A Caratheodory Extension for Signed Measures

Contents

1 Introduction

%DO THIS NEXT
%TALK ABOUT PURPOSE OF PAPER
%FORMAT THE UNFINISHED PAPER FIRST

2 The Original Caratheodory Extension

2.1 The Caratheodory Extension Defined Using Outer-Measures

%SWITCH UP DEFINITIONS TO SOMETHING SYMMETRIC AND TALK ABOUT ISSUES IN REMARK

Definition 2.1.1. Given a nonempty set X, we define an **outer measure**

$\mu^* : \mathcal{P}(X) \to [0, +\infty]$ as a function that satisfies the following properties:

(i) (Null Empty Set) $\mu^*(\varnothing) = 0$.

%CALL THIS PROPERTY "COUNTABLE DISJOINT SUBADDITIVITY"
(ii) (Countable Subadditivity) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ is a collection of disjoint subsets, then

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Source: Folland § 1.4 (page 28)

%MAKE REMARK ABOUT MONOTONICITY PROPERTY AND DISCUSS EQUIVALENT DEFINITION

%MAKE SURE FONT IS CORRECT WITH THIS EXAMPLE
Examples 2.1.2.

(i) Given a measure $\mu : \mathcal{M} \to [0, +\infty]$, we call $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ defined by

$$\mu^*(F) := \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M},\ \bigcup_{n\in\mathbb{N}} E_n \supset F\right\},$$

the induced outer measure of μ. Note that this is a special case of part (ii), and hence we outsource the verification that it is an outer measure there.

(ii) (*Source: Folland 1.10 Proposition page 29*) Given $\mathcal{E} \subset \mathcal{P}(X)$ with $\varnothing, X \in \mathcal{E}$, a function $\rho : \mathcal{E} \to [0, +\infty]$ such that $\rho(\varnothing) = 0$, we define the **induced outer measure** ρ^* on ρ to be the function $\mathcal{P}(X) \to [0, +\infty]$ defined by

$$\rho^*(F) := \inf\left\{\sum_{n=1}^{\infty} \rho(E_n) : \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{n\in\mathbb{N}} E_n \supset F\right\} \quad (2.1.1)$$

We find this is an outer measure since

$$\varnothing \in \mathcal{E} \implies E_n := \varnothing, \text{ for all } n \in \mathbb{N} \text{ gives us } \{E_n\}_{n\in\mathbb{N}} \supset \varnothing \text{ such that} \sum_{n=1}^{\infty} \rho(E_n) = 0,$$

$$\implies \rho^*(\emptyset) \leq \sum_{n=1}^{\infty} \rho(\emptyset) = 0,$$
$$\implies \rho^*(\emptyset) = 0.$$

and given $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\begin{aligned}
\rho^*\left(\bigcup_{n\in\mathbb{N}} F_n\right) &= \inf\left\{\sum_{m=1}^{\infty} \rho(E_m) : \{E_m\}_{m\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{m\in\mathbb{N}} E_m \supset \bigcup_{n\in\mathbb{N}} F_n\right\} \\
&\leq \inf\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \rho(E_{n,m}) : \{E_{n,m}\}_{m\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{m\in\mathbb{N}} E_{n,m} \supset F_n \text{ for every } n \in \mathbb{N}\right\} \\
&\leq \sum_{n=1}^{\infty} \inf\left\{\sum_{m=1}^{\infty} \rho(E_{n,m}) : \{E_{n,m}\}_{m\in\mathbb{N}} \subset \mathcal{E},\ \bigcup_{m\in\mathbb{N}} E_{n,m} \supset F_n\right\} \\
&= \sum_{n=1}^{\infty} \rho^*(F_n),
\end{aligned}$$

hence we have countable subadditivity.

(iii)
%FIND MORE EXAMPLES (INCLUDING LEBESGUE EXAMPLE)
%CONSIDER GIVING OUTER-JORDAN CONTENT AS EXAMPLE

Definition 2.1.3. *(The Caratheodory Criterion for Outer Measures)* Given an outer measure μ^* on X, we state that $F \subset X$ satisfies the **Caratheodory Criterion with respect to μ^*** (or equivalently state that F is **μ^*-measurable**) if for every $E \subset X$, we have

$\mu^*(E) = \mu^*(E \cap F) + \mu^*\left(E \cap F^c\right)$, for all $E \subset X$. $\quad (2.1.2)$
Source: Folland § 1.4 (page 29)
%ALSO STATE EQUIVALENT CONDITION TO THE CARTHEODORY CRITERION FOR OUTER MEASURES

%DO THIS NEXT, AND POSSIBLY ADD $\left(\rho^*\right)(E) = \rho(E)$ FOR ALL $E \in \mathcal{E}$
%SHOW IN REMARK THAT THE CARTHEODORY CRITERION IS NONEMPTY, CLOSED UNDER COMPLEMENTS, FINITE UNIONS, AND SATISFIES THE DISJOINT FINITE ADDITIVITY PROPERTY

Theorem 2.1.4. *(The Original Caratheodory's Theorem)* Given an outer measure μ^* on X, and the collection $\mathcal{M}$ of all subsets of X satisfying the Cartheodory Criterion (i.e. condition (2.2.2)), we find $\mathcal{M}$ is a σ-algebra, and $\mu : \mathcal{M} \to [0, +\infty]$ defined by $\mu := \mu^*|\mathcal{M}$ is a complete measure.
Source: Folland 1.11 Theorem (page 29)

%GIVE REMARK ABOUT PROVING THIS THEOREM DESPITE FOLLAND PROVING IT

Proof. To begin, we show $\mathcal{M}$ is a σ-algebra. First, note that for every $E \subset X$, we have

$$\mu^*(E) = 0 + \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*\left(E \cap \varnothing^c\right),$$

hence $\varnothing \in \mathcal{M} \implies \mathcal{M} \neq \varnothing$. Next, observe that if $F \in \mathcal{M}$, then for all $E \subset X$, we have

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*\left(E \cap F^c\right) = \mu^*\left(E \cap F^c\right) + \mu^*\left(E \cap F^{cc}\right),$$

which shows $F^c \in \mathcal{M}$. Finally, suppose $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$. Then by countable subadditivity, we have

$$\mu^*(E) \leq \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right),$$

and to show that $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$, it remains to show that

$$\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) \leq \mu^*(E).$$

Note that we have two cases.

Case 1. Suppose $\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) < +\infty$. Observe that for all $E \subset X$ and $N \geq 1$, we have

$$
\begin{aligned}
\mu^*(E) &= \mu^*(E \cap F_1) + \mu^*\left(E \cap F_1^c\right) \\
&= \mu^*(E \cap F_1) + \mu^*\left(\left(E \cap F_1^c\right) \cap F_2\right) + \mu^*\left(\left(E \cap F_1^c\right) \cap F_2^c\right) \\
&= \mu^*((E \cap (F_1 \cup F_2)) \cap F_1) + \mu^*\left((E \cap (F_1 \cup F_2)) \cap F_1^c\right) + \mu^*\left(E \cap F_1^c \cap F_2^c\right) \\
&= \mu^*(E \cap (F_1 \cup F_2)) + \mu^*\left(E \cap F_1^c \cap F_2^c\right) \\
&= \mu^*(E \cap (F_1 \cup F_2)) + \mu^*\left(\left(E \cap F_1^c \cap F_2^c\right) \cap F_3\right) + \mu^*\left(\left(E \cap F_1^c \cap F_2^c\right) \cap F_3^c\right) \\
&= \mu^*\left(\left(E \cap \left(\bigcup_{n=1}^{3} F_n\right)\right) \cap \left(\bigcup_{n=1}^{2} F_n\right)\right) + \mu^*\left(\left(E \cap \bigcup_{n=1}^{3} F_n\right) \cap \left(\bigcup_{n=1}^{2} F_n\right)^c\right) + \mu^*\left(E \cap \bigcap_{n=1}^{3} F_n^c\right) \\
&= \mu^*\left(E \cap \left(\bigcup_{n=1}^{3} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{3} F_n^c\right)\right) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{N} F_n^c\right)\right),
\end{aligned}
$$

and moreover for all $N_0 \geq 1$ and for all $E \subset X$, we have

$$
\begin{aligned}
&\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) = \mu^*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1\right) + \mu^*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1^c\right) \\
&= \mu^*(E \cap F_1) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1^c\right]\right) \\
&= \mu^*(E \cap F_1) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1^c\right] \cap F_2\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus F_1\right] \cap F_2^c\right) \\
&= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right) \\
&= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right)
\end{aligned}
$$

$$+\mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right] \cap F_3\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right] \cap F_3^c\right)$$

$$= \mu^*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu^*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right)$$

$$+\mu^*\left(E \cap \left[F_3 \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right) + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{3} F_n\right)\right]\right)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$= \sum_{N=1}^{N_0-1}\left[\mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu^*\left(E \cap \left[\bigcup_{n=1}^{N_0} F_n \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right]\right)$$

$$= \sum_{N=1}^{N_0-1}\left[\mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] + \mu^*\left(E \cap \left[F_{N_0} \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right]\right)$$

$$= \sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right),$$

hence by monotonicity we have

$$\sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) = \mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) \le \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) < +\infty.$$

It follows that we take $N_0 \to +\infty$ and find that for any $E \subset X$, we find by countable subadditivity that

$$\sum_{N=1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) \le \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right)$$

$$\implies \lim_{N_0 \to +\infty} \sum_{N=N_0+1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) = 0.$$

$$\begin{aligned}
\Longrightarrow \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) &= \mu^*\left(\bigcup_{N=1}^{\infty}\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right) \\
&\leq \sum_{N=1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) \\
&= \lim_{N_0 \to +\infty}\left[\sum_{N=1}^{N_0} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right) + \sum_{N=N_0+1}^{\infty} \mu^*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n\right)\right]\right)\right] \\
&= \lim_{N_0 \to +\infty} \mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right),
\end{aligned}$$

and since $E \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right) \subset E \cap \left(\bigcap_{n=1}^{N_0} F_n^c\right)$, for all $N_0 \geq 1$, we conclude that

$$\begin{aligned}
\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) &\leq \lim_{N_0 \to +\infty}\left[\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right)\right)\right] \\
&\leq \lim_{N_0 \to +\infty}\left[\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcap_{n=1}^{N_0} F_n^c\right)\right)\right] \\
&= \lim_{N_0 \to +\infty}\left[\mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right)\right] \\
&= \lim_{N_0 \to +\infty}\left[\mu^*(E)\right] \\
&= \mu^*(E).
\end{aligned}$$

Case 2. Suppose $\mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) = +\infty$. Then

$$+\infty = \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) \leq \mu^*(E), \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right)$$

$$\implies \mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) = +\infty,$$

$$\implies \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right) \leq \mu^*(E).$$

Next, we want to show that $\mu := \mu^*|\mathcal{M}$ is a complete measure. First, observe that $\mu(\varnothing) = \mu^*(\varnothing) = 0$ by definition and for disjoint additivity, we find for disjoint $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty} F_n\right) \leq \sum_{n=1}^{\infty} \mu^*(F_n),$$

%CITE THE REMARK WHERE THIS IS VERIFIED
by countable subadditivity, and by finite disjoint subadditivity and monotonicity we have

$$\sum_{n=1}^{N_0} \mu^*(F_n) = \mu^*\left(\bigcup_{n=1}^{N_0} F_n\right) \leq \mu^*\left(\bigcup_{n=1}^{\infty} F_n\right) \implies \sum_{n=1}^{\infty} \mu^*(F_n) \leq \mu^*\left(\bigcup_{n=1}^{\infty} F_n\right).$$

Finally, to show completeness of the measure, given $N \in \mathcal{M}$ such that $\mu^*(N) = 0$ and $S \subset N$, we find that given $E \subset X$, we by find countable subadditivity that

$$\mu^*(E) \leq \mu^*(E \cap S) + \mu\left(E \cap S^c\right),$$

and since $E \cap S \subset S \subset N$ and $E \cap S^c \subset E$, we find by monotonicity that

$$\mu^*(E \cap S) + \mu^*\left(E \cap S^c\right) \leq \mu^*(N) + \mu^*(E) = \mu^*(E) \implies S \in \mathcal{M}. \quad \square$$

2.2 Applying the Definition to Premeasures

Definition 2.2.1. Given an algebra $\mathcal{A} \subset \mathcal{P}(X)$ we define a **premeasure** to be a function $\mu_0 : \mathcal{A} \to [0, +\infty]$ such that

(i) $\mu_0(\varnothing) = 0.$

(ii) Given a countable family $\mathcal{F} \subset \mathcal{A}$ of pairwise disjoint sets such that $\cup \mathcal{F} \in \mathcal{A}$, we have

$$\mu_0(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_0(F).$$

Source: Folland § 1.4 (page 30)

Definition 2.2.2.

