Notes on Coalgebras on Measurable Spaces and Coalgebraic Modal Logic

Contents

1. Finitary Functor Preliminaries

1.1 Filtered Categories and the General Definition of Finitary Functors

Defintiion 1.1.1. A category $\mathcal{D}$ is **filtered** if every finite diagram of $\mathcal{D}$ has a cocone. A **filtered colimit** is a colimit of a diagram $D : \mathcal{D} \to \mathcal{A}$, where $\mathcal{D}$ is filtered. A **directed colimit** is a colimit of a diagram having a directed poset as its diagram scheme. A functor $F : \mathcal{A} \to \mathcal{B}$, is called **finitary** if F preserves all filtered colimits.

%define equivalent wikipedia definition of filtered category
%prove that they are equivalent
%provide examples of filtered categories

%possibly change lemma into a theorem
Lemma 1.1.2. A functor is finitary if and only if it preserves directed colimits.

Proof.

%FOR THIS IMPLICATION, ACTUALLY USE OTHER DEFINITION OF FILTERED CATEGORY
$\Longrightarrow$ Suppose $F : \mathcal{A} \to \mathcal{B}$ is finitary. Let $(I, \leq)$ be a directed poset and $D : (I, \leq) \to \mathcal{A}$ be a diagram. It suffices to prove that $(I, \leq)$ is filtered. Given a finite diagram $E : \mathrm{C}_0 \to (I, \leq)$, we want to show that there exists $c \in I$ that forms a cocone on E. We shall prove this result by induction on $n := |\mathrm{Mor}(\mathrm{C}_0)|$. If $n = 0$, we have the empty category and the cone exists vacuously. For the inductive step, we set $n + 1 := |\mathrm{Mor}(\mathrm{C}_0)|$, and we choose a C_0-morphism $f : X \to Y$ to omit, and have two cases.
%actually talk about a finite category in terms of a graph and a quotient of that graph generating it

Case 1. Suppose $f : X \to Y$ is the identity morphism id_X of an object with no other morphisms $X \to Y$ or $Y \to X$, between it for any $Y \neq X$. Let $\mathrm{C}_0{}'$ be the subcategory of C_0 with the object X (and all arrows associated to it) eliminated, and note by the inductive hypothesis that the subdiagram $E|\mathrm{C}_0{}'$ has cocone
%use the max argument

Case 2. Suppose $f : X \to Y$ is some nonidentity morphism (and note that since we do not have the scenario in Case, we can always choose such a morphism).

%use directed property

$\Longleftarrow$ Suppose $F : \mathcal{A} \to \mathcal{B}$ preserves directed colimits. Let $D : \mathfrak{D} \to \mathcal{A}$ be a diagram with $\mathfrak{D}$ filtered.
%PROVE THIS RESULT
%choose a colimit of a directed diagram that is a subdiagram of D and show that it is
%POSSIBLY USE THE ADJOINT FUNCTOR THOEREM

1.2 Finitary Functors on $\mathcal{S}et$ and Examples

%define fintary set functors in terms of boundedness and then in terms of filtered colimits

%possibly give examples of such functors in Set

Definition 1.2.1. A set functor F is **finitely bounded**, if for each element $x \in FX$, there exists a finite subset $M \subset X$ such that $x \in Fi[FM]$, where $i : M \hookrightarrow X$ is the inclusion map i.e.

$$FX = \bigcup_{i:M \hookrightarrow X, M \text{ finite}} Fi[FM].$$

%possibly give examples of such functors in Set

Definition 1.2.2. We define a **presentation** of a set functor F as an equational signature Σ and a natural epi-transformation $\epsilon : P_\Sigma \twoheadrightarrow F$. If Σ is finitary (i.e., consisting of only n-ary function symbols), we call the presentation **finitary**. We call F **finitely presentable** if such a finitary presentation exists.

%find sections of papers where this is defined
%possibly give examples of such functors in Set

We now give some examples of finitary functors on Set:

%example involving $\mathcal{P}_f$

Example 1.2.3. *(Coinduction on $\mathcal{P}_f$)*

Final Coalgebra $\nu.\,\mathcal{P}_f$:

Infinite trees (up to $\approx$):

$\Sigma := \{\emptyset, [-]_1, \ldots, [-]_k, \ldots\}$
= signature of single k-ary function symbol for every k

$t, t' \in T_\Sigma$

$t \approx t' \iff \forall n(\partial_n t = \partial_n t') \iff \forall n(\ell_n(t) \approx {}_n \ell_n(t))$

$\partial_n t :=$ quotient when you apply $\mathcal{P}_f$ n-times

The final coalgebra is $(\nu. H_\Sigma / \approx, \tau')$

$\tau' : \nu. H_\Sigma / \approx \ \to \mathcal{P}_f(\nu. H_\Sigma / \approx)$

$[t] \mapsto \{[t_1], \ldots, [t_n]\}$
$(t_1, \ldots, t_n) = \mathrm{tail}_\Sigma(t)$

Well-defined since $\mathcal{P}_f(\nu. H_\Sigma / \approx) = H_\Sigma(\nu. H_\Sigma / \approx) / \approx_1$

$t \approx t' \implies \{t_1, \ldots, t_n\} = \{t_1{}', \ldots, t_m{}'\}$

(follow from the fact that $\ell_n(t) \approx {}_n \ell_n(t')$)

Question: What are extensional trees?

Answer:

extensional: All subtrees are pairwise not isomorphic

strongly extensional: All iterates of subtrees are not pairwise isomorphic

Refer to index

%example involving $\mathfrak{D} + m$

Example 1.2.4. *(Coinduction on $\mathfrak{D} + m$)*

Next, we want to use our understanding of the finite power set functor $\mathcal{P}_f$ to make sense fo the finite probability measure functor $\mathfrak{D}$, and its variants $\mathfrak{D} + m$:

Consider the polynomial functor $H_\Sigma(X) := \mathbb{R}X + 1,\ \Sigma := \{r(-) : r \in \mathbb{R}\} \cup \{*\}$

Then $\nu.\, H_\Sigma$ is T_Σ, or $\mathbb{R}^{\leq\omega}$, and we can define thge infinite sums partial function

Sum : $T_\Sigma \to \mathbb{R}$ (equivalently, a whole function $T_\Sigma \to \mathbb{R} + 1$)

$$\text{Sum}(\sigma) = \begin{cases} 0 & \text{if } \sigma = * \\ \sigma(0) + \text{Sum}(\sigma') & \text{if } \lim\limits_{n \to \infty} \sum\limits_{k=0}^{n} \sigma(k) \text{ exists} \\ \uparrow & \text{otherwise} \end{cases}$$

$* \mapsto 0$

$\sigma(0) + \text{Sum}(\sigma')$

The work comes from showing that

Example 1.2.5. $\sigma(0) = 1/6,\ \sigma' = 1/6\sigma$

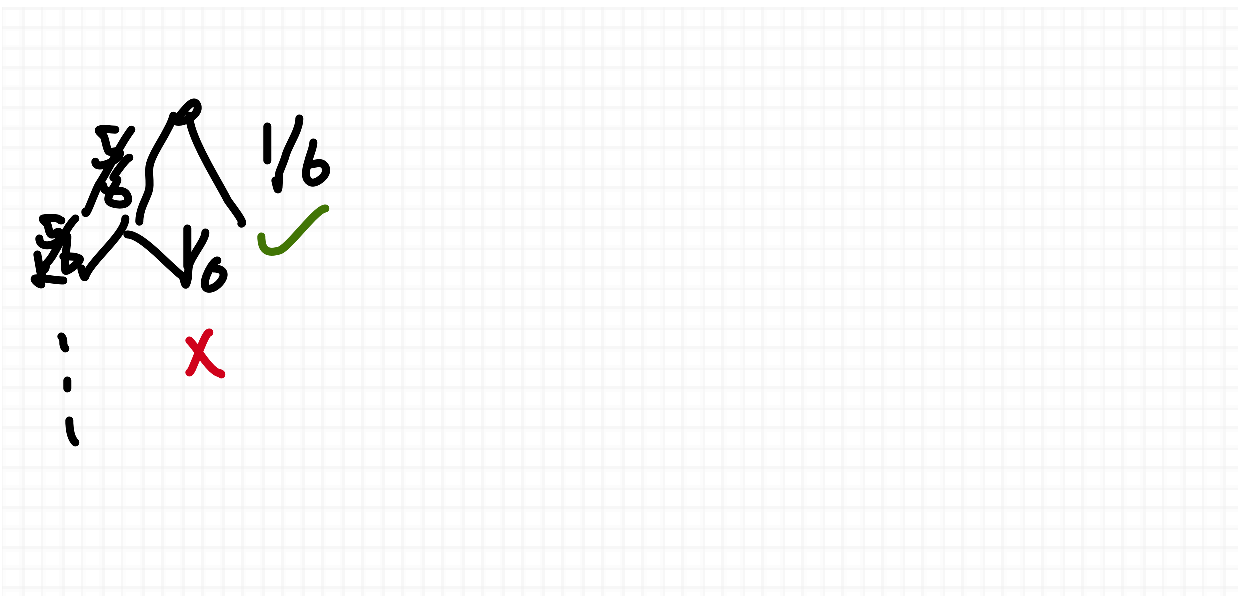
$\sigma = \left(1/6, 1/6^2,\ \dots\right)$

Sum$(\sigma) = 1/6 + $ Sum$(\sigma') = 1/6 + 1/6 \cdot$ Sum(σ)

Sum$(\sigma) = 1/5 = 1/6/(1 - 1/6)$

%Check out Corecursive algebras (ch. 7)

Example 1.1.8. Simlar problem where we want to find the probability of the event E, where the first 1 of the dice is an odd number (after rolling the dice infinitely many times)



Pr$[E] = 1/6 + 5/6^2$Pr$[E]$
Pr$[E] = 6/11$

Q: How do we talk about this in general for Markov Chains?

