \hookrightarrow \cdots.$$

Proof. Analogous to the proof of **Proposition 3.11** (though I may provide details in a later draft if I find any different aspect of the proof that is worth mentioning). $\square$

§ Chain Complexes

When referring to an object in $\mathbf{Ab}^{(\mathbb{N}_0,\geq)}$ (resp. $\mathbf{Ab}^{(\mathbb{N}_0,\leq)}$), i.e. a cochain of abelian groups (resp. a chain of abelian groups), we write $A_\bullet$ (resp. $A^\bullet$) in an effort to adhere to (and somewhat generalize) conventional homological algebra notation.

Definition 5.8. We define the category **Ch(Ab)** of *chain complexes* as the subcategory of $\mathbf{Ab}^{(\mathbb{N}_0,\geq)}$ consisting of cochains of abelian groups

$$\cdots \xrightarrow{\partial_{n+1}} A_{n+1} \xrightarrow{\partial_n} A_n \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_0} A_0,$$

such that im $\partial_{n+1} \subset$ ker ∂_n, or equivalently $\partial_{n+1}\partial_n = 0$, and all morphisms between them. We define the category **CoCh(Ab)** as the subcategory of $\mathrm{Ab}^{(\mathbb{N}_0,\leq)}$ consisting of cochains of *cochain complexes* as the chains (which we denote $A^\bullet$) of abelian groups

$$A_1 \xrightarrow{\delta_0} A_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} A_n \xrightarrow{\delta_n} A_{n+1} \xrightarrow{\delta_{n+1}} \cdots,$$

such that im $\delta_{n-1} \subset$ ker δ_n, or equivalently $\delta_{n-1}\delta_n = 0$, and alll possible morphisms between them.

Definition 5.9. We define a cochain $A_\bullet$ to be a *sub-cochain* of $B_\bullet$ (and write $A_\bullet \leq B_\bullet$ if $A_n \leq B_n$ (i.e. A_n is a subgroup of B_n) for every $n \in \mathbb{N}_0$. A particular case of subchain complex is ker ϕ of a morphism $\phi\colon (f_n)_{n\in\mathbb{N}_0} \to (f'_n)_{n\in\mathbb{N}_0}$ to the cochain, i.e., the sub-cochain defined by ker $(\phi)_n$ for every $n \geq 0$. We furthermore define We shall define *quotient cochains* $B_\bullet/A_\bullet$ by the induced cochain of quotient groups B_n/A_n for every $n \in \mathbb{N}_0$, with the sequence of maps being the induced quotient maps that result. We define *sub-chains* $B^\bullet \leq A^\bullet$, kernels, and *quotient chains* $A^\bullet/B^\bullet$ analogously.

Remark 5.10. It's straightforward to see that "isomorphism theorems" hold that are analogous to the isomorphism theorems that hold in the category **Grp**. The analogous *First Isomorphism Theorem* will prove very important to deriving the homological and cohomological results in this text.

Remark 5.11. Note that the previous definition is a special case of the notion of "subojects" and "quotient objects" of what is called *abelian categories*, but the even more general case is outside the scope of this paper and we work on this specific case, instead, for intuitive purposes. I invite the interested reader to refer to Chapter 1 of the *Tensor Categories* book.

§ Milnor's Limit

Definition 5.12. Given a sequence $(\psi_n\colon A_n \twoheadrightarrow A_{n-1})_{n\geq 1}$ of epimorphisms

$$\cdots \xrightarrow{\psi_{n+1}} A_n \xrightarrow{\psi_n} A_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_3} A_2 \xrightarrow{\psi_2} A_1 \xrightarrow{\psi_1} A_0.$$

in **Ab**, we define

$$\partial\colon \prod_n A_n \to \prod_n A_n$$

by

$$\partial\colon (a_n)_{n\geq 0} \mapsto (a_n - f_n(a_{n+1}))_{n\geq 1}.$$

Definition 5.13. Given a sequence $(\psi_n\colon A_n \twoheadrightarrow A_{n-1})_{n\geq 1}$ of epimorphisms

$$\cdots \xrightarrow{\psi_{n+1}} A_n \xrightarrow{\psi_n} A_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_3} A_2 \xrightarrow{\psi_2} A_1 \xrightarrow{\psi_1} A_0.$$

in **Ab**, we define

$$\varprojlim_n{}^1 A_n := \text{coker } \partial = \prod_n A_n/(\varprojlim_n A_n),$$

which is the abelian group B that makes

$$0 \to \varprojlim_n A_n \xrightarrow{\partial|_{\varprojlim_n A_n}} \prod_n A_n \xrightarrow{[-]_{\varprojlim_n A_n}} B \to 0$$

a short exact sequence.

