Notes on Induction, Recursion, Coinduction and Corecursion

Contents

0. Some Motivational Ideas

0.1 Free Equational Models

Definition 0.1. A **signature** Σ is a set n-ary symbols (usually functions and relations). We call a signature equation if it only consists of function symbols. We call an equational signature **finitarily bounded** if the arities of the function symbols can be bounded for some n.

Note. The amount of symbols themselves could be infinite, but the amount of arities for which there are symbols are finitarily bounded.

<u>Examples.</u>

1. The language of groups $\left(\cdot, (-)^{-1}, 1\right)$ and the language of rings $(+, -, \cdot, 1, 0)$ are finitarily bounded signatures because they're finite

2. The language of real-valued streams has the signature $\left(\mathbb{R}, \text{tail}\right)$ with every $x \in \mathbb{R}$ as a constant symbol. Obvious example of a signature that is finitarily bounded but not finite

3. R-module signatures $(R, +, (r \cdot (-))_{r \in R})$ (with the ring R a collection of constant symbols) is finitarily bounded because there is at most a 2-ary with $+$.

Definition 0.2. Given a finitarily bounded equational signature, we can define a **polynomial functor** $P_\Sigma : \text{Set} \to \text{Set}$ as follows:

$$P_\Sigma(X) = N_0 + N_1 X^2 + \cdots + N_k X^k,$$

where,

- $(-) + (-) : \text{Set}^2 \to \text{Set}$ refers to the coproduct (disjoint union) bifunctor in Set.

- $N_j \subset \Sigma$ $(0 \leq j \leq k)$ refers to the set of j-ary function symbols (NOTE: does not need to be finite) and $N_j(-) : \text{Set} \to \text{Set}$ refers to the following functor

$$N_j X := N_j \times X,$$

and for $f : X \to Y$, we have $N_j(f) = \text{id}_{N_j} \times f : N_j \times X \to N_j \times Y$.

- $(-)^j : \text{Set} \to \text{Set}$ refers to the functor that sends a set X to the j-product tuple X^j, and the function $f : X \to Y$ to

$$f^j := f \times \cdots \times f : X^j \to Y^j$$

Note: The definition of a "polynomial functor" can be extended in a straightforward way to *any* equational signature without losing any algebraic or coalgebraic properties but for aesthetic reasons, these notes only define polynomial functors for "finitarily bounded polynomials"

Definition 0.3. A Σ-algebra (A, α) is a P_Σ-algebra i.e. a set A and a function $\alpha : P_\Sigma A \to A$, with any Σ-homomorphisms $f : (A, \alpha) \to (B, \beta)$ being functions $f : A \to B$ such the following diagram commutes

$$
\begin{tikzcd}
P_\Sigma A \arrow[r, "\alpha"] \arrow[d, "P_\Sigma f"'] & A \arrow[d, "f"] \\
P_\Sigma B \arrow[r, "\beta"] & B
\end{tikzcd}
$$

We call a Σ-algebra **equational** if the signature Σ is equational.

<u>Examples of equational Σ-algebras:</u>

1. groups, rings, and fields (structures themselves and not the language)

NOTE: Not every Σ-structure/model in the corresponding signature is NOT necessarily a group, ring, field, etc.

2. R-modules (and by extension vector spaces)

3. Any Σ-structure, for any equationa. Σ.

Remark. By definition, the category Alg_Σ of Σ-algebras is precisely the category of structures/models of signature Σ.

This gives us the framework to talk about "free" Σ-algebras.

Definition 0.4. Given a signature Σ and a set X, the free Σ-algebra $\mathcal{A}(X) = (A(X), \alpha(X))$ generated by X is the Σ-algebra such that for all Σ-algebras (B, β) and functions $f : X \to B$ there exists a unique Σ-homomorphism $f' : \mathcal{A}(X) \to (B, \beta)$ such that

$$
\begin{tikzcd}
 & B \\
X \arrow[ur, "\forall f"] \arrow[r, hook, "i"'] & A(X) \arrow[u, "\exists! f'"']
\end{tikzcd}
$$

commutes in Set.

Remark. In the categorical sense $\mathcal{A}(X)$ is a "free object" generated by X with respect to the forgetful functor $U : \mathrm{Alg}_\Sigma \to \mathrm{Set}$ because $(\mathcal{A}(X), i)$, where $i : X \hookrightarrow A(X) = U\mathcal{A}(X)$, is the initial object in the comma category X / U.

$$
\begin{tikzcd}
 & U(B, \beta) & & (B, \beta) \\
X \arrow[ur, "\forall f"] \arrow[r, hook, "i"'] & UA(X) \arrow[u, "Uf'"'] & & A(X) \arrow[u, "\exists! f'"']
\end{tikzcd}
$$

Theorem 0.5. Given a finitary equational signature Σ, there always exists a free Σ-algebra

$\mathcal{A}(x)$ generated by a set X.

We equivalently reformulate the theorem as follows:

Proposition 0.6. Given a finitary equational signature Σ and a set X, the comma category X/U has an initial object.

Outline of the proof. We want to verify the following two claims:

Claim 1. $X/U \cong \mathsf{Alg}_{P_\Sigma + X}$ where the functor $P_\Sigma(-) + X : \mathsf{Set} \to \mathsf{Set}$ is defined by $Y \mapsto P_\Sigma(Y) + X$ and $f \mapsto P_\Sigma(f) + \mathrm{id}_X$.

The comma category has objects of the form $\big((A, \alpha), i : X \to A\big)$. Define $G : X/U \to \mathsf{Alg}_{P_\Sigma + X}$ as follows:

$$\big((A, \alpha), i : X \to A\big) \mapsto (A, (\alpha, i))$$

where $(\alpha, i) : P_\Sigma(A) + X \to A$ by mapping for $w \in P_\Sigma(A)$ to $\alpha(w)$ and mapping $w \in X$ to $i(x)$.

Note that $P_\Sigma(-) + X$ is also a polynomial functor, more specifically, the polynomial functor with respect to the signature $\Sigma_X := \Sigma + X$, wiht the elements of X treated in Σ_X as constant symbols.

Claim 2. Any polynomial functor preserves ω-colimits.

After these claims are established, we take $C := \mathrm{Colim}_{n \in \omega} P_\Sigma^n(0)$, which induces an initial algebra by the following ω-iterate construction:

$$0 \xrightarrow{!} P_\Sigma 0 \xrightarrow{P_\Sigma !} P_\Sigma^2 0 \xrightarrow{P_\Sigma^2 !} P_\Sigma^3 0 \to \cdots$$

$\gamma : P_\Sigma(C) \to C$. □

%expand on this outline, and talk about nuance of how the functor preserves ω-colimits but not necessarily other limits and colimits.

Goals:

Prove a more general conjecture.

Conjecture 0.7. Given a theory set-functor $F : \mathrm{Set} \to \mathrm{Set}$ that is a sub-functor of a polynomial functor P_Σ of an equational finitary signature Σ, there exists a free F-algebra generated by any set X.

"co-finitary" as a dual notion of a finitary functor. (for definitions of finitary, refer to on finitary functors and their presentation)

Definition 0.8. A finitary functor F is a quotient functor of a polynomial functor P_Σ.

0.2. Cofree Coalgebras

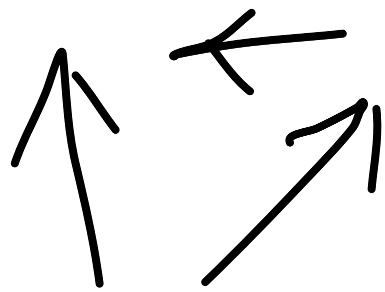
Definition 0.9. Let $F : \mathrm{Set} \to \mathrm{Set}$ be a functor, and let Γ be a set. We call $\Gamma_\#$ a cofree F-coalgebra cogenerated by Γ (with a map $\eta_{\Gamma\#} : \Gamma_\# \to \Gamma$) if

$\forall (A, \alpha)\ \forall g : A \to \Gamma$

$$\begin{tikzcd} A \arrow[r, "\exists! g^\#"] \arrow[d, "\forall g"'] & \Gamma_\# \arrow[dl, "\eta"] \\ \Gamma & \end{tikzcd}$$

$\exists! g^\# : (A, \alpha) \to (\Gamma_\#$

This construction comes from reversing the arrows on a free algebra construction



$\Gamma_{\#}$ is a terminal object of a category, but what category is it the terminal object of?

What I claim is that it's the terminal object of the dual of the following comma category:
$\Gamma\,/\,U^{op}$

U^{op} : $\mathrm{Coalg}_F^{op} \to \mathrm{Set}^{op}$ takes morphisms $f^{op} : (A, \alpha) \leftarrow (B, \beta)$ and maps it to $(Uf)^{op} : A \leftarrow B$, where U is the forgetful functor.

free alg	cofree coalg.
F/U	Γ/U^{op}
obj	obj
$(A, \ldots)$	$(C, \ldots)$

$A \in \mathrm{Alg}_{F}$

$T : \Gamma \to UA$

mor. $f : (A, T) \to (B, \sigma)$

$$\begin{tikzcd} UA \arrow[r, "Uf"] & UB \\ & \Gamma \arrow[ul, "T"] \arrow[u, "\sigma"'] \end{tikzcd}$$

$(C, \eta) \in \mathrm{Coalg}_{F}$

$\eta : U^{op} C \to$

mor. $g : (C, \eta) \leftarrow (D,$

in Coalg_{F}^{op}

$$\begin{tikzcd} U^{op} C \arrow[dr, "\eta"'] & U^{op} D \arrow[l, "g"'] \arrow[d, "\varepsilon"] \\ & \Gamma \end{tikzcd}$$

in Set^{op}

The cofree coalgebra $\Gamma_\#$ is initial in Γ / U^{op}, but final in $\left(\Gamma / U^{op}\right)^{op}$.

Claim. If U^{op} is the "co-forgetful" functor with respect to a polynomial functor P_Σ, then

$$\left(\Gamma / U^{op}\right)^{op} \cong U / \Gamma \cong \text{Coalg}_{P_\Sigma(-)\times\Gamma}$$

I plan to prove this next Friday.

0.3. Brief Introduction to Coinduction and Corecursion

Definition 0.10. Let $F : \text{Set} \to \text{Set}$ be a functor, and $R \subset S \times T$, for coalgebras (S, ξ) and (T, λ), is a bisimulation if $\exists \gamma : R \to F(R)$ such that the projections $\pi_1 : R \to S$ and $\pi_2 : R \to T$ are F-coalg. homomorphisms.

$$\begin{tikzcd}
S \arrow[d, "\xi"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dotted, "\exists\gamma"] & T \arrow[d, "\lambda"] \\
F(S) & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] & F(T)
\end{tikzcd}$$

NOTE: A bisimulation is a dual of congruence relation (on algebras).

Theorem 0.11. *(Coinduction proof principle)* $R \subset B \times B$ bisumlation on final F-coalg. (B, β). Then

$$R \subset \Delta := \{(b, b) : b \in B\}$$

Proof. Note that the projection $\pi_1 : (R, \gamma) \to (B, \beta)$ is the unique morphism from (R, γ) to (B, β) in F-coalg., since (B, β) is final in Coalg_F so $\pi_1(r) = \pi_2(r)$, for all $r \in R$. □

<u>Examples:</u>

Example 0.12. $\left(\mathbb{N} \cup \{+\infty\}, \text{pre}\right)$ $(\text{pre} : \mathbb{N} \cup \{+\infty\} \to \mathbb{N} \cup \{+\infty\} + 1)$
coalgebra $F : X \mapsto X + 1$. Define a relation $x \sim y \Longleftrightarrow x + 1 = y + 1$.

NOTE: $R = \Delta$, but proving that coinductively takes a bit of thought.

We want to find $\gamma : R \to R + 1$ that makes this relation into a bisimulation. We find that setting $\gamma(r) := \begin{cases} * & \pi_1(r) = 0 \\ \text{pre} \times \text{pre}(r) & \text{else} \end{cases}$

$$r \mapsto (\text{pre}(\pi_1(r)), \text{pre}(\pi_2(r)))$$

Note that this function is well defined since

$$\pi_2(r) = 0 \Longrightarrow \pi_1(r) \sim 0 \Longrightarrow \pi_1(r) + 1 = 0 + 1 \Longrightarrow \pi_1(r) = \text{pre}(\pi_1(r) + 1) = \text{pre}(1) = 0.$$

gives us the following commutative diagram

$$\begin{tikzcd}
\mathbb{N} \cup \{+\infty\} & R \arrow[l] \arrow[r] \arrow[d] & \mathbb{N} \cup \{+\infty\} \\
 & R+1 &
\end{tikzcd}$$

$\text{pre} : \mathbb{N} \cup \{+\infty\} \to (\mathbb{N} \cup \{+\infty\}) + 1$

$n = 0 \implies n \mapsto *$
$0 < n < +\infty \implies \text{pre}(n) \mapsto "n-1"$
$n = +\infty \implies \text{pre}(n) = +\infty$

Lambek's Lemma. If (B, β) is a final F-coalgebra, then $\beta : B \to F(B)$ is an isomorphism.

Jan Rutin: Method of Coalgebra

Example 0.13. Let A be a set. Suppose $F : X \mapsto A \times X$. The final F-coalgebra is $(A^\omega, \langle \text{head}, \text{tail} \rangle)$, where

$\text{head}(\sigma) = \sigma(0)$
$\text{tail}(\sigma) = (\sigma(1), \sigma(2), \dots)$

We can define a bisimulation as follows:

$\sigma \sim \tau \iff \sigma(0) = \tau(0)$ and $\text{tail}(\sigma) \sim \text{tail}(\tau)$

We can define the relation this way, since $\Delta := \left\{ (\sigma, \tau) \in A^\omega \times A^\omega : \sigma = \tau \right\}$ follows this criteria.

Fact 1. If R is bisimulation on $\left(A^\omega, \langle \text{head}, \text{tail} \rangle\right)$, then R satisfies the following properties hold

(i) $\sigma(0) = \tau(0)$

(ii) $\left(\text{tail}(\sigma), \text{tail}(\tau)\right) \in R$

And it follows that $R = \Delta$.