A: We can do so using the probability functor $\mathcal{D} + 2$ (for the above example), or $\mathcal{D} + m$ for m distinct "end" states.

$$\mathcal{D} \cong \mathcal{P}_f((-) \times (0, +\infty)) / \text{normalization}$$
$$\cong P_\Sigma((-) \times (0, +\infty)) / \simeq$$

$x := ((x_1, a_1), \ldots, (x_k, a_k))$

$y :=$
$\in P_\Sigma(X \times (0, \infty))$

$x \simeq y \iff \mu(x, \cdot\,) = \mu(y, \cdot\,)$

$\mu(x, \cdot\,) : X \to [0, 1]$

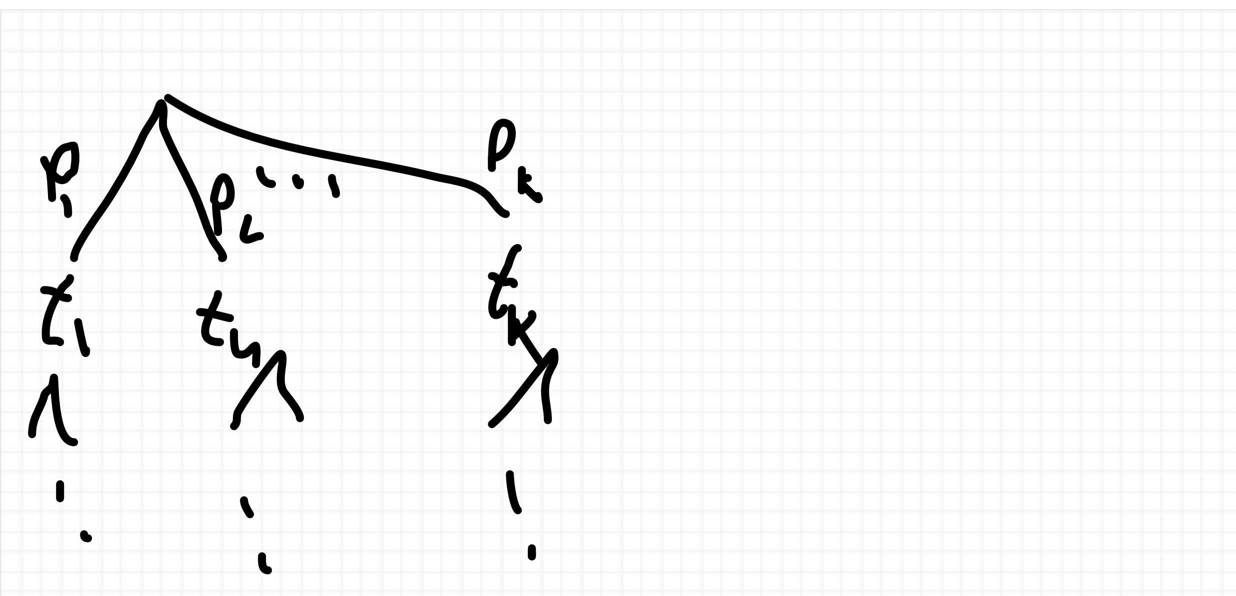
$$\mu(x, q) = \begin{cases} \sum_{x_s = q} a_s \,/\, (a_1 + \ \cdots\ + a_k) & \text{if } q = x_s \ 1 \le s \le k \\ 0 & \text{otherwise} \end{cases}$$

Using Theorem 4.3.26, we find $\nu.\,(\mathfrak{D} + m)$ is $(T_{\Sigma+m}, \tau')$ with

$t \approx t' \iff \partial_n(t) = \partial_n(t')$

with $\tau' : T_{\Sigma+m} / \approx \ \to \mathfrak{D} + m(T_{\Sigma+m} / \approx)$ defined to be inverse tree tupling up to $\approx$

$[t] \mapsto \mu(t, \cdot\,) := \mu\big(\text{tail}(t), \cdot\,\big)$



$\mu(t, t_s) = p_s$

%rephrase this

Next Time: This allows us to define corecursively LongRun $: \nu.\,(\mathfrak{D} + m) \to \mathfrak{D}(\nu.\,(\mathfrak{D} + m))$

%check out this paper
Hella Hansen+Larry Moss paper

Previous Talk Correction:

%figure out how to reorganize this correction on
Consider the polynomial functor $H_\Sigma(X) := \mathbb{R}X + 1,\ \Sigma := \{r(-) : r \in \mathbb{R}\} \cup \{*\}$

Then $\nu.\,H_\Sigma$ is T_Σ, or $\mathbb{R}^{\leq\omega}$, and we can define thge infinite sums partial function

Sum $: T_\Sigma \to \mathbb{R}$ (equivalently, a whole function $T_\Sigma \to \mathbb{R} + 1$)

$$\text{Sum}(\sigma) = \begin{cases} 0 & \text{if } \sigma = * \\ \sigma(0) + \text{Sum}(\sigma') & \text{if } \lim\limits_{n \to \infty} \sum\limits_{k=0}^{n} \sigma(k) \text{ exists} \\ \uparrow & \text{otherwise} \end{cases}$$

$T_\Sigma := \mathbb{R}^{\leq\omega}$

$\mathbb{R} \subset \mathbb{R}^{\leq\omega}$

%talk about super-finitary functors; On Fintary Functors 3.27-3.34 (page 21-24)

Theorem 1.2.6. Given a Set-functor F, the following are equivalent:

1. F is finitary.
2. F is finitely bounded.
3. F is finitely presentable.

1.3 Bisimulations and Corecursion on Set-Polynomial Functors

To start, here's a theorem about such polynomials worth mentioning:

Theorem 1.3.1. Polynomial functors (defined in this more general context) preserve ω-colimits and ω^{op}-limits.

%prove this

Corollary 1.3.2. There exists initial Σ-algebras (or P_Σ-algebras) and final Σ-coalgebras (or P_Σ-coalgebras).

Given a functor F, we shall denote $\mu. F$ as the initial F-algebra and $\nu. F$ as the final coalgebra.

Now that we know a final Σ-coalgebra exists, we can derive bisimulations, a more explicit characterization of the final coalgebra, coinduction, and corecursion.

Remark. Given an equational signature Σ, note that a Σ-coalgebra (S, α) we find $\alpha : S \to P_\Sigma S$ can be represented as $\alpha = \sum_{i \in \mathbb{N}} \alpha_i$ of partial functions

$$\alpha_i = \langle o_i, tr_i \rangle : S \to \Sigma_i \times S^i,\ o_i : S \to \Sigma_i,\ tr_i : S \to S^i.$$

Proposition 1.3.3. *(bisimulations)* Given $s, t \in S$, we find that

$$\begin{aligned} s \sim t &\iff o_i(s) = o_i(t) \text{ and } tr_i(s) \sim tr_i(t) \left(\text{on } S^i\right), \\ &\iff o_i(s) = o_i(t) \text{ and } \forall k \le i(tr_{i,k}(s) \sim tr_{i,k}(t)) \text{ where } tr_{i,k} := \pi_k \circ tr_i. \end{aligned}$$

%prove this

Theorem 1.3.4. *(coinduction on Σ)* The final Σ-coalgebra (T_Σ, τ_Σ) is the set of infinite Σ-trees and the map $\tau_\Sigma := \langle \text{head}_\Sigma, \text{tail}_\Sigma \rangle$ of inverse tree tupling with

$$\begin{aligned} &\text{head}_\Sigma(t) = \text{top vertex coloring } (\in \Sigma) \\ &\text{tail}_\Sigma(t) = \text{the } i\text{-tuple of } \Sigma\text{-trees consisting of all the branches of head}_\Sigma(t), \\ &\qquad\qquad \text{where } i := \text{arity}\left(\text{head}_\Sigma(t)\right). \end{aligned}$$

We find given $t = t'$, we have

$$t = t' \iff \text{head}_\Sigma(t) = \text{head}_\Sigma(t') \text{ and } \text{tail}_\Sigma(t) \sim \text{tail}_\Sigma(t') \left(\text{in } T_\Sigma^i\right).$$

Theorem 1.3.5. *(corecursion on n-ary T_Σ operations)* Given a family $f_{i,0} : T_\Sigma^n \to \Sigma_i,\ f_{i,1} : T_\Sigma^n \to \left(T_\Sigma^n\right)^i$ of partial functions indexed by $i \in \mathbb{N}$ s.t.

$$\alpha := \sum_{i \in \mathbb{N}} \langle f_{i,0}, f_{i,1} \rangle : T_\Sigma^n \to P_\Sigma T_\Sigma^n,$$

forms a Σ-coalgebra, there is a uniquely determined operation $g : T_\Sigma^n \to T_\Sigma$ defined by

$\text{head}_\Sigma(g(\vec{t})) = f_{i,0}(\vec{t}),$
$\text{tail}_\Sigma(g(\vec{t})) = g(f_{i,0}(\vec{t})),$

for some $i \in \mathbb{N}$.