(i) We state that an outer measure $(\mu_0)^*$ (as defined in **Definition 2.1.1**) is **σ-finite** if there exists a sequence $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ such that $X = \bigcup_{n \in \mathbb{N}} E_n$ and $(\mu_0)^*(E_n) < +\infty$, for each $n \in \mathbb{N}$.

(ii) We state that a premeasure μ_0 (as defined in **Definition 2.2.1**) is **σ-finite** if there exists a sequence $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ of sets such that $X = \bigcup_{n \in \mathbb{N}} E_n$ and $\mu_0(E_n) < +\infty$, for each $n \in \mathbb{N}$.

Source: Folland 1.14 Theorem (page 31)

Remark. Note that the terminology Definition 2.2.2. (ii) is used in Folland in 1.14 Theorem to describe the premeasure μ_0, which Folland then does not attempt to define. Although it is unclear whether the text meant to refer the to μ_0 or the measure μ that exists generated by μ_0, which the theorem says exists, and to clarify I plan on both looking up the errata as well as starting a discussion on Stackoverflow.

But whether or not Folland meant to invoke this definition or not, the next proposition below resolves the confusion by stating that an outer-measure and/or pre-measure being σ-finite is logically equivalent to the measure generated by such outer-measure and/or pre-measure.
%START FORUM IN STACKEXCHANGE

%TALK ABOUT WHY THIS NEXT PROPOSITION IS NOT

Proposition 2.2.3. Suppose μ_0 is a premeasure $(\mu_0)^*$ is the induced outer measure on μ_0, as defined in $(2.1.1)$ and in **Examples 2.1.2** *(i)*. Then the following are equivalent:

(i) μ_0 is σ-finite (as a premeasure).

(ii) The measure $\mu := (\mu_0)^* | \mathcal{M}$ defined (and proved to be a measure) in **Theorem 2.1.4** is σ-finite.

(iii) $(\mu_0)^*$ is σ-finite (as an outer measure).

Proof. We shall prove (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii). Choose $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $X = \bigcup_{n\in\mathbb{N}} E_n$ and $\mu_0(E_n) < +\infty$, for every $n \in \mathbb{N}$. It immediately follows by $(2.1.1)$ that

$$\mu(E_n) = (\mu_0)^*(E_n) \le \mu_0(E_n) < +\infty,$$

and we conclude that our desired conclusion is reached by $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$.

(ii) $\implies$ (iii). Immediate from the fact that $\mathcal{M} \subset \mathcal{P}(X)$ and $\mu = (\mu_0)^*|\mathcal{M}$.

(iii) $\implies$ (i). Choose $\{X_m\}_{m\in\mathbb{N}} \subset \mathcal{P}(X)$ such that $X = \bigcup_{m\in\mathbb{N}} E_m$ and $(\mu_0)^*(X_m) < +\infty$. Using $(2.1.1)$, we can for each $m \in \mathbb{N}$, choose $\{E_{m,n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} E_{m,n} \supset X_m$ and

$$\left(\mu^*\right)(X_m) \le \sum_{n=1}^{\infty} \mu_0(E_{m,n}) < +\infty.$$

Then for all $m, n \in \mathbb{N}$, we have

$$\mu_0(E_{m,n}) < +\infty,$$

hence, since $\{E_{m,n}\}_{m,n\in\mathbb{N}}$ is a countable family in $\mathcal{A}$ such that

$$X = \bigcup_{m\in\mathbb{N}} E_m \subset \bigcup_{m,n\in\mathbb{N}} E_{m,n} \subset X \implies X = \bigcup_{m,n\in\mathbb{N}} E_{m,n},$$

our conclusion that μ_0 is σ-finite is reached. □
%CHECK THIS PROOF AND BOLD AND ITALICIZE EVERYTHING
%THEN MAKE SURE THAT THIS SECTION IS COMPLETE

Theorem 2.2.4. *(Caratheodory Extension for the Induced Outer-Measure)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a premeasure on $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ that extends μ_0, namely, $\mu = \mu^*|\mathcal{M}$.

Moreover, if ν is another measure that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < +\infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.
Source: Folland 1.14 Theorem (page 31)

%GIVE REMARK ABOUT PROVING THIS THEOREM DESPITE FOLLAND PROVING IT

%FINISH THIS PROOF
Proof. First, note that for all $A \in \mathcal{A}$, we have

$$\mu(A) = \mu_0(A),$$

since by definition we have $\mu(A) = \mu^*(A) \leq \mu_0(A)$ and for all $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} A_n \supset A$, we have by countable subadditivity and monotonicity

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \geq \mu^*(A) = \mu(A),$$

giving us $\mu(A) \geq \mu_0(A)$.

%POSSIBLY OUTSOURCE THIS AS A PROPERTY OF OUTER-MEASURE

Next, we find that $\mathcal{M}$ satisfies the Caratheodory Criterion, since $\mathcal{M}$ is the intersection of all σ-algebras containing $\mathcal{A}$ and we showed using **Theorem 2.1.4** that the set of all subsets of X satisfying the Caratheodory Criterion is a σ-algebra.

Note that given any other ν extending μ_0, we have

$$\nu(A) = \mu_0(A) = \mu(A), \text{ for all } A \in \mathcal{A}, \quad (2.2.1)$$

and it follows that for any $E \in \mathcal{M}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} A_n \supset E$, we have by countable subadditivity of the induced outer measure ν^* on ν, we have

$$\nu(E) = \nu^*(E) \leq \sum_{n=1}^{\infty} \nu^*(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu_0(A) = \sum_{n=1}^{\infty} \mu(A_n),$$

hence we have

$$\nu(E) \leq \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A} \text{ and } \bigcup_{n\in\mathbb{N}} A_n \supset E\right\} = \mu^*(E) = \mu(E). \quad (2.2.2)$$

Using Lemma 3.3.3 of this paper (which is proved without using any arguments that utilize any result dependent on this theorem), we find that if μ is σ-finite, we find that μ is the unique extension of μ_0, and it immediately follows that $\nu(E) = \mu(E)$, for all $E \in \mathcal{M}$ (whether E is finite with respect to μ or not). □
%EXAMINE WHETHER THE ARGUMENT IS STATED CORRECTLY

2.3 The Caratheodory Extension Defined Using Inner-Measures

%NEXT CITE SOURCES
%AND AFTER THAT MAKE NOTES OF
Definition 2.3.1. Given a nonempty set X, we define an **inner measure** on $\mu_* : \mathcal{P}(X) \to [0, +\infty]$ as a function that satisfies the following properties:

(i) (Null Empty Set) $\mu_*(\varnothing) = 0$.

(ii) (Disjoint Superadditivity) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ is a collection of disjoint subsets, then

$$\mu_*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \sum_{n=1}^{\infty} \mu_*(E_n).$$

%MAKE REMARK ABOUT ADDITIONAL PROPERTIES ON WIKIPEDIA AND MENTION σ-FINITE CONDITIONS

%NEXT FIGURE OUT WHAT TO DO WITH ORGANIZING THESE PROPERTIES
(iii)
%POSSIBLY MENTION THE MONOTONIC PROPERTY, THE LIMIT PROPERTIES, \SIGMA-FINITE PROPERTY, AND/OR SHOW THAT THESE PROPERTIES ARE IMPLIED BY THE PRIOR PROPERTIES IN LEMMA 2.3.2

Source: Modification of definition given in "Inner Measure" (from [Wikipedia link here])

Examples 2.3.3.

(i) (Source: Modification of definition given in "Inner Measure"; from [Wikipedia link here])
Given a premeasure $\mu_0 : \mathcal{A} \to [0, +\infty]$, we define the **induced inner measure** $(\mu_0)_*$ on μ_0 to be the function $\mathcal{P}(X) \to [0, +\infty]$ defined by

$$(\mu_0)_*(E) := \sup\{\mu_0(A) : A \in \mathcal{A},\ A \subset E\}. \quad (2.3.1)$$

We find this is an inner measure since

$$A \subset \varnothing \implies A = \varnothing \implies \mu_0(A) = 0 \text{ for all } A \subset \varnothing \implies (\mu_0)_*(\varnothing) = 0,$$

and for disjoint $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\sum_{n=1}^{\infty} (\mu_0)_*(E_n) = \sup\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

so it suffices to show that given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \mu_0(A_n) \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

Given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, note that $\{A_n\}_{n\in\mathbb{N}}$ is disjoint. For every $N \geq 1$, we find $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and $\bigcup_{n=1}^{N} A_n \subset \bigcup_{n\in\mathbb{N}} E_n$, so we have

$$\sum_{n=1}^{N} \mu_0(A_n) = \mu_0\left(\bigcup_{n=1}^{N} A_n\right) \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

hence

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \sup_{N\geq 1}\left\{\sum_{n=1}^{N} \mu_0(A_n)\right\} \leq (\mu_0)_*\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

%GIVE A SECOND EXAMPLE INVOLVING AN "INNER-MEASURE" INDUCED BY THE OUTER MEASURE
%THEN ALSO MAYBE GIVE HALMOS EXAMPLE INVOLVING THE STANDARD LEBESGUE MEASURE
%CONSIDER GIVING INNER-JORDAN CONTENT AS EXAMPLE

Definition 2.3.4. *(The Caratheodory Criterion for Inner Measures)* Given an inner measure μ_* on X, we state that $F \subset X$ satisfies the **Caratheodory Criterion** with respect to μ_* (or equivalently state that F is **μ_*-measurable**) if for every $E \subset X$, we have

$$\mu_*(E) = \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X. \quad (2.3.2)$$

Source: Analogue of Folland § 1.4 (page 29) applied to inner measures.