Proof. Assume $(\sigma, \tau) \in R$. Observe that since $R \subset \Delta$, we have $\sigma = \tau$, then it immediately follows that (i) holds, and for (ii), observe that

$$
\begin{tikzcd}
A^{\omega} \arrow[d, "\langle h, t\rangle"] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & A^{\omega} \arrow[d, "\langle h, t\rangle"] \\
A \times A^{\omega} & A \times R \arrow[l, "\mathrm{id} \times \pi_1"'] \arrow[r, "\mathrm{id} \times \pi_2"] & A \times A^{\omega}
\end{tikzcd}
$$

We find that $\pi_R(\gamma(\sigma, \tau)) \in R$, and we want to show

$$\pi_R(\gamma(\sigma, \tau)) = \left(\mathrm{tail}(\sigma), \mathrm{tail}(\tau)\right) \;\; \left(\pi_R : A \times R \to R\right)$$

$$\mathrm{id}_A \times \pi_1(\gamma(\sigma, \tau)) = (\langle h, t\rangle \circ \pi_1)((\sigma, \tau)) = \langle h, t\rangle(\sigma) = \left(\sigma(0), \mathrm{tail}(\sigma)\right),$$

similarly, we have

$$\mathrm{id}_A \times \pi_2(\gamma(\sigma, \tau)) = \left(\tau(0), \mathrm{tail}(\tau)\right)$$

and it follow that

$$\pi_R(\gamma(\sigma, \tau)) = \pi_R\left(\sigma(0), \left(\mathrm{tail}(\sigma), \mathrm{tail}(\tau)\right)\right) = \left(\mathrm{tail}(\sigma), \mathrm{tail}(\tau)\right). \quad \square$$

Fact 2. If $\left(S, \langle o, \mathrm{tr}\rangle\right)$ is F-coalgebra (we call such coalgebras stream systems), then

$$s \sim t \iff [[s]] = [[t]],$$

where $[[-]] : (S, \langle o, tr\rangle) \to \left(A^{\omega}, \langle \mathrm{head}, \mathrm{tail}\rangle\right)$ is the unique F-coalgebra homomorphism from $(S, \langle o, tr\rangle) \to \left(A^{\omega}, \langle \mathrm{head}, \mathrm{tail}\rangle\right)$ and $s \sim t$ is defined to be the condition that $(s, t) \in R$ for *some* bisimulation relation R.

Corollary of Fact 2. For $\sigma, \tau \in A^\omega$ $\sigma \sim \tau \iff \sigma = \tau$.

I'll give an example next time.

NOTE: We only look typically look at congruence in $\left(\mathbb{N}, \text{succ}\right)$.

1. Recursion

1.1 Definition and Basic Properties of Congruence Relations

Definition 1.1. A congruence relation $R \subset S \times T$ of two F-algebra (S, α), (T, τ) is a relation such that $\gamma : FR \to R$ such that $\pi_1 : R \to S$, $\pi_2 : R \to T$ are algebra homomorphisms.

$$\begin{tikzcd}
S & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & T \\
F(S) \arrow[u, "\alpha"] & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[u, dotted, "\exists\gamma"'] & F(T) \arrow[u, "\beta"']
\end{tikzcd}$$

Theorem 1.2. *(Principle of General Induction part i)* F : Set $\to$ Set a functor

1. If (I, α) is an initial F-algebra and $R \subset I^2$ is a congruence relation on then $R \supset \Delta := \{(x, x) : x \in I\}$

2. Δ is the smallest congruence relation on (I, α).

1.2 Examples of Congruence Relations

Example 1.3. If G is a group and $\sim$ is an equivalence relation such that $G/\sim$ is a group, then $\sim \subset G^2$ is a congruence: Note that G can be expressed as a $(*, 1)$-algebra (The functor is $P(X) = X^2 + 1$). We find find that

$$\begin{tikzcd}
G^2+1 \arrow[d, "*+1"'] & \sim^2+1 \arrow[l, "\pi_1^2+1"'] \arrow[r, "\pi_2^2+1"] \arrow[d, "x^2+(1,1)"] & G^2+1 \arrow[d] \\
G & \sim \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G
\end{tikzcd}$$

$(a_1, b_1) * (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$ $(1,1)$

NOTE: A congruence relation is not necessarily an equivalence relation.

NOTE: May be able to be generalized to "congruence spans"

Example 1.3. Let $\Sigma_{nat} := \{S, 0\}$ (polynomai functor $P_{\Sigma_{nat}} : X + 1$) be an arithmatic language with a single 1-ary function symbol and constant symbol 0, respectively. The initial algebra $(\mathbb{N}, \text{succ} + 0)$. We find the relation

$x \sim y \iff x = y = 0$ or $x = \text{succ}(x'),\ y = \text{succ}(y')$ and $x' \sim y'$

is a well-defined congruence relation and in fact equal to Δ by the usual principle of mathematical induction.

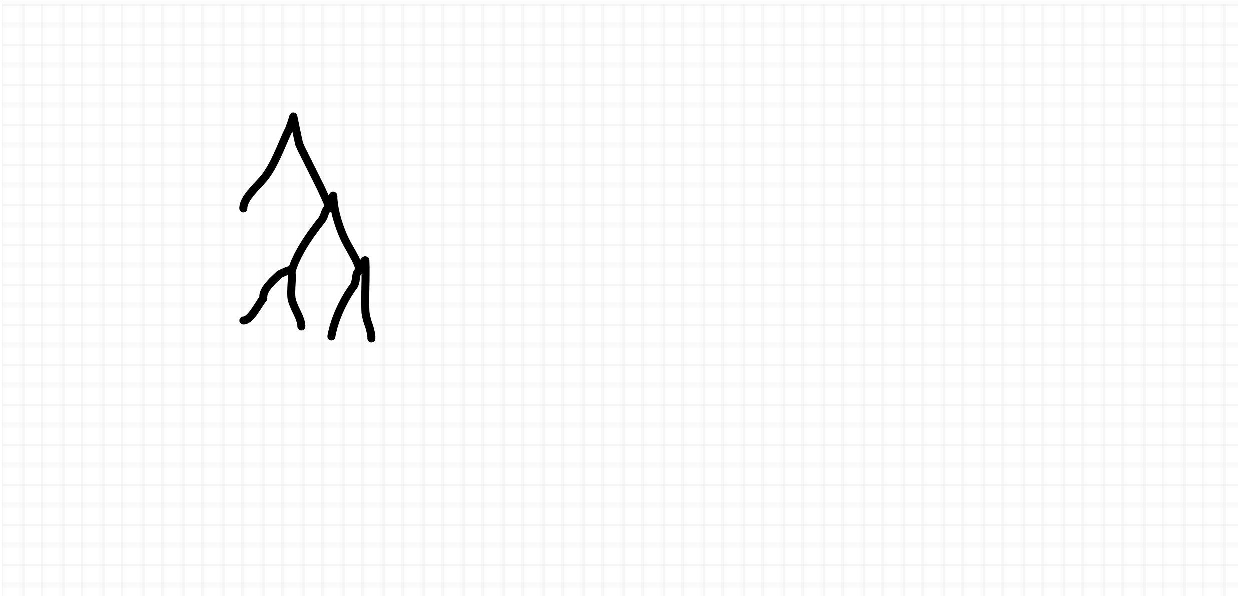
NOTE: Containing the diagonal is equivalent to being reflexive

$x = y \iff (x, y) \in \Delta \iff x \sim y$

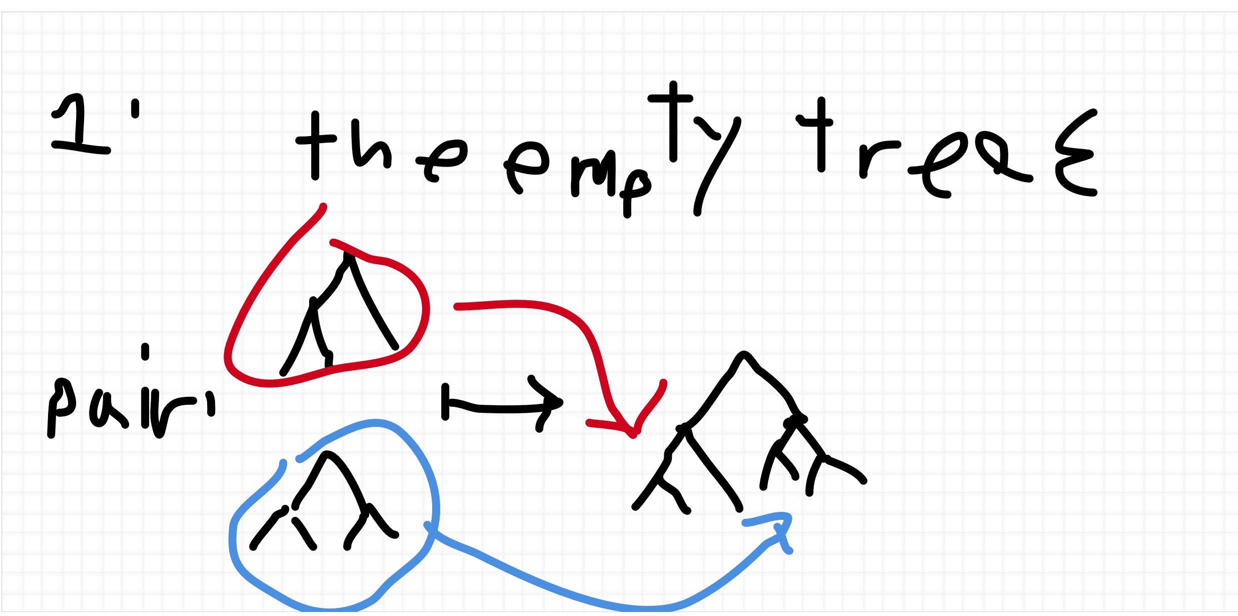
Definition by recursion allows us to define a unique function $g : \mathbb{N} \to A$--given $a_0 \in A$ and $f : A \to A$--such that

$g(0) = a_0, \ g(n+1) = f(g(n))$

Example 1.4. Let $\Sigma_{groups} := \{*, 1\}$. Note the algebra T of binary trees



with the operation pair $+ 1 : T^2 + 1 \to T$ defined by:



is the initial algebra on Σ_{groups}. We find the smallest congruence relation such that

$$x \sim y \iff x = y = 1 \text{ or } x = \text{pair}(x_1, x_2),\ y = \text{pair}(y_1, y_2) \text{ and } x_1 \sim y_1 \text{ and } x_2 \sim y_2$$

is a well-defined congruence relation on $\left(T, \text{pair} + 1\right)$, which is in fact Δ, and this gives us a "principal of mathematical induction" on T.

fact: A variant of the recursion theorem (more specifically primitive recursion) tells us that given $a_1 \in A$ and $h : A^2 \to A$ there exists a unique function $g : T \to A$ such that:

$$g(1) = \epsilon$$
$$g\left(\text{pair}(t_1, t_2)\right) = h(g(t_1), g(t_2))$$

To prove that a function like this exists, we show that if $x \sim y$, then $g(x) = g(y)$

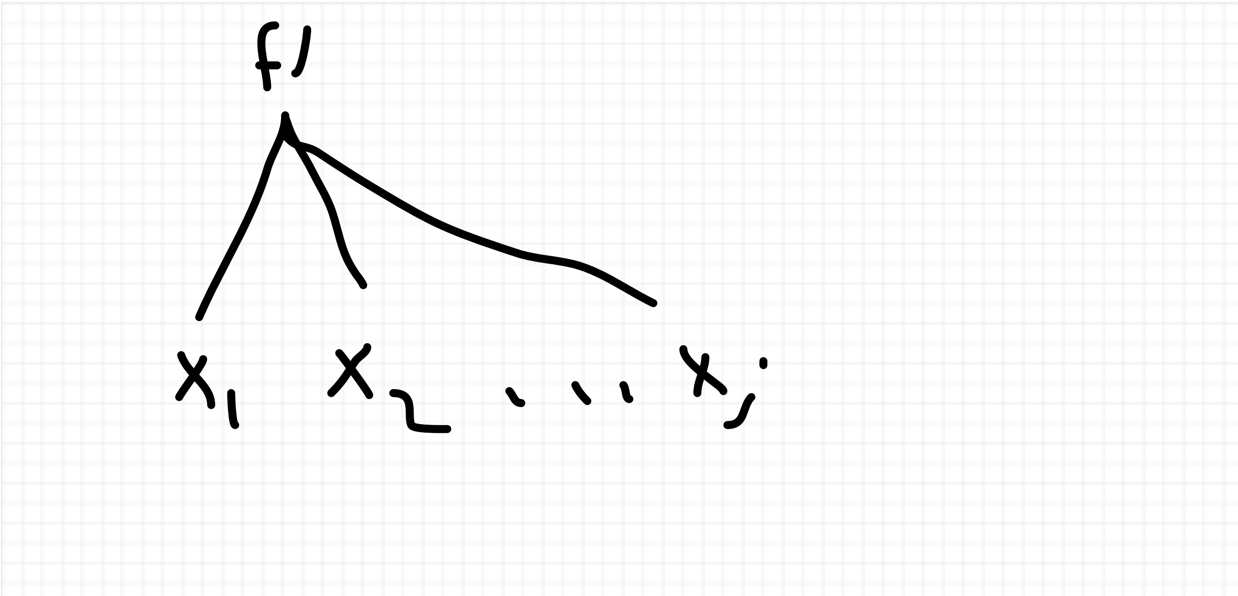
Example 1.5. Let $\left(\mathbb{N}, \left(\text{succ}, 0\right)\right)$ and the relation $\{(n, n+1) : n \in \mathbb{N}\} \cup \Delta$. This is a congruence relation but not symmetric.

Example 1.6. Let $\Sigma := \cup_{j=0}^{k} \Sigma_j := \left\{j\text{-ary function symbols}\right\}$ be an arbitrary finitary signature. The Σ-algebras are on the functor

$$P_\Sigma(X) = \Sigma_0 + \Sigma_1 X + \cdots + \Sigma_k X^k$$

We find that the initial P_Σ-algebra is the set term_Σ and $\alpha : P_\Sigma(X) \to X$ defined by

$$\alpha : \left(f^j, (x_1, \ldots, x_j)\right) \mapsto f^j(x_1, \ldots, x_j)$$



The analogous congruence relation we get from $\left(\text{Term}_{\Sigma}, \alpha\right)$ is the smallest one such that

$$x \sim y \iff x = y = c \in \Sigma_0 \text{ or } x = f^j(x_1, \dots, x_j),\ y = f^j(y_1, \dots y_j) \text{ and } x_1 \sim y_1, \dots, x_j \sim y_j \text{ for some } f^j \in \Sigma \setminus \Sigma_0$$

which ends up being (again) all of Δ, and this gives us yet another instance of the principal of mathematical induction (commonly called "structural induction") on the set term_{Σ}.