1.3 Bisimulations and Corecursion on $\mathfrak{Set}$-Finitary Functors

%REORGANIZE THIS SECTION

Bisimulations of Finitary Functors:

Given a finitary set functor F, note that F preserves weak pullbacks
%doesn't preserve weak pullbacks

Examples.

$\varnothing \mapsto \varnothing$

$A \mapsto *$

$\epsilon : P_\Sigma \twoheadrightarrow F$

Let's assume that F preserves weak pullbacks. Let $s \sim t$ denote the relation on (S, α) (defined to be a P_Σ-coalgebra (note that $\left(S, \alpha^\epsilon\right)$ is an F-coalgebra) if there exists a bisimulation on S. We know that

Let T_Σ be the infinite Σ-trees

$$
\begin{tikzcd}
T_\Sigma \arrow[r, "\tau_\Sigma^\epsilon"] \arrow[d, "\approx"'] & FT_\Sigma \arrow[d, "F \approx"] \\
T_\Sigma / \approx \arrow[r, "\overline{\tau_\Sigma^\epsilon}"'] & FT_\Sigma / \approx
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(S, \alpha^\epsilon\right) \arrow[r, "{[[-]]^{P_\Sigma}}"] \arrow[dr, "{[[-]]^F}"'] & \left(T_\Sigma, \tau_\Sigma^\epsilon\right) \arrow[d, "\approx"] \\
& \left(T_\Sigma / \approx, \overline{\tau_\Sigma^\epsilon}\right)
\end{tikzcd}
$$

We know that $[[-]]^{P_\Sigma}$ is a P_Σ-homomorphism (but using naturality of ϵ, we can show that $[[-]]^{P_\Sigma}$ is also a F-homomorphism)

For $\alpha : S \to P_\Sigma S$, define $\alpha^\epsilon := \epsilon_S \alpha : S \to FS$

τ_Σ is inverse tree tupling of T_Σ

$\tau_\Sigma^\epsilon = \epsilon_{T_\Sigma} \tau_\Sigma$

Define $[[-]]^F : \left(S, \alpha^\epsilon\right) \to (\nu. F, \tau_F) \cong \left(T_\Sigma / \approx, \overline{\tau_\Sigma^\epsilon}\right)$

$$
\begin{aligned}
s \sim t &\iff [[s]]^F = [[t]]^F \\
&\iff \approx \circ [[s]]^{P_\Sigma} = \approx \circ [[t]]^{P_\Sigma} \\
&\iff \overline{\tau_\Sigma^\epsilon} \approx [[s]]^{P_\Sigma} = \overline{\tau_\Sigma^\epsilon} \approx [[t]]^{P_\Sigma}
\end{aligned}
$$

$$\iff F \approx \tau_\Sigma^\epsilon[[s]]^{P_\Sigma} = F \approx \tau_\Sigma^\epsilon[[t]]^{P_\Sigma}$$
$$\iff F \approx \epsilon_{T_\Sigma}\tau_\Sigma[[s]]^{P_\Sigma} = F \approx \epsilon_{T_\Sigma}\tau_\Sigma[[t]]^{P_\Sigma}$$
$$\iff F \approx \epsilon_{T_\Sigma}F[[-]]^{P_\Sigma}\alpha(s) = F \approx \epsilon_{T_\Sigma}F[[-]]^{P_\Sigma}\alpha(t)$$

Long Run Probability:

LongRun : $\nu.(\mathfrak{D} + m) \to (\mathfrak{D} + m)(\nu.(\mathfrak{D} + m))$

$\mathfrak{D}(\nu.(\mathfrak{D} + m)) \subset (\mathfrak{D} + m)(\nu.(\mathfrak{D} + m) \cong \nu.(\mathfrak{D} + m)$

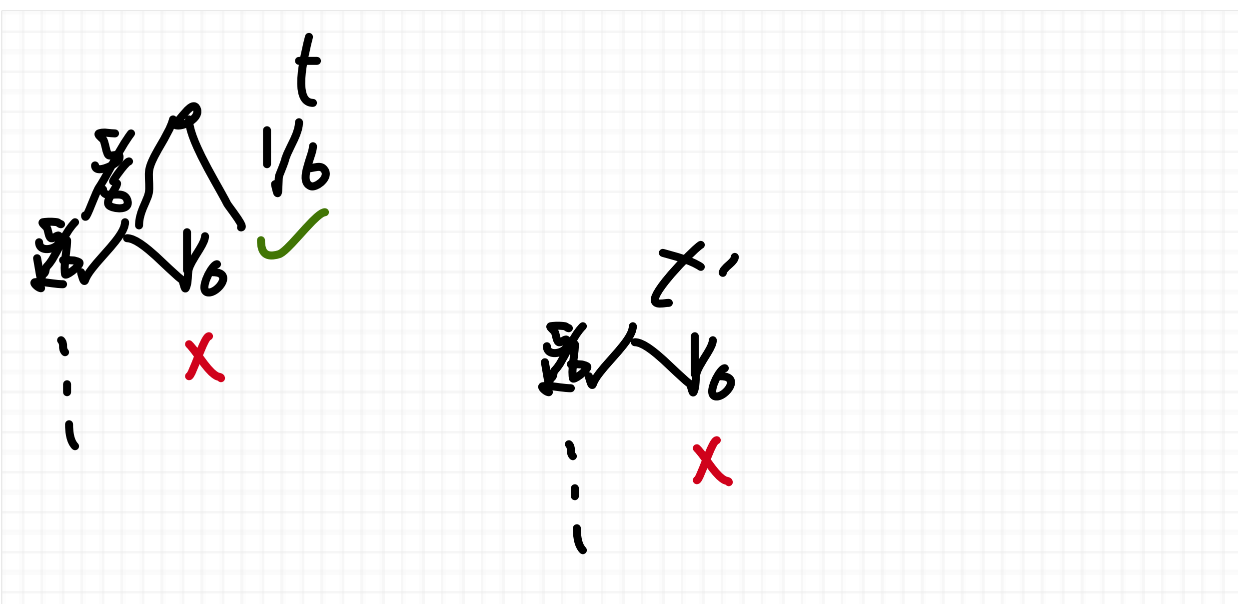
$\nu.(\mathfrak{D} + m) = T_{\Sigma+m} /\approx$

$[t] \in T_{\Sigma+m} /\approx$, $t \in T_{\Sigma+m}$

$$\text{LongRun}([t]) = \begin{cases} \delta_k & \text{if } t = k \ (1 \le k \le m) \\ \sum_{t' \in \text{tail}(t)} \mu(t,t') \cdot \text{LongRun}([t']) & \text{else} \end{cases}$$

NOTE: It is clear how one would define a function by corecursion on $T_{\Sigma+m}$ (and this LongRun function utilizes the same case structure for $T_{\Sigma+m}$)

Example 1.4.1. Let t be the following tree



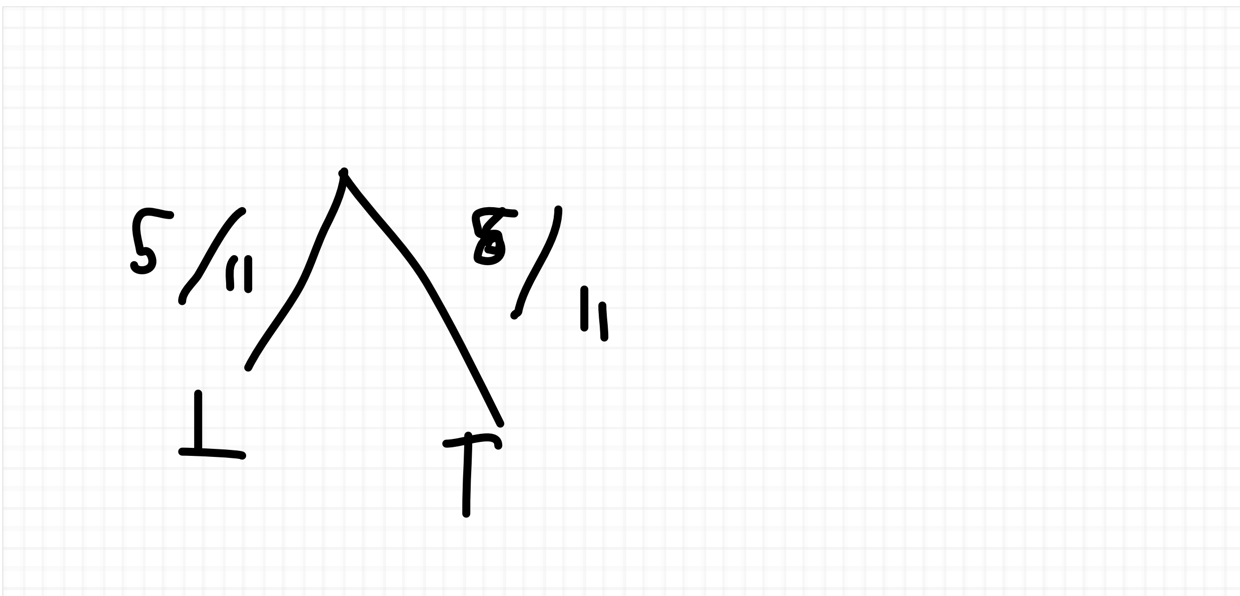
Note that in this example, we can set $m = 2$ (since there are two end states in this example), and let $\perp$ be failure and $\top$ be success. Using this definition of LongRun, we get that

$$\begin{aligned}\text{LongRun}([t]) &= 5/6 \cdot \text{LongRun}([t']) + 1/6 \cdot \text{LongRun}(\top) \\ &= 5/6\text{LongRun}([t']) + 1/6 \cdot \delta_{\top}\end{aligned}$$

$$\begin{aligned}\text{LongRun}([t']) &= 5/6 \cdot \text{LongRun}([t]) + 1/6 \cdot \text{LongRun}(\perp) \\ &= 5/6 \cdot \text{LongRun}([t]) + 1/6 \cdot \delta_{\perp}\end{aligned}$$