%MAKE REMARKS AND BETTER ORGANIZE THESE DERIVATIONS
Note that given $F \subset X$ since disjoint superadditivity automatically gives us

$$\mu_*(E) \geq \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X,$$

we find the Cartheodory criterion for inner measures is equivalent to showing that

$$\mu_*(E) \leq \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right), \text{ for all } E \subset X. \quad (2.3.3)$$

Moreover, we find that given disjoint $\{F_1, \ \ldots, F_{N_0}\}$ satisfying Cartheodory's Criterion, we find by repeated use of $(2.3.2)$, we find for all $E \subset X$ that

$$\begin{aligned}
\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right) &= \mu_*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1\right) + \mu_*\left(\left[E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)\right] \cap F_1^c\right) \\
&= \mu_*(E \cap F_1) + \mu_*\left(E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right) \\
&= \mu_*(E \cap F_1) + \mu_*\left(\left[E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right] \cap F_2\right) + \mu_*\left(\left[E \cap \left(\bigcup_{n=2}^{N_0} F_n\right)\right] \cap F_2^c\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_*(E \cap F_1) + \mu_*(E \cap F_2) + \mu_*\left(E \cap \left(\bigcup_{n=3}^{N_0} F_n\right)\right) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_n) + \mu_*\left(E \cap \left(\bigcup_{n=N_0}^{N_0} F_n\right)\right) \\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_n) + \mu_*(E \cap F_{N_0}) \\
&= \sum_{n=1}^{N_0-1} \mu_*(E \cap F_{N_0}),
\end{aligned}
$$

and it follows that for arbitrary (not necessarily disjoint) $\{F_1, \ldots, F_{N_0}\}$ satisfying Cartheodory's Criterion, we find that

$$
\left\{F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right), F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right), \ldots, F_{N_0} \setminus \left(\bigcup_{n=1}^{N_0-1} F_n\right)\right\},
$$

forms a disjoint partition of $\bigcup_{n=1}^{N_0} F_n$, hence

$$
\begin{aligned}
&\mu_*(E) = \mu_*(E \cap F_1) + \mu_*\left(E \cap F_1^c\right) \\
&= \mu_*(E \cap F_1) + \mu_*\left(\left[E \cap F_1^c\right] \cap F_2\right) + \mu_*\left(\left[E \cap F_1^c\right] \cap F_2^c\right) \\
&= \mu_*(E \cap F_1) + \mu_*(E \cap [(F_1 \cup F_2) \setminus F_1]) + \mu_*\left(E \cap \left(F_1^c \cap F_2^c\right)\right) \\
&= \mu_*(E \cap F_1) + \mu_*(E \cap (F_2 \setminus F_1)) + \mu_*\left(\left[E \cap \left(F_1^c \cap F_2^c\right)\right] \cap F_3\right) + \mu_*\left(\left[E \cap \left(F_1^c \cap F_2^c\right)\right] \cap F_3^c\right) \\
&= \mu_*\left(E \cap \left[F_1 \setminus \left(\bigcup_{n=1}^{0} F_n\right)\right]\right) + \mu_*\left(E \cap \left[F_2 \setminus \left(\bigcup_{n=1}^{1} F_n\right)\right]\right) + \mu_*\left(E \cap \left[F_3 \setminus \left(\bigcup_{n=1}^{2} F_n\right)\right]\right) \\
&\quad + \mu_*\left(E \cap \left(\bigcap_{n=1}^{3} F_1^c\right)\right) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

$$= \sum_{N=1}^{N_0} \left[\mu_* \left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] + \mu_* \left(E \cap \left(\bigcap_{n=1}^{N_0} F_n^c \right) \right)$$
$$= \sum_{N=1}^{N_0} \left[\mu_* \left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right)$$
$$= \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right).$$

%SHOW COUNTEREXAMPLE OF FAILURE WITHOUT INNER-MEASURE σ-FINITE CONDITION

Theorem 2.3.5. *(A Caratheodory Theorem for Inner Measures)* Given a σ-finite inner measure μ_* on X, and the collection $\mathcal{M}$ of all subsets of X satisfying the Cartheodory Criterion (i.e. condition $(2.3.2)$), we find μ_* is a σ-algebra, and $\mu : \mathcal{M} \to [0, +\infty]$ defined by $\mu := \mu_* | \mathcal{M}$ is a complete measure.

Source: Analogue of Folland 1.11 Theorem (page 29) applied to inner measures.

Proof. To begin, we show $\mathcal{M}$ is a σ-algebra. First, note that for every $E \subset X$, we have

$$\mu_*(E) = 0 + \mu_*(E) = \mu_*(E \cap \varnothing) + \mu_*\left(E \cap \varnothing^c\right),$$

hence $\varnothing \in \mathcal{M} \implies \mathcal{M} \neq \varnothing$. Next, observe that if $F \in \mathcal{M}$, then for all $E \subset X$, we have

$$\mu_*(E) = \mu_*(E \cap F) + \mu_*\left(E \cap F^c\right) = \mu_*\left(E \cap F^c\right) + \mu_*\left(E \cap F^{cc}\right),$$

which shows $F^c \in \mathcal{M}$. Finally, suppose $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$. Given $E \subset X$, first, note that for all $N_0 \geq 1$, we have

$$\mu_*(E) = \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_* \left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right),$$

hence

$$\mu_*(E) = \lim_{N_0 \to +\infty} [\mu_*(E)] = \lim_{N_0 \to +\infty} \left[\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) \right].$$

In the case where $\mu_*(E) < +\infty$, we find by Disjoint Superadditivity that

$$\begin{aligned}
\mu_*(E) &= \lim_{N_0 \to +\infty} \left[\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) \right] \\
&= \lim_{N_0 \to +\infty} \sum_{N=1}^{N_0} \left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] + \lim_{N_0 \to +\infty} \mu_*\left(\bigcap_{n=1}^{N_0} \left[E \cap F_n^c \right] \right) \\
&= \sum_{N=1}^{\infty} \left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] + \mu_*\left(\bigcap_{n=1}^{\infty} \left[E \cap F_n^c \right] \right) \\
&\leq \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n \right) \right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n \right)^c \right),
\end{aligned}$$

%COMPLETE THIS CASE USING σ-FINITE CONDITION
In the case where $\mu_*(E) = +\infty$, we find that

$$\lim_{N_0 \to +\infty} \left[\mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) \right] = +\infty,$$

hence either

$$\lim_{N_0 \to +\infty} \sum_{N=1}^{N_0} \left[\mu_*\left(E \cap \left[F_N \setminus \left(\bigcup_{n=1}^{N-1} F_n \right) \right] \right) \right] = \lim_{N_0 \to +\infty} \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right) \right) = +\infty \text{ or}$$

$$\lim_{N_0 \to +\infty} \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n \right)^c \right) = +\infty$$

$$\mu_*(E), \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right) \geq \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) = +\infty$$

$$\implies \mu_*(E) = +\infty = \mu_*\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu_*\left(E \cap \left(\bigcup_{n=1}^{N_0} F_n\right)^c\right).$$

In both cases, we have shown that condition (2.3.2) holds for $\bigcup_{n=1}^{\infty} F_n$, and $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$ immediately follows.

Next, we want to show that $\mu := \mu_*|\mathcal{M}$ is a complete measure. First, observe that $\mu(\emptyset) = \mu_*(\emptyset) = 0$ by definition and for disjoint additivity, we find for disjoint $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$, in the case where

$$\mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) < +\infty,$$

we have for all $N_0 \geq 1$, we find by finite disjoint additivity that

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{N_0} \mu_*(F_n) + \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right),$$

hence we have

%MENTION MONOTONIC PROPERTY USED TO JUSTIFY THIS

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} F_n\right) &= \lim_{N_0 \to +\infty}\left[\sum_{n=1}^{N_0} \mu_*(F_n) + \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right)\right] \\
&= \lim_{N_0 \to +\infty} \sum_{n=1}^{N_0} \mu_*(F_n) + \lim_{N_0 \to +\infty} \mu_*\left(\bigcup_{n=N_0+1}^{\infty} F_n\right) \\
&= \sum_{n=1}^{\infty} \mu_*(F_n) + \mu_*\left(\bigcap_{N_0=1}^{\infty}\left[\bigcup_{n=N_0+1}^{\infty} F_n\right]\right)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \mu_*(F_n) + \mu_*(\varnothing)$$
$$= \sum_{n=1}^{\infty} \mu(F_n).$$

In the case where

$$\mu_*\left(\bigcup_{n=1}^{\infty} F_n\right) = +\infty,$$

we find that
%FINISH THIS ARGUMENT

Next, to show completeness of the measure, given $N \in \mathcal{M}$ such that $\mu_*(N) = 0$ and $S \subset N$, we that for all $E \subset X$, we find by monotonicity that since $E \cap S \subset E \cap N \subset N$, we have

$$0 \leq \mu_*(E \cap N), \mu_*(E \cap S) \leq \mu_*(N) = 0 \Longrightarrow \mu_*(E \cap N) = \mu_*(E \cap S) = 0,$$

hence $(2.2.3)$ is satisfied, and we conclude that $S \in \mathcal{M}$.

%SHOW THAT σ-FINITE CONDITION IS SATISFIED (OR COME UP WITH A COUNTEREXAMPLE WHERE IT'S NOT SATISFIED POSSIBLY USING SOME KIND OF WEIRD INNER SUM CONDITION)
□

%SHOW THAT σ-FINITENESS OF PREMEASURES IS EQUIVALENT TO σ-FINITENESS OF INDUCED INNER MEASURES
%TRY THIS EXAMPLE AND TRY TO SHOW THAT ONLY INFINITE AND MEASURE ZERO SETS SATISFY THE CRITERION, AND HENCE THE MEASURE FAILS TO BE σ-FINITE

$$\mu_*(A) := \sup\left\{\int \phi dm : \phi \text{ is of the form } \sum_{k=1}^{N} a_k \mathbb{1}_{I_k} \leq \mathbb{1}_A \text{ for intervals } I_1, \ldots, I_k\right\}$$

%WORK OUT THIS EXAMPLE DURING THE NEXT DRAFT

Theorem 2.3.6. *(A Caratheodory Extension for the Induced Inner-Measure)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a σ-finite premeasure on $\mathcal{A}$. Then $\mu := \mu_*|\mathcal{M}$ defines a uniquely determined σ-finite measure that extends μ_0.

Source: Analogue of Folland 1.14 Theorem (page 31) applied to inner-measures
%POSSIBLY FINISH THIS THEOREM HERE AND FIGURE OUT IF MORE NEEDS TO BE SAID

Proof. First, given $A \in \mathcal{A}$, we find for all $B \in \mathcal{A}$ such that $B \subset A$ we have by monotonicity

$$\mu_0(B) = \mu_*(B) \leq \mu_*(A) = \mu_0(A) \Longrightarrow \mu_*(A) \leq \mu_0(A).$$

It follows that since A itself is in $\mathcal{A}$ and $A \subset A$, we have $\mu_*(A) \geq \mu_0(A)$, hence

$$\mu(A) = \mu_*(A) = \mu_0(A),$$

and we've shown $\mu_*|\mathcal{M}$ is an extension of μ_0.

Next, we find that $\mathcal{M}$ satisfies the Caratheodory Criterion (for inner measures), since $\mathcal{M}$ is the intersection of all σ-algebras containing $\mathcal{A}$ and we showed using **Theorem 2.3.5** that the set $\mathcal{M}'$ of all subsets of X satisfying the Caratheodory Criterion is a σ-algebra and μ (and more generally $\mu_*|\mathcal{M}'$) is a measure.
%ACTUALLY SHOW THAT ALL $\mathcal{A}$ SATISFIES THE CARATHEODORY CRITERION

Uniqueness follows immediately from **Theorem 2.2.4** and the fact that μ_0 is σ-finite, since in that situation μ must agree with the outer measure $(\mu_0)^*$ induced by μ_0. □

%MAKE COROLLARY ABOUT JORDAN CONTENT

3 Inner and Outer Single Dimension Signed Measures

3.1 Some Properties of the Jordan Decomposition

%BEGIN CITING SOURCES IN THIS SECTION NEXT

Assume that μ, ν are signed ($\mathbb{R}$-valued) measures on a nonempty σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$.

Definition 3.1.1.