This leads to the questions: What does "recursion" look like? So generalizing the fact from example 1.4, we get the following generalized analogue.

Theorem 1.7. *(Definition by Structural Recursion)* Given a set A with $f^j_A : A^j \to A$ j-ary functions indexed by $f^j \in \Sigma$, there exists exists a unique function $g : \text{term}_{\Sigma} \to A$ such that

$$\forall f^j \in \Sigma, \quad g\left(f^j(x_1, \dots, x_j)\right) = f^j_A(g(x_1), \dots, g(x_j)) \qquad (*)$$

Lemma 1.8. Let $g : \text{term}_{\Sigma} \to A$ be a function. The following are equivalent.

(i) the condition $(*)$ holds

(ii) $g : \left(\text{term}_{\Sigma}, \alpha\right) \to \mathcal{A}$ is a Σ-homomorphism, where $\mathcal{A} := \left(A, \left(f^j_A\right)_{f^j \in \Sigma}\right)$ is a Σ-

structure.

Proof. It holds by definition of a Σ-structure.

$$\begin{tikzcd}
P_\Sigma \mathrm{term}_\Sigma \quad (F^j(x_1, \ldots, x_j)) \arrow[r, "P_\Sigma g"] \arrow[d, "\alpha"'] \arrow[d, phantom, shift left=14ex, "\downarrow"] & P_\Sigma A \quad (f^j_A(a_1, \ldots \arrow[d, "(f^j_A)_{f^j \in \Sigma}"] \\
\mathrm{term}_\Sigma \quad F^j(x_1, \ldots, x_j) \arrow[r, "g"] & A \quad f^j_A(a_1, \ldots
\end{tikzcd}$$

□

Proof of Theorem 1.7. By **Lemma 1.8** and initiality of $\left(\mathrm{Term}_\Sigma, \alpha\right)$. □

The Takeaway: We can formulate definition by recursion more generally from the unique maps from the intial F-algebra (I, α) to any other F-algebra (B, β).

For coalgebras, we can formulate definition by corecursion with the unique maps going to the final coalgebra (F, τ).

1.3 General Principle of Mathematical Induction and Definition by Recursion

F is an endofunctor on Set

Notation: $\Delta_A := \{(a, a) : a \in A\}$

Theorem 1.9. *(Principal of Mathematical Induction Version I)* Every congruence relation R on an initial F-algera (I, α) contains the diagonal Δ_I, or equivalently, the diagonal Δ_I is the $\subset$-smallest congruence relation.

Conjecture 1.10. *(Principal of Mathematical Induction Version II)* If there is an initial algebra (I, α) then for every algebra (B, β), there exists a $\subset$-smallest congruence relation $\sim$ on B. Moreover, this congruence relation is a subset of the diagonal Δ_B.

%need functor to preserve intersections (possibly)

Example 1.11.

(i) Let's look at $\Sigma := \{0, S\}$, and we find for any algebra $(B, S_B + 0_B)$, we can define $\sim$ as follows:

(remember that $S_B + 0_B : B + 1 \to B$)

$b \sim c \Longleftrightarrow b = c = 0_B$ or $b = S_B(b')$ and $c = S_B(c')$ and $b' \sim c'$

(ii) Similar things for the other signatures in previous examples.

Example 1.12. Let's look at the initial algebra $\left(\mathbb{N}, 0, \text{ succ}\right)$ of $P_{\Sigma_{nat.}}(X) := X + 1$. It can be shown that for any congruence R on $\mathbb{N}$ that

$(0, 0) \in R,$
$(n, m) \in R \Longrightarrow \left(\text{succ}(n), \text{succ}(m)\right) \in R.$

We shall show (using this idea) that if $P \subset \mathbb{N}$ that is contains 0 and is closed under succ then $P = \mathbb{N}$.

Given $P \subset \mathbb{N}$, we shall show that $\Delta_P := \{(p, p) : p \in P\}$ is a congruence relation on $\mathbb{N}$.

$$1 \mapsto 1$$
$$(p,p) \mapsto p$$

$$\begin{tikzcd}
\mathbb{N}+1 \arrow[d, "\mathrm{succ}+0"'] & \Delta_P+1 \arrow[l, "\pi_1+1"'] \arrow[r, "\pi_2+1"] \arrow[d, dashed, "r"] & \mathbb{N}+1 \arrow[d, "\mathrm{succ}+0"] \\
\mathbb{N} & \Delta_P \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{N}
\end{tikzcd}$$

$r:\ 1 \mapsto (0,0)$

$(p,p) \mapsto (\mathrm{succ}\,p, \mathrm{succ}\,p)$

Since Δ_P is a congruence relation and $\Delta_P \subset \Delta_{\mathbb{N}}$, we find by *Theorem 1.9*, we have $\Delta_P = \Delta_{\mathbb{N}} \implies P = \mathbb{N}$.

Theorem 1.12. *(Definition by Recursion Version I)* Given an initial F-algebra (I, α) and F-algebra (B, β), then $\exists! : \langle\!\langle - \rangle\!\rangle : (I, \alpha) \to (B, \beta)$ with the property that for all $a \in FI$, we have

$\langle\!\langle \alpha(a) \rangle\!\rangle = \beta(F\langle\!\langle - \rangle\!\rangle(a))$.

$$\begin{tikzcd} FI \arrow[r, dashed, "F\langle\!\langle - \rangle\!\rangle"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\ I \arrow[r, dashed, "\exists! \langle\!\langle - \rangle\!\rangle"'] & B \end{tikzcd}$$

Proof. Immediate by (I, α) being initial. □

Notation: We shall denote the unique map $(I, \alpha) \to (B, \beta)$ by $\langle\!\langle - \rangle\!\rangle$, and more specifically $\langle\!\langle - \rangle\!\rangle_B$ if context is needed.

Conjecture 1.13. *(Definition by Recursion Version II)* If there is an initial F-algebra (I, α), then for every F-algebra (B, β), we have

$$\forall x, y \in I \; \langle\!\langle x \rangle\!\rangle = \langle\!\langle y \rangle\!\rangle \iff \langle\!\langle x \rangle\!\rangle \sim \langle\!\langle y \rangle\!\rangle,$$

where $\sim$ is the smallest congruence relation in *Conjecture 1.10*.

Remark. This result implies that for some map $h : (I, \alpha) \to \; \sim$ (it turns out that $h(x) = (\langle\!\langle x \rangle\!\rangle, \langle\!\langle x \rangle\!\rangle)$), we have

$$\sim = h[I]$$

The Takeaway: The dual notion of the usual definition by corecursion that we often use is not as well-known (and perhaps not as useful) as our usual definition by recursion, and we shall see that we have two notions of coinduction and corecursion where the second notion is more useful than the first.

2. Corecursion

2.1 General Principle of Matheamtical Coinduction and Definition by Corecursion

Theorem 2.1. *(Principal of Mathematical Coinduction Version I)* Every bisimulation R on a final F-coalgebra (T, τ) is a subset of the diagonal Δ_T.

Moreover, If F preserves weak pullbacks (or more generally weak "kernel pairs"), then Δ_T is the $\subset$-largest bisimulation.

Theorem 2.2. *(Principal of Mathematical Coinduction Version II)* If there exists a final F-coalgebra (T, τ) and F preserves weak pullbacks, then every F-coalgebra (B, β) has a $\subset$-largest bisimulation on B.

%H. Peter Gumm's paper on weaker condition for coinduction

We shall prove coinduction through first proving the "definitions by corecursion"

Theorem 2.3. *(Definition by Corecursion Version I)* Given an final F-coalgebra (T, τ) and F-coalgebra (B, β), then $\exists! : [[-]] : (B, \beta) \to (T, \tau)$ with the property that for all $b \in B$, we have

$\tau[[b]] = (F[[-]] \circ \beta)(b)$.

$$\begin{tikzcd}
T \arrow[d, "\tau"'] & & B \arrow[ll, dashed, "\exists! [[-]]"'] \arrow[d, "\beta"] \\
FT & & FB \arrow[ll, dashed, "F[[-]]"']
\end{tikzcd}$$

Proof. Immediate by (T, τ) being final. ◻

Notation: Denote $[[-]] : (B, \beta) \to (T, \tau)$ as the unique F-coalgebra morphism (by finality), and we shall write $[[-]]_B$ if further context is needed.

Remark: Since $\tau[[-]] : B \to FT$, as opposed to a function $\langle\langle - \rangle\rangle \alpha : FI \to B$, for $\alpha : FI \to I$ we find that this notion of corecursion does not target T, and hence takes a back seat to the second version of "Definition by Corecursion".

Theorem 2.4. *(Definition by Corecursion Version II)* Given a final F-coalgebra (T, τ) and a coalgebra (B, β) we find

$b \sim c \implies [[b]] = [[c]],$

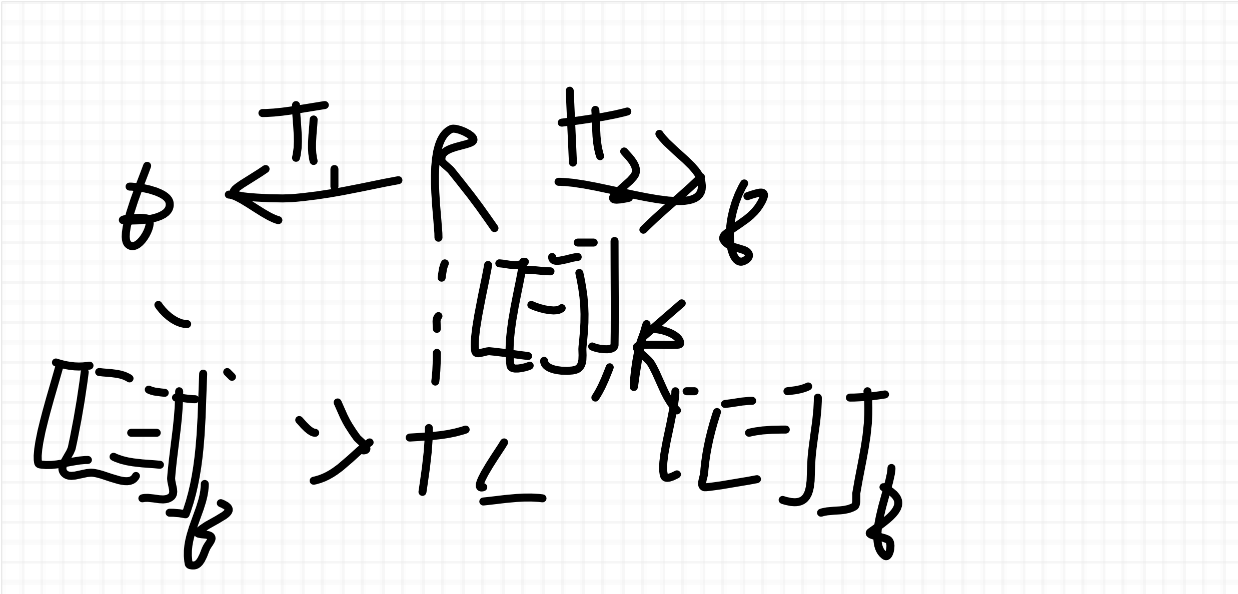
where $\sim$ is defined as follows:

$b \sim c \iff \exists$ bisimulation R s.t. $(b, c) \in R$

The converse $b \sim c \impliedby [[b]] = [[c]]$ holds if F preserves weak pullbacks.

Proof.

$\implies$ Suppose $b \sim c$, and choose a bismulation R and $\gamma : R \to FR$ such that $(b, c) \in R$. We find that $[[-]]_B \circ \pi_1$ and $[[-]]_B \circ \pi_2$ are both morphisms $(R, \gamma) \to (T, \tau)$, so equality follows by finality



and we have

$$[[b]]_B = [[\pi_1(b,c)]]_B = [[(b,c)]]_R = [[\pi_2(b,c)]]_B = [[c]]_B.$$

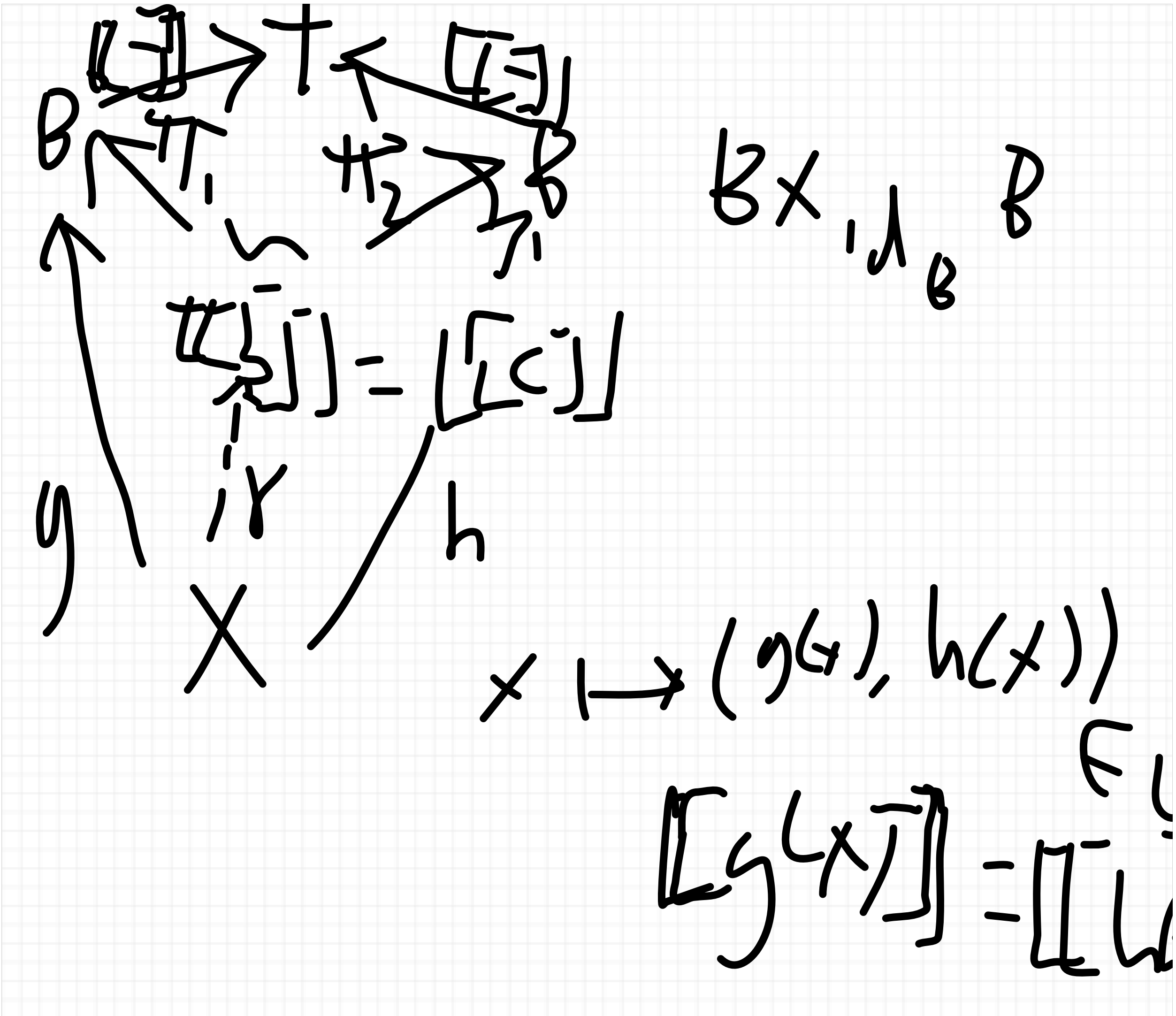
To prove $\Longleftarrow$ we'll use the following lemma:

Lemma 2.5. If F preserves weak pullbacks, then the relation $\sim$ defined by

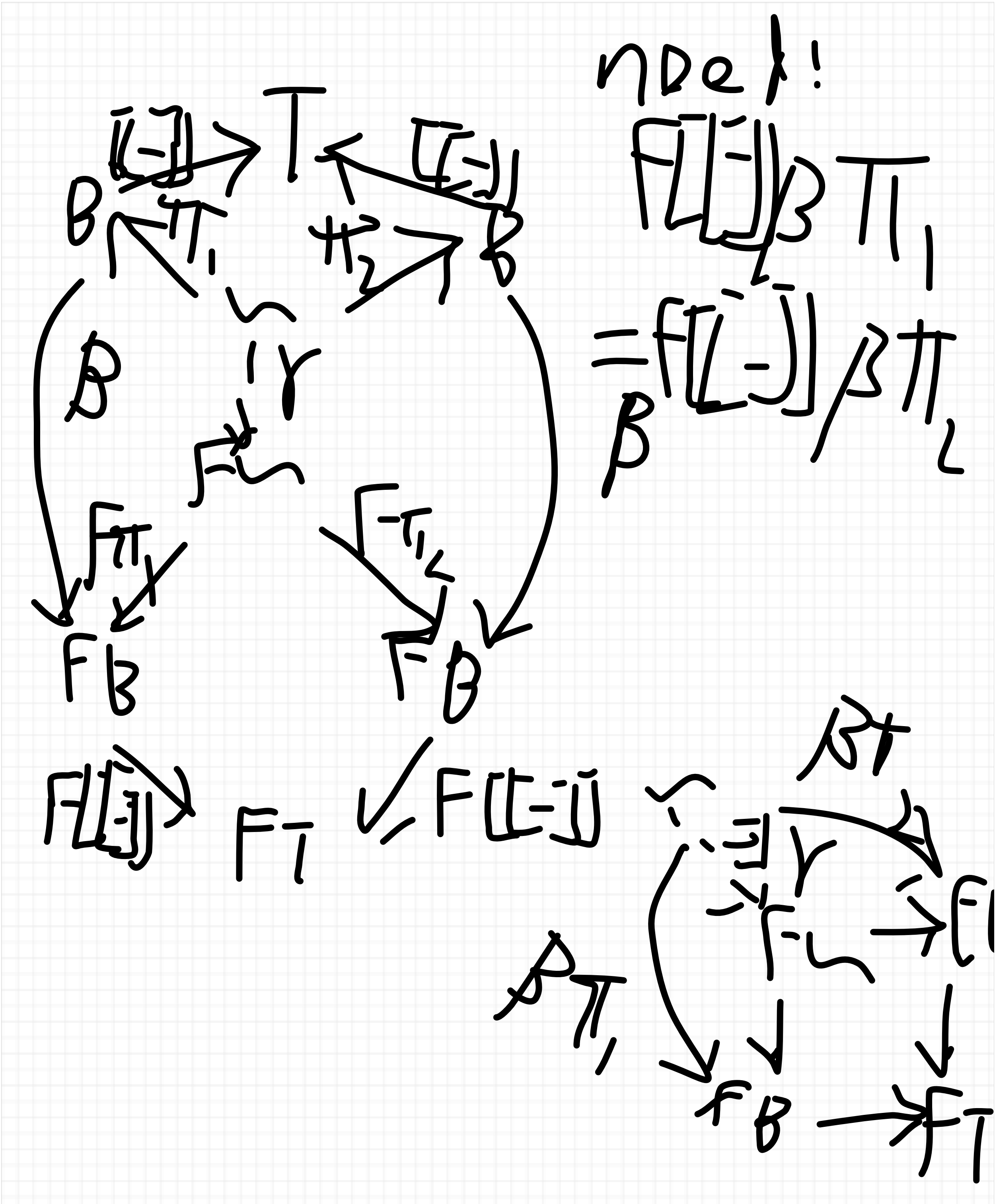
$$b \sim c \Longleftrightarrow [[b]] = [[c]]$$

is a bismulation.

Proof. Note that $\sim$ is the pullback of $B \xrightarrow{[[-]]} T \xleftarrow{[[-]]} B$ (note that such a pullback where $\rightarrow$ is the same as $\leftarrow$ is known as a "kernel pair"), as the diagram below shows:



If $F[[-]]\beta\pi_1 = F[[-]]\beta\pi_2$, then we find by the preservation of weak pullbacks and the diagrams below that there exists the necessary $\gamma :\sim\ \to F \sim$ that satisfies the bisimulation condition, hence we need to show that $F[[-]]\beta\pi_1 = F[[-]]\beta\pi_2$.



We find that given $x \sim y$, we have

$x \sim y \implies [[x]] = [[y]]$

$\implies F([[-]]) \circ \beta(\pi_1(x, y)) = F([[-]]) \circ \beta(x) = \tau([[x]]) = \tau([[y]]) = F([[-]]) \circ \beta(y)$

$$= F([[-]]) \circ \beta(\pi_2(x, y)) \qquad \square$$

Now we prove $\Longleftarrow$ of *Theorem 2.4*. Using *Lemma 2.5*, we find that the relation

$b \sim c \Longleftrightarrow [[b]] = [[c]]$

is itself a bisimulation, and our conclusion follows. $\square$

2.2 Examples of Bisimulations, Coinduction, and Corecursion

Example 2.7. First, we have coalgebras on the functor $S \mapsto A \times S$ for an arbitrary set A. The final coagelbra is the coalgebra of streams $\left(A^\omega, \langle(-)(0), (-)'\rangle\right)$ with

NOTE: A^ω is the set of sequences on A

$\langle(-)(0), (-)'\rangle(\sigma) = (\sigma(0), \sigma'),$ where $\sigma' := (\sigma(1), \sigma(2), \ldots)$ (we often like to call σ' the "stream derivative")

We can check using the diagram on bisimulations that for any $(S, \langle o, tr\rangle)$ ($o : S \to A$ $tr : S \to S$ gives us $\langle o, tr\rangle : S \to A \times S$), we have

$s \sim t \Longleftrightarrow o(s) = o(t)$ and $tr(b) \sim tr(c)$

This gives us the following notions of coinduction and corecursion on streams

<u>Coinduction on Streams:</u>

$\sigma, \tau \in A^\omega$

$\sigma = \tau \Longleftrightarrow \sigma(0) = \tau(0)$ and $\sigma' \sim \tau'$

<u>Corecursion on Streams:</u>

Given $o : \left(A^\omega\right)^n \to A$ and $tr : \left(A^\omega\right)^n \to \left(A^\omega\right)^n$ (gives us a coalgebra $\left(\left(A^\omega\right)^n, \langle o, tr\rangle\right)$), then $\exists!$ operation $g := [[-]]_{\left(A^\omega\right)^n} : \left(A^\omega\right)^n \to A^\omega$ with the following property:

$\eta := \langle o, tr\rangle,$

$\vec{t} \sim \vec{s} \Longleftrightarrow Fg\eta(\vec{t}) = Fg\eta(\vec{s})$

$g(\vec{\sigma})(0) = o(\vec{\sigma})$
$g(\vec{\sigma})' = g(tr(\vec{\sigma}))$

If we set $A := \mathbb{R}$, we get an interesting mode of arithmetic called "Stream Calculus". There's a few operations defined corecursively in a format "Stream Differential equations" (SDE's) for short:

<u>Some Operations:</u>

$c \in \mathbb{R},\ \ \overline{c} := (c, c, c, \ \dots)$ has corecursive definition

$\overline{c}(0) = c,\ \overline{c}' = \overline{c}$

Another operation is pointwise addition and pointwise multiplication $+, \ \cdot$:

$\sigma + \tau = (\sigma(0) + \tau(0), \sigma(1) + \tau(1), \ \dots)$
$\sigma \cdot \tau = (\sigma(0) \cdot \tau(0), \sigma(1) \cdot \tau(1), \ \dots)$

$+, \ \cdot : \left(A^{\omega}\right)^2 \to A^{\omega}$

$(\sigma + \tau)(0) = \sigma(0) + \tau(0)$
$(\sigma \cdot \tau)(0) = \sigma(0) \cdot \tau(0)$

$(\sigma + \tau)' = \sigma' + \tau'$
$(\sigma \cdot \tau)' = \sigma' \cdot \tau'$

The above definitions by corecursion give us unique maps $\oplus, \otimes : \left(A^{\omega}\right)^2 \to A^{\omega}$ such that

$\sigma \oplus \tau = \sigma + \tau$
$\sigma \otimes \tau = \sigma \cdot \tau$

$o(\sigma, t) = \sigma(0) + \tau(0)$
$tr(\sigma, \tau) = (\sigma', \tau')$

$\langle (-)(0), (-)' \rangle (\sigma \oplus \tau) = F(\oplus)(\langle o, tr \rangle (\sigma, \tau))$

$F(\oplus) = id_A \times \oplus : A \times T^2 \to A \times T$

$(\sigma \oplus \tau)(0) = o(0) = \sigma(0) + \tau(0)$
$(\sigma \oplus \tau)' = \oplus(tr(\sigma, \tau)) = \oplus(\sigma', \tau') = \sigma' \oplus \tau'$

Let's look at the convolution product $\times : \left(A^\omega\right)^2 \to A^\omega$

$$(\sigma \times \tau)(n) = \sum_{k=0}^{n} \sigma(k)\tau(n-k)$$

$(\sigma \times \tau)(0) = \sigma(0) \cdot \tau(0)$
$(\sigma \times \tau)' = \sigma' \cdot \tau + \sigma \cdot \tau'$

Generalize the notion of SDE's to "second order SDE's"

$\sigma(0) = 0 \;\; \sigma'(0) = 1 \quad \sigma'' = \sigma' + \sigma$

Turns that σ = Fib = $(0, 1, 1, 2, 3, 5, 8, \ldots)$

Example 2.8. Automata $S \mapsto 2 \times S^A$

$(S, \langle o, tr \rangle) \quad o : S \to 2, \; tr : S \to S^A$

We find first off that

$s \sim t \iff o(s) = o(t)$ and $\forall a \in A(tr(s)(a) \sim tr(t)(a))$.

Next, the final coalgebra is $\left(\mathcal{P}\left(A^*\right), \langle(-)(\epsilon), d\rangle\right)$, where

$\lambda(\epsilon) = 1 \iff \epsilon \in \lambda \qquad \forall \lambda \in \mathcal{P}\left(A^*\right)$
$d(\lambda)(a) = \lambda_a := \left\{w \in A^* : aw \in \lambda\right\}$

<u>Coinduction on $\mathcal{P}\left(A^*\right)$:</u>

$\lambda = \kappa \iff \lambda(\epsilon) = \kappa(\epsilon)$ and $\forall a \in A(\lambda_a \sim \kappa_a)$.

<u>Corecursion on $\mathcal{P}\left(A^*\right)$:</u>

Given $o : \mathcal{P}(A^*)^n \to A$ and $tr : \mathcal{P}(A^*)^n \to \mathcal{P}(A^*)^n$ $\exists!$ operation $g : \mathcal{P}(A^*)^n \to \mathcal{P}(A^*)$ such that

$$g(\vec{\lambda})(\epsilon) = o(\vec{\lambda}),$$
$$d(g(\vec{\lambda})) = g(tr(\vec{\lambda})).$$

Some Operations:

Constants: $0 := \varnothing,\ 1 := \{\epsilon\},\ b := \{b\}$

$$0(\epsilon) = 0,\ 0_a = 0$$
$$1(\epsilon) = 1 \quad 1_a = 0$$
$$b(\epsilon) = 0 \quad b_a = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else.} \end{cases}$$

Sums and Products:

$$\kappa + \lambda := \kappa \cup \lambda \quad \kappa \cdot \lambda := \{vw : v \in \kappa,\ w \in \lambda\}$$

$$(\kappa + \lambda)(\epsilon) = \max\{\kappa(\epsilon), \lambda(\epsilon)\} \quad (\kappa + \lambda)_a = \kappa_a + \lambda_a$$
$$\kappa \cdot \lambda = \min\{\kappa(\epsilon), \lambda(\epsilon)\} \quad (\kappa \cdot \lambda)_a = \begin{cases} \kappa_a \cdot \lambda & \text{if } \kappa(\epsilon) = 0, \\ \kappa_a \cdot \lambda + \lambda_a & \text{else.} \end{cases}$$

Kleene Star:

$$\kappa^* := \bigcup_{n>0} \kappa^n \ \left(\kappa \in \mathcal{P}(A^*)\right), \text{ where } \kappa^0 := 1 \text{ and } \kappa^{n+1} := \kappa \cdot \kappa^n,$$
$$\kappa^*(\epsilon) = 1 \ \left(\kappa^*\right)_a = \kappa_a \cdot \kappa^*.$$

Example 2.9.

$$F : X \mapsto X + 1$$

(S, β) $\beta : S \to S + 1$ β partial function on S with $\beta(s) \uparrow$ when $\beta(s) = *$.