Claim:

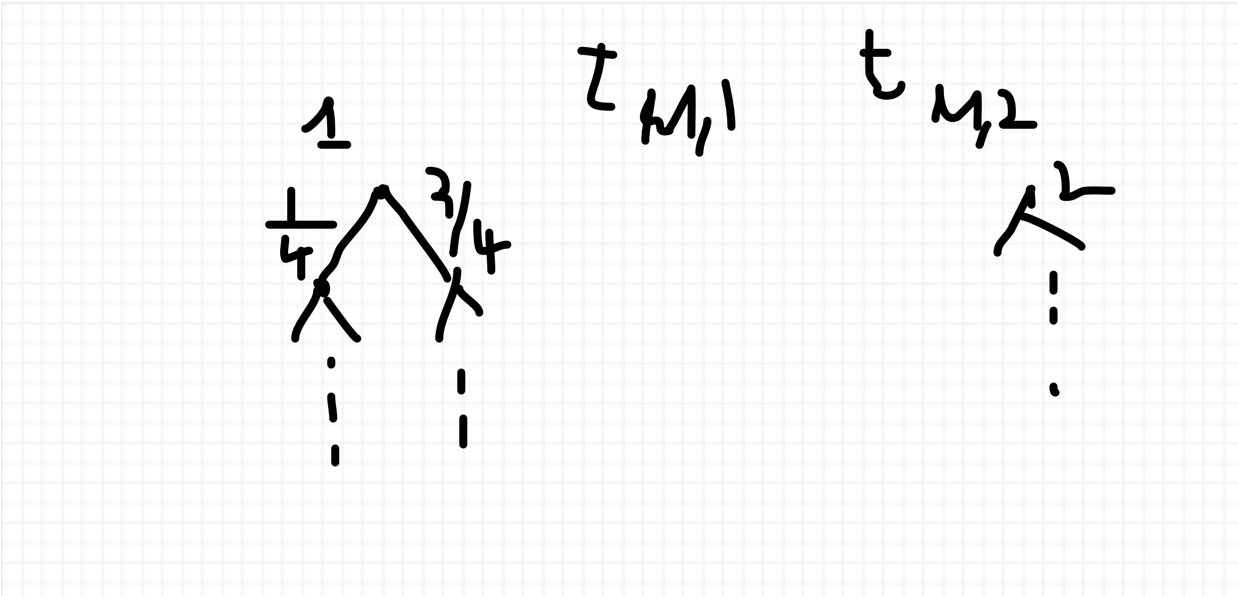
We can determine that using linear systems if we regard LongRun($[t]$) as a probability vector.



Example 1.4.2. Let's consider the markov chain

$$M = \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix}$$

In $T_{\Sigma+m}/\cong$, we can talk about M as the following two probability trees $t_{M,1},\ t_{M,2}$



To find LongRun($[t_{M,1}]$) $=$ LongRun($[t_{M,2}]$), we solve for the linear system we get by applying the coalgebraic definition, which is

LongRun($[t_{M,1}]$) $= 1/4$LongRun($[t_{M,1}]$) $+ 3/4$LongRun($[t_{M,2}]$)

LongRun($[t_{M,2}]$) $= 2/3$LongRun($[t_{M,1}]$) $+ 1/3$LongRun($[t_{M,2}]$)

The solution is the probability vector that solves the matrix system

$$Mw = w$$
$$(M - I)w = 0$$

The question that I next want to answer is how we go about defining operations on $T_{\Sigma+m}/\cong$ corecursively. The key to this puzzle is finding out more about the bisimulations on $(\mathfrak{D} + m)$-coalgebras, or more generally, any finitary functor F.

%WRITE IDEA OF CORECURSION AND BISIMULATIONS HERE

Corecursion on Finitary Functors

Let F be a finitary functor. Let $\epsilon : P_\Sigma \twoheadrightarrow F$ be the finitary presentation and for any P_Σ-coalgebra (S, α), let $[[-]]^{P_\Sigma} : (S, \alpha) \to (\nu.\, P_\Sigma, \tau_{P_\Sigma})$ be the canonical P_Σ-coalgebra homomorphism, and $[[-]]^F : \left(S, \alpha^\epsilon\right) \to (\nu.F, \tau_F)$ be the canonical F-coalgebra homomorphism (where $\alpha^\epsilon := \epsilon_S \alpha : S \to FS$ is an F-coalgebra structure map).

It can be shown using naturality of ϵ that the function $[[-]]^{P_\Sigma} : S \to \nu.P_\Sigma(\cong T_\Sigma)$ is an F-coalgebra homomorphism $\left(S, \alpha^\epsilon\right) \to \left(\nu.P_\Sigma, \tau^\epsilon_{P_\Sigma}\right)(\cong (T_\Sigma, \tau_\Sigma))$. Moreover, we have

$$(\nu.F, \tau_F) \cong \left(T_\Sigma / \approx, \overline{\tau^\epsilon_\Sigma}\right),$$

where $\approx$ is the relation

$x \approx y \iff \partial_n x = \partial_n y$ for all n.
%explain the relation $\partial_n x = \partial_n y$

and $\overline{\tau^\epsilon_\Sigma} : T_\Sigma / \approx \; \to F(T_\Sigma / \approx)$ is the canonical coalgebra structure that makes $\approx$ into an F-coalgebra homomorphism

$$\begin{tikzcd}
T_\Sigma \arrow[r, "\tau^\epsilon_\Sigma"] \arrow[d, "\approx"'] & FT_\Sigma \arrow[d, "F\approx"] \\
T_\Sigma / \approx \arrow[r, "\overline{\tau^\epsilon_\Sigma}"'] & FT_\Sigma / \approx
\end{tikzcd}$$

We find by finality that since $\approx \circ [[-]]^{P_\Sigma} : \left(S, \alpha^\epsilon\right) \to \left(T_\Sigma / \approx, \overline{\tau^\epsilon_\Sigma}\right)$ is an F-coalgebra homomorphism, we have $[[-]]^F = \approx \circ [[-]]^{P_\Sigma}$, i.e. the commutative diagram

$$\begin{tikzcd}
\left(S, \alpha^{\epsilon}\right) \arrow[r, "{[[-]]^{P_\Sigma}}"] \arrow[dr, "{[[-]]^{F}}"'] & \left(T_\Sigma, \tau_\Sigma^{\epsilon}\right) \arrow[d, "\approx"] \\
& \left(T_\Sigma / \approx, \overline{\tau_\Sigma^{\epsilon}}\right)
\end{tikzcd}$$

This gives us the following theorem:

Notation: Given an P_Σ coalgebra (S, α), let

$s \sim_{P_\Sigma} t \iff \exists$a bisimulation between them in (S, α)
$s \sim_F t \iff \exists$a bisimulation between them in $\left(S, \alpha^{\epsilon}\right)$

Theorem 1.4.3. Suppose a finitary functor F preserves weak-pullbacks, and (S, α) is a P_Σ-coalgebra (and hence $\left(S, \alpha^{\epsilon}\right)$ is an F-coalgebra). Then for $s, t \in S$.

$$s \sim_{P_\Sigma} t \Longrightarrow s \sim_F t$$

Proof. F preserving weak pullbacks guarentees the largest bisimulation $s \sim_F t \iff [[s]]^F = [[t]]^F$ exists on S. It follows that

$$\begin{aligned}
s \sim_{P_\Sigma} t &\Longrightarrow [[s]]^{P_\Sigma} = [[t]]^{P_\Sigma} \\
&\Longrightarrow [[s]]^F =\approx [[s]]^{P_\Sigma} =\approx [[s]]^{P_\Sigma} = [[t]]^F \\
&\Longrightarrow s \sim_F t.
\end{aligned}$$ □

Corollary 1.4.4. *(A Principle of Coinduction on Finitary Functors)* Suppose a finitary functor F preserves weak-pullbacks. Then for $[s], [t] \in T_\Sigma / \approx$, we have

$$s \sim_{P_\Sigma} t \Longrightarrow [s] = [t].$$

This means that when F has the property of preserving weak pullbacks, we can use bisimulations on P_Σ to get bisimulations on F.

2. Categorical Measure Theory Preliminaries

2.? Some Concepts

Definition 1. A **polish space** is a topological space that is metrizable with a complete metric, and which has a countable dense subset. (see page 13, Doberkat)

Definition 2. A **separable measurable space** $(M, \mathcal{M})$ has a countable set $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ of generators which separate points, i.e. given $x \neq x' \in M$, $\exists A_n$ containing exactly one of them. (see page 20, Doberkat)

%define analytic space

Definition 3. The initial σ-algebra $\mathcal{M}^* \subset \Delta(M, \mathcal{M})$ that makes all evaulation maps $\mu \mapsto \mu(E)$ for $E \in \mathcal{M}$ measurable is called the **weak*-σ-algebra**, (see page 32, Doberkat) where

$$\Delta(M, \mathcal{M}) := \left\{\mu : \mathcal{M} \to [0,1] : \mu \text{ is a subprobability measure}\right\}.$$

We shall further denote $\mathcal{S}(M, \mathcal{M})$ as the σ-algebra $\left(\Delta(M, \mathcal{M}), \mathcal{M}^*\right)$.

