A Caratheodory Extension for Signed Measures

Contents

1. Introduction

2. The Original Caratheodory Extension
2.1 The Caratheodory Extension Defined Using Outer-Measures
2.2 Applying the Definitions to Premeasures
2.3 The Caratheodory Extension Defined Using σ-Finite Inner-Measures

3. Inner and Outer Single Dimension Signed Measures
3.1 Some Properties of the Hahn-Banach Decomposition
3.2 Single Dimension Signed Premeasures and the Inner Hahn-Banach Decomposition
3.3 The Caratheodory Extension for $\mathbb{R}$-valued Premeasures
3.4 The Caratheodory Extension for $\mathbb{C}$-valued Premeasures
3.5 Outer Single Dimension Signed Measures

4. The Caratheodory Extension for Vector-Valued Measures
4.1 Definitions and Properties of Vector-Valued Measures
4.2 Vector-Valued Premeasure Definition and Examples
4.3 The Caratheodory Extension in the Vector Valued Case

5. Conclusion

1 Introduction

%WRITE INTRODUCTION IN NEXT DRAFT

2 The Original Caratheodory Extension

2.1 The Caratheodory Extension Defined Using Outer-Measures

Definition 2.1.1. Given a nonempty set X, we define an **outer measure** $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ as a function that satisfies the following properties:

(i) (Null Empty Set) $\mu^*(\varnothing) = 0$.

(ii) (Countable Subadditivity) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ is a collection of disjoint subsets, then

$$\mu_*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n=1}^{\infty} \mu_*(E_n).$$

%Folland page 28 definition of outer-measure

Examples 2.1.2.
%Folland page 29 1.10 proposition

%TALK ABOUT INDUCED OUTTER MEASURE FROM MEASURE AND THE FACT THAT AS A RESULT, A MEASURE HAS COUNTABLE SUBADDITIVITY
(i)

%TALK ABOUT HOW THIS IS A MORE GENERAL CASE OF PART (I)
(ii) Given $\mathcal{E} \subset \mathcal{P}(X)$ with $\varnothing, X \in \mathcal{E}$, a function $\rho : \mathcal{E} \to [0, +\infty]$ such that $\rho(\varnothing) = 0$, we define the **induced outer measure** ρ^* on ρ to be the function $\mathcal{P}(X) \to [0, +\infty]$ defined by

$$\rho^*(F) := \inf\left\{\sum_{n=1}^{\infty} \rho(E_n) : \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{n\in\mathbb{N}} E_n \supset F\right\} \quad (2.1.1)$$

We find this is an outer measure since

$$\varnothing \in \mathcal{E} \implies E_n := \varnothing, \text{ for all } n \in \mathbb{N} \text{ gives us } \{E_n\}_{n\in\mathbb{N}} \supset \varnothing \text{ such that} \sum_{n=1}^{\infty} \rho(E_n) = 0,$$
$$\implies \rho^*(\varnothing) \leq \sum_{n=1}^{\infty} \rho(\varnothing) = 0,$$
$$\implies \rho^*(\varnothing) = 0.$$

and given $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\begin{aligned}\rho^*\left(\bigcup_{n\in\mathbb{N}} F_n\right) &= \inf\left\{\sum_{m=1}^{\infty} \rho(E_m) : \{E_m\}_{m\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{m\in\mathbb{N}} E_m \supset \bigcup_{n\in\mathbb{N}} F_n\right\} \\ &\leq \inf\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \rho(E_{n,m}) : \{E_{n,m}\}_{m\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{m\in\mathbb{N}} E_{n,m} \supset F_n \text{ for every } n \in \mathbb{N}\right\}\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \inf\left\{\sum_{m=1}^{\infty} \rho(E_{n,m}) : \{E_{n,m}\}_{m\in\mathbb{N}} \subset \mathcal{E}, \bigcup_{m\in\mathbb{N}} E_{n,m} \supset F_n\right\}$$
$$= \sum_{n=1}^{\infty} \rho^*(F_n),$$

hence we have countable subadditivity.

%FIND MORE EXAMPLES (INCLUDING LEBESGUE EXAMPLE)

Definition 2.1.3. *(The Caratheodory Criterion for Outer Measures)* Given an outer measure μ^* on X, we state that $F \subset X$ satisfies the **Caratheodory Criterion with respect to μ^*** (or equivalently state that F is **μ^*-measurable**) if for every $E \subset X$, we have

$\mu^*(E) = \mu^*(E \cap F) + \mu^*\left(E \cap F^c\right)$, for all $E \subset X$. $\quad (2.1.2)$
%Folland page 29 1.10 proposition
%ALSO STATE EQUIVALENT CONDITION TO THE CARTHEODORY CRITERION FOR OUTER MEASURES

%SHOW THAT THE CARTHEODORY CRITERION IS NONEMPTY, CLOSED UNDER COMPLEMENTS, FINITE UNIONS, AND SATISFIES THE DISJOINT FINITE ADDITIVITY PROPERTY

Theorem 2.1.4. *(The Original Caratheodory's Theorem)* Given an outer measure μ^* on X, and the collection $\mathcal{M}$ of all subsets of X satisfying the Cartheodory Criterion (i.e. condition $(2.2.2)$), we find μ^* is a σ-algebra, and $\mu : \mathcal{M} \to [0, +\infty]$ defined by $\mu := \mu^*|\mathcal{M}$ is a complete measure.
%Folland page 29 1.11 Theorem

%FINISH THIS PROOF
Proof. To begin, we show $\mathcal{M}$ is a σ-algebra. First, note that for every $E \subset X$, we have

$$\mu^*(E) = 0 + \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*\left(E \cap \varnothing^c\right),$$

hence $\varnothing \in \mathcal{M} \implies \mathcal{M} \neq \varnothing$. Next, observe that if $F \in \mathcal{M}$, then for all $E \subset X$, we have

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*\left(E \cap F^c\right) = \mu^*\left(E \cap F^c\right) + \mu^*\left(E \cap F^{cc}\right),$$

which shows $F^c \in \mathcal{M}$. Finally, suppose $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$. Then by countable subadditivity, we have

$$\mu^*(E) \leq \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right),$$

and to show that $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$, it remains to show that

$$\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) \leq \mu^*(E).$$

%DIVIDE IT INTO TWO CASES WITH INFINITY AND FINITE

Observe that for all $E \subset X$ and $N \geq 1$, we have

%MOVE THIS OVER TO THE FIRST ARGUMENT

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap F_1) + \mu^*\left(E \cap F_1^c\right) \\
&= \mu^*(E \cap F_1) + \mu^*\left(\left(E \cap F_1^c\right) \cap F_2\right) + \mu^*\left(\left(E \cap F_1^c\right) \cap F_2^c\right) \\
&= \mu^*((E \cap (F_1 \cup F_2)) \cap F_1) + \mu^*\left((E \cap (F_1 \cup F_2)) \cap F_1^c\right) + \mu^*\left(E \cap F_1^c \cap F_2^c\right) \\
&= \mu^*(E \cap (F_1 \cup F_2)) + \mu^*\left(E \cap F_1^c \cap F_2^c\right) \\
&= \mu^*(E \cap (F_1 \cup F_2)) + \mu^*\left(\left(E \cap F_1^c \cap F_2^c\right) \cap F_3\right) + \mu^*\left(\left(E \cap F_1^c \cap F_2^c\right) \cap F_3^c\right) \\
&= \mu^*\left(\left(E \cap \left(\bigcup_{n=1}^{3} F_n\right)\right) \cap \left(\bigcup_{n=1}^{2} F_n\right)\right) + \mu^*\left(\left(E \cap \bigcup_{n=1}^{3} F_n\right) \cap \left(\bigcup_{n=1}^{2} F_n\right)^c\right) + \mu^*\left(E \cap \bigcap_{n=1}^{3} F_n^c\right) \\
&= \mu^*\left(E \cap \left(\bigcup_{n=1}^{3} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{3} F_n^c\right)\right) \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
&= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{N} F_n^c\right)\right),
\end{aligned}$$

and moreover for all $N_0 \geq 1$ and for all $E \subset X$, we have

$$\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) = \mu^*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1\right) + \mu^*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1^c\right)$$

$$= \mu^*(E \cap F_1) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1^c\right]\right)$$

$$= \mu^*(E \cap F_1) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1^c\right] \cap F_2\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1\right] \cap F_2^c\right)$$

$$= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right)$$

$$= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right)$$

$$+\mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right] \cap F_3\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right] \cap F_3^c\right)$$

$$= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right)$$

$$+\mu^*\left(E \cap \left[F_3 \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{3} F_n\right)\right]\right)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$= \sum_{N=1}^{N_0-1}\left[\mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right]\right)$$

$$= \sum_{N=1}^{N_0-1}\left[\mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu^*\left(E \cap \left[F_{N_0} \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right]\right)$$

$$= \sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right),$$

hence by monotonicity we have

$$\sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) = \mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) \leq \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right).$$