The final coalgebra is $\left(\mathbb{N}^\top, \text{pre}\right)$ with $\mathbb{N}^\top := \mathbb{N} \cup \{+\infty\}$ and pre is the predecessor.

What can we say about the bisimulations on any (S, β)?

NOTE: F preserves weak pullbacks. (more generally, P_Σ preserves pullbacks in general)

We derive the following:

$$\begin{aligned}
b \sim c &\iff [[b]] = [[c]] \\
&\iff \text{pre}[[b]] = \text{pre}[[c]] \\
&\iff F[[-]] \circ \beta(b) = F[[-]] \circ \beta(c) \\
&\iff \beta(b) = \beta(c) = * \text{ or } \beta(b) \sim \beta(c) \\
&\iff \beta(b) = \beta(c) = * \text{ or } \beta^2(b) = \beta^2(c) = * \text{ or } \beta^2(b) \sim \beta^2(c) \\
&\iff \cdots \\
&\iff \beta^n(b) = \beta^n(c) = * \text{ for some } n \geq 0 \text{ or } \beta^n(b) \sim \beta^n(c) \ \left(\beta^n(b), \beta^n(c) \in \mathbb{N}\right) \ \forall n \\
&\iff \beta^n(b) = \beta^n(c) = * \text{ eventually, or both never equal } *
\end{aligned}$$

<u>Coinduction on $\mathbb{N}^\top$:</u>

$$\begin{aligned}
n = m &\iff \text{pre}(n) = \text{pre}(m) = * \text{ or } \text{pre}(n) \sim \text{pre}(m) \\
&\iff \text{pre}^k(n) = \text{pre}^k(m) = * \text{ for some } k \text{ or } n = m = +\infty
\end{aligned}$$

<u>Corecursion on $\mathbb{N}^\top$:</u>

We can define an operation $g : \left(\mathbb{N}^\top\right)^n \to \mathbb{N}^\top$ given a partial function $\beta : \left(\mathbb{N}^\top\right)^n \to \left(\mathbb{N}^\top\right)^n$, and g is uniquely determined and has the following property:

$$\text{pre}(g(\vec{x})) = \begin{cases} * & \text{if } \beta(\vec{x}) \uparrow \\ g(\beta(\vec{x})) & \text{else} \end{cases} \qquad ((1))$$

$$\tau[[x]] = F[[-]] \circ \beta$$

Notation: for any partial function β we write $\beta(x) \uparrow \iff \beta(x) = *$

We find from $((1))$ that

$$g(\vec{x}) = \begin{cases} n & \text{if } \beta^{n+1}\vec{x} \uparrow \ (n \geq 0) \\ +\infty & \text{else} \end{cases} \qquad ((2))$$

Some Operations:

Sum Operation: We're going to define $+ : \mathbb{N}^2 \to \mathbb{N}$ using corecursion. First we define a partial function $\beta_+ : \mathbb{N}^2 \to \mathbb{N}^2$ (it's a whole function $\mathbb{N}^2 \to \mathbb{N}^2 + 1$ as follows:

$$\beta_+(n,m) = \begin{cases} * & \text{if } n = m = 0 \\ \big(\text{pre}(n), m\big) & \text{if } n > 0 \\ \big(n, \text{pre}(m)\big) & \text{if } n = 0 \text{ and } m > 0 \end{cases}$$

We find that the $+$ operation has the following property

$$\text{pre}(n+m) = \begin{cases} * & \text{if } n = m = 0 \\ \text{pre}(n) + m & \text{if } n > 0 \\ n + \text{pre}(m) & \text{if } n = 0 \text{ and } m > 0 \end{cases}$$

And it holds that $+$ is the canonical operation on $\big(\mathbb{N}^2, \beta\big)$ to $\big(\mathbb{N}^\top, \text{pre}\big)$ such that

$$(n_1, m_1) \sim (n_2, m_2) \iff n_1 + m_1 = n_2 + m_2$$

Multiplication Opertation: Defined a partial function $\gamma : \mathbb{N}^3 \to \mathbb{N}^3$ as follows

$$\gamma(n,m,p) = \begin{cases} * & \text{if } n = 0 \text{ or } m = 0 \text{ and } p = 0 \\ \big(n, m, \text{pre}(p)\big) & \text{if } n = 0 \text{ or } m = 0 \text{ and } p > 0 \\ \big(\text{pre}(n), m, p + \text{pre}(m)\big) & \text{if } n > 0 \text{ and } m > 0 \end{cases}$$

$$n \cdot m = [[(n,m,0)]]_\gamma$$

$$\text{pre}[[(n,m,p)]]_\gamma = \begin{cases} * & \text{if } n = 0 \text{ or } m = 0 \text{ and } p = 0 \\ [[\gamma(n,m,p)]]_\gamma & \text{else} \end{cases}$$

In general:

$$[[(n,m,p)]]_\gamma = n \cdot m + p$$

$$(n,m,p) \sim (n',m',p') \iff n \cdot m + p = n' \cdot m' + p'$$

$\text{pre}(n \cdot m + p) = n$

$(n_1, m_1, 0) \sim (n_2, m_2, 0) \iff n_1 \cdot m_1 = n_2 \cdot m_2$

$\beta_\times : \mathbb{N}^2 \to \mathbb{N}^2$

$$\beta_\times(n, m) = \begin{cases} * & \text{if } n = 0 \text{ or } m = 0 \\ & \text{if } n > 0 \end{cases}$$

Want:

$\cdot : \mathbb{N}^2 \to \mathbb{N}$

$$\text{pre}(n \cdot m) = \begin{cases} * & \text{if } n = 0 \text{ or } m = 0 \\ \text{pre}(m) \cdot n + \text{pre}(m) & m > 0 \end{cases}$$

Recall: $\mathbb{N}^\top := \mathbb{N} \cup \{+\infty\}$

In general, if we're given a partial function $\beta : \left(\mathbb{N}^\top\right)^n \to \left(\mathbb{N}^\top\right)^n$ then there is a uniquely determined $g : \left(\mathbb{N}^\top\right)^n \to \mathbb{N}^\top$ such that

$$g(\overrightarrow{m}) = \begin{cases} * & \text{if } \beta(\overrightarrow{m}) \uparrow \\ g(\beta(\overrightarrow{m})) & \text{otherwise} \end{cases}$$

%correct mistake and prove how this example applies to more general analysis

Example 2.10. $F : \ X \mapsto A \times X + 1$

$(S, \langle o, tr \rangle)$ $\langle o, tr \rangle : S \to A \times S + 1$ (we can think of $\langle o, tr \rangle$ as a partial function on $A \times S$)

Equaivalently, we can define $o : S \to A$ as a partial function and $tr : S \to S$ as a complete function.

<u>The Final Coalgebra:</u>

Exists due to limit preservation of the ω^{op} chain

$1 \leftarrow F1 \leftarrow F2 \leftarrow F3 \leftarrow \ \cdots$

$$1 \leftarrow A+1 \leftarrow A^2+A+1 \leftarrow A^3+A^2+A+1 \leftarrow \cdots$$
$$A\times 1+1 \quad A\times(A+1)+1$$

Limit ends up being $\sum_{\alpha\le\omega} A^\alpha := A^{\le\omega}$, and the final coalgebra $\left(A^{\le\omega}, \langle(-)(0),\ (-)'\rangle\right)$ ends up being the set of full streams and partial streams (ones with the first α entries for some $\alpha < \omega$), with the partial function $\sigma(0)$ (undefined if $\sigma = *$) and

$$\sigma' = \begin{cases} * & \text{if } \sigma = * \in A^0 = 1 \\ (\sigma(1),\ \ldots, \sigma(\alpha-1)) & \text{if } \sigma \in A^\alpha \text{ for some finite } 0 < \alpha \in \omega \\ \sigma' & \text{if } \sigma \in A^\omega \end{cases}$$

Given $(S, \langle o, tr\rangle)$ and $s, t \in S$, we find that

$$s \sim t \iff o(s) = o(t) \text{ and } tr(s) \sim tr(t) \text{ or } o(s), o(t) \text{ are undefined}$$
$$\iff \cdots$$
$$\iff \begin{cases} o(s) = o(t) \text{ and } tr(s) \sim tr(t) & \text{if } \forall n\left(o\left(tr^n(s)\right),\ o\left(tr^n(t)\right) \downarrow\right) \\ \exists n\left(o\left(tr^n(s)\right), o\left(tr^n(t)\right) \uparrow\right) \text{and} & \text{otherwise} \\ \forall k < n\left(o\left(tr^k(s)\right) = o\left(tr^k(t)\right)\right) & \end{cases}$$

$$tr(s) \sim tr(t) \iff o(tr(s)) = o(tr(t)) \text{ and } tr^2(s) \sim tr^2(t) \text{ or } o(tr(s)), o(tr(t)) \text{ are undefined}$$

<u>Coinduction on $A^{\le\omega}$:</u>

For $\sigma, \tau \in A^{\le\omega}$,

$$\sigma = \tau \iff \sigma \sim \tau \iff \sigma(0) = \tau(0) \text{ and } \sigma' \sim \tau' \text{ or } \sigma(0), \tau(0) \uparrow$$

<u>Corecursion on $A^{\le\omega}$:</u>

Given a partial function $f_1 : \left(A^{\le\omega}\right)^n \to A$ and a function $f_2 : \left(A^{\le\omega}\right)^n \to \left(A^{\le\omega}\right)^n$, there exists a unique operation $g : \left(A^{\le\omega}\right)^n \to A^{\le\omega}$ such that

$g(\vec{\sigma})(0) = f_1(\vec{\sigma})$ (note: may be undefined)
$g(\vec{\sigma})' = g(f_2(\vec{\sigma}))$

Some Operations:

Sum Operation: Truncates the length of the larger stream, and adds up all the entries.

$\left(A^{\leq\omega}\right)^2 \to A^{\leq\omega}$

$\sigma, \tau \in A^{\leq\omega}$, $(\sigma+\tau)(n) = \sigma(n) + \tau(n)$ where we can define it, for the minimum length of the stream

$(\sigma+\tau)(0) = \sigma(0) + \tau(0)$ undefined if either $\sigma(0)\uparrow$ or $\tau(0)\uparrow$

$$(\sigma+\tau)' = \begin{cases} * & \text{if either } \sigma \text{ or } \tau \text{ is } * \\ \sigma' + \tau' & \text{otherwise} \end{cases}$$

Example 2.11.

$F : X \mapsto A \times X^2$

$(S, \langle o, tr_1, tr_2\rangle)$ $\langle o, tr_1, tr_2\rangle : S \to A \times S^2$ $o : S \to A,\ tr_1, tr_2 : S \to S$

The Final Coalgebra:

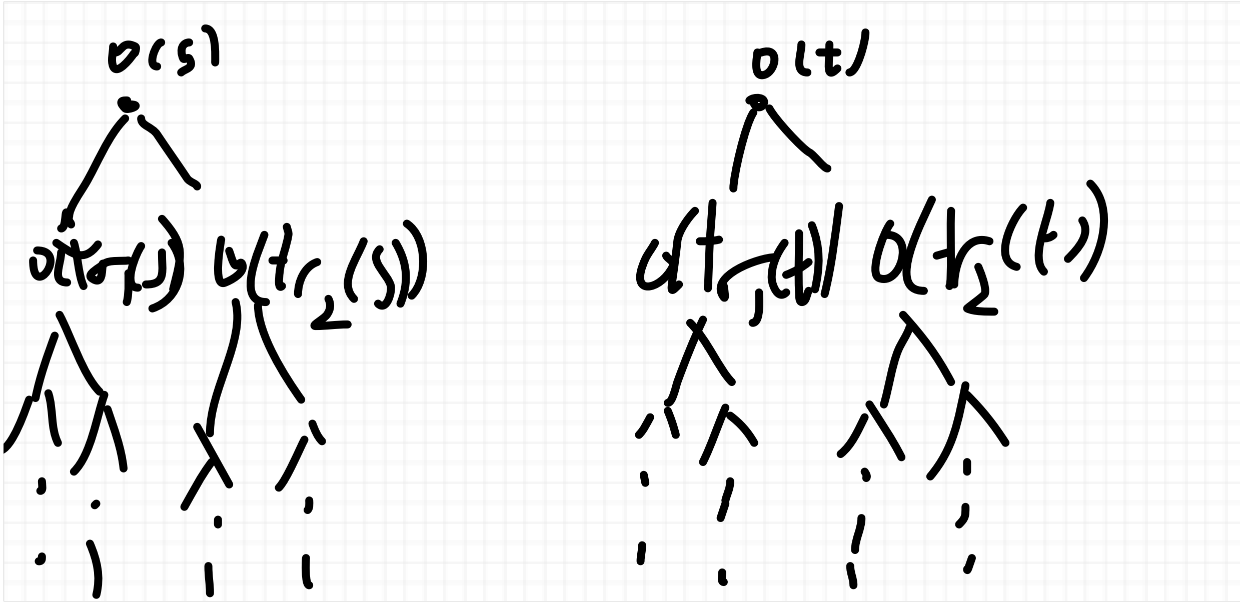
The final coalgebra $(\nu.F, \tau)$ exists due to ω^{op} limit preservation

We'll derive what exactly the final coalgebra is using bisimulations. Given $s, t \in S$

$$\begin{aligned} s \sim t &\iff [[s]] = [[t]] \\ &\iff \tau[[s]] = \tau[[t]] \\ &\iff F[[\langle o, tr_1, tr_2\rangle(s)]] = F[[\langle o, tr_1, tr_2\rangle(s)]] \\ &\iff o(s) = o(t) \text{ and } tr_1(s) \sim tr_1(t) \text{ and } tr_2(s) \sim tr_2(t) \\ &\iff \cdots \end{aligned}$$