Definition 4. A **stochastic relation** $K : (M, \mathcal{M}) \rightsquigarrow (N, \mathcal{N})$ between the measurable spaces $(M, \mathcal{M})$ and $(N, \mathcal{N})$ is an $\mathcal{M} - \mathcal{N}^*$-measurable map. $K : (M, \mathcal{M}) \to \mathcal{S}(N, \mathcal{N})$. (page 33, Doberkat)

$K : (M, \mathcal{M}) \rightsquigarrow (N, \mathcal{N})$ will be denoted $\mathcal{K} = (M, N, K)$.

A few things worth noting:

- The function $\mathcal{S} :\ (M, \mathcal{M}) \mapsto \left(\Delta(M, \mathcal{M}), \mathcal{M}^*\right)$ is an endofunctor on Meas if we define $f : (X, \mathcal{A}) \to (Y, \mathcal{B}) \mapsto \mathcal{S}(f)$, where $\mathcal{S}(f)(\mu)$, for $\mu \in \mathcal{S}(X, \mathcal{A})$ is the pushfoward measure

$$\mathcal{S}(f)(\mu)(B) := \mu\left(f^{-1}[B]\right),\ B \in \mathcal{B}.$$ (page 34, Doberkat)

- Given stochastic relations $\mathcal{K} = (X, Y, K)$, $\mathcal{L} = (A, B, L)$, we define a stochastic morphism $F := (f, g) : \mathcal{K} \to \mathcal{L}$ as follows:

$f : X \twoheadrightarrow A, g : Y \twoheadrightarrow B$ measurable epimorphisms such that

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "K"'] & A \arrow[d, "L"] \\
\mathcal{S}(Y) \arrow[r, "\mathcal{S}(g)"'] & \mathcal{S}(B),
\end{tikzcd}
$$

commutes.

Definition 5. If X is an analytic space with ρ an equivalence relation on X, we say ρ is smooth if $\exists$sequence $(A_n)_{n \in \mathbb{N}}$ of Borel sets such that

$x \, \rho \, x' \iff \forall(n \in \mathbb{N})(x \in A_n \iff x' \in A_n)$.

The sequence $(A_n)_{n \in \mathbb{N}}$ is said to determine ρ.

Corollary 6. ρ is smooth $\implies \rho \subset X \times X$ is Borel.

%prove that later

Definition 7. We define $\mathcal{B}(X)/\rho$ to be the largest σ-algebra $\mathcal{C}$ rendering the natural projection $\eta_\rho : X \to X/\rho$ a $\mathcal{B}(X) - \mathcal{C}$-measurable map.

NOTE: $\mathcal{B}(X/\rho)$ coincides with $\mathcal{B}(X)/\rho$. Moreover X is analytic $\implies X/\rho$ is analytic.

%show what this means and show that definition 7 is a well-defined concept

Defintion 8. Let ρ be a smooth equivalence relation.

a. $A \subset X$ is called ***ρ*-invariant** iff $x \in A$ and $x \, \rho \, x' \implies x' \in A$.

b. Denote by $\Sigma(\mathcal{B}(X), \rho)$ the σ-algebra of ρ-invariant Borel subsets of X.

(page 47, Doberkat)

%show that $\Sigma(\mathcal{B}(X), \rho)$ is a σ-algebra on X.

Definition 9. Given a stochastic relation $\mathcal{K} = (X, Y, K)$ with analytic spaces X and Y, a pair $\mathfrak{c} = (\alpha, \beta)$ of smooth equivalence relations and α and β on X and Y, respectively, is a **congruence** if $K(x)(B) = K(x')(B)$, whenever $x \; \alpha \; x'$ and $B \in \Sigma(\mathcal{B}(Y), B)$ is invariant. (page 53-54, Doberkat)

Proposition 9. TFAE. (page 54, Doberkat)

a. $\mathfrak{c}$ is a congruence for $\mathcal{K}$

b. $\mathcal{K}^{|} : (X, \Sigma(\mathcal{B}(X), \alpha) \rightsquigarrow (Y, \Sigma(B(Y), \beta))$ is a stochastic relation, where $K^{|}(x)$ is the restriction of $K(x)$ to $\Sigma(\mathcal{B}(Y), \beta)$.

%check that both of these definitions are congruences in the conventional sense

Definition 10. The factor stochastic relation $\mathcal{K} / \mathfrak{c}$ is defined canonically through $\mathcal{K} / \mathfrak{c} := (X / \alpha, Y / \beta, K_{\mathfrak{c}})$, where

$$K_{\mathfrak{c}}([x]_\alpha)(B) := K(x)\left(\eta_\beta^{-1}[B]\right)$$
$$(= (\mathcal{S}(\eta_\beta) \circ K)(x)(B).$$

We find that $\eta_{\mathfrak{c}} := (\eta_\alpha, \eta_\beta)$ is an epimorphism $\mathcal{K} \twoheadrightarrow \mathcal{K} / \mathfrak{c}$ with kernel $\mathfrak{c}$.

2.? The Giry Monad

3. Bisimulations in a General Category

3.1 Spans and Cospans in a Category

3.2 Bisimulations as Spans

Bisimulations as Spans:

Definition. A span for a category $\mathbb{C}$ is a $\mathbb{C}$-diagram of the form $A \leftarrow C \rightarrow B$. A cospan is span in $\mathbb{C}^{op}$

Definition. Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be an endofunctor. Let (A, α) and (B, β) be F-coalgebras. A bisimulation (C,γ) is some coalgebra that forms a span $(A,\alpha) \leftarrow (C,\gamma) \rightarrow (B,\beta)$

NOTE: Aczel-Mendler's Definition.

It's worth asking how this definition of a bisimulation relates to the definition in Set.

NOTE: For other versions, see Sam Staton’s paper from about 2010

Proposition. Suppose that F is an endofunctor on Set. Let (A,α) and (B,β) be F-coalgebras. (A,α) and (B,β) $\iff \exists(R,\gamma)$ such that $R \subset A \times B$ and

$$(A,\alpha) \xleftarrow{\pi_1} (R,\gamma) \xrightarrow{\pi_2} (B,\beta)$$

form a span.

Outline of the proof.

$\Longleftarrow$ Trivial.

$\Longrightarrow$ Suppose they're bisimilar and choose the bisimulation. $(A,\alpha) \xleftarrow{f} (C,\gamma) \xrightarrow{g} (B,\beta)$. We can define

$$aRb \iff \exists c(a = f(c),\ b = g(c))$$

$\gamma'((a,b)) = F\phi \circ \gamma(c_{a,b})$, for some choice of c s.t. $f(c) = a,\ g(c) = b$

Next question: How do we relate the intuition of bisimulations in Set to the abstract definition in similar categories, like finitely complete ones.

Proposition. Suppose $\mathbb{C}$ is a finitely complete category. Then there exists a span that is the solution of the cospan $A \rightarrow C \leftarrow B$ that is a subobject of $A \times B$.

Outline of the Proof. Note that the pullback P of $A \rightarrow C \leftarrow B$ is a subobject of $A \times B$ (verifying this is done by a diagram chase on the (mono)morphism $u : P \hookrightarrow A \times B$.

Proposition. Let $\mathbb{C}$ be a finite complete category and let F be an endofunctor on $\mathbb{C}$ preserving weak pullbacks. If there is a terminal coalgebra (T,τ), then every pair of coalgebra $(A,\alpha),(B,\beta)$ is bisimimilar and has a largest bisimulation (with respect to the relation $u \leq v \iff v$ factors u) a subobject of $A \times B$.

Outline of the proof. Note that

$$(A,\alpha) \xrightarrow{[[-]]_A} (T,\tau) \xleftarrow{[[-]]_B} (B,\beta)$$

is a cospan in Coalg_F. Let L be the $\mathbb{C}$-pullback of $A \xrightarrow{[[-]]_A} T \xleftarrow{[[-]]_B} B$ and preservation of weak pullbacks induces a largest pullback (L,γ), which is the largest subobject of $A \times B$ that satisfies this diagram. □

Larsen & Skou's paper on probabilistic bisimulation
Rutten and de Vink (I think) have another early paper on this topic
Papers by Prakash Panangaden and his student Josee Desharnais

3.3 Some Important Examples

3.4 Finitary Functors in a General Category

Defintiion 3.1. A category $\mathfrak{D}$ is **filtered** if every finite diagram $\mathfrak{D}$ has a cocone. A **filtered colimit** is a colimit of a diagram $D : \mathfrak{D} \to \mathcal{A}$, where $\mathfrak{D}$ is filtered. A **directed colimit** is a colimit of a diagram having a directed poset as its diagram scheme. A functor $F : \mathcal{A} \to \mathcal{B}$, is called **finitary** if $\mathcal{A}$ has all its filtered colimits and F preserves all filtered colimits.

Lemma 3.2. A functor is finitary if and only if it preserves directed colimits.

%give examples of finitary functors in this sense (in the more general categorical setting)

Definition 3.3. An object $C \in \mathbb{C}$ is called **finitely presentable (fp)** if its hom-functor $\mathbb{C}(C,-)$ is finitary and **finitely generated (fg)** if $\mathbb{C}(C,-)$ preserves directed colimits of monos (i.e. all connecting morphisms in $\mathbb{C}$ are monic). Subobjects $m : M \hookrightarrow C$ of $C \in \mathbb{C}$ with M finitely generated are called **finitely generated subobjects**.