Remark 5.14 The definition of Milnor's Limit can be defined analogously in the category $\mathbf{Ab}^{(\mathbb{N}_0,\geq)}$ (refer to **Definition 5.9** in the previous section), which we do in the proof of **Theorem 6.5**, or any category with well-defined quotient objects over subobjects.

6 Homology Functor

§ Functoriality of Homologies

Theorem 6.1.

1. $C_\bullet$ and $H_\bullet$ are well-defined functors from **Top** to **Ch(Ab)** with induced maps $C_\bullet(f)\colon C_\bullet(X) \to C_\bullet(Y)$, $H_\bullet(f)\colon H_\bullet(X) \to H_\bullet(Y)$ for every continuous map $f\colon X \to Y$.

2. $C^\bullet$ and $H^\bullet$ are contravariant functors from **Top** to **CoCh(Ab)** with induced maps $C^\bullet(f)\colon C^\bullet(Y) \to C^\bullet(X)$, $H^\bullet(f)\colon C^\bullet(Y) \to C^\bullet(X)$ for every continuous map $f\colon X \to Y$.

§ Limit and Colimit Preservation Properties

Theorem 6.3. The homology functor $H_\bullet^{CW}\colon \mathbf{CW} \to \mathbf{Ch}(\mathbf{Ab})$ preserves direct limits, i.e., given a chain of injections $i_j X_{j-1} \xrightarrow{i_j} X_j$ of CW complexes

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{j-1}} X_{j-1} \xrightarrow{i_j} X_j \xrightarrow{i_{j+1}} \cdots ,$$

we have

$$H_n(\varinjlim_j X_j) = \varinjlim_j H_n(X_j),$$

for every $n \geq 1$.

Lemma 6.4.

1. The cellular chain complex functor $C_\bullet^{CW}\colon \mathbf{CW} \to \mathbf{Ch}(\mathbf{Ab})$ preserves direct limits.

2. Given a chain of injections $i_j X_{j-1} \xrightarrow{i_j} X_j$ of CW complexes we have

$$\varinjlim_j \ker \partial_{n,j} = \ker \partial_n \text{ and } \varinjlim_j \operatorname{im} \partial_{n,j} = \operatorname{im} \partial_n,$$

for every $n \geq 1$, where $\partial_{n,j}\colon C_{n+1}(X_j) \to C_n(X_j)$ $(j \geq 0)$ and $\partial_n\colon C_{n+1}(X) \to C_n(X)$ are the boundary maps

Proof. Without loss of generality, suppose $X_0 \subset X_1 \subset \cdots \subset X_j \subset \cdots$. Set $X := \varinjlim_j X_j = \bigcup_{j \in \mathbb{N}_0} X_j$

1. Given $n \geq 1$, note that $C_n(X)$ is the free abelian group generated by the n-cells of X, while $\varinjlim_j C_n(X_j)$ is the free abelian group generated by the union of all generators of $C_n(X_j)$, which also consists of the n-cells of X

2. Follows from previous part that $\partial_{n,j}$ are restriction maps of ∂_n to X_j, which gives us

$$\varinjlim_j \ker\, \partial_{n,j} = \bigcup_j \ker\, \partial_{n,j} = \ker\, \partial_n \text{ and } \varinjlim_j \operatorname{im}\, \partial_{n,j} = \bigcup_j \operatorname{im}\, \partial_{n,j} = \operatorname{im}\, \partial_n.$$