It follows that we take $N_0 \to +\infty$ and find that for any $E \subset X$, we find by countable subadditivity that

$$\begin{aligned}
&\sum_{N=1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) \leq \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) \\
&\implies \lim_{N_0 \to +\infty} \sum_{N=N_0+1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) = 0. \\
&\implies \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) = \mu^*\left(\bigcup_{N=1}^{\infty}\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right) \\
&\qquad \leq \sum_{N=1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) \\
&\qquad = \lim_{N_0 \to +\infty}\left[\sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) + \sum_{N=N_0+1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] \\
&\qquad = \lim_{N_0 \to +\infty} \mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right),
\end{aligned}$$

and we conclude that

$$\begin{aligned}
\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) &\leq \lim_{N_0 \to +\infty}\left[\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right)\right)\right] \\
&= \lim_{N_0 \to +\infty}\left[\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{N_0} F_n^c\right)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N_0 \to +\infty} \left[\mu^* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu^* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) \right] \\
&= \lim_{N_0 \to +\infty} \left[\mu^*(E) \right] \\
&= \mu^*(E).
\end{aligned}
$$

Next, we want to show that $\mu := \mu^*|\mathcal{M}$ is a complete measure. First, observe that $\mu(\varnothing) = \mu^*(\varnothing) = 0$ by definition and for disjoint additivity, we find for disjoint $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$, we have

$$
\mu^* \left(\bigcup_{n=1}^{\infty} F_n \right) \leq \sum_{n=1}^{\infty} \mu^*(F_n),
$$

by countable subadditivity, and by finite disjoint subadditivity and monotonicity we have

$$
\sum_{n=1}^{N_0} \mu^*(F_n) = \mu^* \left(\bigcup_{n=1}^{N_0} F_n \right) \leq \mu^* \left(\bigcup_{n=1}^{\infty} F_n \right) \implies \sum_{n=1}^{\infty} \mu^*(F_n) \leq \mu^* \left(\bigcup_{n=1}^{\infty} F_n \right).
$$

Finally, to show completeness of the measure, given $N \in \mathcal{M}$ such that $\mu^*(N) = 0$ and $S \subset N$, we find that given $E \subset X$, we by find countable subadditivity that

$$
\mu^*(E) \leq \mu^*(E \cap S) + \mu \left(E \cap S^c \right),
$$

and since $E \cap S \subset S \subset N$ and $E \cap S^c \subset E$, we find by monotonicity that

$$
\mu^*(E \cap S) + \mu^* \left(E \cap S^c \right) \leq \mu^*(N) + \mu^*(E) = \mu^*(E) \implies S \in \mathcal{M}. \quad \square
$$

2.2 Applying the Definition to Premeasures

Definition 2.2.1. Given an **algebra** $\mathcal{A} \subset \mathcal{P}(X)$ we define a premeasure to be a function $\mu_0 : \mathcal{A} \to [0, +\infty]$ such that

(i) $\mu_0(\varnothing) = 0.$

(ii) Given a countable family $\mathcal{F} \subset \mathcal{A}$ of pairwise disjoint sets such that $\cup \mathcal{F} \in \mathcal{A}$, we have

$\mu_0(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_0(F).$
%Folland page 30 definition of premeasure

Definition 2.2.2.

(i) We state that an outer measure $(\mu_0)^*$ (as defined in **Definition 2.1.1**) is **σ-finite** if there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ such that $X = \bigcup_{n\in\mathbb{N}} E_n$ and $(\mu_0)^*(E_n) < +\infty$, for each $n \in \mathbb{N}$.

(ii) We state that a premeasure μ_0 (as defined in **Definition 2.2.1**) is **σ-finite** if there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ of sets such that $X = \bigcup_{n\in\mathbb{N}} E_n$ and $\mu_0(E_n) < +\infty$, for each $n \in \mathbb{N}$.
%define σ-finite premeasure

%MAKE NOTE ABOUT HOW THIS WAS NOT MENTIONED IN FOLLAND

Proposition 2.2.3. Suppose μ_0 is a premeasure $(\mu_0)^*$ is the induced outer measure on μ_0, as defined in $(2.1.1)$ and in **Examples 2.1.2** *(i)*. Then the following are equivalent:

(i) μ_0 is σ-finite (as a premeasure).

(ii) The measure $\mu := (\mu_0)^*|\mathcal{M}$ defined (and proved to be a measure) in **Theorem 2.1.4** is σ-finite.

(iii) $(\mu_0)^*$ is σ-finite (as an outer measure).

%show that these definitions of σ-finite are equivalent
%PROVE IT

Theorem 2.2.4. *(Caratheodory Extension for the Induced Outer-Measure)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a premeasure on $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ that extends μ_0, namely, $\mu = \mu^*|\mathcal{M}$.

Moreover, if ν is another measure that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < +\infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a

measure on $\mathcal{M}$.
%Folland page 31 1.14 Theorem

%FINISH THIS PROOF
Proof. First, note that for all $A \in \mathcal{A}$, we have

$$\mu(A) = \mu_0(A),$$

since by definition we have $\mu(A) = \mu^*(A) \leq \mu_0(A)$ and for all $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup\limits_{n\in\mathbb{N}} A_n \supset A$, we have by countable subadditivity and monotonicity

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \geq \mu^*(A) = \mu(A),$$

giving us $\mu(A) \geq \mu_0(A)$.

%POSSIBLY OUTSOURCE THIS AS A PROPERTY OF OUTER-MEASURE

Next, we find that $\mathcal{M}$ satisfies the Caratheodory Criterion, since $\mathcal{M}$ is the intersection of all σ-algebras containing $\mathcal{A}$ and we showed using **Theorem 2.1.4** that the set of all subsets of X satisfying the Caratheodory Criterion is a σ-algebra.

Note that given any other ν extending μ_0, we have

$$\nu(A) = \mu_0(A) = \mu(A), \text{ for all } A \in \mathcal{A}, \quad (2.2.1)$$

and it follows that for any $E \in \mathcal{M}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup\limits_{n\in\mathbb{N}} A_n \supset E$, we have by countable subadditivity of the induced outer measure ν^* on ν, we have

$$\nu(E) = \nu^*(E) \leq \sum_{n=1}^{\infty} \nu^*(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu_0(A) = \sum_{n=1}^{\infty} \mu(A_n),$$

hence we have

$$\nu(E) \le \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A} \text{ and } \bigcup_{n\in\mathbb{N}} A_n \supset E \right\} = \mu^*(E) = \mu(E). \quad (2.2.2)$$

%INSTEAD USE LEMMA 3.3.3 AND DELETE THIS PART OF PROOF
Suppose $\mu(E) < +\infty$. Since we can choose a sequence $\{A_{j,n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\sum_{n=1}^{\infty} \mu(A_{j,n}) \searrow \mu(E) \text{ as } j \to +\infty,$$

note that since we have

$$\mu(E) \le \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) \le \sum_{n=1}^{\infty} \mu(A_{j,n}), \text{ for all } j \ge 1,$$

we have

$$\lim_{j\to+\infty} \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) = \inf_{j\ge 1} \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) = \mu(E),$$

and to show $\nu(E) = \mu(E)$, it remains to show that

$$\lim_{j\to+\infty} \left| \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) - \nu(E) \right| = 0. \quad (2.2.3)$$

Since for each $j \ge 1$, we have $\bigcup_{n=1}^{N} A_{j,n} \in \mathcal{A}$ and $\bigcup_{n=1}^{N} A_{j,n} \nearrow \bigcup_{n\in\mathbb{N}} A_{j,n}$ we find by (2.2.1) that

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) = \lim_{N\to+\infty} \nu\left(\bigcup_{n=1}^{N} A_{j,n}\right) = \lim_{N\to+\infty} \mu\left(\bigcup_{n=1}^{N} A_{j,n}\right) = \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right),$$

hence we have

$$\lim_{j \to +\infty} \nu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) = \lim_{j \to +\infty} \mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) = \mu(E) < +\infty$$
$$\Longrightarrow \nu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) < +\infty, \text{ e.v., for } j \geq 1,$$
$$\Longrightarrow \nu(E) \leq \inf_{j\geq 1} \nu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) < +\infty,$$

and we conclude by (2.2.2) that

$$\begin{aligned}\lim_{j \to +\infty} \left|\mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) - \nu(E)\right| &= \lim_{j \to +\infty} \left(\nu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) - \nu(E)\right) \\ &\leq \lim_{j \to +\infty} \left(\mu\left(\bigcup_{n\in\mathbb{N}} A_{j,n}\right) - \mu(E)\right) \\ &= 0,\end{aligned}$$

which shows (2.2.3).

It then follows from the fact that $\mu(E) < +\infty \Longrightarrow \nu(E) = \mu(E)$ that if μ_0 is σ-finite, then we can choose $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \nearrow X$ and $\mu(A) < +\infty$, and we conclude uniqueness of μ from the fact that for all $E \in \mathcal{M}$, we find that since $E \cap A_n \nearrow E$, we have

$$\begin{aligned}\mu(E\cap A_n) \leq \mu(A_n) < +\infty &\Longrightarrow \nu(E\cap A_n) = \mu(E \cap A_n), \text{ for all } n \in \mathbb{N}, \\ &\Longrightarrow \nu(E) = \lim_{n \to +\infty} \nu(E\cap A_n) = \lim_{n \to +\infty} \mu(E \cap A_n) = \mu(A). \quad \square\end{aligned}$$

2.3 The Caratheodory Extension Defined Using Inner-Measures

Definition 2.3.1. Given a nonempty set X, we define an **inner measure** on $\mu_* : \mathcal{P}(X) \to [0, +\infty]$ as a function that satisfies the following properties:

(i) (Null Empty Set) $\mu_*(\varnothing) = 0$.