%give examples of this

Clearly, every fp object is fg, but not conversely in general.

Notation: For a category $\mathcal{A}$ we denote by $\mathcal{A}_{\mathsf{fp}}$ and $\mathcal{A}_{\mathsf{fg}}$ the full subcategories of $\mathcal{A}$ respresenting (up to isomorphism) all finitely presentable and finitely generated objects, respectively.

Definition 3.4. We call $\mathcal{A}$ a **locally finitely presentable (lfp)** category, if it is cocomplete, $\mathcal{A}_{\text{fp}}$ is small, and every object is a colimit of a filtered diagram in $\mathcal{A}_{\text{fp}}$.

Defintiion 3.5. for a functor F, if for every object X of $\mathcal{A}$, every finitely generated subobject of FX in $\mathcal{B}$ is factorized through the image by F of a finitely generated subobject of X in $\mathcal{A}$, then we call F **finitely bounded**.

$$\begin{tikzcd}
\forall Y \arrow[r, hook, "\text{f.g.}"'] \arrow[d, "\exists u"'] & FX & X \\
FZ \arrow[ur, hook] & \exists Z \arrow[ur, hook, "\text{f.g.}"'] &
\end{tikzcd}$$

%give examples of this, and show that this definition generalizes the one in Set
%define locally finitely generated functors and read up its uses; A New Foundation for Finitary Corecursion section 3 (page 11-17)
%talk about strict locally finite presentability and equivalent concepts; On Finitary Functors 3.9-3.16 (page 11-14)

Theorem 3.6. A functor between strictly lfp categories is finitary iff it is finitely bounded.

4. Coalgebraic Modal Logic on Set

4.1 Modal Logic and Kripke Semantics

Overview of Kripke Models:

Nondeterministic Kripke Models:

For a family $\mathcal{O}$ of n-ary modal operation symbols, Kripke models are usually defined as $\mathfrak{K} = (S, R_\tau, [[-]]_{\mathfrak{K}})$, where:

- S is a state space.

- $R_\tau := (R_\Delta)_{\Delta\in\mathcal{O}},\ R_\Delta : S \to \mathcal{P}\left(S^{\mathrm{ar}(\Delta)}\right)$.
- Some valuation map $[[-]]_{\mathcal{R}} : P \to \mathcal{P}(S)$.

Modal Logic on this system consists of variables and connectives

$\mathfrak{M}(\tau, P):\ P \mid \top \mid \neg \mid\ \wedge\ \mid\ \vee \mid \Delta \in \mathcal{O}$.

We can extend $[[-]]_{\mathcal{R}}$ to a function $\mathfrak{M}(\tau, P) \to \mathcal{P}(S)$ using recursion on the connectives, and define a satisfaction relation $\vDash$ defined for $\mathcal{R} = (S, R_\tau, [[-]]_{\mathcal{R}})$ as follows:

$\mathcal{R}, s \vDash p \iff s \in [[p]]_{\mathcal{R}}$,
$\mathcal{R}, s \vDash \top$ (atomic cases)

$\mathcal{R}, s \vDash \neg\phi \iff \mathcal{R}, s \nvDash \phi \iff s \in [[\neg\phi]]_{\mathcal{R}} := \mathcal{P}(P) \setminus [[\phi]]_{\mathcal{R}}$
$\mathcal{R}, s \vDash \phi_1 \wedge \phi_2 \iff \mathcal{R}, s \vDash \phi_1$ and $\mathcal{R}, s \vDash \phi_2 \iff s \in [[\phi_1 \wedge \phi_2]]_{\mathcal{R}} := [[\phi_1]]_{\mathcal{R}} \wedge [[\phi_2]]_{\mathcal{R}}$
$\mathcal{R}, s \vDash \Delta(\phi_1, \ldots, \phi_{\mathrm{ar}(\Delta)}) \iff \exists(s_1, \ldots, s_{\mathrm{ar}(\Delta)}) \in R_\Delta(s) : \mathcal{R}, s_i \vDash \phi_i$ for $1 \leq i \leq \mathrm{ar}(\Delta)$.

We then define

$\mathrm{Th}_{\mathcal{R}}(s) := \{\phi \in \mathfrak{M}(\tau, P) : \mathcal{R}, s \vDash \phi\}$

and **call it the theory of states s in $\mathcal{R}$.**

We now turn to the semantical consequence relation $\vDash$. In the context of modal logic, one distinguishes between two different such relations: If ϕ and ψ are modal formulas, one calls ψ a **global consequence** of ϕ, if

$\forall s \in S\ (\mathcal{R}, s \vDash \phi) \implies \forall s \in S(\mathcal{R}, s \vDash \psi)$

for all $\mathcal{R} \in \mathrm{Mod}(\mathcal{O}, P)$. The class of models, which globally satisfy ϕ is a subclass of the models of ψ.

For ϕ, ψ, we say that ψ is a **local consequence** of ϕ if

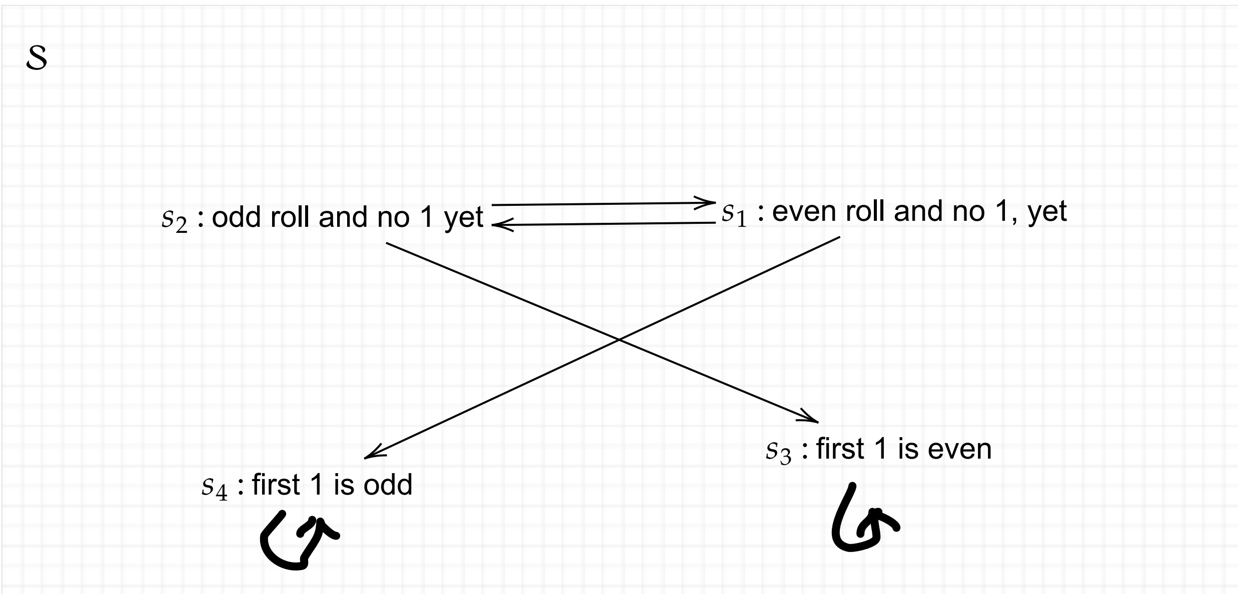
$\forall s \in S(\mathcal{R}, s \vDash \phi \implies \mathcal{R}, s \vDash \psi)$

for all $\mathcal{R} \in \mathrm{Mod}(\mathcal{O}, P)$. If ψ is a local consequence of ϕ, we write $\phi \vDash \psi$.

First we'll do standard Kripke Models

Example 1. This is a standard kripke model $\mathcal{S}$ that models rolling a dice until the first one comes up after sume number of rolls. We first set

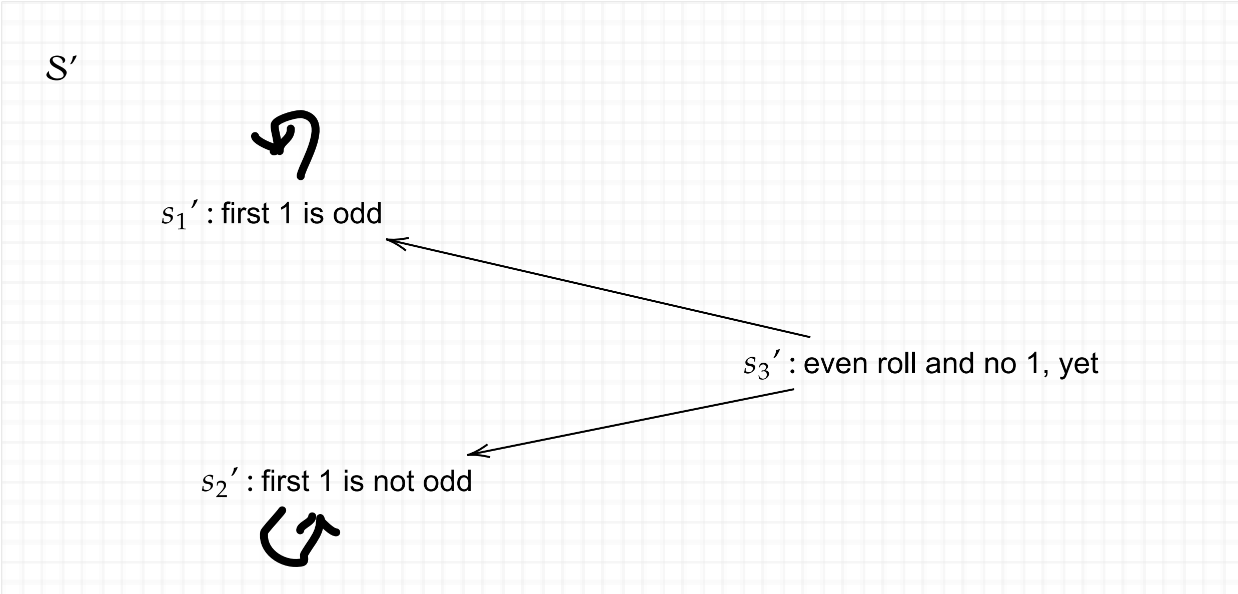
$P = \{T, U, F\}$, where T denotes "true", F denotes "false", and U denotes "undetermined".