□

Proof of Theorem 6.3. Without loss of generality, suppose $X_0 \subset X_1 \subset \cdots \subset X_j \subset \cdots$. Set $X := \varinjlim_j X_j = \bigcup_{j \in \mathbb{N}_0} X_j$. Note that $\varinjlim_j H_\bullet(X_j)$ is the chain-complex limit of the diagram

$$\ker\, \partial_{\bullet,0}/\operatorname{im}\, \partial_{\bullet+1,0} \to \ker\, \partial_{\bullet,1}/\operatorname{im}\, \partial_{\bullet+1,1} \to \cdots \to \ker\, \partial_{\bullet,j}/\operatorname{im}\, \partial_{\bullet+1,j} \to \cdots.$$

We find by **Lemma 6.4** that

$$\varinjlim_j \ker\, \partial_{\bullet,j} = \ker\, \partial_\bullet \text{ and } \varinjlim_j \operatorname{im}\, \partial_{\bullet,j} = \operatorname{im}\, \partial_\bullet.$$

it follows that for every $k \geq 0$

$$\varinjlim_j (\ker\, \partial_{\bullet,j}/\operatorname{im}\, \partial_{\bullet+1,k}) = \ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,k}.$$

Since $\operatorname{im}\, \partial_{n,0} \subset \operatorname{im}\, \partial_{n,1} \subset \cdots \subset \operatorname{im}\, \partial_{n,j} \subset \cdots$, we get the following induced chain of chain complexes

$$\begin{tikzcd}
\ker\, \partial_\bullet \arrow[d, "{[-]_0}"] \arrow[dr, "{[-]_1}"] \arrow[drrr, "{[-]_{j-1}}"] & & & & \\
\ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,0} \arrow[r, dashed, "\overline{[-]_1}"] & \ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,1} \arrow[r, dashed, "\overline{[-]_2}"] & \cdots \arrow[r, dashed, "\overline{[-]_j}"] & \ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,j} \arrow[r, dashed, "\overline{[-]_j}"] & \cdots.
\end{tikzcd}$$

It remains to show that

$$\varinjlim_j H_\bullet(X_j) = \varinjlim_j \ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,j} = H_\bullet(X).$$

Set $P_\bullet := \varinjlim_j \ker\, \partial_\bullet/\operatorname{im}\, \partial_{\bullet+1,j}$. First note that $\ker\, \partial_\bullet, \ker\, \partial_{j,\bullet}$ ($j \in \mathbb{N}_0$) are cochains of free abelian groups and that there exist epimorphisms $q_j \colon \ker\, \partial_\bullet \twoheadrightarrow \ker\, \partial_{j,\bullet}$ that split the inclusion map $\ker\, \partial_{j,\bullet} \hookrightarrow \ker\, \partial_\bullet$, which gives us $\ker\, \partial_\bullet = \ker\, q_j \oplus \ker\, \partial_{j,\bullet}$. Moreover, note that $\ker\, \partial_\bullet/\operatorname{im}\, \partial_{j,\bullet+1} \cong \ker\, q_j \oplus H_\bullet(X_j)$, and there exists an epimorphism $\widetilde{q_j} \colon \ker\, q_j \oplus H_\bullet(X_j) \twoheadrightarrow H_\bullet(X_j)$ that splits the inclusion map $H_\bullet(X_j) \hookrightarrow \ker\, q_j \oplus H_\bullet(X_j)$. Given a cocone $(A_\bullet, \phi_j)$, and observe that we get a cocone $(A_\bullet, \phi_j \circ q_j)$ on the diagram

$$\begin{tikzcd}
A_\bullet & & & & \\
H_\bullet(X_0) \arrow[r] \arrow[u, "\phi_0"] & H_\bullet(X_1) \arrow[r] \arrow[ul, "\phi_1"'] & \cdots \arrow[r] & H_\bullet(X_j) \arrow[r] \arrow[ulll, "\phi_j"'] & \cdots \\
\ker q_0 \oplus H_\bullet(X_0) \arrow[r] \arrow[u, "\widetilde{q_0}"] & \ker q_1 \oplus H_\bullet(X_1) \arrow[r] \arrow[u, "\widetilde{q_1}"] & \cdots \arrow[r] & \ker q_j \oplus H_\bullet(X_j) \arrow[r] \arrow[u, "\widetilde{q_j}"] & \cdots \\
\ker \partial_\bullet / \mathrm{im}\ \partial_{0,\bullet+1} \arrow[r] \arrow[u, "\sim"] & \ker \partial_\bullet / \mathrm{im}\ \partial_{1,\bullet+1} \arrow[r] \arrow[u, "\sim"] & \cdots \arrow[r] & \ker \partial_\bullet / \mathrm{im}\ \partial_{j,\bullet+1} \arrow[r] \arrow[u, "\sim"] & \cdots .
\end{tikzcd}$$