(i) We state that μ is **positive** if $\mu(E) \geq 0$, for every $E \in \mathcal{M}$.

(ii) We state that a set $E \in \mathcal{M}$ is **positive for μ** if $\mu(E') \geq 0$ for every $E' \in \mathcal{M}$ such that $E' \subset E$, **negative for μ** if $\mu(E') \leq 0$ for every $E' \in \mathcal{M}$ such that $E' \subset E$, and a **μ-null set** if

$\mu(E') = 0$ for every $E' \in \mathcal{M}$ such that $E' \subset E$.

(iii) We state that two measures μ, ν on a σ-algebra $\mathcal{M}$ are **mutually singular** (and write $\mu \perp \nu$) if there exists disjoint sets $E, F \in \mathcal{M}$ such that $E \cup F = X$ and E is μ-null and F is ν-null.

(iv) The decomposition (P, N) consisting of $P, N \in \mathcal{M}$ such that P is positive for μ, N is negative for μ, $X = P \cup N$ and $P \cap N = \varnothing$ is called a **Hahn Decomposition**.
Source: Folland § 3.1 (page 86-87)

%DEFINE σ-FINITE IN SIGNED MEASURE SETTING AND MAKE REMARK ABOUT HOW FOLLAND DOESN'T CLARIFY THAT

Theorem 3.1.2. *(Jordan Decomposition Theorem)* Given a (signed $\mathbb{R}$-valued) measure μ, there exists two unique positive measures μ^+, μ^- such that $\mu^+ \perp \mu^-$ and $\mu = \mu^+ - \mu^-$.
Source: Folland 3.4 Theorem (page 87).

%TALK ABOUT HOW PROOF IS DONE IN FOLLAND BUT AN ALTERNATIVE PROOF WILL BE PROVIDED BY SHOWING THAT CARATHEODORY EXTENSION IS IN FACT A MORE GENERAL RESULT

Definition 3.1.3.

(i) The unique positive measures μ^+, μ^- that exist by the **Jordan Decomposition Theorem** (given in **Theorem 3.1.2**) such that $\mu^+ \perp \mu^-$ and $\mu = \mu^+ - \mu^-$ is called the **Jordan decomposition of $\boldsymbol{\mu}$** with μ^+ called the **positive variation of $\boldsymbol{\mu}$** and μ^- called the **negative variation of $\boldsymbol{\mu}$**.

(ii) Define the **total varation $|\boldsymbol{\mu}|$** of a (signed $\mathbb{R}$-valued measure) μ to be

$|\mu| := \mu^+ + \mu^-$.

(iii) We call μ **finite** (resp. **$\boldsymbol{\sigma}$-finite**) if $|\mu|$ (as a regular measure) is finite (resp. σ-finite).
Source: Folland § 3.1 (page 87-88).

%MAKE NOTE ON EQUIVALENT CONDITIONS OF μ BEING FINITE

Proposition 3.1.4. *(Properties of the Jordan Decomposition)*

(i) μ^+, μ^- are Jordan Decompositions of $\mu := \mu^+ - \mu^-$ if and only if there exists a Hahn Decomposition (P, N) of μ such that

$\mu^+ = \mu((-) \cap P)$ and $\mu^- = -\mu((-) \cap N)$. $\quad (3.1.1)$

Moreover, if μ^+, μ^- are Jordan decompositions of $\mu := \mu^+ - \mu^-$, then:

(ii) $E \in \mathcal{M}$ is μ-null iff $|\mu|(E) = 0$

(iii) $\mu \perp \nu$ iff $|\mu| \perp \nu$ iff $\mu^+ \perp \nu$ and $\mu^- \perp \nu$.

(iv) For all $E \in \mathcal{M}$, we have

$$\mu^+(E) = \sup\{\mu(F) : F \in \mathcal{M}, F \subset E\},$$
$$\mu^-(E) = -\inf\{\mu(F) : F \in \mathcal{M}, F \subset E\}, \tag{3.1.2}$$
$$\begin{aligned} |\mu|(E) &= \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E\right\} \\ &= \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\}. \end{aligned} \tag{3.1.3}$$

Source: Folland § 3.1 Exercise 2 and 7 (page 88)

Note that originally given in *Folland § 3.1 Exercise 7 (page 88)* is the property

$$|\mu|(E) = \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E\right\},$$

as opposed to $(3.1.1)$ (which we verify in the proof below). We do this because taking the supremum of the set

$$\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\},$$

moreso corresponds to the sets we take the supremum of when defining important definitions in Definition 3.2.2, which we do specifically in equations $(3.2.3)$ and $(3.2.2)$, and we find this perspective is helpful when verifying $(3.3.1)$ in the proof of

%DO THIS DISCUSSION

%DISCUSS/EXPLORE CHANGE FROM $\bigcup_{j=1}^{n} E_j = E$ TO $\bigcup_{j=1}^{n} E_j \subset E$, AND JUSTIFY WHY THIS CHANGE IS A GOOD IDEA

Proof.

(i) Observe that

$$
\begin{aligned}
&\mu^+ = \mu((-) \cap P), \mu^- = -\mu((-) \cap N)), \text{ for Hahn Decomposition } (P, N) \text{ of } \mu; \\
\iff &\mu(E) = \mu(E \cap P) + \mu(E \cap N) \\
&\quad = \mu^+(E) - \mu^-(E), \qquad \text{for Hahn Decompositon } (P, N) \text{ of } \mu; \\
&\mu^+(N) = \mu(P \cap N) = 0 = -\mu(P \cap P) = \mu^-(P), \\
\iff &\mu^+ \perp \mu^- \text{ and } \mu = \mu^+ - \mu^-; \\
\iff &\mu^+, \mu^- \text{ is a Jordan Decomposition of } \mu.
\end{aligned}
$$

(ii) Let (P, N) be a Hahn Decomposition of μ, and observe that for $E \in \mathcal{M}$

$$
\begin{aligned}
E \text{ is } \mu\text{-null} &\iff \mu(F) = 0 \text{ for all } F \in \mathcal{M} \text{ such that } F \subset E; \\
&\iff \mu^+(F) = \mu(F \cap P) = 0 \text{ and } \mu^-(F) = \mu(F \cap N) = 0 \text{ for all } F \in \mathcal{M} \text{ such that } F \subset E; \\
&\iff 0 \le \mu(F \cap P) \le \mu(E \cap P) = 0 \text{ and } 0 \le -\mu(F \cap N) \le -\mu(E \cap N) = 0, \\
&\qquad \text{for all } F \in \mathcal{M} \text{ such that } F \subset E; \\
&\iff \mu^+(E) = \mu(E \cap P) = \mu^-(E) = -\mu(E \cap N) = 0; \\
&\iff |\mu|(E) = \mu^+(E) + \mu^-(E) = 0.
\end{aligned}
$$

%CHECK OVER THIS

(iii) Observe that

$$
\begin{aligned}
\mu \perp \nu &\implies \exists E, F \in \mathcal{M} \text{ s.t. } E \text{ is } \mu\text{-null}, \ F \text{ is } \nu\text{-null}, \text{ and } E \cup F = X, \ E \cap F = \varnothing; \\
&\implies \exists E, F \in \mathcal{M} \text{ s.t. } \mu^+(E') = \mu(E' \cap P_\mu) = \mu^-(E') = -\mu(E' \cap N_\mu) = 0, F \text{ is } \nu\text{-null}, \\
&\qquad \text{and } E \cup F = X, \ E \cap F = \varnothing, \\
&\qquad \text{for all } E' \in \mathcal{M} \text{ s.t. } E' \subset E, \text{ for Hahn-Decomposition } (P_\mu, N_\mu) \text{ of } \mu; \\
&\implies \exists E, F \in \mathcal{M} \text{ s.t. } E \text{ is } \nu^+\text{-null and } \nu^-\text{-null}, \ F \text{ is } \nu\text{-null}, \text{ and } E \cup F = X, \ E \cap F = \varnothing; \\
&\implies \mu^+ \perp \nu \text{ and } \mu^- \perp \nu \\
&\implies \exists E_+, F_+, E_-, F_- \in \mathcal{M} \text{ s.t. } E_+ \text{ is } \mu^+\text{-null}, \ E_- \text{ is } \mu^-\text{-null}, \ F_+ \text{ and } F_- \text{ is } \nu\text{-null}, \\
&\qquad \text{and } E_+ \cup F_+ = E_- \cup F_- = X, \ E_+ \cap F_+ = E_- \cap F_- = \varnothing;
\end{aligned}
$$

$$
\begin{aligned}
&\implies \exists E_+, F_+, E_-, F_- \in \mathcal{M} \text{ s.t. } 0 \le |\mu|(E') \le |\mu|(E_+ \cap E_-) \le \mu^+(E_+) + \mu^-(E_-) = 0, \\
&\qquad 0 \le |\nu(F')| \le |\nu|(F') \le |\nu(F' \cap P_\nu)| + |\nu(F' \cap N_\nu)| \le 2|\nu(F_+ \cup F_-)| \\
&\qquad\quad \le 2(|\nu(F_+)| + |\nu(F_-)|) = 0, \text{ for all } E', F' \in \mathcal{M} \text{ s.t. } E' \subset E_+ \cap E_-,\ F' \subset F_+ \cup F_-, \\
&\qquad \text{and } (E_+ \cap E_-) \cup (F_+ \cup F_-) = (E_+ \cup F_+ \cup F_-) \cap (E_- \cup F_- \cup F_+) = X, \\
&\qquad\qquad (E_+ \cap E_-) \cap (F_+ \cup F_-) = [F_+ \cap (E_+ \cap E_-)] \cup [F_- \cap (E_+ \cap E_-)] = \varnothing; \\
&\implies \exists E_+, F_+, E_-, F_- \in \mathcal{M} \text{ s.t. } E_+ \cap E_- \text{ is } |\mu|\text{-null},\ F_+ \cup F_- \text{ is } \nu\text{-null}, \\
&\implies |\mu| \perp \nu \\
&\implies \exists E, F \in \mathcal{M} \text{ s.t. } E \text{ is } |\mu|\text{-null},\ F \text{ is } \nu\text{-null}, \text{ and } E \cup F = X,\ E \cap F = \varnothing; \\
&\implies \exists E, F \in \mathcal{M} \text{ s.t. } 0 \le |\mu(E')| \le |\mu|(E) = 0,\ F \text{ is } \nu\text{-null}, \text{ and } E \cup F = X,\ E \cap F = \varnothing, \\
&\qquad \text{for all } E' \in \mathcal{M} \text{ s.t. } E' \subset E \\
&\implies \exists E, F \in \mathcal{M} \text{ s.t. } E \text{ is } \mu\text{-null},\ F \text{ is } \nu\text{-null}, \text{ and } E \cup F = X,\ E \cap F = \varnothing; \\
&\implies \mu \perp \nu.
\end{aligned}
$$

(iv) Note that given $E \in \mathcal{M}$, we find for all $F \in \mathcal{M}$ such that $F \subset E$, we have

$$
\begin{aligned}
&\mu(F) = \mu^+(F) - \mu^-(F) \le \mu^+(F) \le \mu^+(E), \\
&-\mu(F) = -\mu^+(F) + \mu^-(F) \le \mu^-(F) \le \mu^-(E),
\end{aligned}
$$

which gives us

$$
\begin{aligned}
\mu^+(E) &\ge \sup\{\mu(F) : F \in \mathcal{M}, F \subset E\}, \\
\mu^-(E) &\ge \sup\{-\mu(F) : F \in \mathcal{M}, F \subset E\} \\
&= -\inf\{\mu(F) : F \in \mathcal{M}, F \subset E\}
\end{aligned}
$$