$tr_1(s) \sim tr_1(t) \iff \cdots$

$tr_2(s) \sim tr_2(t) \iff \cdots$

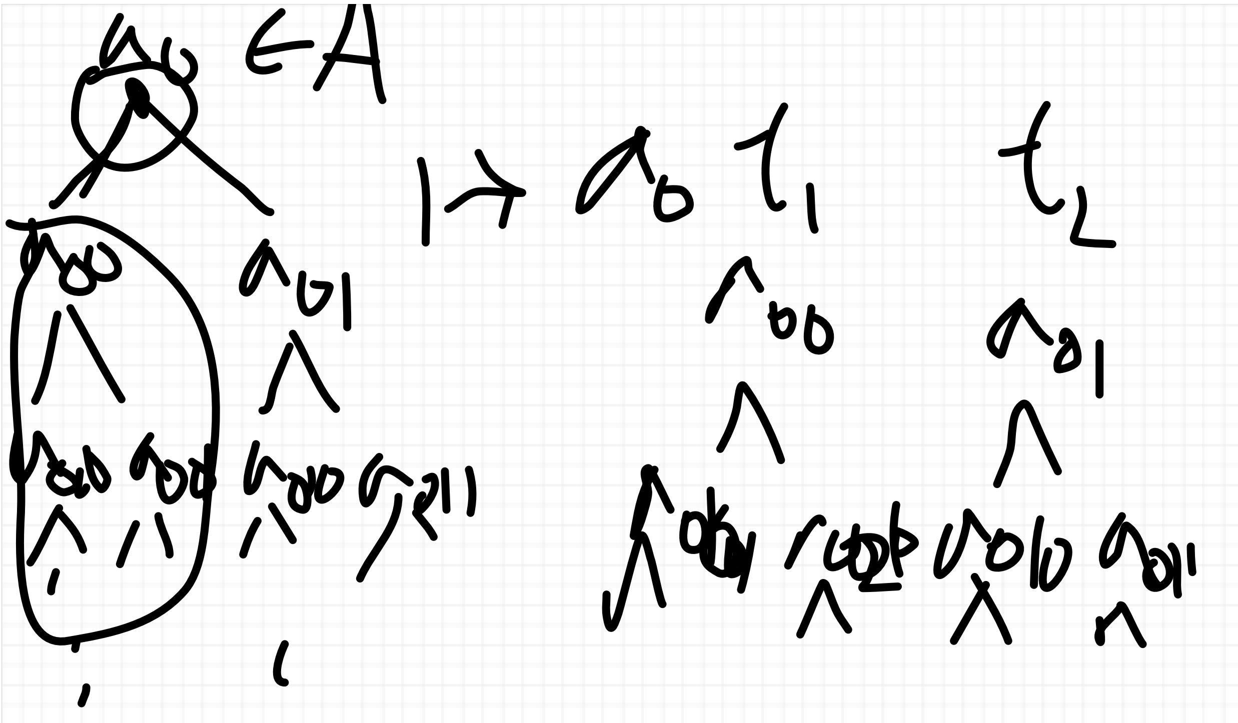


One can characterize the final coalgebra as a

$\nu.F := T_{2,A} := \left\{\text{infinite binary trees with } A\text{-coloring on vertices}\right\}$

$\tau : T_{2,A} \to A \times T_{2,A}^2$ inverse tree tupling

$\tau(t) = (a_0, t_1, t_2) = \left(\text{head}(t), \text{tail}_1(t), \text{tail}_2(t)\right)$



Coinduction on $T_{2,A}$:

For $t, t' \in T_{2,A}$

$t = t' \iff$ head(t) = head(t') and tail$_1(t) \sim$ tail$_1(t')$, tail$_2(t) \sim$ tail$_2(t')$

Corecursion on $T_{2,A}$:

Given $f_1 : T^n_{2,A} \to A,\ f_2, f_3 : T^n_{2,A} \to T^n_{2,A}$, we define a uniquely determined operation $g : T^n_{2,A} \to T_{2,A}$ such that

head$(g(\vec{t})) = f_1(\vec{t})$
tail$_1(g(\vec{t})) = g(f_2(\vec{t}))$
tail$_2(g(\vec{t})) = g(f_3(\vec{t}))$

Example 2.12. $F : X \mapsto AX^3 + BX + 2$

The Final Coalgebra: Exists due to colimit preservation of ω^{op}-chain

$1 \leftarrow F1 \leftarrow F^2 1 \leftarrow F^3 1 \leftarrow \cdots$

We shall derive what the final coalgebra $(\nu.F, \tau)$ is via a similar process to before by deriving the bisimulation.

First, we want to characterize a coalgebra (S, α) amd their bisimulations, we find

$$\alpha := \alpha_1 + \alpha_2 + \alpha_3 : S \to AS^3 + BS + 2,$$

where

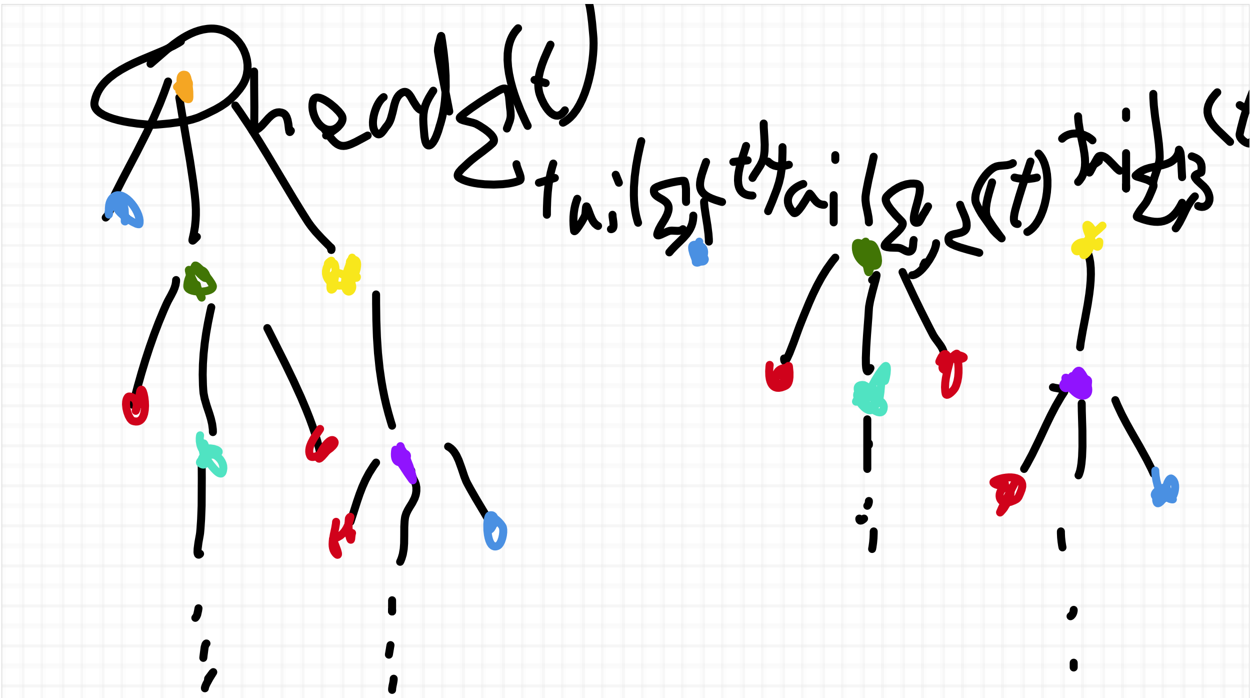
$$\alpha_1 := \langle o_1, tr_{1,1}, tr_{1,2}, tr_{1,3} \rangle : S \to AS^3 \quad o_1 : S \to A,\ tr_{1,i} : S \to S\ i = 1, 2, 3$$
$$\alpha_2 := \langle o_2, tr_2 \rangle : S \to BS$$
$$\alpha_3 := o_3 : S \to 2$$

are partial functions.

We find for $s, t \in S$

$$\begin{aligned}
s \sim t &\iff [[s]] = [[t]] \\
&\iff \tau[[s]] = \tau[[t]] \\
&\iff F[[\alpha_1 s + \alpha_2 s + \alpha_3 s]] = F[[\alpha_1 t + \alpha_2 t + \alpha_3 t]] \\
&\iff \begin{cases} o_1(s) = o_1(t),\ tr_{1,1}(s) \sim tr_{1,1}(t),\ tr_{1,2}(s) \sim tr_{1,2}(t), & \\ tr_{1,3}(s) \sim tr_{1,3}(t) & \text{if } \alpha(s) = \alpha_1(s) \in AS^3 \\ o_2(s) = o_2(t),\ tr_1(s) \sim tr_2(t) & \text{if } \alpha(s) = \alpha_2(s) \in BS \\ o_3(s) = o_3(t) & \text{if } \alpha(s) = \alpha_3(s) \in 2 \end{cases}
\end{aligned}$$

Applying this bisimulation property irritively, we see intuitively that objects in $\nu.F$ are completely characterized by the Σ-trees and the bisimulation between any two objects is established iff the o_i map is the same and agree and the branches are bisimilar, where $\Sigma := A + B + 2$.



It turns out that, similar to before, the final coalgebra $\nu.F$ consists of exactly these Σ-trees (i.e. trees with Σ-colorings that branch out for each arity-symbol until a constant is reached) and the operation τ is inverse tree tupling, mapping a tree t to the stem vertex $\text{head}_{\Sigma}(t)$ and the finite product of trees $\text{tail}_{\Sigma}(t)$ that branch out from t (see the illustration above), i.e.

$\nu.F = T_\Sigma = \left\{\text{infinite } \Sigma\text{-trees}\right\}$

$\tau = \tau_\Sigma = \langle \text{head}_{\Sigma}, \text{tail}_{\Sigma} \rangle$

$\text{head}_{\Sigma}(t) := o_i(t)$ for $i \in \{0, 1, 3\}$ s.t. $\alpha(t) = \alpha_i(t)$

$\text{tail}_{\Sigma}(t) := \left(\text{tail}_{\Sigma,1}(t), \ldots, \text{tail}_{\Sigma,i}(t)\right)$

$\left(\text{note that } i = \text{arity}(o_i(t)) \text{ and } \text{tail}_{\Sigma,k}(t) := \text{the } k\text{th branch of } t \text{ for } 1 \leq k \leq n\right)$

We call this τ_Σ operation "inverse tree tupling" in this context and the ore general context of an arbitrary Σ (see next section).

<u>Coinduction on T_Σ:</u> Given $t, t' \in T_\Sigma$

$t = t' \iff \text{head}_{\Sigma}(t) = \text{head}_{\Sigma}(t')$ and $\text{tail}_{\Sigma}(t) \sim \text{tail}_{\Sigma}(t')$

<u>Corecursion on T_Σ:</u> Given partial functions

$f_{0,0} : T^n_\Sigma \to 2,\ f_{1,0} : T^n_\Sigma \to A, f_{1,1} : T^n_\Sigma \to T^n_\Sigma, f_{3,0} : T^n_\Sigma \to B,\ f_{3,1} : T^n_\Sigma \to \left(T^n_\Sigma\right)^3$

s.t. $f_{0,0} + \langle f_{1,0}, f_{1,1}\rangle + \langle f_{3,0}, f_{3,1}\rangle : T^n_\Sigma \to F\left(T^n_\Sigma\right),$

is a well-defined function, we have a unique function $g : T^n_\Sigma \to T_\Sigma$ such that

$\tau_\Sigma(g(\vec{t})) = Fg \circ \alpha$
$\tau_\Sigma(g(\vec{t})) = \langle f_{i,0}(\vec{t}), f_{i,1}(\vec{t})\rangle,$ for some $i,$ i.e.

$\text{head}_\Sigma(g(\vec{t})) = f_{i,0}(\vec{t}),\ \text{tail}_\Sigma(g(\vec{t})) = f_{i,1}(\vec{t}).$

2.3 Coinduction and Corecursion on an Equational Signature Σ

We shall generalize all the previous examples into a general coalgebra concept involving any signature Σ of n-ary function symbols. Before we do that, we shall generalize the notion of polynomial functor.

Definition 2.13. We shall define a polynomial functor P_Σ for any equational signature Σ (recall: a signature of entirely n-ary function symbols) as follows:

$P_\Sigma(X) := \sum_{i \in \mathbb{N}} \Sigma_i \times X^i,$ where $\Sigma_i \subset \Sigma$ is the set of n-ary function symbols.

A theorem about such polynomials worth mentioning:

Theorem 2.14. Polynomial functors (defined in this more general context) preserve ω-colimits and ω^{op}-limits.

Corollary 2.15. There exists initial Σ-algebras (or P_Σ-algebras) and final Σ-coalgebras (or P_Σ-coalgebras).

Given a functor F, we shall denote $\mu. F$ as the initial F-algebra and $\nu. F$ as the final coalgebra.

Now that we know a final Σ-coalgebra exists, we can derive bisimulations, a more explicit characterization of the final coalgebra, coinduction, and corecursion.

Remark. Given an equational signature Σ, note that a Σ-coalgebra (S, α) we find $\alpha : S \to P_\Sigma S$ can be represented as $\alpha = \sum_{i \in \mathbb{N}} \alpha_i$ of partial functions

$$\alpha_i = \langle o_i, tr_i \rangle : S \to \Sigma_i \times S^i,\ o_i : S \to \Sigma_i,\ tr_i : S \to S^i.$$

Proposition 2.16. *(bisimulations)* Given $s, t \in S$, we find that

$$\begin{aligned} s \sim t &\iff o_i(s) = o_i(t) \text{ and } tr_i(s) \sim tr_i(t)\ \left(\text{on } S^i\right), \\ &\iff o_i(s) = o_i(t) \text{ and } \forall k \leq i(tr_{i,k}(s) \sim tr_{i,k}(t)) \text{ where } tr_{i,k} := \pi_k \circ tr_i. \end{aligned}$$

Theorem 2.17. *(coinduction on Σ)* The final Σ-coalgebra (T_Σ, τ_Σ) is the set of infinite Σ-trees and the map $\tau_\Sigma := \langle \text{head}_\Sigma, \text{tail}_\Sigma \rangle$ of inverse tree tupling with

$$\begin{aligned} \text{head}_\Sigma(t) &= \text{top vertex coloring } (\in \Sigma) \\ \text{tail}_\Sigma(t) &= \text{the } i\text{-tuple of } \Sigma\text{-trees consisting of all the branches of head}_\Sigma(t), \\ &\quad\ \text{where } i := \text{arity}\left(\text{head}_\Sigma(t)\right). \end{aligned}$$

We find given $t = t'$, we have

$$t = t' \iff \text{head}_\Sigma(t) = \text{head}_\Sigma(t') \text{ and } \text{tail}_\Sigma(t) \sim \text{tail}_\Sigma(t')\ \left(\text{in } T^i_\Sigma\right).$$

Theorem 2.18. *(corecursion on n-ary T_Σ operations)* Given a family $f_{i,0} : T^n_\Sigma \to \Sigma_i,\ f_{i,1} : T^n_\Sigma \to \left(T^n_\Sigma\right)^i$ of partial functions indexed by $i \in \mathbb{N}$ s.t.

$$\alpha := \sum_{i \in \mathbb{N}} \langle f_{i,0}, f_{i,1} \rangle : T^n_\Sigma \to P_\Sigma T^n_\Sigma,$$

forms a Σ-coalgebra, there is a uniquely determined operation $g : T^n_\Sigma \to T_\Sigma$ defined by

$$\begin{aligned} \text{head}_\Sigma(g(\vec{t})) &= f_{i,0}(\vec{t}), \\ \text{tail}_\Sigma(g(\vec{t})) &= g(f_{i,0}(\vec{t})), \end{aligned}$$

for some $i \in \mathbb{N}$.