$S := \{s_1, s_2, s_3\}$
$\alpha : s_1 \mapsto \{s_2, s_4\},\ s_2 \mapsto \{s_1, s_3\},\ s_3 \mapsto \{s_3\},\ s_4 \mapsto \{s_4\}$

$[[T]]_{\mathcal{S}} = \{s_1\},\ [[F]]_{\mathcal{S}} = \{s_3\},\ [[U]] = \{s_1, s_2\}$

Example 2. Let P still be $\{T, F, U\}$:



$S' := \{s_1', s_2', s_3'\}$
$\alpha' :\ s_1' \mapsto \{s_1'\},\ s_2' \mapsto \{s_2\},\ s_3' \mapsto \{s_1', s_2'\}$

Remarks:

1. Homomorphism $f : \mathcal{S} \to \mathcal{S}'$ mapping $s_1, s_2 \mapsto s_3',\ s_3 \mapsto s_2',\ s_4 \mapsto s_1'$

2. Add probabilities to them to make these kripke models stochastic

Example 3. For $P := \{p_{x>q} : q \in \mathbb{Q} \cap [0,1]\}$ let $\mathcal{Q} := (Q, \alpha)$ be defined by

$\alpha(r) := (r, 1] \cap \mathbb{Q}$

$p_{x>q}, r \vDash \mathcal{Q} \iff r > q.$

Example 4. For $P := \{p_{x>q} : q \in \mathbb{Q} \cap [0,1]\}$ let $\mathcal{R} := (R, \xi)$ be defined by

$\xi(r) := (r, 1]$

$p_{x>q}, r \vDash \mathcal{R} \iff r > q.$

Remarks:

1. Talks about all finite collections of open rays and left edpoint open intervals

2. the extension mapping $i : \mathcal{Q} \hookrightarrow \mathcal{R}$ is a kripke model homomorphism.

3. One can probably verify by structural induction, and cases on an endpoints vs. points in between that $\mathcal{Q}$ is logically equivalent to $\mathcal{R}$; however, it can also be shown that $\mathcal{Q}$ and $\mathcal{R}$ are neither behaviorally equivalent nor bisimilar.

%standard kripke model examples in regular modal logic with diamond modality, as well as examples where Hennessy-Milner fails
%examples of stochastic kripke models

4.2 Coalgebraic Interpretation of Kripke Models

Kripke Models Done Coalgebraically:

Nondeterministic Case:

Can be discussed as a coalgebra for the following $F \in \mathrm{End}\big(\mathrm{Set}\big)$:

$$F = \left[\prod_{\Delta \in \mathcal{O}} \mathcal{P}\big(-^{\mathrm{ar}(\Delta)}\big)\right] \times \mathcal{P}(P)$$

We find that an F-coalgebra (S, α) has the data

$$\alpha := ((\alpha_\Delta)_{\Delta \in \mathcal{O}}, \alpha_P) : S \to F(S),$$

and it can be shown that for the category $\mathrm{Mod}(\mathcal{O}, P)$ of objects as nonstochastic Kripke Models $\mathcal{R}$ and morphisms $h : (S, R_\tau, [[-]]_{\mathcal{R}}) \to (S', R'_\tau, [[-]]_{\mathcal{R}'})$ Kripke Model homomorphisms, i.e., functions $h : S \to S'$ such that

$$\forall \Delta \in \mathcal{O}(R'_\Delta(h(s)) = h[R_\Delta(s)]) \text{ and } \forall (p \in P)([[p]]_{\mathcal{R}'} = h[[[p]]_{\mathcal{R}}]),$$

we have

$$\mathrm{Mod}(\tau, P) \cong \mathrm{Coalg}_F \cong \left(\prod_{\Delta \in \mathcal{O}} \mathrm{Coalg}_{\mathcal{P} \circ (-)^{\mathrm{ar}(\Delta)}}\right) \times \mathrm{Coalg}_{\mathcal{P}(P)},$$

%NOTE: Last isomorphism claim is incorrect where the product $\times$ and Π to the right refers to the product category in Cat.

The one point of concern is $\alpha_P : S \to \mathcal{P}(P)$ defined by $\alpha_P(s) \subset P$, which on the surface looks different from $[[-]]_{\mathfrak{R}} : P \to \mathcal{P}(S)$, but we find that that in Set, we have

$$\mathrm{Hom}_{\mathrm{Set}}(P, \mathcal{P}(S)) \cong \mathrm{Hom}_{\mathrm{Set}}(P \times S, 2) \cong \mathrm{Hom}_{\mathrm{Set}}(S, \mathcal{P}(P))$$

giving us the one-to-one correspondence:

$$\alpha_P : S \to \mathcal{P}(P) \longleftrightarrow [[-]]_{\alpha,P} : P \to \mathcal{P}(S),\ s \in [[p]]_{\alpha,P} \Longleftrightarrow p \in \alpha_P(s).$$

Applying This to ◊-Modal Logic:

Specifically, ◊-modal logic has the alphabet

$$\mathfrak{M}(\tau, P):\ P \mid \top \mid \neg \mid \wedge \mid \vee \mid \Diamond$$

with $\Diamond\phi$ being the formula "it's possible that ϕ". From this, we can derive all the other connectives of modal logic, including the other modal operator $\Box$, from the listed connectives, as follows:

$\perp := \neg\top$
$\phi \subset \psi := \neg\phi \vee \psi,$
$\Box\phi := \neg\Diamond\neg\phi$
$\phi \to \psi := \Box(\phi \subset \psi)$

and kripke models of ◊-modal logic are the coalgebras (S, α) for $\alpha := (\alpha_\Diamond, \alpha_P) : S \to \mathcal{P}(S) \times \mathcal{P}(P)$.

4.3 Finitely Branching Hennessy-Milner Theorem and Consequences

Behavioral Equivalence, Strong Bisimulations, and Logical Equivalence:

Definition. A morphism $f : \mathfrak{R} \to \mathfrak{R}'$ between two Kripke models $\mathfrak{R} = (S, (R_\Delta)_{\Delta \in O}, [[-]]_{\mathfrak{R}}), \mathfrak{R}' = (S', (R'_\Delta)_{\Delta \in O}, [[-]]_{\mathfrak{R}'})$ in the category KrMod(O, P), i.e., a morphism $f : S \to S'$ in Set such that

1. $R'_\Delta \circ f = \mathcal{P}f \circ R_\Delta$, or $(R'_\Delta \circ f)(s) = f[R_\Delta(s)]$ for all $s \in S$,
2. $[[-]]_{\mathfrak{R}'} = \mathcal{P}f \circ [[-]]_{\mathfrak{R}}$, or $[[p]]_{\mathfrak{R}'} = f[[[p]]_{\mathfrak{R}}]$ for all $p \in P$,

is defined to be a **strong morphism** if f is surjective (or equivalently, an epimorphism in Set).

%note difference between "strong morphism" and epimorphism

Definition. Two Kripke models $\mathcal{R} = (S, (R_\Delta)_{\Delta \in \mathcal{O}}, [[-]]_\mathcal{R})$ and $\mathcal{R}' = (S', (R'_\Delta)_{\Delta \in \mathcal{O}}, [[-]]_{\mathcal{R}'})$ are:

1. **Logically equivalent** if

$$\left\{\mathrm{Th}_{\mathcal{R}}(s) : s \in S\right\} = \left\{\mathrm{Th}_{\mathcal{R}'}(s') : s' \in S'\right\},$$

i.e., if for every $s \in S$, there exists $s' \in S'$ such that $\mathrm{Th}_{\mathcal{R}}(s) = \mathrm{Th}_{\mathcal{R}'}(s')$.

2. **Bisimilar** if there exists a span $\mathcal{R} \xleftarrow{f} \mathcal{M} \xrightarrow{f'} \mathcal{R}'$ of strong morphisms $f,\ f'$ in the category $\mathrm{KrMod}(\mathcal{O}, P)$ that is nontrivial in the sense that there exists some m in the state space M of $\mathcal{M}$ such that $f(m) = s$ and $f'(m) = s'$ (i.e. the set $\mathcal{M}$ is not empty).

3. **Behaviorly Equivalent** if there exists a cospan $\mathcal{R} \xrightarrow{g} \mathcal{N} \xleftarrow{g'} \mathcal{R}'$ of strong morphisms g, g' in the category $\mathrm{KrMod}(\mathcal{O}, P)$.