Moreover, we find by part (i) of this proposition that there exists Hahn Decompositions (P, N) such that $(3.1.1)$ holds, and it follows that

$$
\begin{aligned}
\mu^+(E) &= \mu(E \cap P) \\
&\le \sup\{\mu(F) : F \in \mathcal{M}, F \subset E\}, \\
\mu^-(E) &= -\mu(E \cap N) \\
&\le \sup\{-\mu(F) : F \in \mathcal{M}, F \subset E\} \\
&= -\inf\{\mu(F) : F \in \mathcal{M}, F \subset E\},
\end{aligned}
$$

hence the verification of $(3.1.2)$ is complete, and it remains to show $(3.1.3)$. Note that since

$$
\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E \right\}
$$

$$\subset \left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E \right\},$$

we find that

$$\sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E \right\}$$
$$\leq \sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E \right\},$$

and for all disjoint $E_1, \ldots, E_n \in \mathcal{M}$ such that $\bigcup_{j=1}^{n} E_j \subset E$, we find that for

$$E_{n+1} := E \setminus \left(\bigcup_{j=1}^{n} E_j \right),$$

we have $\bigcup_{j=1}^{n+1} E_j = E$ and

$$\sum_{j=1}^{n} |\mu(E_j)| \leq \sum_{j=1}^{n+1} |\mu(E_j)|,$$

giving us

$$\sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j = E \right\}$$
$$\geq \sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E \right\},$$

and it remains to show that

$$|\mu|(E) = \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\}.$$

For all disjoint $E_1, \ldots, E_n \in \mathcal{M}$ such that $\bigcup_{j=1}^{n} E_j \subset E$, we have

$$\sum_{j=1}^{n} |\mu(E_j)| = \sum_{j=1}^{n} |\mu^+(E_j) - \mu^-(E_j)| \le \sum_{j=1}^{n} \left[\mu^+(E_j) + \mu^-(E_j)\right] = \sum_{j=1}^{n} |\mu|(E_j) = |\mu|\left(\bigcup_{j=1}^{n} E_j\right)$$
$$\le |\mu|(E),$$

hence

$$|\mu|(E) \ge \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\}.$$

Next, using the previously mentioned Hahn Decomposition (P, N), we find for

$$E_1 := E \cap P, E_2 := E \cap N, \bigcup_{j=1}^{2} E_j \subset E,$$
we have

$$|\mu|(E) = \mu^+(E) + \mu^-(E) = |\mu(E \cap P)| + |\mu(E \cap N)| = \sum_{j=1}^{2} |\mu(E_j)|$$
$$\le \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\},$$

and the verification of $(3.1.3)$ is complete. □
%FIND SIMILAR ARGUMENTS IN NEXT TWO SECTIONS
%PROVE PROPERTIES GIVEN FOLLAND 3.1 EXERCISE 7 ON PAGE 88

3.2 Single Dimension Signed Premeasures and the Induced Jordan Decomposition

Note that from this point on, sources will not be cited, since every the results from this point forward were independently conjectured and originally verified in this paper.

It is my belief that a result at least similar to this one has been proven before, but that the result is very old and hence the literature that proves such a result is hard to pinpoint. In any case, to bridge the generational gap between the understanding of signed premeasures potentially back then and signed premeasures now, we show that these results hold, showing first in this section that the so-called "induced Jordan decomposition", along with the "induced total variation", are inner measures. Then in the next section, we show the Caratheodory Extension holds for σ-finite $\mathbb{R}$-valued premeasures.

Defintion 3.2.1. Given an algebra $(E, \mathcal{P})$, we define a **signed ($\mathbb{R}$-valued) premeasure** $\mu_0 : \mathcal{A} \to [-\infty, +\infty]$ to be a function such that

(i) $\mu_0(\varnothing) = 0$.
(ii) μ_0 assumes at most one of the values $\pm\infty$.
(iii) given a countable family $\mathcal{F} \subset \mathcal{A}$ of pairwise disjoint sets, we have $\sum_{F \in \mathcal{F}} \mu_0(F)$ either converging absolutely or diverging to $\pm\infty$. Moreover, if $\cup \mathcal{F} \in \mathcal{A}$, we have

$$\mu_0(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_0(F).$$

Remark. Note that the above definition is an extension of standard premeasures in the positive-valued setting (as defined in Definition 2.2.1). This is easy to see since condition (i) of each definition of premeasure is identical, condition (ii) of the above definition is satisfied since such measures are positive-valued (and hence cannot take on the value of $-\infty$), and by condition (ii) of Definition 2.2.1 we have implies condition (iii) since a countable collection $\mathcal{F} \subset \mathcal{A}$ of pairwise disjoint sets, we have $\sum_{F \in \mathcal{F}} \mu_0(F) \in [0, +\infty]$, and if $\cup \mathcal{F} \in \mathcal{A}$, we have

$$\mu_0(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_0(F).$$

%GIVE SOME EXAMPLES IN NEXT DRAFT
%IN PARTICULAR, GIVE FUNCTIONS OF BOUNDED VARIATION AND CADLAG FUNCTIONS AS EXAMPLES

Definition 3.2.2. Let μ_0 be a (signed $\mathbb{R}$-valued) premeasure μ_0.

(i) We define the **induced Jordan decomposition $(\mu_0)^+, (\mu_0)^- : \mathrm{P}(\mathrm{X}) \to [0, +\infty]$ of μ_0** as so:

$$(\mu_0)^+(E) := \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset E\}, \qquad (3.2.1)$$
$$(\mu_0)^-(E) := -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E\}.$$

(ii) We define the **induced total variation $|\mu_0| : \mathrm{P}(\mathrm{X}) \to [0, +\infty]$ of μ_0** as so:

$$|\mu_0|(E) = \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\}. \quad (3.2.2)$$

%FIGURE OUT WHETHER NEXT LEMMA IS NECESSARY
Lemma 3.2.3.
%GIVE THE IMPORTANT PROPERTIES OF THE INNER JORDAN DECOMPOSITION
%POSSIBLY DELETE THIS

Lemma 3.2.4. For every signed ($\mathbb{R}$-valued) premeasure μ_0, we find that the induced Jordan decomposition $(\mu_0)^+, (\mu_0)^-$ and inner total variation $|\mu_0|$ are inner measures.
%LEMMA ABOUT INNER JORDAN AND INNER TOTAL VARIATIONS BEING PREMEASURES

Proof. Note that since $\varnothing \in \mathcal{A}$, we have

$$(\mu_0)^+(\varnothing) = (\mu_0)^-(\varnothing) = |\mu_0|(\varnothing) = 0,$$
%SHOW THIS IN MORE DETAIL

and it remains to show that $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ satisfy the properties of disjoint superadditivity. For disjoint $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$, we find that

$$\sum_{n=1}^{\infty} (\mu_0)^+(E_n) = \sup\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$
$$\sum_{n=1}^{\infty} (\mu_0)^-(E_n) = -\inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}, A_n \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

$$\sum_{n=1}^{\infty} |\mu_0|(E) = \sup\left\{\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})| : \{A_{n,1}, \dots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}\ A_{n,1}, \dots A_{n,N_n} \text{ are disjoint, } \bigcup_{j=1}^{N_n} A_{n,j} \subset E_n \text{ for all } n \in \mathbb{N}\right\},$$

so it suffices to show that:

(i) For all $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \mu_0(A_n) \leq (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$
$$-\sum_{n=1}^{\infty} \mu_0(A_n) \leq (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

(ii) For all $\{A_{n,1}, \dots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{j=1}^{N_n} A_{n,j} \subset E_n$, for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} |\mu_0(A_{n,j})| \leq |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right).$$

To show *(i)*, given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subset E_n$, for all $n \in \mathbb{N}$, note that $\{A_n\}_{n\in\mathbb{N}}$ is disjoint. It follows that for every $N \geq 1$, we find $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and $\bigcup_{n=1}^{N} A_n \subset \bigcup_{n\in\mathbb{N}} E_n$, so we have

$$\inf\left\{\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\} \le \mu_0\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu_0(A_n)$$
$$\le \sup\left\{\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\},$$
$$\implies \sum_{n=1}^{N} \mu_0(A_n) \le (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

$$-\sum_{n=1}^{N} \mu_0(A_n) \le \sup\left\{-\mu_0(A) : A \in \mathcal{A}, A \subset \bigcup_{n\in\mathbb{N}} E_n\right\} = (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

hence

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \lim_{N\to+\infty} \sum_{n=1}^{N} \mu_0(A_n) \le (\mu_0)^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$
$$-\sum_{n=1}^{\infty} \mu_0(A_n) = \lim_{N\to+\infty} -\sum_{n=1}^{N} \mu_0(A_n) \le (\mu_0)^-\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

completing our verification that *(i)* holds.

To show *(ii)*, given $\{A_{n,1}, \dots, A_{n,N_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_{n,1}, \dots, A_{n,N_n}$ are disjoint and $\bigcup_{j=1}^{N_n} A_{n,j} \subset E_n$, for all $n \in \mathbb{N}$, note that $\left\{\bigcup_{j=1}^{N_n} A_{n,j}\right\}_{n\in\mathbb{N}}$ is disjoint. It follows that for every $N \ge 1$, we find $\bigcup_{n=1}^{N} \bigcup_{j=1}^{N_n} A_{n,j} \subset \bigcup_{n\in\mathbb{N}} E_n$, and $\bigcup_{n=1}^{N} \{A_{n,1}, \dots, A_{n,N_n}\}$ is a finite disjoint collection, so we have

$$\sum_{n=1}^{N} \sum_{j=1}^{N_n} |\mu_0(A_{n,j})| \le \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \dots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset \bigcup_{n\in\mathbb{N}} E_n\right\}$$
$$= |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

hence we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}|\mu_0(A_{n,j})| = \sup_{N\geq 1}\left\{\sum_{n=1}^{N}\sum_{j=1}^{N_n}|\mu_0(A_{n,j})|\right\} \leq |\mu_0|\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

completing our verification that *(ii)* holds. □

%IN NEXT DRAFT DEFINE A COMPLEX-VALUED PREMEASURE AND ITS INNER JORDAN DECOMPOSITIONS AND INNER TOTAL VARIATION

3.3 The Caratheodory Extension for $\mathbb{R}$-valued Premeasures

In this section, we assume that all premeasures are $\mathbb{R}$-valued signed premeasures.

Definition 3.3.1. *(The Caratheodory Criterion for the Induced Jordan Decomposition)* Given a premeasure μ_0, we state that $F \subset X$ satisfies the **Caratheodory criterion for the induced Jordan decomposition $(\mu_0)^+, (\mu_0)^-$** if it satisfies the Caratheodory criterion for both $(\mu_0)^+$ and $(\mu_0)^-$ as inner measures (see **Definition 2.3.4**), i.e., for every $E \subset X$, we have

$$(\mu_0)^+(E) = (\mu_0)^+(E\cap F) + (\mu_0)^+\left(E\cap F^c\right),$$
$$(\mu_0)^-(E) = (\mu_0)^-(E\cap F) + (\mu_0)^-\left(E\cap F^c\right),$$

or equivalently $(2.3.3)$ holds for both $(\mu_0)^+$ and $(\mu_0)^-$.