Take the cocone $(P_\bullet, \xi_j)$. We find that there exists a solution $u\colon P_\bullet \to A_\bullet$ of $(A_\bullet, \phi_j \circ q_j)$ such that

$$\begin{tikzcd}
& A_\bullet & \\
& P_\bullet \arrow[u, dashed, "u"] & \\
\ker \partial_\bullet / \mathrm{im}\ \partial_{\bullet+1,j} \arrow[rr] \arrow[uur] \arrow[ur, "\xi_j"'] & & \ker \partial_\bullet / \mathrm{im}\ \partial_{\bullet+1,j} \arrow[uul] \arrow[ul, "\xi_{j+1}"']
\end{tikzcd}$$

commutes (up to isomorphism), hence u is a solution of $(A_\bullet, \phi_j)$, and we have shown $\varinjlim_j H_\bullet(X_j) = P_\bullet$.

note that $(H_\bullet(X), [-]'_j)$ forms a cocone, hence there exists a solution $v\colon H_\bullet(X) \to P_\bullet$ with the following diagram commuting

$$\begin{tikzcd}
P_\bullet & & & & & \\
& \ker \partial_\bullet / \mathrm{im}\ \partial_{0,\bullet+1} \arrow[r, "\overline{[-]_1}"] \arrow[ul, "\xi_0"'] \arrow[dl, "{[-]'_0}"'] & \ker \partial_\bullet / \mathrm{im}\ \partial_{1,\bullet+1} \arrow[r, "\overline{[-]_2}"] \arrow[ull, "\xi_1"'] \arrow[dll, "{[-]'_1}"'] & \cdots \arrow[r, "\overline{[-]_j}"] & \ker \partial_\bullet / \mathrm{im}\ \partial_{j,\bullet+1} \arrow[r, "\overline{[-]_j}"] \arrow[ullll, "\xi_j"'] \arrow[dllll, "{[-]'_j}"'] & \cdots \\
H_\bullet(X) \arrow[uu, dashed, "v"] & & & & &
\end{tikzcd}$$

We find that v is an isomorphism since the cocone $(H_\bullet(X), [-]'_j)$ is an epimorphism so v is also an epimorphism and hence surjective. Moreover, since $P_\bullet = \varinjlim_j H_j(X)$, we find the diagram

$$\begin{tikzcd}
\varinjlim_j H_\bullet(X_j) \arrow[ddd, "\sim"'] & & & & & \\
& H_\bullet(X_0) \arrow[r] \arrow[ul] & H_\bullet(X_1) \arrow[r] \arrow[ull] & \cdots \arrow[r] & H_\bullet(X_j) \arrow[r] \arrow[ullll] & \cdots \\
& \ker\, q_0 \oplus H_\bullet(X_0) \arrow[r] \arrow[u, "q_0"] & \ker\, q_1 \oplus H_\bullet(X_1) \arrow[r] \arrow[u, "q_1"] & \cdots \arrow[r] & \ker\, q_j \oplus H_\bullet(X_j) \arrow[r] \arrow[u, "q_j"] & \cdots \\
P_\bullet & \ker\, \partial_\bullet / \mathrm{im}\, \partial_{0,\bullet+1} \arrow[r] \arrow[u, "\sim"] \arrow[dl, two heads] & \ker\, \partial_\bullet / \mathrm{im}\, \partial_{1,\bullet+1} \arrow[r] \arrow[u, "\sim"] \arrow[dll, two heads] & \cdots \arrow[r] & \ker\, \partial_\bullet / \mathrm{im}\, \partial_{j,\bullet+1} \arrow[r] \arrow[u, "\sim"] \arrow[dllll, two heads] & \cdots \\
H_\bullet(X) \arrow[u, two heads, "v"] & & & & &
\end{tikzcd}$$

commutes, and it follows that $\ker\, v \subset \ker\, q_j$, for every j, and we conclude that $\ker\, v = 0$, since $\bigcap_j \ker\, q_j = 0$. $\square$