(ii) (Disjoint Superadditivity) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ is a collection of disjoint subsets, then

$$\mu_*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \sum_{n=1}^{\infty} \mu_*(E_n).$$

(iii)
%POSSIBLY MENTION THE MONOTONIC PROPERTY, THE LIMIT PROPERTIES, \SIGMA-FINITE PROPERTY, AND/OR SHOW THAT THESE PROPERTIES ARE IMPLIED BY THE PRIOR PROPERTIES IN LEMMA 2.3.2

Examples 2.3.3.

(i) Given a premeasure $\mu_0 : \mathcal{A} \to [0, +\infty]$, we define the **induced inner measure** $(\mu_0)_*$ on μ_0 to be the function $\mathcal{P}(X) \to [0, +\infty]$ defined by

$$(\mu_0)_*(E) := \sup\{\mu_0(A) : A \in \mathcal{A},\ A \subset E\}. \quad (2.3.1)$$

We find this is an inner measure since

$A \subset \varnothing \implies A = \varnothing \implies \mu_0(A) = 0$ for all $A \subset \varnothing \implies (\mu_0)_*(\varnothing) = 0,$

and for disjoint $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\sum_{n=1}^{\infty} (\mu_0)_*(E_n) = \sup\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

so it suffices to show that given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \mu_0(A_n) \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

Given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, note that $\{A_n\}_{n\in\mathbb{N}}$ is disjoint. For every $N \geq 1$, we find $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and $\bigcup_{n=1}^{N} A_n \subset \bigcup_{n\in\mathbb{N}} E_n$, so we have

$$\sum_{n=1}^{N} \mu_0(A_n) = \mu_0\left(\bigcup_{n=1}^{N} A_n\right) \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

hence

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \sup_{N\geq 1}\left\{\sum_{n=1}^{N} \mu_0(A_n)\right\} \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

Definition 2.3.4. *(The Caratheodory Criterion for Inner Measures)* Given an inner measure μ_* on X, we state that $F \subset X$ satisfies the **Caratheodory Criterion** with respect to μ_* (or equivalently state that F is **μ_*-measurable**) if for every $E \subset X$, we have

$$\mu_*(E) = \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X. \quad (2.3.2)$$

%MAKE REMARKS AND BETTER ORGANIZE THESE DERIVATIONS
Note that given $F \subset X$ since disjoint superadditivity automatically gives us

$$\mu_*(E) \geq \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X,$$

we find the Cartheodory criterion for inner measures is equivalent to showing that

$$\mu_*(E) \leq \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X. \quad (2.3.3)$$

Moreover, we find that given disjoint $\{F_1, \ldots, F_{N_0}\}$ satisfying Cartheodory's Criterion, we find by repeated use of (2.3.2), we find for all $E \subset X$ that

$$\begin{aligned}
\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) &= \mu_*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1\right) + \mu_*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1^c\right) \\
&= \mu_*(E \cap F_1) + \mu_*\left(E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right) \\
&= \mu_*(E \cap F_1) + \mu_*\left(\left[E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right] \cap F_2\right) + \mu_*\left(\left[E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right] \cap F_2^c\right) \\
&= \mu_*(E \cap F_1) + \mu_*(E \cap F_2) + \mu_*\left(E \cap \left(\bigcup_{n=3}^{N_0} F_n\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_n) + \mu_*\left(E \cap \left(\bigcup_{n=N_0}^{N_0} F_n\right)\right)\\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_n) + \mu_*(E \cap F_{N_0})\\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_{N_0}),
\end{aligned}
$$

and it follows that for arbitrary (not necessarily disjoint) $\{F_1, \ldots, F_{N_0}\}$ satisfying Cartheodory's Criterion, we find that

$$
\left\{F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right), F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right), \ldots, F_{N_0} \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right\},
$$

forms a disjoint partition of $\displaystyle\bigcup_{n=1}^{N_0} F_n$, hence

$$
\begin{aligned}
&\mu_*(E) = \mu_*(E \cap F_1) + \mu_*\left(E \cap F_1^c\right)\\
&= \mu_*(E \cap F_1) + \mu_*\left(\left[E \cap F_1^c\right] \cap F_2\right) + \mu_*\left(\left[E \cap F_1^c\right] \cap F_2^c\right)\\
&= \mu_*(E \cap F_1) + \mu_*(E \cap [(F_1 \cup F_2) \setminus F_1]) + \mu_*\left(E \cap \left(F_1^c \cap F_2^c\right)\right)\\
&= \mu_*(E \cap F_1) + \mu_*(E \cap (F_2 \setminus F_1)) + \mu_*\left(\left[E \cap \left(F_1^c \cap F_2^c\right)\right] \cap F_3\right) + \mu_*\left(\left[E \cap \left(F_1^c \cap F_2^c\right)\right] \cap F_3^c\right)\\
&= \mu_*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu_*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right) + \mu_*\left(E \cap \left[F_3 \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right)\\
&\quad + \mu_*\left(E \cap \left(\bigcap_{n=1}^{3} F_1^c\right)\right)\\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&= \sum_{N=1}^{N_0} \left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu_*\left(E \cap \left(\bigcap_{n=1}^{N_0} F_n^c\right)\right)
\end{aligned}
$$

$$= \sum_{N=1}^{N_0} \left[\mu_* \left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right)$$

$$= \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right).$$

%SHOW COUNTEREXAMPLE OF FAILURE WITHOUT INNER-MEASURE σ-FINITE CONDITION

Theorem 2.3.5. *(A Caratheodory Theorem for Inner Measures)* Given a σ-finite inner measure μ_* on X, and the collection $\mathcal{M}$ of all subsets of X satisfying the Cartheodory Criterion (i.e. condition (2.2.2)), we find μ_* is a σ-algebra, and $\mu : \mathcal{M} \to [0, +\infty]$ defined by $\mu := \mu_* | \mathcal{M}$ is a complete measure.
%Analogue of Folland page 29 1.11 Theorem for Inner-Measures

Proof. To begin, we show $\mathcal{M}$ is a σ-algebra. First, note that for every $E \subset X$, we have

$$\mu_*(E) = 0 + \mu_*(E) = \mu_*(E \cap \varnothing) + \mu_*\left(E \cap \varnothing^c\right),$$

hence $\varnothing \in \mathcal{M} \implies \mathcal{M} \neq \varnothing$. Next, observe that if $F \in \mathcal{M}$, then for all $E \subset X$, we have

$$\mu_*(E) = \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right) = \mu_*\left(E \cap F^c\right) + \mu_*\left(E \cap F^{cc}\right),$$

which shows $F^c \in \mathcal{M}$. Finally, suppose $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$. Given $E \subset X$, first, note that for all $N_0 \geq 1$, we have

$$\mu_*(E) = \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right),$$

hence

$$\mu_*(E) = \lim_{N_0 \to +\infty} [\mu_*(E)] = \lim_{N_0 \to +\infty} \left[\mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) \right].$$

In the case where $\mu_*(E) < +\infty$, we find by Disjoint Superadditivity that

$$
\begin{aligned}
\mu_*(E) &= \lim_{N_0 \to +\infty}\left[\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right)\right] \\
&= \lim_{N_0 \to +\infty}\sum_{N=1}^{N_0}\left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \lim_{N_0 \to +\infty}\mu_*\left(\bigcap_{n=1}^{N_0}\left[E \cap F_n^c\right]\right) \\
&= \sum_{N=1}^{\infty}\left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu_*\left(\bigcap_{n=1}^{\infty}\left[E \cap F_n^c\right]\right) \\
&\leq \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right),
\end{aligned}
$$

%COMPLETE THIS CASE USING σ-FINITE CONDITION
In the case where $\mu_*(E) = +\infty$, we find that

$$
\lim_{N_0 \to +\infty}\left[\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right)\right] = +\infty,
$$

hence either

$$
\lim_{N_0 \to +\infty}\sum_{N=1}^{N_0}\left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] = \lim_{N_0 \to +\infty}\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) = +\infty \text{ or}
$$

$$
\lim_{N_0 \to +\infty}\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right) = +\infty
$$

$$
\mu_*(E), \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right) \geq \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) = +\infty
$$

$$
\implies \mu_*(E) = +\infty = \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right).
$$

In both cases, we have shown that condition (2.3.2) holds for $\bigcup_{n=1}^{\infty} F_n$, and $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$ immediately follows.