%MAKE REMARKS ABOUT OTHER WAYS TO FORMULATE THIS CRITERION IN NEXT DRAFT (IN PARTICULAR LOOK AT THE CARTHEODORY CRITEREON FOR $|\mu_0|$)

Theorem 3.3.2. *(A Caratheodory Theorem for the Induced Jordan Decomposition)* For every σ-finite premeasure μ_0 on an algebra $\mathcal{A}$, there exists a σ-algebra $\mathcal{M} \supset \mathcal{A}$ such that the induced Jordan decompositions $(\mu_0)^+, (\mu_0)^-$ and induced total variation $|\mu_0|$, restricted to $\mathcal{M}$ all form complete measures $\mu_+ := (\mu_0)^+|\mathcal{M},\ \mu_- := (\mu_0)^-|\mathcal{M},$ and $\mu_{tot.} := |\mu_0|\ |\mathcal{M}$.

Proof. Let $\mathcal{M}$ be the set of all $F \subset X$ that satisfy both the Caratheodory criterion for the inner Jordan decomposition and the Caratheodory criterion of $|\mu_0|$ as an inner measure.

First, we shall show that $\mathcal{A} \subset \mathcal{M}$ by showing that given $A \in \mathcal{A}$, we find $(2.3.3)$ holds for

$(\mu_0)^+, (\mu_0)^-, |\mu_0|$. To show that (2.3.3) holds for $(\mu_0)^+, (\mu_0)^-$, we find that given $E \subset X$, we find that for all $A_0 \in \mathcal{A}$ such that $A_0 \subset E$, we have $A_0 \cap A, A_0 \cap A^c \in \mathcal{A}$, such that $A_0 \cap A \subset E \cap A, A_0 \cap A^c \subset E \cap A^c$, hence

$$\begin{aligned}
&\mu_0(A_0) = \mu_0(A_0 \cap A) + \mu_0\left(A_0 \cap A^c\right) \leq (\mu_0)^+(E \cap A) + (\mu_0)^+\left(E \cap A^c\right), \\
&\implies (\mu_0)^+(E) \leq (\mu_0)^+(E \cap A) + (\mu_0)^+\left(E \cap A^c\right),
\end{aligned}$$

$$\begin{aligned}
\mu_0(A_0) &= \mu_0(A_0 \cap A) + \mu_0\left(A_0 \cap A^c\right) \\
&\geq \inf\{\mu_0(A_1) : A_1 \in \mathcal{A}, A_1 \subset E \cap A\} + \inf\left\{\mu_0(A_2) : A_2 \in \mathcal{A}, A_2 \subset E \cap A^c\right\}, \\
\implies &\inf\{\mu_0(A_0) : A_0 \in \mathcal{A}, A_0 \subset E\} \\
&\geq \inf\{\mu_0(A_1) : A_1 \in \mathcal{A}, A_1 \subset E \cap A\} + \inf\left\{\mu_0(A_2) : A_2 \in \mathcal{A}, A_2 \subset E \cap A^c\right\}, \\
\implies &(\mu_0)^-(E) \leq (\mu_0)^-(E \cap A) + (\mu_0)^-\left(E \cap A^c\right).
\end{aligned}$$

To show that (2.3.3) holds for $|\mu_0|$, for all $E \subset X$, we find that for all disjoint $A_1, \ldots, A_N \in \mathcal{A}$ such that $\bigcup_{j=1}^{N} A_j \subset E$, we have disjoint collections

$$\{A_1 \cap A, A_2 \cap A, \ldots, A_N \cap A\}, \left\{A_1 \cap A^c, A_2 \cap A^c, \ldots, A_N \cap A^c\right\} \subset \mathcal{A},$$

such that

$$\bigcup_{j=1}^{N} [A_j \cap A] \subset E \cap A, \ \bigcup_{j=1}^{N} \left[A_j \cap A^c\right] \subset E \cap A^c,$$

hence we have

$$\begin{aligned}
\sum_{j=1}^{N} |\mu_0(A_j)| &= \sum_{j=1}^{N} \left|\mu_0(A_j \cap A) + \mu_0\left(A_j \cap A^c\right)\right| \\
&\leq \sum_{j=1}^{N} |\mu_0(A_j \cap A)| + \sum_{j=1}^{N} \left|\mu_0\left(A_j \cap A^c\right)\right| \\
&\leq |\mu_0|(E \cap A) + |\mu_0|\left(E \cap A^c\right), \\
\implies |\mu_0|(E) &\leq |\mu_0|(E \cap A) + |\mu_0|\left(E \cap A^c\right).
\end{aligned}$$

Next, take $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_{tot.}$ to each be the set of all $F \subset X$ that satisfy the Caratheodory criterion for $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ as inner measures (see **Definition 2.3.4**), respectively. We find by **Theorem 2.3.5** that $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_{tot.}$ are all σ-algebras and that $(\mu_0)^+|\mathcal{M}_+, (\mu_0)^-|\mathcal{M}_-, |\mu_0|\ |\mathcal{M}_{tot.}$ are complete measures. Then since

$$\mathcal{M} = \mathcal{M}_+ \cap \mathcal{M}_- \cap \mathcal{M}_{tot.},$$

we find $\mathcal{M}$, as an intersection of σ-algebras, is itself a σ-algebra; so follows that the further restriction of $(\mu_0)^+, (\mu_0)^-, |\mu_0|$ to $\mu_+, \mu_-, \mu_{tot.}$, respectively, are all measures; and it remains to show that $\mu_+, \mu_-, \mu_{tot.}$ are all complete measures in $\mathcal{M}$.

To do this, note that given $E \in \mathcal{M}$, we find for all $A \in \mathcal{A}$ and $A \subset E$, we have $\mu_0(A), -\mu_0(A) \leq |\mu_0(A)|$ and $\{A\}$ a family of disjoint sets with is union a subset of E, using (3.2.1) and (3.2.2), we have

$$\begin{aligned}
\mu_+(E) &= \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset E\} \\
&\leq \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\} \\
&= \mu_{tot.}(E), \\
\mu_-(E) &= \sup\{-\mu_0(A) : A \in \mathcal{A}, A \subset E\} \\
&\leq \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\} \\
&= \mu_{tot.}(E),
\end{aligned}$$

hence we have

$$0 \leq \mu_+, \mu_- \leq \mu_{tot.}, \quad (3.3.1)$$

and it shall suffice to show that $\mu_{tot.}$ is complete. For all $N \in \mathcal{M}$ such that $\mu_{tot.}(N) = 0$ and $S \subset N$, observe by (3.3.1) that $\mu_+(N) = \mu_-(N) = 0$, hence we find from the fact that $(\mu_0)^+|\mathcal{M}_+, (\mu_0)^-|\mathcal{M}_-, |\mu_0|\ |\mathcal{M}_{tot.}$ are complete that $S \in \mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_{tot.}$ and

$$(\mu_0)^+|\mathcal{M}_+(S) = (\mu_0)^-|\mathcal{M}_-(S) = |\mu_0|\ |\mathcal{M}_{tot.} = 0,$$

so we have $S \in \mathcal{M}$ and our conclusion that $\mu_{tot.}$ is complete is reached. $\square$

Lemma 3.3.3. *(Uniqueness of measures extending signed premeasure)* For every σ-finite signed ($\mathbb{R}$-valued) premeasure μ_0 on an algebra $\mathcal{A}$, any ($\mathbb{R}$-valued) signed measure $\mu : \sigma(\mathcal{A}) \to [-\infty, +\infty]$ extending μ_0 (if it exists) is uniquely determined.

Remark. Note that if $\{A_n\}_{n\in\mathbb{N}}$ is a $\subset$-increasing sequence, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to+\infty} \mu(A_n), \quad (3.3.2)$$

and if $\{B_n\}_{n\in\mathbb{N}}$ is a $\subset$-decreasing sequence such that $|\mu(B_n)| < +\infty$ eventually, then

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to+\infty} \mu(B_n). \quad (3.3.3)$$

Important to note that this is stated in *Folland § 3.1 3.1 Proposition (page 86)*, though the proof is left to the reader as it usually is with Folland's book, so we'll go ahead and take care of it, here.

(3.3.2) can be shown from the fact that

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A_n', \text{ and } \bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} A_j', \text{ for all } n \in \mathbb{N}, \text{ where } A_n' := A_n \setminus \left(\bigcup_{j=1}^{n-1} A_j\right),$$

and $\{A_n'\}_{n\in\mathbb{N}}$ is a family of pairwise disjoint sets, and we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n'\right) = \sum_{n=1}^{\infty} \mu(A_n') = \lim_{n\to+\infty} \sum_{j=1}^{n} \mu(A_j') = \lim_{n\to+\infty} \mu\left(\bigcup_{j=1}^{n} A_j'\right).$$

(3.3.3) can be shown from the fact that $|\mu(B_{n_0})| < +\infty$, for some $n_0 \geq 1$, and

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\geq n_0} B_n,$$

hence by (3.3.2), which we previously verified, we find that since $\{B_{n_0} \setminus B_n\}_{n\in\mathbb{N}}$ is a $\subset$-increasing sequence, we conclude that

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigcap_{n\geq n_0} B_n\right) = \mu\left(B_{n_0} \setminus \left(\bigcup_{n\geq n_0} [B_{n_0} \setminus B_n]\right)\right) = \mu(B_{n_0}) - \mu\left(\bigcup_{n\geq n_0} [B_{n_0} \setminus B_n]\right)$$
$$= \lim_{n \to +\infty} [\mu(B_{n_0}) - \mu(B_{n_0} \setminus B_n)] = \lim_{n \to +\infty} \mu(B_n).$$

Now we are ready to prove the above lemma.

Proof of Lemma 3.3.3. Let $\mu, \nu : \sigma(\mathcal{A}) \to \mathbb{R}$ be measures extending μ_0. First, we shall prove this in the case where μ_0 is finite. Let $\mathcal{M}_0$ be the set of all $E \in \sigma(\mathcal{A})$ such that $\mu(E) = \nu(E)$, and note that it shall suffice by the *Monotone Class Lemma* to show that $\mathcal{M}_0$ is a monotone class containing $\mathcal{A}$, since we'd have $\sigma(\mathcal{A}) = \mathcal{M}_0$ (as a result of $\sigma(\mathcal{A}) \supset \mathcal{M}_0$ by construction and $\sigma(\mathcal{A}) \subset \mathcal{M}_0$ since $\sigma(\mathcal{A})$ is the smallest monotone class) and uniqueness would immediately follow.

First, observe that $\mathcal{A} \subset \mathcal{M}_0$ is immediate by μ, ν both extending μ_0. Next, we want to show $\mathcal{M}_0$ is closed under monotone unions and intersections. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_0$ be a $\subset$-increasing sequence and $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_0$ be a $\subset$-decreasing sequence and observe that since

$$\mu(E_n) = \nu(E_n) \text{ and } \mu(F_n) = \nu(F_n), \text{ for all } n \in \mathbb{N},$$

and

$$|\mu(F_1) + \mu\left(F_1^c\right)| = |\mu(X)| = |\mu_0(X)| < +\infty,$$
$$\implies |\mu(F_1)| = |\nu(F_1)| < +\infty,$$

we find that

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n \to +\infty} \mu(E_n) = \lim_{n \to +\infty} \nu(E_n) = \nu\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$
$$\mu\left(\bigcap_{n\in\mathbb{N}} F_n\right) = \lim_{n \to +\infty} \mu(F_n) = \lim_{n \to +\infty} \nu(F_n) = \nu\left(\bigcup_{n\in\mathbb{N}} F_n\right),$$

and our conclusion is reached.

Now we shall prove this lemma in the general case. Using σ-finiteness of μ_0, choose a $\subset$-increasing sequence $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $|\mu_0(X_n)| < +\infty$ and $X_n \nearrow X$ as $n \to +\infty$.