2.4 Coinduction and Corecursion on Finitary Functors

Definition 2.19. We define a **presentation** *of a* **set functor** F as a signature Σ and a natural epi-transformation $\epsilon : P_\Sigma \twoheadrightarrow F$. If Σ is **finitary** (i.e., consisting of only n-ary function symbols), we call the presentation finitary.

Remark. A quotient object Q of a category $\mathbb{C}$. An object Q such that $\exists$ an epimorphism $e : C \twoheadrightarrow Q$. So quotient functor F of H is a functor such that $\exists$ an epi-transformation $\epsilon : H \twoheadrightarrow F$, and a finitary functor F is a quotient of a polynomial functor P_Σ, for some signature Σ.

Example 2.20.

1. The Asczel-Mendler functor is the set functor $(-)^{3/2}$ given by

$$(X)^{3/2} = \Big\{(a, b, c) \in X^3 : (a = b) \text{ or } (a = c) \text{ or } (b = c)\Big\}.$$

Given $(a, b, c) \in X^3$ and $g : X \to Y$ we find $(g)^{3/2}(a, b, c) := (g(a), g(b), g(c)) \in (Y)^{3/2}$. We can set

$$\Sigma := \Sigma_2 = \{\sigma_1, \sigma_2, \sigma_3\} \Big(\text{ i.e., three binary function symbols}\Big)$$

$$\epsilon : P_\Sigma \twoheadrightarrow (-)^{3/2}$$
$$\epsilon_X : \ P_\Sigma(X) \to (X)^{3/2}, \ \sigma_1(x, y) \mapsto (x, x, y), \ \sigma_2(x, y) \mapsto (x, y, x), \ \sigma_3(x, y) \mapsto (y, x, x)$$

2. The finite power set functor is the set functor $\mathcal{P}_f(X) := \Big\{X_0 \subset X : X_0 \text{ is finite}\Big\}$ with $\mathcal{P}_f(g) : X \to Y$ defined by $\mathcal{P}_f(g)(A) := g[A]$ for $A \in \mathcal{P}_f(X)$. Define $\Sigma := \{[-]_1, [-]_2, [-]_3, \ \dots\ \}$ with $[-]_n$ an n-ary function symbol. Define $\epsilon : H_\Sigma \to \mathcal{P}_f$ by

$$\epsilon_X : [x_1, \ \dots, x_n]_n \mapsto \{x_1, \ \dots, x_n\}.$$

Important Note. We find that every P_Σ-coalgebra (A, α) induces a F-coalgebra $(A, \epsilon_A \alpha)$.

$$\epsilon_A \alpha : A \to FA$$

Proposition 2.21.

1. Given the coalgebra (T_Σ, τ_Σ) of trees we find that $(T_\Sigma, \epsilon_{T_\Sigma} \circ \tau_\Sigma)$ is weakly terminal.

2. We find that the smallest quotient coalgebra $C :=$ Colim Quo(T_Σ) (where Quo(T_Σ) is the set of quotients coalgebras of T_Σ) is a terminal coalgebra.

If it is additionally worth noting that a finitary functor F preserves weak pullbacks, and hence important bisimulation properties are inherited, and coinductive and corecursive principles become more concrete.

%correct problem with finitary functor not preserving weak pullbacks

Definition 2.22. Given a signature Σ, we call an equation of the form

$$\sigma(\vec{x}) = \tau(\vec{y})\ \sigma, \tau \in \Sigma,$$

basic. Given a set $\mathcal{E}$ of basic equations over a fixed set X_0 of variables we define $F := P_\Sigma / \mathcal{E}$ as follows: Given a set X, we form the smallest equivalence $\approx$ on $H_\Sigma X$ given by all basic equations

$$\sigma(s(\vec{x})) = \tau(s(\vec{y})),$$

for every $s : X_0 \to X$ and every $\sigma(\vec{x}) = \tau(\vec{y}) \in \mathcal{E}$. Then $FX = P_\Sigma X$ mod $\approx$ with $g : X \to Y$ giving us

$$Fg : \ \sigma(\vec{x}) \mapsto \sigma(g(\vec{x})).$$

Note: The above relation $\approx$ consists of all equations obtained by a restricted form of equational logic.

Theorem 2.23. Every functor of the form $P_\Sigma / \mathcal{E}$ is finitary, and every finitary functor F is naturally isomorphic to $P_\Sigma / \mathcal{E}$ for some $\mathcal{E}$ set of basic equations.

Note that we can generalize this to not just a theory containing only basic equations but an arbitrary equational theory T.

Corollary 2.24. Given an equational signature Σ and an equational theory T, we find P_Σ / T is finitary, and P_Σ / T is naturally isomorphic to some functor $P_{\Sigma'} / \mathcal{E}$ for some signature Σ' and $\mathcal{E}$

%incorrect

%figure out way to generalize "finitary" functors

%move to chapter 3

%talk about coinduction on finitary coalgebras

Conclusion. As a result of what we derived, we can certainly look at interesting ways to talk about coinduction and corecursion in specific examples but this is where the discussion of finitary functors as it pertains to coalgebras stop, and we shall shift gears to free algebras.

2.5 Generalizations on Multisorted Signatures and Multivariable Polynomial Functors

Notation: For this section, we shall let Σ be a multisorted equational signature with sorts $1, \ldots, n$. We'll define $\Sigma_k \subset \Sigma$ as set of all function symbols that are $\vec{m} \to k$-sorted for some $\vec{m} \in \mathbb{N}^n$. Next, we define

$$\mathbf{X} := (X_1, \ldots, X_n), \ \ \mathbf{X}^\alpha := X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n}, \ \ \alpha = (\alpha_1, \ldots, \alpha_n);$$

$\overrightarrow{\Sigma_\alpha} := (\Sigma_{\alpha,1}, \ldots, \Sigma_{\alpha,n}) \in \text{Set}^n$, $\Sigma_{\alpha,n} \subset \Sigma$ is the set of all $\alpha \to n$-sorted function symbols;

$$\overrightarrow{\Sigma_\alpha} \times \mathbf{X}^\alpha := \left(\Sigma_{\alpha,1} \times \mathbf{X}^\alpha, \ldots, \Sigma_{\alpha,n} \times \mathbf{X}^\alpha\right).$$

$$(\Sigma_{\alpha,1}, \ldots, \Sigma_{\alpha,n}) \neq \Sigma_{\alpha,1} \times \cdots \times \Sigma_{\alpha,n}$$

Now we classify Σ as a multivariable polynomial endofunctor on Set^n

Definition 2.25. We define the **multivariable polynomial functor** $\vec{P}_\Sigma : \text{Set}^n \to \text{Set}^n$ on Σ as follows:

$$\begin{aligned}\vec{P}_\Sigma(X_1, \ldots, X_n) &:= (P_{\Sigma_1}(X_1, \ldots, X_n), \ldots, P_{\Sigma_n}(X_1, \ldots, X_n)) \\ &= \sum_{\alpha \in \mathbb{N}^n} \overrightarrow{\Sigma_\alpha} \times \mathbf{X}^\alpha\end{aligned}$$

We shall call $P_{\Sigma_1} : \text{Set}^n \to \text{Set}$ the **component (multivariable) polynomial functor**.

%include limit and colimit preservation somewhere

$1 \leftarrow \vec{P}_{\Sigma} 1 \leftarrow \cdots$

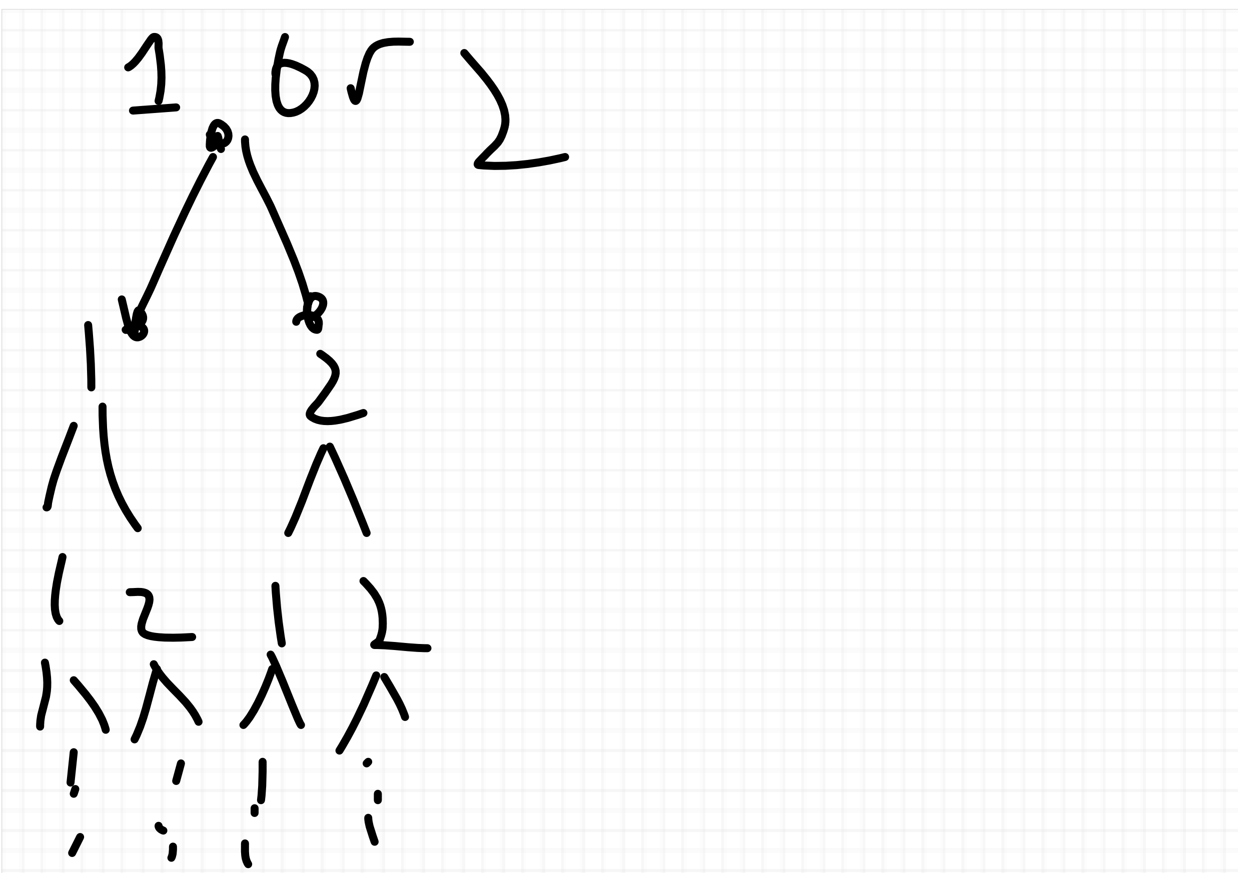
$1 \rightarrow \vec{P}_{\Sigma} 1 \rightarrow \cdots$

Example 2.26. We shall start with the two-variable analogue of stream calculus and let F be the functor

$$F\colon (X, Y) \mapsto \vec{\Sigma} XY,$$

for some $\vec{\Sigma} := (\Sigma_1, \Sigma_2)$. Note that $F = P_{\Sigma}$, for $\Sigma = \Sigma_1 + \Sigma_2$. In the particular case that $\vec{\Sigma} = (\mathbb{R}, \mathbb{R})$, we get the iteration

$$1 \leftarrow (\mathbb{R}, \mathbb{R})1^2 \leftarrow (\mathbb{R}, \mathbb{R})\mathbb{R}^2 = \left(\mathbb{R}^3, \mathbb{R}^3\right) \leftarrow (\mathbb{R}, \mathbb{R})\left(\mathbb{R}^3\right)^2 = \left(\mathbb{R}^7, \mathbb{R}^7\right) \leftarrow \cdots$$



%draw picture

I conjecture that the final F-coalgebra is a 2-D vector $\vec{T} := (T_1, T_2)$ of binary trees (with T_i being the Σ-trees with the top coloring with sort $(1,1) \rightarrow i$), with map

$$\langle \text{head}, \text{tail} \rangle : \vec{T} \rightarrow (\mathbb{R}, \mathbb{R})(T_1 \times T_2) = (\mathbb{R}(T_1 \times T_2), \mathbb{R}(T_1 \times T_2)) \\ = (\mathbb{R}, \mathbb{R}) \times (T_1 \times T_2, T_1 \times T_2)$$

$\text{head} = \left(\text{head}_1, \text{head}_2\right) : \vec{T} \rightarrow (\mathbb{R}, \mathbb{R})$
$(i = 1, 2)$ $\text{head}_i : T_i \rightarrow \mathbb{R}$, $\text{head}_i(t_i) = \text{stem of } t_i$

$\text{tail} = \left(\text{tail}_1, \text{tail}_2\right) : \vec{T} \rightarrow (T_1 \times T_2, T_1 \times T_2)$,
$(i = 1, 2)$ $\text{tail}_i : T_i \rightarrow T_1 \times T_2$, $\text{tail}_i(t_i) = \text{branches of } t_i = (t_{i,1}, t_{i,2})$

Conjecture 2.27. *(coinduction on Σ)* The final Σ-coalgebra $\left(\overrightarrow{T_\Sigma}, \tau_\Sigma\right)$ is the set of infinite Σ-tree vectors with $\overrightarrow{T_\Sigma} := (T_{\Sigma,1}, \ldots, T_{\Sigma,n})$ where $T_{\Sigma,i}$ is the set of infinite Σ-trees with the top vertex coloring in Σ_i and the map $\tau_\Sigma := \langle \text{head}_\Sigma, \text{tail}_\Sigma \rangle$ of inverse tree tupling with

$\text{head}_\Sigma = \left(\text{head}_{\Sigma,1}, \ldots, \text{head}_{\Sigma,n}\right) : (T_{\Sigma,1}, \ldots, T_{\Sigma,n}) \rightarrow (\Sigma_1, \ldots, \Sigma_n)$
$(i = 1, \ldots, n)$ $\text{head}_{\Sigma,i}(t_i) = \text{top vertex coloring } (\in \Sigma_i)$

$\text{tail}_\Sigma = \left(\text{tail}_{\Sigma,1}, \ldots, \text{tail}_{\Sigma,n}\right)$
$(i = 1, \ldots, n)$ $\text{tail}_{\Sigma,i}(t_i) =$ the $\alpha \in \mathbb{N}^n$ tuple of Σ-trees consisting of all the branches of $\text{head}_\Sigma(t_i)$,
where $\alpha := \text{arity}\left(\text{head}_\Sigma(t_i)\right)$, i.e., the α such that t_i is sort $\alpha \rightarrow t_i$

%talk about finitary functors of multivariable polynomials

%do module and vector space example, then category example, 2-category example, then finite product/coproduct/monoidal category, cartesian closed category

3. Applications to Equational Theories

3.1 Equational Σ-Model Categories as a Split Subcategory of a P_Σ-Algebra Category

As usual, let Σ be an equational signature (with possibly many sorts).