Next, we want to show that $\mu := \mu_*|\mathcal{M}$ is a complete measure. First, observe that $\mu(\varnothing) = \mu_*(\varnothing) = 0$ by definition and for disjoint additivity, we find for disjoint $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$, in the case where

$$\mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) < +\infty,$$

we have for all $N_0 \geq 1$, we find by finite disjoint additivity that

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{N_0} \mu_*(F_n) + \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right),$$

hence we have

%MENTION MONOTONIC PROPERTY USED TO JUSTIFY THIS

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} F_n\right) &= \lim_{N_0 \to +\infty}\left[\sum_{n=1}^{N_0} \mu_*(F_n) + \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right)\right] \\
&= \lim_{N_0 \to +\infty} \sum_{n=1}^{N_0} \mu_*(F_n) + \lim_{N_0 \to +\infty} \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right) \\
&= \sum_{n=1}^{\infty} \mu_*(F_n) + \mu_*\left(\bigcap_{N_0=1}^{\infty}\left[\bigcup_{n=N_0+1}^{\infty} F_n\right]\right) \\
&= \sum_{n=1}^{\infty} \mu_*(F_n) + \mu_*(\varnothing) \\
&= \sum_{n=1}^{\infty} \mu(F_n).
\end{aligned}$$

In the case where

$$\mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) = +\infty,$$

we find that
%FINISH THIS ARGUMENT

Next, to show completeness of the measure, given $N \in \mathcal{M}$ such that $\mu_*(N) = 0$ and $S \subset N$, we that for all $E \subset X$, we find by monotonicity that since $E \cap S \subset E \cap N \subset N$, we have

$$0 \leq \mu_*(E \cap N), \mu_*(E \cap S) \leq \mu_*(N) = 0 \Longrightarrow \mu_*(E \cap N) = \mu_*(E \cap S) = 0,$$

hence $(2.2.3)$ is satisfied, and we conclude that $S \in \mathcal{M}$.

%SHOW THAT σ-FINITE CONDITION IS SATISFIED (OR COME UP WITH A COUNTEREXAMPLE WHERE IT'S NOT SATISFIED POSSIBLY USING SOME KIND OF WEIRD INNER SUM CONDITION)
□

%SHOW THAT σ-FINITENESS OF PREMEASURES IS EQUIVALENT TO σ-FINITENESS OF INDUCED INNER MEASURES
%TRY THIS EXAMPLE AND TRY TO SHOW THAT ONLY INFINITE AND MEASURE ZERO SETS SATISFY THE CRITERION, AND HENCE THE MEASURE FAILS TO BE σ-FINITE

$$\mu_*(A) := \sup\left\{\int \phi dm : \phi \text{ is of the form } \sum_{k=1}^{N} a_k \mathbb{1}_{I_k} \leq \mathbb{1}_A \text{ for intervals } I_1, \ldots, I_k\right\}$$

%WORK OUT THIS EXAMPLE DURING THE NEXT DRAFT

Theorem 2.3.6. *(A Caratheodory Extension for the Induced Inner-Measure)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a σ-finite premeasure on $\mathcal{A}$. Then $\mu := \mu_*|\mathcal{M}$ defines a uniquely determined σ-finite measure that extends μ_0.
%POSSIBLY FINISH THIS THEOREM HERE AND FIGURE OUT IF MORE NEEDS TO BE SAID
%Analogue of Folland page 31 1.14 Theorem for Inner-Measures

Proof. First, given $A \in \mathcal{A}$, we find for all $B \in \mathcal{A}$ such that $B \subset A$ we have by monotonicity

$$\mu_0(B) = \mu_*(B) \leq \mu_*(A) = \mu_0(A) \Longrightarrow \mu_*(A) \leq \mu_0(A).$$

It follows that since A itself is in $\mathcal{A}$ and $A \subset A$, we have $\mu_*(A) \geq \mu_0(A)$, hence

$$\mu(A) = \mu_*(A) = \mu_0(A),$$

and we've shown $\mu_*|\mathcal{M}$ is an extension of μ_0.

Next, we find that $\mathcal{M}$ satisfies the Caratheodory Criterion (for inner measures), since $\mathcal{M}$ is the intersection of all σ-algebras containing $\mathcal{A}$ and we showed using **Theorem 2.3.5** that the set $\mathcal{M}'$ of all subsets of X satisfying the Caratheodory Criterion is a σ-algebra and μ (and more generally $\mu_*|\mathcal{M}'$) is a measure.
%ACTUALLY SHOW THAT ALL $\mathcal{A}$ SATISFIES THE CARATHEODORY CRITERION

Uniqueness follows immediately from **Theorem 2.2.4** and the fact that μ_0 is σ-finite, since in that situation μ must agree with the outer measure $(\mu_0)^*$ induced by μ_0. □

%MAKE COROLLARY ABOUT JORDAN DECOMPOSITION

3 Inner and Outer Single Dimension Signed Measures

3.1 Some Properties of the Jordan Decomposition

Assume that μ, ν are signed ($\mathbb{R}$-valued) measures on a nonempty σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$.

Definition 3.1.1.

(i) We state that μ is **positive** if $\mu(E) \geq 0$, for every $E \in \mathcal{M}$.

(ii) We state that a set $E \in \mathcal{M}$ is a **μ-null set** if $\mu(E') = 0$ for every $E' \in \mathcal{M}$ such that $E' \subset E$.

(iii) We state that two measures μ, ν on a σ-algebra $\mathcal{M}$ are **mutually singular** (and write $\mu \perp \nu$) if there exists disjoint sets $E, F \in \mathcal{M}$ such that $E \cup F = X$ and E is μ-null and F is ν-null.

%DEFINE σ-FINITE IN SIGNED MEASURE SETTING AND MAKE REMARK ABOUT HOW FOLLAND DOESN'T CLARIFY THAT

Theorem 3.1.2. *(Jordan Decomposition Theorem)* Given a (signed $\mathbb{R}$-valued) measure μ, there exists two unique positive measures μ^+, μ^- such that $\mu^+ \perp \mu^-$ and $\mu = \mu^+ - \mu^-$.

%TALK ABOUT HOW PROOF IS DONE IN FOLLAND BUT AN ALTERNATIVE PROOF WILL BE PROVIDED BY SHOWING THAT CARATHEODORY EXTENSION IS IN FACT A MORE GENERAL RESULT

Definition 3.1.3. Define the **total varation $|\mu|$** of a (signed $\mathbb{R}$-valued measure) μ to be

$$|\mu| := \mu^+ + \mu^-.$$

Proposition 3.1.4. *(Properties of the Jordan Decomposition)* If μ^+, μ^- are Jordan decompositions of $\mu := \mu^+ - \mu^-$, then:

(i) $E \in \mathcal{M}$ is μ-null iff $|\mu|(E) = 0$

(ii) $\mu \perp \nu$ iff $|\mu| \perp \nu$ iff $\mu^+ \perp \nu$ and $\mu^- \perp \nu$.

(iii) For all $E \in \mathcal{M}$, we have

$$\begin{aligned}
\mu^+(E) &= \sup\{\mu(F) : F \in \mathcal{M}, F \subset E\},\\
\mu^-(E) &= -\inf\{\mu(F) : F \in \mathcal{M}, F \subset E\},\\
|\mu|(E) &= \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \dots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E\right\}\\
&= \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \dots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\}.
\end{aligned}$$

%PROVE THIS PROPOSITION IN NEXT DRAFT

%DISCUSS/EXPLORE CHANGE FROM $\bigcup_{j=1}^{n} E_j = E$ TO $\bigcup_{j=1}^{n} E_j \subset E$, AND JUSTIFY WHY THIS CHANGE IS A GOOD IDEA

%CITE HAHN BANACH DECOMPOSITION OF FOLLAND
%STATE AND PROVE PROPERTIES GIVEN FOLLAND 3.1 EXERCISE 7 ON PAGE 88

3.2 Single Dimension Signed Premeasures and the Induced Jordan Decomposition

Defintion 3.2.1. Given an algebra $(E, \mathcal{P})$, we define a **signed ($\mathbb{R}$-valued) premeasure** $\mu_0 : \mathcal{A} \to [-\infty, +\infty]$ to be a function such that

(i) $\mu_0(\emptyset) = 0$.
(ii) μ_0 assumes at most one of the values $\pm\infty$.
(iii) given a countable family $\mathcal{F} \subset \mathcal{A}$ of pairwise disjoint sets, we have $\sum_{F \in \mathcal{F}} \mu_0(F)$ either converging absolutely or diverging to $\pm\infty$. Moreover, if $\cup \mathcal{F} \in \mathcal{A}$, we have

$$\mu_0(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_0(F).$$

%SHOW THAT THIS DEFINITION AGREES WITH THE GENERAL DEFINITION OF PREMEASURES
%GIVE SOME EXAMPLES IN NEXT DRAFT
%IN PARTICULAR, GIVE FUNCTIONS OF BOUNDED VARIATION AND CADLAG FUNCTIONS AS EXAMPLES

Definition 3.2.2. Let μ_0 be a (signed $\mathbb{R}$-valued) premeasure μ_0.