Theorem. *(Hennessy-Milner)* let $\mathcal{R}$ and $\mathcal{R}'$ be finitely branching Kripke models. The following are equivalent.

1. $\mathcal{R}$ and $\mathcal{R}'$ are logically equivalent.
2. $\mathcal{R}$ and $\mathcal{R}'$ are strongly bisimilar.
3. $\mathcal{R}$ and $\mathcal{R}'$ are behaviorly equivalent.

%figure out how to generalize theorem and proof to various settings
%try to come up with generalization to concrete categories (and refer to notes for general categorical generalization)

Proving the Hennessy-Milner Theorem:

Theorem. *(Hennessy-Milner)* let $\mathcal{R}$ and $\mathcal{R}'$ be finitely branching Kripke models. The following are equivalent.

1. $\mathcal{S}$ and $\mathcal{S}'$ are logically equivalent.

2. $\mathcal{S}$ and $\mathcal{S}'$ are strongly bisimilar.
3. $\mathcal{S}$ and $\mathcal{S}'$ are behaviorly equivalent.

First, let's prove the easy stuff

Lemma. If $\Phi : \mathcal{S} \to \mathcal{S}'$ is a kripke model morphism, then $\mathrm{Thm}_{\mathcal{S}}(s) = \mathrm{Thm}_{\mathcal{S}'}(\Phi(s))$.

Corollary.

1. If s and s' are states in $\mathcal{S}$ and $\mathcal{S}'$ respectively such that $s \sim s'$, then $\mathrm{Thm}_{\mathcal{S}}(s) = \mathrm{Thm}_{\mathcal{S}'}(s')$.

2. If $\Phi : \mathcal{S} \to \mathcal{S}'$ is a strong morphism, then they are logically equivalent.

3. 1. $\Longrightarrow$ 2. and 1. $\Longrightarrow$ 3. of the Hennessy-Milner Theorem.

Now to prove finitely branching kripke models give us the converse of the Hennessy-Milner Theorem.

Theorem. $\mathcal{W} := (W, \delta)$ defined by

$$W := \left\{ \mathrm{Thm}_{(S,\alpha)}(s) : s \in S \text{ and } \alpha : S \to \mathcal{P}_f(S) \times \mathcal{P}(P) \text{ and } S \text{ is countable} \right\}$$

δ is a final $\mathcal{P}_f \times \mathcal{P}(P)$-coalgebra.

Theorem. The relation $\mathrm{Thm}_{\mathcal{S}}(s) = \mathrm{Thm}_{\mathcal{S}'}(s')$ for s and s' states in $\mathcal{P}_f \times \mathcal{P}(P)$-coalgebras $\mathcal{S} := (S, \alpha)$ and $\mathcal{T} := (\mathcal{T}, \tau)$, respectively defines a bisimulation on S.

Corollary. 2. $\Longrightarrow$ 1.

4.4 Potential Ways to Generalize the Hennessy-Milner Theorem

Generalizations of the Hennessy-Milner Theorem:

Hennessy-Milner Theorem does not generalize to arbitrary Kripke models (in Set, that is), not even countably branching Kripke models, in spite of the countable power-set functor $\mathcal{P}_\omega$ having an initial algebra.

The closest thing to a generalization of the Hennessy-Milner theorem (for countably branching Kripke models) is the following theorem:

Theorem. For the modification of modal logic with countable disjunctions (and therefore countable conjunctions using negation plus DeMorgan's Laws) and states $s, s' \in S$ for some countably branching Kripke model $\mathcal{K} := (S, \alpha)$ (i.e., $\alpha : S \to \mathcal{P}(S) \times \mathcal{P}_\omega(S)$), we have $\text{Thm}_{\mathcal{K}}(s) = \text{Thm}_{\mathcal{K}}(s') \iff s \sim s'$.

(and it's worth noting that proving this theorem is pretty similar to proving the finite case, and likely generalizes to modal logic with arbitrary κ-length disjunctions in a Kripke model that is up to κ-length branching)

A good question is: Where do we go from here? Well there's two possibilities that come to mind:

1. Try to see whether analogous Hennessy-Milner Theorems work for Functors on the Category of Measurable spaces, or other categories where it makes sense to talk about states and theories (i.e., any concrete category)

%define concrete category

2. Find other ways to talk about bisimulations on varous coalgebraic settings (whether they're directly or indirectly related to modal logic). For example, $\mathcal{P}_\omega$ (and more generally $\mathcal{P}_\kappa$, for any cardinal κ) preserves the limit for some ordinal chain (i.e., the sequence $\left(\mathcal{P}_\kappa^\alpha(1) | \alpha \in \text{ON}\right)$ has some fixed point), and also preserves weak pullbacks. Therefore, bisimulation conditions exist. They may not be the ones we find "sexy" (in the sense of a kind of some Hennessy-Milner-type construction), but one can still find rather simple bisimulation conditions that yield some kind of coinductive condition on a given coalgebraic structure. I plan to do this much for any finitary functor.

5. Stochastic Coalgebraic Modal Logic

5.1 Stochastic Kripke Semantics and Coalgebraic Interpretation

Stochastic Kripke Models:

For a family $\mathcal{O}$ of n-ary modal operation symbols, Kripke models are usually defined as $\mathcal{K} = ((S, \mathcal{A}), K_\tau, [[-]]_{\mathcal{K}})$, where:

- S is a state (measurable) space $(S, \mathcal{A})$.

- $K_\tau := (K_\Delta)_{\Delta\in\mathrm{O}}$, a family of stochastic relations $K_\Delta : S \rightsquigarrow S^{\mathsf{ar}(\Delta)}$
- Some valuation map $[[-]]_{\mathcal{K}} : P \to \mathcal{A}$.

Modal Logic on this system consists of variables and connectives

$\mathfrak{K}(\tau, P):\ P \mid \top \mid \neg \mid \wedge \mid \vee \mid \Delta_q,\ \Delta \in \mathrm{O},\ q \in \mathbb{Q} \cap [0,1]$

%correct syntax

Stochastic Case:

Can be discussed as a coalgebra for the following $G \in \mathsf{End}\big(\mathsf{Meas} \times \mathsf{Set}\big)$:

$$G = \left(\left[\prod_{\Delta\in\mathrm{O}} \mathfrak{S}\left(-^{\mathsf{ar}(\Delta)}\right)\right], \mathcal{P}(P)\right)$$

We find that an G-coalgebra (S, α) has the data

$\gamma := ((\gamma_\Delta)_{\Delta\in\mathrm{O}}, \gamma_P) : (S, \mathcal{A}) \to G(S, \mathcal{A})$.

NOTE: G has an additional Set input, but the Set input doesn't change the output, and the Set output is a set function $\gamma_P : S \to \mathcal{P}(P)$.

It can be shown that for the category $\mathsf{StMod}(\mathrm{O}, P)$ of objects as stochastic Kripke models $\mathcal{K}$ and morphisms $h : (S, K_\tau, [[-]]_{\mathcal{K}}) \to (S', K'_\tau, [[-]]_{\mathcal{K}'})$ stochastic Kripke model homomorphisms, i.e., functions $h : S \to S'$ such that

$\forall \Delta \in \mathrm{O}(K'_\Delta(h(s)) = h[K_\Delta(s)])$ and $\forall(p \in P)([[p]]_{\mathcal{K}'} = h[[[p]]_{\mathcal{K}}])$,

we have

$$\mathsf{StMod}(\tau, P) \cong \mathsf{Coalg}_G \cong \left(\prod_{\Delta\in\mathrm{O}} \mathsf{Coalg}_{\mathfrak{S}\circ(-)^{\mathsf{ar}(\Delta)}}\right) \times \mathsf{Coalg}_{\mathcal{P}(P)}.$$

%NOTE: Last isomorphism claim is incorrect

Note that $\mathsf{Coalg}_{\mathfrak{S}\circ(-)^{\mathsf{ar}(\Delta)}}$, for every $\Delta \in \mathrm{O}$ is a coalgebra of a Meas-endofunctor and $\mathsf{Coalg}_{\mathcal{P}(P)}$ is a coalgebra of a Set-endofunctor. It would be ideal if there was a way find an endofunctor

G that was completely in Meas, but the exponential trick that was used to make nondeterministic Kripke model into a Set-endofunctor is harder (and may not be able to be) employed for Stochastic Kripke models, since given a state S, we must be able to express any arbitrary Set-function $[[-]]_{\mathcal{K}} : P \to \mathcal{A}$, i.e. an element of $\mathrm{Hom}_{\mathrm{Meas}}((P, \mathcal{P}(P)), (\mathcal{A}, \mathcal{P}(\mathcal{A}))$ as some $(S, \mathcal{A})$-measurable function to some σ-algebra $(\mathcal{P}(P), [[\mathcal{A}]])$, such that

$$\mathrm{Hom}_{\mathrm{Meas}}((S, \mathcal{A}), (\mathcal{P}(P), [[\mathcal{A}]])) \cong \mathrm{Hom}_{\mathrm{Meas}}((P, \mathcal{P}(P)), (\mathcal{A}, \mathcal{P}(\mathcal{A})),$$

and it is not clear there exists a σ-algebra $[[\mathcal{A}]]$ that leads to this isomorphism. In fact, it holds that the category Meas does not contain all its exponential objects.

5.2 Hennessy-Milner Theorem for Stochastic Kripke Models

5.3 μCSL and Analogous Results

%come up with contents based on reading course