Note that for all $n \in \mathbb{N}$, we find $\mu_{n,0} : \mathcal{A} \to [-\infty, +\infty]$ defined by $\mu_{n,0} := \mu_0((-) \cap X_n)$ is a finite premeasure such that $\mu_n, \nu_n : \sigma(\mathcal{A}) \to [-\infty, +\infty]$ defined by $\mu_n := \mu((-) \cap X_n), \nu_n((-) \cap X_n)$ are finite measures that extend $\mu_{n,0}$, since for all $n \in \mathbb{N}$ we have

$$|\mu_n(X)| = |\mu(X \cap X_n)| = |\nu(X \cap X_n)| = |\nu_n(X)| = |\mu_{0,n}(X)| = |\mu_0(X \cap X_n)| = |\mu(X_n)| < +\infty,$$

and for all $A \in \mathcal{A}$, we have

$$\mu_n(A) = \mu(A \cap X_n) = \mu_0(A \cap X_n) = \mu_{0,n}(A) = \mu_0(A \cap X_n) = \nu(A \cap X_n) = \nu_n(A).$$

Then for every $E \in \sigma(\mathcal{A})$, noting that $\{E \cap X_n\}_{n \in \mathbb{N}}$ is a $\subset$-increasing sequence such that $E \cap X_n \nearrow E$, we have

$$\mu(E) = \lim_{n \to +\infty} \mu(E \cap X_n) = \lim_{n \to +\infty} \nu(E \cap X_n) = \nu(E). \ \square$$

Theorem 3.3.4. *(A Caratheodory Extension for the Induced Jordan Decomposition)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{M} := \sigma(\mathcal{A})$, and μ_0 is a σ-finite premeasure on $\mathcal{A}$. Define

$$\mu_+ := (\mu_0)^+|\mathcal{M}, \ \mu_- := (\mu_0)^-|\mathcal{M}, \text{ and } \mu_{tot.} := |\mu_0|\,|\mathcal{M}.$$

Then $\mu_+ \perp \mu_-$, and there exists a uniquely determined σ-finite measure on $\mathcal{M}$ that extends μ_0 defined by $\mu := \mu_+ - \mu_-$, with its Jordan decomposition uniquely determined by $\mu^+ = \mu_+$ and $\mu^- = \mu_-$. Moreover, we have

$$\mu^+ + \mu^- = |\mu| = \mu_{tot.} = \mu_+ + \mu_-. \quad (3.3.4)$$

Proof. First, we shall prove that μ_+, μ_- forms a decomposition such that $\mu := \mu_+ - \mu_-$ extends μ_0. To do this, we have two cases.

Case 1. Suppose $|\mu_0(X)| < +\infty$, i.e. we look at the case where μ_0 is finite. To prove that $\mu := \mu_+ - \mu_-$ extending μ_0, it shall suffice to show that μ_0 satisfies the property

$$\mu_0(F) = \mu_+(F) - \mu_-(F), \text{ for every } F \in \mathcal{A}.$$

Let $F \in \mathcal{A}$. Choose $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that

$$\bigcup_{n\in\mathbb{N}} A_n, \bigcup_{n\in\mathbb{N}} B_n \subset F, \text{ and}$$

$$\begin{aligned} &0 \le \mu_0(A_n) \nearrow \mu_+(F), \\ &0 \le -\mu_0(B_n) \nearrow \mu_-(F), \text{ as } n \to +\infty. \end{aligned} \tag{3.3.5}$$

Note that since for each $n \geq 1$, we have

$$\mu_0(A_n) \le \mu_+(A_n), \ -\mu_0(B_n) \le \mu_-(B_n),$$

we find that

$$\mu_+(A_n) \nearrow \mu_+(F), \ \mu_-(B_n) \nearrow \mu_-(F), \text{ as } n \to +\infty,$$

and it follows that

$$\begin{aligned} \lim_{n\to+\infty} \mu_+(F \setminus A_n) &= \lim_{n\to+\infty} [\mu_+(F) - \mu_+(A_n)] = \mu_+(F) - \mu_+(F) = 0, \\ \lim_{n\to+\infty} \mu_-(F \setminus B_n) &= \lim_{n\to+\infty} [\mu_-(F) - \mu_-(B_n)] = \mu_-(F) - \mu_-(F) = 0, \end{aligned}$$

hence

$$\begin{aligned} \lim_{n\to+\infty} \mu_+(F \setminus (A_n \cup B_n)) &\le \lim_{n\to+\infty} \mu_+(F \setminus A_n) = 0, \\ \lim_{n\to+\infty} \mu_-(F \setminus (A_n \cup B_n)) &\le \lim_{n\to+\infty} \mu_-(F \setminus B_n) = 0. \end{aligned}$$

Then we have

$$-\mu_-(F \setminus (A_n \cup B_n)) \le \mu_0(F \setminus (A_n \cup B_n)) \le \mu_+(F \setminus (A_n \cup B_n)), \text{ for all } n \ge 1,$$

$$\Longrightarrow \lim_{n\to+\infty} \mu_0(F \setminus (A_n \cup B_n)) = 0,$$

and it follows that

$$\begin{aligned} \mu_0(F) &= \lim_{n\to+\infty} [\mu_0(A_n \cup B_n) + \mu_0(F \setminus (A_n \cup B_n))] \\ &= \lim_{n\to+\infty} [\mu_0(A_n \cup B_n)]; \end{aligned} \tag{3.3.6}$$

$$\lim_{n\to+\infty} \mu_0\left(A_n \cap B_n^c\right) = \lim_{n\to+\infty} [\mu_0(A_n \cup B_n) - \mu_0(B_n)]$$

$$
\begin{aligned}
&= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] + \lim_{n \to +\infty} [-\mu_0(B_n)] \\
&= \mu_0(F) + \mu_-(F);
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n \to +\infty} \mu_0\left(A_n^c \cap B_n\right) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) - \mu_0(A_n)] \\
&= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] - \lim_{n \to +\infty} [\mu_0(A_n)] \\
&= \mu_0(F) - \mu_+(F);
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n \to +\infty} \mu_0(A_n \triangle B_n) &= \lim_{n \to +\infty} \left[\mu_0\left(A_n \cap B_n^c\right)\right] + \lim_{n \to +\infty} \left[\mu_0\left(A_n^c \cap B_n\right)\right] \\
&= 2\mu_0(F) - [\mu_+(F) - \mu_-(F)];
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n \to +\infty} \mu_0(A_n \cap B_n) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n) - \mu_0(A_n \triangle B_n)] \\
&= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] - \lim_{n \to +\infty} [\mu_0(A_n \triangle B_n)] \qquad (3.3.7) \\
&= [\mu_+(F) - \mu_-(F)] - \mu_0(F).
\end{aligned}
$$

Then by (3.3.5), (3.3.6), and (3.3.7), we conclude that

$$
\begin{aligned}
\mu_0(F) &= \lim_{n \to +\infty} [\mu_0(A_n \cup B_n)] \\
&= \lim_{n \to +\infty} [\mu_0(A_n)] + \lim_{n \to +\infty} [\mu_0(B_n)] - 2 \lim_{n \to +\infty} [\mu_0(A_n \cap B_n)] \\
&= [\mu_+(F) - \mu_-(F)] - 2 \cdot ([\mu_+(F) - \mu_-(F)] - \mu_0(F)) \\
&= 2\mu_0(F) - [\mu_+(F) - \mu_-(F)],
\end{aligned}
$$

$$\implies \mu_0(F) = \mu_+(F) - \mu_-(F).$$

Next, to prove $\mu_+ \perp \mu_-$ (and more generally that μ_+, μ_- is the Jordan decomposition of μ) that since

$$\mu_0(X) = \mu_+(X) - \mu_-(X), \quad (3.3.8)$$

and

$$|\mu_+(X) - \mu_-(X)| = |\mu_0(X)| < +\infty,$$

we find μ_+, μ_- are finite. Next, choose a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that

$$\mu_+(X) \leq \mu_0(A_n) + 2^{-n}, \text{ for all } n \in \mathbb{N}, \text{ and} \qquad (3.3.9)$$

$\mu_0(A_n) \nearrow \mu_+(X)$ as $n \to +\infty$,

and note that since for all $n \in \mathbb{N}$, we have

$$\mu_0(A_n) = \mu_+(A_n) - \mu_-(A_n), \qquad (3.3.10)$$
$$0 \le \mu_0(A_n) \le \mu_+(A_n) \le \mu_+(X),$$

we find by (3.3.9) and (3.3.10) that

$$\lim_{n \to +\infty} \mu_+(A_n) = \lim_{n \to +\infty} \mu_0(A_n) = \mu_+(X),$$

$$\lim_{n \to +\infty} \mu_+\left(A_n^c\right) = \mu_+(X) - \lim_{n \to +\infty} [\mu_+(A_n)] = 0.$$

Define

$$P := \limsup_n A_n = \bigcap_{N \in \mathbb{N}} \bigcup_{n \ge N} A_n, \quad (3.3.11)$$
$$N := P^c.$$

We want to show that P, N make up a Hahn Decomposition, i.e., noting that P, N partition X, it remains to show that N is μ_+-null and P is μ_--null. First, note that

$$N = \liminf_n A_n^c = \bigcup_{n \in \mathbb{N}} \bigcap_{n \ge N} A_n^c.$$

Then since μ_+ is finite, we have

$$\mu_+(N) = \sup_{N \in \mathbb{N}} \mu_+\left(\bigcap_{n \ge N} A_n^c\right) = \sup_{N \in \mathbb{N}}\left[\inf_{n \ge N} \mu_+\left(A_N^c \cap A_{N+1}^c \cap \cdots \cap A_n^c\right)\right]$$
$$\le \sup_{N \in \mathbb{N}}\left[\inf_{n \ge N} \mu_+\left(A_n^c\right)\right] = \sup_{N \in \mathbb{N}}\left[\lim_{n \to +\infty} \mu_+\left(A_n^c\right)\right] = 0,$$

and hence N is μ_+-null. Next, observe by (3.3.8), (3.3.9) and (3.3.10) we find that for all $n \in \mathbb{N}$, we have

$$\mu_+(X) \le \mu_0(A_n) + 2^{-n} \le \mu_+(A_n) + 2^{-n},$$

$$\implies \mu_+(X) - \mu_0(A_n) \leq 2^{-n}$$
$$\mu_+\left(A_n^c\right) = \mu_+(X) - \mu_+(A_n) \leq 2^{-n},$$

$$\begin{aligned}\implies \mu_-(A_n) &= \mu_-(X) - \mu_-\left(A_n^c\right)\\ &= \mu_-(X) - \left[\mu_+\left(A_n^c\right) - \mu_0\left(A_n^c\right)\right]\\ &= \mu_-(X) - \mu_+\left(A_n^c\right) + \mu_0(X) - \mu_0(A_n)\\ &= [\mu_+(X) - \mu_0(A_n)] - \mu_+\left(A_n^c\right)\\ &= O\left(2^{-n}\right),\end{aligned}$$

$$\implies \sum_{n=1}^{\infty} \mu_-(A_n) < +\infty,$$

and it immediately follows that we have $\mu_-(P) = 0$ by applying $(3.3.11)$ to the *First Borel-Cantelli Lemma*, and our conclusion that P is μ_--null is reached, finishing *Case 1*.