(i) We define the **induced Jordan decomposition $(\mu_0)^+, (\mu_0)^- : \mathrm{P(X)} \to [0, +\infty]$ of μ_0** as so:

$$(\mu_0)^+(E) := \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset E\}, \qquad (3.2.1)$$
$$(\mu_0)^-(E) := -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E\}.$$

(ii) We define the **induced total variation $|\mu_0|$ of μ_0** as so:

$$|\mu_0|(E) = \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\}.$$

Lemma 3.2.3.
%GIVE THE IMPORTANT PROPERTIES OF THE INNER JORDAN DECOMPOSITION
%POSSIBLY DELETE THIS

Lemma 3.2.4. For every signed ($\mathbb{R}$-valued) premeasure μ_0, we find that the induced Jordan decomposition $(\mu_0)^+, (\mu_0)^-$ and inner total variation $|\mu_0|$ are inner measures.
%LEMMA ABOUT INNER JORDAN AND INNER TOTAL VARIATIONS BEING PREMEASURES

Proof. Note that since $\varnothing \in \mathcal{A}$, we have

$$(\mu_0)^+(\varnothing) = (\mu_0)^-(\varnothing) = |\mu_0|(\varnothing) = 0,$$
%SHOW THIS IN MORE DETAIL

and it remains to show that $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ satisfy the properties of disjoint superadditivity. For disjoint $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\sum_{n=1}^{\infty}(\mu_0)^+(E_n) = \sup\left\{\sum_{n=1}^{\infty}\mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

$$\sum_{n=1}^{\infty}(\mu_0)^-(E_n) = -\inf\left\{\sum_{n=1}^{\infty}\mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

$$\sum_{n=1}^{\infty}|\mu_0|(E) = \sup\left\{\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}|\mu_0(A_{n,j})| : \{A_{n,1}, \ldots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}\ A_{n,1}, \ldots A_{n,N_n} \text{ are disjoint}, \bigcup_{j=1}^{N_n} A_{n,j} \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

so it suffices to show that:

(i) For all $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}\mu_0(A_n) \le (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

$$-\sum_{n=1}^{\infty}\mu_0(A_n) \le (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

(ii) For all $\{A_{n,1}, \dots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{j=1}^{N_n} A_{n,j} \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})| \leq |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

To show *(i)*, given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, note that $\{A_n\}_{n\in\mathbb{N}}$ is disjoint. It follows that for every $N \geq 1$, we find $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and $\bigcup_{n=1}^{N} A_n \subset \bigcup_{n\in\mathbb{N}} E_n$, so we have

$$\begin{aligned}
\inf\left\{\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\} \leq \mu_0\left(\bigcup_{n=1}^{N} A_n\right) &= \sum_{n=1}^{N} \mu_0(A_n) \\
&\leq \sup\left\{\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\},
\end{aligned}$$

$$\implies \sum_{n=1}^{N} \mu_0(A_n) \leq (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

$$-\sum_{n=1}^{N} \mu_0(A_n) \leq \sup\left\{-\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\} = (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

hence

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \lim_{N\to+\infty} \sum_{n=1}^{N} \mu_0(A_n) \leq (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$
$$-\sum_{n=1}^{\infty} \mu_0(A_n) = \lim_{N\to+\infty} -\sum_{n=1}^{N} \mu_0(A_n) \leq (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

completing our verification that *(i)* holds.

To show *(ii)*, given $\{A_{n,1}, \dots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_{n,1}, \dots, A_{n,N_n}$ are disjoint and

$\bigcup_{j=1}^{N_n} A_{n,j} \subset E_n$, for all $n \in \mathbb{N}$, note that $\left\{\bigcup_{j=1}^{N_n} A_{n,j}\right\}_{n\in\mathbb{N}}$ is disjoint. It follows that for every $N \geq 1$, we find $\bigcup_{n=1}^{N} \bigcup_{j=1}^{N_n} A_{n,j} \subset \bigcup_{n\in\mathbb{N}} E_n$, and $\bigcup_{n=1}^{N} \{A_{n,1}, \ldots, A_{n,N_n}\}$ is a finite disjoint collection, so we have

$$\begin{aligned}\sum_{n=1}^{N}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})| &\leq \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset \bigcup_{n\in\mathbb{N}} E_n\right\} \\ &= |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right),\end{aligned}$$

hence we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})| = \sup_{N\geq 1}\left\{\sum_{n=1}^{N}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})|\right\} \leq |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

completing our verification that *(ii)* holds. □

%IN NEXT DRAFT DEFINE A COMPLEX-VALUED PREMEASURE AND ITS INNER JORDAN DECOMPOSITIONS AND INNER TOTAL VARIATION

3.3 The Caratheodory Extension for $\mathbb{R}$-valued Premeasures

In this section, we assume that all premeasures are $\mathbb{R}$-valued signed premeasures.

Definition 3.3.1. *(The Caratheodory Criterion for the Induced Jordan Decomposition)* Given a premeasure μ_0, we state that $F \subset X$ satisfies the **Caratheodory criterion for the induced Jordan decomposition $(\mu_0)^+, (\mu_0)^-$** if it satisfies the Caratheodory criterion for both $(\mu_0)^+$ and $(\mu_0)^-$ as inner measures (see **Definition 2.3.4**), i.e., for every $E \subset X$, we have

$$\begin{aligned}(\mu_0)^+(E) &= (\mu_0)^+(E \cap F) + (\mu_0)^+\left(E \cap F^c\right), \\ (\mu_0)^-(E) &= (\mu_0)^-(E \cap F) + (\mu_0)^-\left(E \cap F^c\right),\end{aligned}$$

or equivalently (2.3.3) holds for both $(\mu_0)^+$ and $(\mu_0)^-$.

%MAKE REMARKS ABOUT OTHER WAYS TO FORMULATE THIS CRITERION IN NEXT DRAFT (IN PARTICULAR LOOK AT THE CARTHEODORY CRITEREON FOR $|\mu_0|$)

Theorem 3.3.2. *(A Caratheodory Theorem for the Induced Jordan Decomposition)* For every σ-finite premeasure μ_0 on an algebra $\mathcal{A}$, there exists a σ-algebra $\mathcal{M} \supset \mathcal{A}$ such that the induced Jordan decompositions $(\mu_0)^+, (\mu_0)^-$ and induced total variation $|\mu_0|$, restricted to $\mathcal{M}$ all form the complete measures $\mu_+ := (\mu_0)^+|\mathcal{M},\ \mu_- := (\mu_0)^-|\mathcal{M},$ and $\mu_{tot.} := |\mu_0|\ |\mathcal{M}$.

%ADD FINISHING TOUCHES BY SHOWING COMPLETENESS
Proof. Let $\mathcal{M}$ be the set of all $F \subset X$ that satisfy both the Caratheodory criterion for the inner Jordan decomposition and the Caratheodory criterion of $|\mu_0|$ as an inner measure.

First, we shall show that $\mathcal{A} \subset \mathcal{M}$ by showing that given $A \in \mathcal{A}$, we find (2.3.3) holds for $(\mu_0)^+, (\mu_0)^-, |\mu_0|$. To show that (2.3.3) holds for $(\mu_0)^+, (\mu_0)^-$, we find that given $E \subset X$, we find that for all $A_0 \in \mathcal{A}$ such that $A_0 \subset E$, we have $A_0 \cap A, A_0 \cap A^c \in \mathcal{A}$, such that $A_0 \cap A \subset E \cap A, A_0 \cap A^c \subset E \cap A^c$, hence

$$\begin{aligned}
&\mu_0(A_0) = \mu_0(A_0 \cap A) + \mu_0\left(A_0 \cap A^c\right) \leq (\mu_0)^+(E \cap A) + (\mu_0)^+\left(E \cap A^c\right),\\
&\implies (\mu_0)^+(E) \leq (\mu_0)^+(E \cap A) + (\mu_0)^+\left(E \cap A^c\right),
\end{aligned}$$

$$\begin{aligned}
&\mu_0(A_0) = \mu_0(A_0 \cap A) + \mu_0\left(A_0 \cap A^c\right)\\
&\qquad\qquad \geq \inf\{\mu_0(A_1) : A_1 \in \mathcal{A}, A_1 \subset E \cap A\} + \inf\left\{\mu_0(A_2) : A_2 \in \mathcal{A}, A_2 \subset E \cap A^c\right\},\\
&\implies \inf\{\mu_0(A_0) : A_0 \in \mathcal{A}, A_0 \subset E\}\\
&\qquad \geq \inf\{\mu_0(A_1) : A_1 \in \mathcal{A}, A_1 \subset E \cap A\} + \inf\left\{\mu_0(A_2) : A_2 \in \mathcal{A}, A_2 \subset E \cap A^c\right\},\\
&\implies (\mu_0)^-(E) \leq (\mu_0)^-(E \cap A) + (\mu_0)^-\left(E \cap A^c\right).
\end{aligned}$$