Theorem 6.5. Given a sequence of monomorphisms of CW compexes

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X_n \xrightarrow{i_{n+1}} \cdots .$$

there is an exact sequence

$$0 \to \varprojlim_j {}^1 H^{n-1}(X_j) \to H^n(\varinjlim_j X_j) \to \varprojlim_j H^n(X_j) \to 0.$$

As usual, we assume without loss of generality that $X_0 \subset X_1 \subset \cdots \subset X_j \subset \cdots$ and set $X := \bigcup_{j \in \mathbb{N}_0} X_j$. We shall prove more generally that there exists the following exact sequence of chain complexes

$$0 \to \varprojlim_j {}^1 H^{\bullet-1}(X_j) \to H^\bullet(X) \to \varprojlim_j H^\bullet(X_j) \to 0.$$

Next, we prove the following lemma:

Lemma 6.6. Given a family $\{X_i\}_{j \in I}$ of CW complexes, we have

$$H^\bullet\left(\bigsqcup_{j \in I} X_j\right) = \prod_{j \in I} H^\bullet(X_i).$$

Proof. It is shown in *Davis-Kirk* that

$$C_\bullet\left(\bigsqcup_{j \in I} X_j\right) = \bigoplus_{j \in I} C_\bullet(X_j),$$

and it is well-known fact that for the dual contravariant functor $(-)^* \colon \mathbf{Ab} \to \mathbf{Ab}$

$$\left(\bigoplus_{j \in I} A_j\right)^* = \prod_{j \in I} (A_j)^*.$$

We find that

$$C^\bullet\left(\bigsqcup_{j\in I} X_j\right) = \left(\bigoplus_{j\in I} C_\bullet(X_j)\right)^* = \prod_{j\in I} C^\bullet(X_j).$$

Since the coboundary maps vary on each of the individual tuples, we find that $\partial_\bullet^* = (\partial_j^*)_{j\in I}$ and hence

$$H^\bullet(X) = \ker\,\partial_\bullet^*/\mathrm{im}\,\partial_{\bullet-1}^* = \left(\prod_{j\in I}\ker\,\partial_{j,\bullet}^*\right)/\mathrm{im}\,\partial_{\bullet-1}^* = \prod_{j\in I} H^\bullet(X).$$

□

Proof of Theorem 6.5. First, we want to show that there exists a monomorphism $\alpha\colon \varprojlim_j{}^1 H^{\bullet-1}(X_j) \rightarrowtail H^\bullet(\varprojlim_j X_j)$. For each $j \in \mathbb{N}_0$, note that we can choose an epimorphism $q_i\colon X \to X_j$ which for each $n \in \mathbb{N}_0$, we choose an n-cell $e^n_{i_{j,n}}$ and map any n-cell in $X \setminus X_j$ to $e^n_{i_{j,n}}$ and keep any n-cell in X_j fixed. We find for the inclusion map $i_j\colon X_j \hookrightarrow X_{j+1}$, we find the diagram

$$\begin{tikzcd}
 & & & X \arrow[dlll, "q_0"'] \arrow[dll, "q_1"] \arrow[d, "q_j"'] \arrow[dr, "q_{j+1}"] & & \\
X_0 \arrow[r, hook, "i_0"'] \arrow[drrr, "\sqcup_{k\in\mathbb{N}_0} i_0"'] & X_1 \arrow[r, hook, "i_1"'] \arrow[drr, "\sqcup_{k\in\mathbb{N}_0} i_2"] & \cdots \arrow[r, hook, "i_{j-1}"'] & X_j \arrow[r, hook, "i_j"'] \arrow[d, "\sqcup_{k\in\mathbb{N}_0} i_j"] & X_{j+1} \arrow[r, hook, "i_{j+1}"'] \arrow[dl, "\sqcup_{k\in\mathbb{N}_0} i_{j+1}"] & \cdots \\
 & & & \bigsqcup_{j\in\mathbb{N}_0} X_j & &
\end{tikzcd}$$