Proposition 3.1.

1. The category $\text{Mod}_\Sigma(T)$ of models of a theory T is a subcategory of Alg_{P_Σ} with the canonical extension functor $J_T : \text{Mod}_\Sigma(T) \hookrightarrow \text{Alg}_{P_\Sigma}$.

2. If T is equational then J_T splits and there exists quotient map $Q_T : \text{Alg}_{P_\Sigma} \to \text{Mod}_\Sigma(T)$.

Note: Analogously to quotient coalgebras, we have the concept of quotient algebra $\alpha : FA \to A$, it is a structure $(\overline{A}, \overline{\alpha})$ and an epimorphism $e : A \twoheadrightarrow \overline{A}$ such that it is also an F-epimorphism $(A, \alpha) \twoheadrightarrow (\overline{A}, \overline{\alpha})$.

Proof.

1. is immediate via the functor J_T sending $(A, \alpha) \in \text{Mod}_\Sigma(T)$ to itself and T-homomorphisms $f : (A, \alpha) \to (B, \beta)$ to itself.

2. We define Q_T as follows. Given (A, α), we send $A \mapsto \overline{A}$, where

$\overline{A} = A \text{ mod } T$,

where A mod T is the smallest relation such that $\forall \vec{a}, \vec{b}, \vec{c} \in A(t(\vec{a}, \vec{b}) \sim s(\vec{a}, \vec{c}))$ for all $t(\vec{x}, \vec{y}) = s(\vec{x}, \vec{z}) \in T$, and we send $\alpha \mapsto \overline{\alpha}$, i.e. we have $\overline{\alpha}([\vec{a}]) = [\alpha(\vec{\alpha})]$

$\forall(x, y)(t(x, y) = s(x, y))$

%show that Σ-model categories $\text{Mod}_\Sigma(T)$ are subcategories of Alg_{P_Σ} and Σ-model category $\text{Mod}_\Sigma(T)$ of an equational theory T is split.

Example 3.2. We find for $\Sigma_{\text{grp}} := \left\{1, (-)^{-1}, *\right\}$ and

$$T_{\text{groups}} := \left\{(x*y)*z = x*(y*z),\ x*1 = x,\ 1*x = x,\ x*x^{-1} = 1\right\},$$

we find that $\text{Mod}_{\Sigma_{\text{grp}}}(T_{\text{groups}}) \cong \text{Grp}$, i.e. is the category of groups and are $P_{\Sigma_{\text{grp}}}$-algebras.

We find $Q_{T_{\text{groups}}} : \left(A, *, (-)^{-1}, 1\right) \mapsto \left(\overline{A}, *, (-)^{-1}, 1\right)$

$a, b, c \in A \mapsto [a], [b], [c] \in \overline{A} \bmod T$

$(a*b)*c \sim a*(b*c)$
$a*1 \sim 1*b$
$a*a^{-1} \sim 1$

For

$$T_{\text{ab}} := T_{\text{groups}} \cup \{x*y = y*x\},$$

we find $\text{Mod}_{\Sigma_{\text{grp}}}(T_{\text{ab}}) \cong \text{Ab}$, i.e. is the category of abelian groups and are $P_{\Sigma_{\text{grp}}}$-algebras.

Now we shall give some many-sorted examples

Example 3.3.

$\Sigma_{\text{Rings}} := \{+, \cdot, -(-), 0, 1\}$

$P_{\Sigma_{\text{Rings}}} := 2X^2 + X + 2$

Let's look at $T_{\text{Fields}} := T_{\text{Rings}} \cup \left\{\forall x\left(x \neq 0 \rightarrow \exists y(xy = 1)\right)\right\}$

$\Sigma_{Fields} := \left\{+, \cdot, -(-), (-)^{-1}, 0, 1, \uparrow\right\}$

$T_{\text{Rings}} \cup \{$

$\forall x\big(x \neq 0 \to xx^{-1} = 1\big)$,
$\forall x\big(x =\uparrow \leftrightarrow (0)^{-1} = x\big)$
$\forall x(x + \ \uparrow = \uparrow)$
$\forall x(x \cdot \ \uparrow = \uparrow)$
$(-(\uparrow) = \uparrow)$
$(\uparrow$

Example 3.4. We find the theory of modules is a two-sorted theory on

$\Sigma_{\text{Mod}} := \{0_X, +_X, -_Y(-), 0_Y, 1_Y, +_Y, -_Y(-), \cdot_Y, \times\}$,
%change to 1, 2 sort
i.e.

$$P_{\Sigma_{\text{Mod}}}(X, Y) := \begin{bmatrix}1\\0\end{bmatrix}XY + \begin{bmatrix}0\\2\end{bmatrix}Y^2 + \begin{bmatrix}1\\0\end{bmatrix}X^2 + \begin{bmatrix}1\\0\end{bmatrix}X + \begin{bmatrix}0\\1\end{bmatrix}Y + \begin{bmatrix}1\\2\end{bmatrix}.$$

We find that for

$$\begin{aligned}
T_{\text{Ab}}(X) &:= \{(x +_X y) +_X z = x +_X (y +_X z),\ x +_X 0_X = x,\ 0_X +_X x = x,\ x -_Y x = 0_X\} \\
T_{\text{Ring}}(Y) &:= T_{\text{Ab}}(Y) \cup \{x \cdot_Y (y \cdot_Y z) = (x \cdot_Y y) \cdot_Y z,\ x \cdot_Y 1_Y = x, 1_Y \cdot_Y x = x, x \cdot_Y y = y \cdot_Y x, \\
&\qquad x \cdot_Y (y +_Y z) = x \cdot_Y y +_Y x \cdot_Y z, (x +_Y y) \cdot_Y z = x \cdot_Y z +_Y y \cdot_Y z\} \\
T_{\text{Mod}}(X, Y) &:= T_{\text{Ab}}(X) \cup T_{\text{Ring}}(Y) \cup \{(x \cdot_Y y) \times z = x \times (y \times z),\ x \times (y +_X z) = x \times y +_X x \times z\},
\end{aligned}$$

we find $\text{Mod}_{\Sigma_{\text{Mod}}}(T_{\text{Mod}}(X, Y))$ of models of $T_{\text{Mod}}(X, Y)$ is NOT isomorphic to the category Mod_R of R-models since the ring R is fixed whereas *any* module of *any* ring could satisfy $T_{\text{Mod}}(X, Y)$.

Example 3.4. We find that category can be described as a two-sorted theory with the following signatures

$\Sigma_{\text{Cat}} := \{1_{(-)}, s, t, \circ, \uparrow_1, \uparrow_2\}$,

where $1_{(-)}$ is a $(0, 1) \to 2$ sorted symbol, s, t are $(0, 1) \to 1$ sorted symbols, and $\circ$ is a $(0, 2) \to 2$ sorted symbol. We find that

$$P_{\Sigma_{\text{Cat}}}(X, Y) = \begin{bmatrix}0\\1\end{bmatrix}Y^2 + \begin{bmatrix}2\\0\end{bmatrix}Y + \begin{bmatrix}0\\1\end{bmatrix}X + \begin{bmatrix}1\\1\end{bmatrix}$$

T_{Cat} is the theory with the following axioms:

$\forall f(s(f) =\uparrow_1 \leftrightarrow f =\uparrow_2)$
$\forall f(t(f) =\uparrow_1 \leftrightarrow f =\uparrow_2)$
$\forall f \forall g(s(f) \neq t(g) \leftrightarrow f \circ g =\uparrow_2)$
$\quad \forall f(f \circ 1_{s(f)} = f)$
$\quad \forall f(1_{t(f)} \circ f = f)$
$\forall f \forall g \forall h((f \circ g) \circ h = f \circ (g \circ h)),$

Theorem. $T_{\text{Cat}} \vdash \forall x(s(1_x) = x \wedge t(1_x) = x)$

$T_{\text{Cat}} \vdash \forall f(f \circ 1_{s(f)} = f)$
$T_{\text{Cat}} \vdash 1_x \circ 1_{s(1_x)} = 1_x$
$T_{\text{Cat}} \vdash \forall x(1_x \circ 1_{s(1_x)} = 1_x)$
$T_{\text{Cat}} \vdash (1_{\uparrow_1} \circ 1_{s(\uparrow_1)} = 1_{\uparrow_1})$
$T_{\text{Cat}} \vdash 1_{\uparrow_1} =\uparrow$

$f \circ g \; s(f) \neq t(g)$

$f \circ$

Better possible axiomatization:
$1_{\uparrow_1} =\uparrow_2$
$\forall f(s(f) =\uparrow_1 \leftrightarrow f =\uparrow_2)$
$\forall f(t(f) =\uparrow_1 \leftrightarrow f =\uparrow_2)$
$\forall f \forall g\big(s(f) \neq t(g) \rightarrow f \circ g =\uparrow_2\big)$
$\forall f \forall g\big(s(f) = t(g) \rightarrow s(f \circ g) = s(g)\big)$
$\forall f \forall g\big(s(f) = t(g) \rightarrow t(f \circ g) = t(f)\big)$
$\quad \forall x(s(1_x) = x)$
$\quad \forall x(t(1_x) = x)$
$\quad \forall f(f \circ 1_{s(f)} = f)$
$\quad \forall f(1_{t(f)} \circ f = f)$
$\forall f \forall g \forall h((f \circ g) \circ h = f \circ (g \circ h)),$

and hence $\text{Mod}_{\Sigma_{\text{Cat}}}(T_{\text{Cat}}) \cong \text{Cat}$. We find similarly that 2-categories can be axiomatized with a 3-sorted signature. We additionally find that we can axiomatize finite cartesian categories in the following way:

$\Sigma_{\text{CCat}} := \Sigma_{\text{Cat}} \cup \{\times, \pi_{(-,-),1}, \pi_{(-,-),2}, \langle -, - \rangle\}$,

with T_{CCat} containing the axioms of T_{Cat} along with

$\forall x \forall y (s(\pi_{(x,y),1}) = x \times y \wedge t(\pi_{(x,y),1}) = x)$
$\forall x \forall y (s(\pi_{(x,y),2}) = x \times y \wedge t(\pi_{(x,y),2}) = y)$
$\forall f \forall g (\langle f, g \rangle = \uparrow_2 \leftrightarrow s(f) \neq s(g))$
%categories and 2-categories

Next, we talk about finitary functors and their relation to models of a theory

Given a presentation $\epsilon : P_\Sigma \twoheadrightarrow F$ of finitary functor F we can consider the category Alg F of all F-algebras as a full subcategory of Alg P_Σ, with the embedding assigning every F-algebra (A, α) to an P_Σ-algebra $(A, \alpha \circ \epsilon_A)$ and defined by the homomorphisms $f \mapsto f$.

Proposition 3.5. Given a presentation $\epsilon : P_\Sigma \twoheadrightarrow F$, the initial algebra $\mu. F$ is the largest quotient of the initial Σ-algebra lying in Alg F.

More concretely, $\mu. F = \text{term}_\Sigma / \sim$, where for two Σ-terms s, t we have

$s \sim t \iff s = t$ can be derived using the standard rules of equational logic for a theory T such that $F = P_\Sigma / T$.

Moreover, if $\mathcal{E}$ is a set of basic equations such that $F = P_\Sigma / \mathcal{E}$, then $\text{Alg}_F \cong \text{Mod}_\Sigma(\mathcal{E})$.

Theorem 3.6. If T is an equational theory, then $\text{Mod}_\Sigma(T)$ contains an initial object, which specifically is the set term_Σ / T of Σ-terms mod T.

Proof.
%prove this

3.2 Free Models of an Equational Theory T

Now, we want to talk about the result that there always exist free models of an equational theory T. Note that such an equational theory T induces the finitary functor $F_T = P_\Sigma / T$, which makes finitary functors once again a useful concept.

Now we are ready to talk about free models of an equational theory. First, recall the following equivalent characterization of a free algebra.

Proposition 3.2. In a category $\mathbb{C}$ with coproducts and endofunctor F, there exists a free algebra generated by $A \in \mathbb{C}$ iff the functor $F + A$ has an initial algebra $\mu. F + A$ and moreover $\mu. F + A$ is the free F-algebra generated by A.

Now we are ready to prove our important theorem. Let T be an equational theory T of a signature Σ. Set $F_T := P_\Sigma \,/\, T$.

Theorem 3.3. Given a set A, there exists a free equational model $\mathcal{M}(A)$ of a theory T (i.e., $\mathcal{M}(A) \vDash T$) generated by A, i.e., there exists a free F_T-algebra generated by $\mathcal{M}(A)$. More specifically, $\mathcal{M}(A)$ consists of every $t \in \text{term}_\Sigma(A)$ mod T.

The proof of the theorem follows pretty straightforwardly from the following observation:

Lemma 3.4. $F_T + A$ is a finitary functor.

3.3 Free Categories