To show that (2.3.3) holds for $|\mu_0|$, for all $E \subset X$, we find that for all disjoint $A_1, \ldots, A_N \in \mathcal{A}$ such that $\bigcup_{j=1}^{N} A_j \subset E$, we have disjoint collections

$$\{A_1 \cap A, A_2 \cap A, \ldots, A_N \cap A\}, \left\{A_1 \cap A^c, A_2 \cap A^c, \ldots, A_N \cap A^c\right\} \subset \mathcal{A},$$

such that

$$\bigcup_{j=1}^{N}[A_j \cap A] \subset E \cap A, \ \bigcup_{j=1}^{N}\left[A_j \cap A^c\right] \subset E \cap A^c,$$

hence we have

$$\begin{aligned}
\sum_{j=1}^{N}|\mu_0(A_j)| &= \sum_{j=1}^{N}\left|\mu_0(A_j \cap A) + \mu_0\left(A_j \cap A^c\right)\right| \\
&\leq \sum_{j=1}^{N}|\mu_0(A_j \cap A)| + \sum_{j=1}^{N}\left|\mu_0\left(A_j \cap A^c\right)\right| \\
&\leq |\mu_0|(E \cap A) + |\mu_0|\left(E \cap A^c\right), \\
\implies |\mu_0|(E) &\leq |\mu_0|(E \cap A) + |\mu_0|\left(E \cap A^c\right).
\end{aligned}$$

Next, take $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_{tot.}$ to each be the set of all $F \subset X$ that satisfy the Caratheodory criterion for $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ as inner measures (see **Definition 2.3.4**), respectively. We find by **Theorem 2.3.5** that $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_{tot.}$ are all σ-algebras and that $(\mu_0)^+|\mathcal{M}_+, (\mu_0)^-|\mathcal{M}_-, |\mu_0|\ |\mathcal{M}_{tot.}$ are measures. Then since

$$\mathcal{M} = \mathcal{M}_+ \cap \mathcal{M}_- \cap \mathcal{M}_{tot.},$$

we find $\mathcal{M}$, as an intersection of σ-algebras, is itself a σ-algebra; so follows that the further restriction of $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ to $\mu_+, \mu_-, \mu_{tot.}$, respectively, are all measures; and it remains to show that $\mu_+, \mu_-, \mu_{tot.}$ are all complete measures in $\mathcal{M}$.
%FINISH PROOF BY SHOWING COMPLETENESS □

Lemma 3.3.3. *(Uniqueness of measures extending signed premeasure)* For every σ-finite signed ($\mathbb{R}$-valued) premeasure μ_0 on an algebra $\mathcal{A}$, any ($\mathbb{R}$-valued) signed measure $\mu : \sigma(\mathcal{A}) \to [-\infty, +\infty]$ extending μ_0 (if it exists) is uniquely determined.

%SHOW IN REMARK THAT IF $\{E_n\}_{n\in\mathbb{N}}$ IS A $\subset$-INCREASING SEQUENCE, THEN

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n \to +\infty} \mu(E_n),$$

AND IF $\{E_n\}_{n\in\mathbb{N}}$ IS A $\subset$-DECREASING SEQUENCE SUCH THAT $|\mu(E_1)| < +\infty$, THEN

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to+\infty} \mu(E_n),$$

Proof. Let $\mu, \nu : \sigma(\mathcal{A}) \to \mathbb{R}$ be measures extending μ_0. First, we shall prove this in the case where μ_0 is finite. Let $\mathcal{M}_0$ be the set of all $E \in \sigma(\mathcal{A})$ such that $\mu(E) = \nu(E)$, and note that it shall suffice by the *Monotone Class Lemma* to show that $\mathcal{M}_0$ is a monotone class containing $\mathcal{A}$, since we'd have $\sigma(\mathcal{A}) = \mathcal{M}_0$ (as a result of $\sigma(\mathcal{A}) \supset \mathcal{M}_0$ by construction and $\sigma(\mathcal{A}) \subset \mathcal{M}_0$ since $\sigma(\mathcal{A})$ is the smallest monotone class) and uniqueness would immediately follow.

First, observe that $\mathcal{A} \subset \mathcal{M}_0$ is immediate by μ, ν both extending μ_0. Next, we want to show $\mathcal{M}_0$ is closed under monotone unions and intersections. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_0$ be a $\subset$-increasing sequence and $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_0$ be a $\subset$-decreasing sequence and observe that since

$$\mu(E_n) = \nu(E_n) \text{ and } \mu(F_n) = \nu(F_n), \text{ for all } n \in \mathbb{N},$$

and

$$|\mu(F_1) + \mu\left(F_1^c\right)| = |\mu(X)| = |\mu_0(X)| < +\infty,$$
$$\implies |\mu(F_1)| = |\nu(F_1)| < +\infty,$$

we find that

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to+\infty} \mu(E_n) = \lim_{n\to+\infty} \nu(E_n) = \nu\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$
$$\mu\left(\bigcap_{n\in\mathbb{N}} F_n\right) = \lim_{n\to+\infty} \mu(F_n) = \lim_{n\to+\infty} \nu(F_n) = \nu\left(\bigcup_{n\in\mathbb{N}} F_n\right),$$

and our conclusion is reached.

Now we shall prove this lemma in the general case. Using σ-finiteness of μ_0, choose a $\subset$-increasing sequence $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $|\mu_0(X_n)| < +\infty$ and $X_n \nearrow X$ as $n \to +\infty$. Note that for all $n \in \mathbb{N}$, we find $\mu_{n,0} : \mathcal{A} \to [-\infty, +\infty]$ defined by $\mu_{n,0} := \mu_0((-) \cap X_n)$ is a finite premeasure such that $\mu_n, \nu_n : \sigma(\mathcal{A}) \to [-\infty, +\infty]$ defined by $\mu_n := \mu((-) \cap X_n), \nu_n((-) \cap X_n)$ are finite measures that extend $\mu_{n,0}$, since for all $n \in \mathbb{N}$ we have

$$|\mu_n(X)| = |\mu(X \cap X_n)| = |\nu(X \cap X_n)| = |\nu_n(X)| = |\mu_{0,n}(X)| = |\mu_0(X \cap X_n)| = |\mu(X_n)| < +\infty,$$

and for all $A \in \mathcal{A}$, we have

$$\mu_n(A) = \mu(A \cap X_n) = \mu_0(A \cap X_n) = \mu_{0,n}(A) = \mu_0(A \cap X_n) = \nu(A \cap X_n) = \nu_n(A).$$

Then for every $E \in \sigma(\mathcal{A})$, noting that $\{E \cap X_n\}_{n \in \mathbb{N}}$ is a $\subset$-increasing sequence such that $E \cap X_n \nearrow E$, we have

$$\mu(E) = \lim_{n \to +\infty} \mu(E \cap X_n) = \lim_{n \to +\infty} \nu(E \cap X_n) = \nu(E). \ \square$$

Theorem 3.3.4. *(A Caratheodory Extension for the Induced Jordan Decomposition)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a σ-finite premeasure on $\mathcal{A}$. Define

$$\mu_+ := (\mu_0)^+|\mathcal{M}, \ \mu_- := (\mu_0)^-|\mathcal{M}, \text{ and } \mu_{tot.} := |\mu_0| \, |\mathcal{M}.$$

Then $\mu_+ \perp \mu_-$, and there exists a uniquely determined σ-finite measure on $\mathcal{M}$ that extends μ_0 defined by $\mu := \mu_+ - \mu_-$, with its Jordan decomposition uniquely determined by $\mu^+ = \mu_+$ and $\mu^- = \mu_-$. Moreover, we have

$$\mu^+ + \mu^- = |\mu| = \mu_{tot.} = \mu_+ + \mu_-. \quad (3.3.1)$$

Proof. First, we shall prove that μ_+, μ_- forms a decomposition such that $\mu := \mu_+ - \mu_-$ extends μ_0. To do this, we have two cases.

Case 1. Suppose $|\mu_0(X)| < +\infty$, i.e. we look at the case where μ_0 is finite. To prove that $\mu := \mu_+ - \mu_-$ extending μ_0, it shall suffice to show that μ_0 satisfies the property

$$\mu_0(F) = \mu_+(F) - \mu_-(F), \text{ for every } F \in \mathcal{A}.$$

Let $F \in \mathcal{A}$. Choose $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that

$$\begin{aligned} &\bigcup_{n \in \mathbb{N}} A_n, \bigcup_{n \in \mathbb{N}} B_n \subset F, \text{ and} \\ &0 \le \mu_0(A_n) \nearrow \mu_+(F), \\ &0 \le -\mu_0(B_n) \nearrow \mu_-(F), \text{ as } n \to +\infty. \end{aligned} \quad (3.3.2)$$

Note that since for each $n \geq 1$, we have

$$\mu_0(A_n) \leq \mu_+(A_n), \ -\mu_0(B_n) \leq \mu_-(B_n),$$

we find that

$$\mu_+(A_n) \nearrow \mu_+(F), \ \mu_-(B_n) \nearrow \mu_-(F), \text{ as } n \to +\infty,$$

and it follows that

$$\begin{aligned} \lim_{n \to +\infty} \mu_+(F \setminus A_n) &= \lim_{n \to +\infty} [\mu_+(F) - \mu_+(A_n)] = \mu_+(F) - \mu_+(F) = 0, \\ \lim_{n \to +\infty} \mu_-(F \setminus B_n) &= \lim_{n \to +\infty} [\mu_-(F) - \mu_-(B_n)] = \mu_-(F) - \mu_-(F) = 0, \end{aligned}$$

hence

$$\begin{aligned} \lim_{n \to +\infty} \mu_+(F \setminus (A_n \cup B_n)) &\leq \lim_{n \to +\infty} \mu_+(F \setminus A_n) = 0, \\ \lim_{n \to +\infty} \mu_-(F \setminus (A_n \cup B_n)) &\leq \lim_{n \to +\infty} \mu_-(F \setminus B_n) = 0. \end{aligned}$$