commutes, and it follows by contravariant functoriality of $H^\bullet$ and **Lemma 6.7** that

$$\begin{tikzcd}
 & & & H^\bullet(X) & & \\
H^\bullet(X_0) \arrow[urrr] & H^\bullet(X_1) \arrow[l] \arrow[urr] & \cdots \arrow[l] & H^\bullet(X_j) \arrow[l] \arrow[u] & H^\bullet(X_{j+1}) \arrow[l] \arrow[ul] & \cdots \arrow[l] \\
 & & & \prod_{j\in\mathbb{N}_0} H^\bullet(X_j) \arrow[ulll] \arrow[ull] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

commutes. Define α_0 by

$$\alpha_0\colon \prod_{j\in\mathbb{N}_0} H^{\bullet-1}(X_j) \xrightarrow{H^{\bullet-1}(\sqcup_{k\in\mathbb{N}_0} i_m)} H^\bullet(X_m) \xrightarrow{H^\bullet(q_m)} H^{\bullet-1}(X) \xrightarrow{\partial_\bullet^*} H^\bullet(X).$$

We find that $\ker\,\alpha_0 = \varprojlim_j H^{\bullet-1}(X_j)$, since using the above diagram and the construction of the map, we get

$$f \in \ker\,\alpha_0 \iff f \in \varprojlim_j H^{\bullet-1}(X_j) \implies \ker\,\alpha_0 = \varprojlim_j H^\bullet(X_j).$$

Then by the *First Isomorphism Theorem*, we find α defined by the induced map

$$\overline{\alpha}\colon \varprojlim_j{}^1 H^{\bullet-1}(X_j) \rightarrowtail H^\bullet(X)$$

is a monomorphism.

Next, we can assume $\varprojlim_j {}^1 H^{\bullet-1}(X_j) \subset H^\bullet(X)$, and we show that

$$\varprojlim_j H^\bullet(X_j) = H^\bullet(X) / \varprojlim_j {}^1 H^{\bullet-1}(X_j),$$

and the conclusion that

$$0 \to \varprojlim_j {}^1 H^{\bullet-1}(X_j) \xrightarrow{\alpha} H^\bullet(X) \xrightarrow{[-]} \varprojlim_j H^\bullet(X_j) \to 0.$$

is a long exact sequence immediately follows. First, note that since $(H^\bullet(X), i_j^*)$ is a cocone of epimorphisms, where $i_j^* \colon H^\bullet(X) \twoheadrightarrow H^\bullet(X_j)$ is the dual map of the inclusion map $i \colon H^\bullet(X_j) \hookrightarrow H^\bullet(X)$, we find that there exists an epimorphism $u \colon H^\bullet(X) \to \varprojlim_j H^\bullet(X_j)$ such that

$$\begin{tikzcd}
H^\bullet(X) \arrow[dd, "\exists u"] \arrow[dr, two heads, "i_0^*"] \arrow[drr, two heads, "i_1^*"] \arrow[drrrr, two heads, "i_j^*"] \arrow[drrrrr, two heads, "i_{j+1}^*"] & & & & & & \\
& H^\bullet(X_0) & H^\bullet(X_1) \arrow[l] & \cdots \arrow[l] & H^\bullet(X_j) \arrow[l] & H^\bullet(X_{j+1}) \arrow[l, two heads] & \cdots \arrow[l] \\
\varprojlim_j H^\bullet(X_j) \arrow[ur] \arrow[urr] \arrow[urrrr] \arrow[urrrrr] & & & & & &
\end{tikzcd}$$

commutes. Recall the map $\alpha_0 \colon \prod_{j \in \mathbb{N}_0} H^{\bullet-1}(X_j) \to H^\bullet(X)$ and the maps $H^\bullet \sqcup_{k \in \mathbb{N}_0} i_j \colon \prod_{j \in \mathbb{N}_0} H^{\bullet-1}(X_j) \to H^{\bullet-1}(X)$, which gives us the following commuting diagram:

$$\begin{tikzcd}
& H^\bullet(X) \arrow[d, "u"] \arrow[ddr] \arrow[ddrrr] \arrow[ddrrrr] & & & & & \\
& \varprojlim_j H^\bullet(X_j) \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] & & & & & \\
& H^\bullet(X_0) & H^\bullet(X_1) \arrow[l] & \cdots \arrow[l] & H^\bullet(X_j) \arrow[l] & H^\bullet(X_{j+1}) \arrow[l] & \cdots \arrow[l] \\
& H^{\bullet-1}(X_0) \arrow[u, "\partial_{0,\bullet}^*"] & H^{\bullet-1}(X_1) \arrow[l] \arrow[u, "\partial_{1,\bullet}^*"] & \cdots \arrow[l] & H^{\bullet-1}(X_j) \arrow[l] \arrow[u, "\partial_{j,\bullet}^*"] & H^{\bullet-1}(X_{j+1}) \arrow[l] \arrow[u, "\partial_{j+1,\bullet}^*"] & \cdots \arrow[l] \\
\prod_{j \in \mathbb{N}_0} H^{\bullet-1}(X_j) \arrow[uuuur, "\alpha_0"] \arrow[ur, "H^\bullet \sqcup_{k \in \mathbb{N}_0} i_0"] \arrow[urr, "H^\bullet \sqcup_{k \in \mathbb{N}_0} i_1"] \arrow[urrrr, "H^\bullet \sqcup_{k \in \mathbb{N}_0} i_j"] \arrow[urrrrr, "H^\bullet \sqcup_{k \in \mathbb{N}_0} i_{j+1}"'] & & & & & &
\end{tikzcd}$$

Recall that $\ker \alpha_0 = \varprojlim_j H^{\bullet-1}(X_j)$, which using the *First Isomorphism Theorem* induces the commuting diagram

$$
\begin{tikzcd}
& \varprojlim_j{}^1 H^{\bullet-1}(X_j) \arrow[d, "u"] \arrow[ddddl, "id"'] \arrow[ddr] \arrow[ddrrr] \arrow[ddrrrr] \\
& \varprojlim_j H^\bullet(X_j) \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] \\
& H^\bullet(X_0) & H^\bullet(X_1) \arrow[l] & \cdots \arrow[l] & H^\bullet(X_j) \arrow[l] & H^\bullet(X_{j+1}) \arrow[l] & \cdots \arrow[l] \\
& H^{\bullet-1}(X_0) \arrow[u, "\partial^*_{0,\bullet}"] & H^{\bullet-1}(X_1) \arrow[l] \arrow[u, "\partial^*_{1,\bullet}"] & \cdots \arrow[l] & H^{\bullet-1}(X_j) \arrow[l] \arrow[u, "\partial^*_{j,\bullet}"] & H^{\bullet-1}(X_{j+1}) \arrow[l] \arrow[u, "\partial^*_{j+1,\bullet}"] & \cdots \arrow[l] \\
\varprojlim_j{}^1 H^{\bullet-1}(X_j), \arrow[ur, "\overline{H^\bullet\sqcup_{k\in\mathbb{N}_0} i_0}"] \arrow[urr, "\overline{H^\bullet\sqcup_{k\in\mathbb{N}_0} i_1}"] \arrow[urrrr, "\overline{H^\bullet\sqcup_{k\in\mathbb{N}_0} i_j}"] \arrow[urrrrr, "\overline{H^\bullet\sqcup_{k\in\mathbb{N}_0} i_{j+1}}"']
\end{tikzcd}
$$

and it follows by commutivity of the diagram that $\ker u = \varprojlim_j{}^1 H^{\bullet-1}(X_j)$ since elements of $\varprojlim_j{}^1 H^{\bullet-1}(X_j)$ don't commute with the maps $\overline{H^\bullet \sqcup_{k\in\mathbb{N}_0} i_j}$ and the cochain $(H^{\bullet-1}(X_j) \leftarrow H^{\bullet-1}(X_{j+1}))_{j\in\mathbb{N}_0}$ unless it maps to zero, and we conclude applying the *First Isomorphism Theorem* again to the epimorphism u that

$$\varprojlim_j H^\bullet(X_j) = H^\bullet(X)/\varprojlim_j{}^1 H^{\bullet-1}(X_j).$$

$\square$

7 Appendix: Sets and Topology Definitions

§ Sets Definitions

§ Topology Definitions

§ Homological Algebra Definitions