Case 2. Suppose $|\mu_0(X)| = +\infty$, i.e., we look at the case where μ_0 is infinite. Choose disjoint $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} X_n = X$, and note that $\mu_{n,0} := \mu_0((-) \cap X_n)$ are finite premeasures. It follows by *Case 1* that for all $n \in \mathbb{N}$, we find

$$\mu_{n,+} := (\mu_{n,0})^+|\mathcal{M},\ \mu_{n,-} := (\mu_{n,0})^-|\mathcal{M},$$

is such that $\mu_{n,+}, \mu_{n,-}$ form a uniquely determined Jordan Decomposition of

$$\mu_n := \mu_{n,+} - \mu_{n,-} \quad (3.3.12)$$

extending $\mu_{n,0}$, i.e., we have

$$\mu_{n,0}(A) = \mu_{n,+}(A) - \mu_{n,-}(A), \text{ for all } A \in \mathcal{A}. \quad (3.3.13)$$

We shall start by proving the following claims:

Claim 1. For every $n \in \mathbb{N}$, we have

$$\mu_{n,+} = \mu_+((-) \cap X_n), \quad (3.3.14)$$
$$\mu_{n,-} = \mu_-((-) \cap X_n).$$

Proof. Note that for every $n \in \mathbb{N}$, and $E \in \sigma(\mathcal{A})$, we find since

$$\begin{aligned}\mu_0(A \cap X_n) \in \{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} &\iff A \in \mathcal{A}, A \subset E \\ &\iff A' = A \cap X_n \in \mathcal{A}, A' \subset E \cap X_n \\ &\iff \mu_0(A') \in \{\mu_0(A') : A' \in \mathcal{A}, A' \subset E \cap X_n\},\end{aligned}$$

we have

$$\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} = \{\mu_0(A') : A' \in \mathcal{A}, A' \subset E \cap X_n\},$$

and it follows that

$$\begin{aligned}\mu_{n,+}(E) &= \sup\{\mu_{n,0}(A) : A \in \mathcal{A}, A \subset E\} \\ &= \sup\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} \\ &= \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset E \cap X_n\} \\ &= \mu_+((-) \cap X_n),\end{aligned}$$

$$\begin{aligned}\mu_{n,-}(E) &= -\inf\{\mu_{n,0}(A) : A \in \mathcal{A}, A \subset E\} \\ &= -\inf\{\mu_0(A \cap X_n) : A \in \mathcal{A}, A \subset E\} \\ &= -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E \cap X_n\} \\ &= \mu_-((-) \cap X_n).\end{aligned}$$ □

Claim 2. We have

$$\mu_+ = \sum_{n=1}^{\infty} \mu_{n,+}, \mu_- = \sum_{n=1}^{\infty} \mu_{n,-}. \quad (3.3.15)$$

Proof. Observe that for all $E \in \sigma(\mathcal{A})$, we find by $(3.3.14)$ (proved by *Claim 1*), we have

$$\begin{aligned}\mu_+(E) &= \sum_{n=1}^{\infty} \mu_+(E \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(E), \\ \mu_-(E) &= \sum_{n=1}^{\infty} \mu_-(E \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(E).\end{aligned}$$ □

Now, we shall proceed to prove the rest of the theorem in this case. To show that μ is an extension of μ_0, observe by $(3.3.13)$, $(3.3.14)$ (proved in *Claim 1*), and $(3.3.15)$ (proved in *Claim 2*) that for all $A \in \mathcal{A}$, we have

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,0}(A) = \sum_{n=1}^{\infty} [\mu_{n,+}(A) - \mu_{n,-}(A)] = \sum_{n=1}^{\infty} [\mu_{n,+}(A)] - \sum_{n=1}^{\infty} [\mu_{n,-}(A)]$$
$$= \mu_+(A) - \mu_-(A) = \mu(A).$$

Next, we show that $\mu_+ \perp \mu_-$, showing that μ_+, μ_- forms a Jordan Decomposition of μ.
Since $\mu_{n,+} \perp \mu_{n,-}$, we can choose a Hahn-Decomposition P_n, N_n such that N_n is $\mu_{n,+}$-null and P_n is $\mu_{n,-}$-null. Set

$$P := \bigcup_{n\in\mathbb{N}} [P_n \cap X_n], N := \bigcup_{n\in\mathbb{N}} [N_n \cap X_n],$$

and note that for all $n \in \mathbb{N}$, we have

$$(P_n \cap X_n) \cup (N_n \cap X_n) = X_n \cap (P_n \cup N_n) = X_n \cap X = X_n,$$
$$(P_n \cap X_n) \cap (N_n \cap X_n) = X_n \cap (P_n \cap N_n) = \varnothing,$$

hence

$$\begin{aligned}
P \cup N &= \bigcup_{n\in\mathbb{N}} [(P_n \cap X_n) \cup (N_n \cap X_n)] \\
&= \bigcup_{n\in\mathbb{N}} [(P_n \cup N_n) \cap X_n] \\
&= \bigcup_{n\in\mathbb{N}} [X \cap X_n] \\
&= \bigcup_{n\in\mathbb{N}} X_n \\
&= X,
\end{aligned}$$
$$\begin{aligned}
P \cap N &= \bigcup_{m,n\in\mathbb{N}} [(P_m \cap X_m) \cap (N_n \cap X_n)] \\
&= \bigcup_{m\neq n\in\mathbb{N}} [P_m \cap N_n \cap (X_m \cap X_n)] \cup \bigcup_{n\in\mathbb{N}} [X_n \cap (P_n \cap N_n)] \\
&= \bigcup_{m\neq n\in\mathbb{N}} [P_m \cap N_n \cap \varnothing] \cup \bigcup_{n\in\mathbb{N}} [X_n \cap \varnothing] \\
&= \varnothing,
\end{aligned}$$

and by (3.3.14) we have

$$\mu_+(N) = \sum_{n=1}^{\infty} \mu_+(N_n \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(N_n) = 0,$$
$$\mu_-(P) = \sum_{n=1}^{\infty} \mu_+(P_n \cap X_n) = \sum_{n=1}^{\infty} \mu_{n,+}(P_n) = 0,$$

and our conclusion that P, N form a Hahn-decomposition has been reached, finishing *Case 2*.

Next, to show uniqueness of μ extending μ_0, as well as the Jordan Decomposition μ_-, μ_+, note that **Lemma 3.3.3** shows that the extension μ of μ_0 (in this case where μ_0 is finite) is uniquely determined. Then given a Jordan Decomposition μ^+, μ^- of μ, we choose a Hahn Decomposition P', N' of μ^+, μ^- and observe that

$$\mu^+ = \mu((-) \cap P'), \; \mu^- = \mu((-) \cap N'). \quad (3.3.16)$$

Next, since for all $E \subset X$, we have

$$\{\mu_0(A) : A \in \mathcal{A}, A \subset E\} \subset \{\mu(F) : F \in \mathcal{M}, F \subset E\},$$

by (3.2.1) and **Proposition 3.1.4** *(iii)* that since

$$\begin{aligned}\mu_+(N') = (\mu_0)^+(N') &= \sup\{\mu_0(A) : A \in \mathcal{A}, A \subset N'\} \\ &\leq \sup\{\mu(F) : F \in \sigma(\mathcal{A}), F \subset N'\} = \mu^+(N') = 0,\end{aligned}$$

$$\begin{aligned}\mu_-(P') = (\mu_0)^-(P') &= -\inf\{\mu_0(A) : A \in \mathcal{A}, A \subset E\} \\ &\leq -\inf\{\mu(F) : F \in \sigma(\mathcal{A}), F \subset P'\} = \mu^-(P') = 0,\end{aligned}$$

we conclude by (3.3.16) that for every $E \in \sigma(\mathcal{A})$, we have

$$\mu_-(E \cap P') \leq \mu_-(P') = 0,$$
$$\mu_+(E \cap N') \leq \mu_+(N') = 0,$$

$$\begin{aligned}\implies \mu_+(E) &= \mu_+(E \cap P') + \mu_+(E \cap N') = \mu_+(E \cap P') + 0 = \mu_+(E \cap P') + \mu_-(E \cap P') = \mu(E \cap P') \\ &= \mu^+(E),\end{aligned}$$

$$\begin{aligned}\mu_-(E) &= \mu_-(E \cap P') + \mu_-(E \cap N') = 0 + \mu_-(E \cap N') = \mu_+(E \cap N') + \mu_-(E \cap P') = \mu(E \cap N') \\ &= \mu^-(E),\end{aligned}$$

and our conclusion is met.

Finally, show that $(3.3.4)$ holds, note from showing that μ_+, μ_- is in fact the Jordan Decomposition of μ, with P, N (as before) defined to be the Hahn decompompomposition of μ_+, μ_-, we have shown that $|\mu| = \mu_+ + \mu_-$, and it remains to show that $|\mu| = \mu_{tot.}$. First, observe by **Proposition 3.1.4** *(iv)* that for all $E \in \mathcal{M}$, we have

$$\begin{aligned}\mu_{tot.}(E) &= \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\}\\ &\leq \sup\left\{\sum_{j=1}^{n} |\mu(E_j)| : E_1, \ldots, E_n \in \mathcal{M} \text{ are disjoint and } \bigcup_{j=1}^{n} E_j \subset E\right\}\\ &= |\mu|(E),\end{aligned}$$

so $|\mu| \leq \mu_{tot.}$. Next, we find that since P, N are disjoint, we have

$$\begin{aligned}&\{\mu_0(A_1) - \mu_0(A_2) : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\}\\ &\subset \{|\mu_0(A_1)| + |\mu_0(A_2)| : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\}\\ &\subset \left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset A\right\},\end{aligned}$$

and since N is μ_+-null and P is μ_--null, find that for all $E \in \mathcal{M}$, we have

$$\begin{aligned}|\mu|(E) &= \mu_+(E) + \mu_-(E)\\ &= \mu_+(E \cap P) + \mu_+(E \cap N) + \mu_-(E \cap P) + \mu_-(E \cap N)\\ &= \mu_+(E \cap P) + \mu_-(E \cap N)\\ &= \sup\{\mu_0(A_1) - \mu_0(A_2) : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\}\\ &\leq \sup\{|\mu_0(A_1)| + |\mu_0(A_2)| : A_1, A_2 \in \mathcal{A}, A_1 \subset A \cap P, A_2 \subset E \cap N\}\\ &\leq \sup\left\{\sum_{j=1}^{N} |\mu_0(A_j)| : A_1, \ldots, A_N \in \mathcal{A} \text{ are disjoint and } \bigcup_{j=1}^{N} A_j \subset E\right\}\\ &= \mu_{tot.}(E),\end{aligned}$$

showing that $|\mu| \geq \mu_{tot.}$, and our conclusion of $|\mu| = \mu_{tot.}$ is met. □

%DO THIS NEXT

%THEN ENUMERATE UNFINISHED 1-10-25 UPDATE

3.4 The Caratheodory Extension for $\mathbb{C}$-valued Premeasures

%SHOW THIS IN THE NEXT DRAFT
%SHOW THAT ONE CAN FORM AN ANALOGOUS EXTENSION FOR $\mathbb{C}$-VALUED MEASURES

3.5 Single Dimension Outer Signed Measures

%SHOW THIS IN THE NEXT DRAFT
%CONSIDER MOVING THIS SECTION TO THE FIRST PART
%TALK ABOUT WHY YOU CAN'T DO SUCH AN EXTENSION WITH OUTER SIGNED MEASURES

%FORMAT BELOW PARTS WHEN READY
%PUT LAST CHAPTER ON HOLD UNTIL THE NEXT DRAFT

4. The Cartheodory Extension for Vector-Valued Measures

4.1 Definitions and Properties of Vector-Valued Measures

%START WRITING IN NEXT DRAFT

4.2 Vector-Valued Premeasure Definition and Examples

%START WRITING IN NEXT DRAFT

4.3 The Cartheodory Extension in the Vector Valued Case

%START WRITING IN NEXT DRAFT

5. Conclusion