Then we have

$$-\mu_-(F \setminus (A_n \cup B_n)) \leq \mu_0(F \setminus (A_n \cup B_n)) \leq \mu_+(F \setminus (A_n \cup B_n)), \text{ for all } n \geq 1,$$

$$\implies \lim_{n \to +\infty} \mu_0(F \setminus (A_n \cup B_n)) = 0,$$

and it follows that

$$\begin{aligned} \mu_0(F) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) + \mu_0(F \setminus (A_n \cup B_n))] \\ &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)]; \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} \lim_{n \to +\infty} \mu_0\left(A_n \cap B_n^c\right) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) - \mu_0(B_n)] \\ &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] + \lim_{n \to +\infty} [-\mu_0(B_n)] \\ &= \mu_0(F) + \mu_-(F); \end{aligned}$$

$$\lim_{n \to +\infty} \mu_0\left(A_n^c \cap B_n\right) = \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) - \mu_0(A_n)]$$

$$= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] - \lim_{n \to +\infty} [\mu_0(A_n)]$$
$$= \mu_0(F) - \mu_+(F);$$

$$\lim_{n \to +\infty} \mu_0(A_n \triangle B_n) = \lim_{n \to +\infty} \left[\mu_0\left(A_n \cap B_n^c\right)\right] + \lim_{n \to +\infty} \left[\mu_0\left(A_n^c \cap B_n\right)\right]$$
$$= 2\mu_0(F) - [\mu_+(F) - \mu_-(F)];$$

$$\begin{aligned}\lim_{n \to +\infty} \mu_0(A_n \cap B_n) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) - \mu_0(A_n \triangle B_n)] \\ &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] - \lim_{n \to +\infty} [\mu_0(A_n \triangle B_n)] \qquad (3.3.4) \\ &= [\mu_+(F) - \mu_-(F)] - \mu_0(F).\end{aligned}$$

Then by (3.3.2), (3.3.3), and (3.3.4), we conclude that

$$\begin{aligned}\mu_0(F) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] \\ &= \lim_{n \to +\infty} [\mu_0(A_n)] + \lim_{n \to +\infty} [\mu_0(B_n)] - 2 \lim_{n \to +\infty} [\mu_0(A_n \cap B_n)] \\ &= [\mu_+(F) - \mu_-(F)] - 2 \cdot ([\mu_+(F) - \mu_-(F)] - \mu_0(F)) \\ &= 2\mu_0(F) - [\mu_+(F) - \mu_-(F)],\end{aligned}$$

$$\Longrightarrow \mu_0(F) = \mu_+(F) - \mu_-(F).$$

Next, to prove $\mu_+ \perp \mu_-$ (and more generally that μ_+, μ_- is the Jordan decomposition of μ) that since

$$\mu_0(X) = \mu_+(X) - \mu_-(X), \quad (3.3.5)$$

and

$$|\mu_+(X) - \mu_-(X)| = |\mu_0(X)| < +\infty,$$

we find μ_+, μ_- are finite. Next, choose a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that

$$\mu_+(X) \le \mu_0(A_n) + 2^{-n}, \text{ for all } n \in \mathbb{N}, \text{ and} \qquad (3.3.6)$$
$$\mu_0(A_n) \nearrow \mu_+(X) \text{ as } n \to +\infty,$$

and note that since for all $n \in \mathbb{N}$, we have

$$\mu_0(A_n) = \mu_+(A_n) - \mu_-(A_n), \qquad (3.3.7)$$

$$0 \le \mu_0(A_n) \le \mu_+(A_n) \le \mu_+(X),$$

we find by (3.3.6) and (3.3.7) that

$$\lim_{n \to +\infty} \mu_+(A_n) = \lim_{n \to +\infty} \mu_0(A_n) = \mu_+(X),$$

$$\lim_{n \to +\infty} \mu_+\left(A_n^c\right) = \mu_+(X) - \lim_{n \to +\infty} [\mu_+(A_n)] = 0.$$

Define

$$P := \limsup_n A_n = \bigcap_{N \in \mathbb{N}} \bigcup_{n \ge N} A_n, \quad (3.3.8)$$
$$N := P^c.$$

We want to show that P, N make up a Hahn Decomposition, i.e., noting that P, N partition X, it remains to show that N is μ_+-null and P is μ_--null. First, note that

$$N = \liminf_n A_n^c = \bigcup_{n \in \mathbb{N}} \bigcap_{n \ge N} A_n^c.$$

Then since μ_+ is finite, we have

$$\begin{aligned}\mu_+(N) &= \sup_{N \in \mathbb{N}} \mu_+\left(\bigcap_{n \ge N} A_n^c\right) = \sup_{N \in \mathbb{N}}\left[\inf{}_{n \ge N} \mu_+\left(A_N^c \cap A_{N+1}^c \cap \cdots \cap A_n^c\right)\right] \\ &\le \sup_{N \in \mathbb{N}}\left[\inf{}_{n \ge N} \mu_+\left(A_n^c\right)\right] = \sup_{N \in \mathbb{N}}\left[\lim_{n \to +\infty} \mu_+\left(A_n^c\right)\right] = 0,\end{aligned}$$

and hence N is μ_+-null. Next, observe by (3.3.5), (3.3.6) and (3.3.7) we find that for all $n \in \mathbb{N}$, we have

$$\mu_+(X) \le \mu_0(A_n) + 2^{-n} \le \mu_+(A_n) + 2^{-n},$$

$$\begin{aligned}\implies & \mu_+(X) - \mu_0(A_n) \le 2^{-n} \\ & \mu_+\left(A_n^c\right) = \mu_+(X) - \mu_+(A_n) \le 2^{-n},\end{aligned}$$

$$\implies \mu_-(A_n) = \mu_-(X) - \mu_-\left(A_n^c\right)$$

$$
\begin{aligned}
&= \mu_-(X) - \left[\mu_+\left(A_n^c\right) - \mu_0\left(A_n^c\right)\right] \\
&= \mu_-(X) - \mu_+\left(A_n^c\right) + \mu_0(X) - \mu_0(A_n) \\
&= [\mu_+(X) - \mu_0(A_n)] - \mu_+\left(A_n^c\right) \\
&= O\left(2^{-n}\right),
\end{aligned}
$$

$$
\implies \sum_{n=1}^{\infty} \mu_-(A_n) < +\infty,
$$

and it immediately follows that we have $\mu_-(P) = 0$ by applying $(3.3.8)$ to the *First Borel-Cantelli Lemma*, and our conclusion that P is μ_--null is reached, finishing *Case 1*.

Case 2. Suppose $|\mu_0(X)| = +\infty$, i.e., we look at the case where μ_0 is infinite. Choose disjoint $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} X_n = X$, and note that $\mu_{n,0} := \mu_0((-)\cap X_n)$ are finite premeasures. It follows by *Case 1* that for all $n \in \mathbb{N}$, we find

$$
\mu_{n,+} := (\mu_{n,0})^+|\mathcal{M},\ \mu_{n,-} := (\mu_{n,0})^-|\mathcal{M},
$$

is such that $\mu_{n,+}, \mu_{n,-}$ form a uniquely determined Jordan Decomposition of

$$
\mu_n := \mu_{n,+} - \mu_{n,-} \quad (3.3.9)
$$

extending $\mu_{n,0}$, i.e., we have

$$
\mu_{n,0}(A) = \mu_{n,+}(A) - \mu_{n,-}(A), \text{ for all } A \in \mathcal{A}. \quad (3.3.10)
$$

We shall start by proving the following claims:

Claim 1. For every $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mu_{n,+} = \mu_+((-)\cap X_n), \quad (3.3.11) \\
&\mu_{n,-} = \mu_-((-)\cap X_n).
\end{aligned}
$$

Proof. Note that for every $n \in \mathbb{N}$, and $E \in \sigma(\mathcal{A})$, we find since

$$
\mu_0(A\cap X_n) \in \{\mu_0(A\cap X_n) : A \in \mathcal{A}, A \subset E\} \iff A \in \mathcal{A}, A \subset E
$$

$$\iff A' = A \cap X_n \in \mathcal{A}, A' \subset E \cap X_n$$
$$\iff \mu_0(A') \in \{\mu_0(A') : A' \in \mathcal{A}, A' \subset E \cap X_n\},$$

we have

$$\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} = \{\mu_0(A') : A' \in \mathcal{A}, A' \subset E \cap X_n\},$$

and it follows that

$$\begin{aligned} \mu_{n,+}(E) &= \sup\{\mu_{n,0}(A) : A \in \mathcal{A}, A \subset E\} \\ &= \sup\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} \\ &= \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset E \cap X_n\} \\ &= \mu_+((-) \cap X_n), \end{aligned}$$

$$\begin{aligned} \mu_{n,-}(E) &= -\inf\{\mu_{n,0}(A) : A \in \mathcal{A}, A \subset E\} \\ &= -\inf\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} \\ &= -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E \cap X_n\} \\ &= \mu_-((-) \cap X_n). \qquad \square \end{aligned}$$

Claim 2. We have

$$\mu_+ = \sum_{n=1}^{\infty} \mu_{n,+}, \mu_- = \sum_{n=1}^{\infty} \mu_{n,-}. \quad (3.3.12)$$

Proof. Observe that for all $E \in \sigma(\mathcal{A})$, we find by $(3.3.11)$ (proved by *Claim 1*), we have

$$\begin{aligned} \mu_+(E) &= \sum_{n=1}^{\infty} \mu_+(E \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(E), \\ \mu_-(E) &= \sum_{n=1}^{\infty} \mu_-(E \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(E). \quad \square \end{aligned}$$

Now, we shall proceed to prove the rest of the theorem in this case. To show that μ is an extension of μ_0, observe by $(3.3.10)$, $(3.3.11)$ (proved in *Claim 1*), and $(3.3.12)$ (proved in *Claim 2*) that for all $A \in \mathcal{A}$, we have

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,0}(A) = \sum_{n=1}^{\infty} [\mu_{n,+}(A) - \mu_{n,-}(A)] = \sum_{n=1}^{\infty} [\mu_{n,+}(A)] - \sum_{n=1}^{\infty} [\mu_{n,-}(A)]$$

$$= \mu_+(A) - \mu_-(A) = \mu(A).$$

Next, we show that $\mu_+ \perp \mu_-$, showing that μ_+, μ_- forms a Jordan Decomposition of μ. Since $\mu_{n,+} \perp \mu_{n,-}$, we can choose a Hahn-Decomposition P_n, N_n such that N_n is $\mu_{n,+}$-null and P_n is $\mu_{n,-}$-null. Set

$$P := \bigcup_{n\in\mathbb{N}} [P_n \cap X_n], N := \bigcup_{n\in\mathbb{N}} [N_n \cap X_n],$$

and note that for all $n \in \mathbb{N}$, we have

$$(P_n \cap X_n) \cup (N_n \cap X_n) = X_n \cap (P_n \cup N_n) = X_n \cap X = X_n,$$
$$(P_n \cap X_n) \cap (N_n \cap X_n) = X_n \cap (P_n \cap N_n) = \varnothing,$$

hence

$$\begin{aligned}
P \cup N &= \bigcup_{n\in\mathbb{N}} [(P_n \cap X_n) \cup (N_n \cap X_n)] \\
&= \bigcup_{n\in\mathbb{N}} [(P_n \cup N_n) \cap X_n] \\
&= \bigcup_{n\in\mathbb{N}} [X \cap X_n] \\
&= \bigcup_{n\in\mathbb{N}} X_n \\
&= X, \\
P \cap N &= \bigcup_{m,n\in\mathbb{N}} [(P_m \cap X_m) \cap (N_n \cap X_n)] \\
&= \bigcup_{m\neq n\in\mathbb{N}} [P_m \cap N_n \cap (X_m \cap X_n)] \cup \bigcup_{n\in\mathbb{N}} [X_n \cap (P_n \cap N_n)] \\
&= \bigcup_{m\neq n\in\mathbb{N}} [P_m \cap N_n \cap \varnothing] \cup \bigcup_{n\in\mathbb{N}} [X_n \cap \varnothing] \\
&= \varnothing,
\end{aligned}$$

and by $(3.3.11)$ we have

$$\mu_+(N) = \sum_{n=1}^{\infty} \mu_+(N_n \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(N_n) = 0,$$
$$\mu_-(P) = \sum_{n=1}^{\infty} \mu_+(P_n \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(P_n) = 0,$$

and our conclusion that P, N form a Hahn-decomposition has been reached, finishing *Case 2*.

Next, to show uniqueness of μ extending μ_0, as well as the Jordan Decomposition μ_-, μ_+, note that **Lemma 3.3.3** shows that the extension μ of μ_0 (in this case where μ_0 is finite) is uniquely determined. Then given a Jordan Decomposition μ^+, μ^- of μ, we choose a Hahn Decomposition P', N' of μ^+, μ^- and observe that

$$\mu^+ = \mu((-) \cap P'),\ \ \mu^- = \mu((-) \cap N'). \quad (3.3.13)$$

Next, since for all $E \subset X$, we have

$$\{\mu_0(A) : A \in \mathcal{A}, A \subset E\} \subset \{\mu(F) : F \in \mathcal{M}, F \subset E\},$$

by $(3.2.1)$ and **Proposition 3.1.4** *(iii)* that since

$$\begin{aligned}\mu_+(N') = (\mu_0)^+(N') &= \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset N'\}\\ &\le \sup\{\mu(F) : F \in \sigma(\mathcal{A}), F \subset N'\} = \mu^+(N') = 0,\end{aligned}$$

$$\begin{aligned}\mu_-(P') = (\mu_0)^-(P') &= -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E\}\\ &\le -\inf\{\mu(F) : F \in \sigma(\mathcal{A}), F \subset P'\} = \mu^-(P') = 0,\end{aligned}$$

we conclude by $(3.3.13)$ that for every $E \in \sigma(\mathcal{A})$, we have

$$\mu_-(E \cap P') \le \mu_-(P') = 0,$$
$$\mu_+(E \cap N') \le \mu_+(N') = 0,$$

$$\begin{aligned}\implies \mu_+(E) &= \mu_+(E \cap P') + \mu_+(E \cap N') = \mu_+(E \cap P') + 0 = \mu_+(E \cap P') + \mu_-(E \cap P') = \mu(E \cap P')\\ &= \mu^+(E),\end{aligned}$$

$$\begin{aligned}\mu_-(E) &= \mu_-(E \cap P') + \mu_-(E \cap N') = 0 + \mu_-(E \cap N') = \mu_+(E \cap P') + \mu_-(E \cap P') = \mu(E \cap P')\\ &= \mu^-(E),\end{aligned}$$

and our conclusion is met.

Finally, show that $(3.3.1)$ holds, note from showing that μ_+, μ_- is in fact the Jordan Decomposition of μ, with P, N (as before) defined to be the Hahn decompompomposition of μ_+, μ_-, we have shown that $|\mu| = \mu_+ + \mu_-$, and it remains to show that $|\mu| = \mu_{tot.}$. First, observe by **Proposition 3.1.4** *(iii)* that for all $E \in \mathcal{M}$, we have

$$\begin{aligned}
\mu_{tot.}(E) &= \left\{ \sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E \right\} \\
&\le \sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E \right\} \\
&= |\mu|(E),
\end{aligned}$$

so $|\mu| \le \mu_{tot.}$. Next, we find that since P, N are disjoint, we have

$$\begin{aligned}
&\{\mu_0(A_1) - \mu_0(A_2) : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\} \\
&\subset \{|\mu_0(A_1)| + |\mu_0(A_2)| : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\} \\
&\subset \left\{ \sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset A \right\},
\end{aligned}$$

and since N is μ_+-null and P is μ_--null, find that for all $E \in \mathcal{M}$, we have

$$\begin{aligned}
|\mu|(E) &= \mu_+(E) + \mu_-(E) \\
&= \mu_+(E \cap P) + \mu_+(E \cap N) + \mu_-(E \cap P) + \mu_-(E \cap N) \\
&= \mu_+(E \cap P) + \mu_-(E \cap N) \\
&= \sup\{\mu_0(A_1) - \mu_0(A_2) : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\} \\
&\le \sup\{|\mu_0(A_1)| + |\mu_0(A_2)| : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\} \\
&\le \sup\left\{ \sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E \right\} \\
&= \mu_{tot.}(E),
\end{aligned}$$

showing that $|\mu| \ge \mu_{tot.}$, and our conclusion of $|\mu| = \mu_{tot.}$ is met. ◻

3.4 The Caratheodory Extension for $\mathbb{C}$-valued Premeasures

%SHOW THIS IN THE NEXT DRAFT
%SHOW THAT ONE CAN FORM AN ANALOGOUS EXTENSION FOR $\mathbb{C}$-VALUED MEASURES

3.5 Single Dimension Outer Signed Measures

%SHOW THIS IN THE NEXT DRAFT
%CONSIDER MOVING THIS SECTION TO THE FIRST PART
%TALK ABOUT WHY YOU CAN'T DO SUCH AN EXTENSION WITH OUTER SIGNED MEASURES

%FORMAT BELOW PARTS WHEN READY
%PUT LAST CHAPTER ON HOLD UNTIL THE NEXT DRAFT

4. The Cartheodory Extension for Vector-Valued Measures

4.1 Definitions and Properties of Vector-Valued Measures

%START WRITING IN NEXT DRAFT

4.2 Vector-Valued Premeasure Definition and Examples

%START WRITING IN NEXT DRAFT

4.3 The Cartheodory Extension in the Vector Valued Case

%START WRITING IN NEXT DRAFT

5. Conclusion