Proving Skorohod's Theorem Using Probability Trees

§ 1 Introduction and Preliminary Results

%COPY AND PASTE PAST SECTION ONE

Let $\mathcal{X}$ be a Banach space, let $\{X_n\}_{n\in\mathbb{N}}, X$ be an $\mathcal{X}$-valued random variable on the probability space $(\Omega, \Sigma, \mathbb{P})$.

Theorem 1.1. *(Skorohod Representation Theorem)* If $X_n \Rightarrow X$, then there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ and r.v.'s $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}$ and $\widetilde{X}$ on $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ such that,

(i) $\widetilde{X}_n \overset{D}{=} X_n$
(ii) $\widetilde{X} \overset{D}{=} X$
(iii) $\widetilde{X}_n \to \widetilde{X}\ \widetilde{\mathbb{P}}$-a.s.
Source: M647 Lecture 1 (revised) Theorem 1.4.5.
%REWRITE THOEREM WITH CONSISTENT NOTATION

Proposition 1.2.. If $X = c$ $\mathbb{P}$-a.s., for some $c \in E$, then $X_n \Rightarrow X \implies X_n \xrightarrow{\mathbb{P}} X$.
%REVISE AND CITE SOURCE

Note: The proposition is in the more general setting of metric-spaced valued random variables, whereas the analogous statement of this proposition metioned in Billingsley is in the real-valued random variable setting. However, the proof (which we shall provide in full detail below) is pretty much the same as in this case.
%EDIT THIS NOTE TO ACCOUNT FOR PROOF NOT BEING MENTIONED

Proposition 1.3. If $X_n \Rightarrow X$ and $f : \mathcal{X} \to \mathcal{Y}$ is a continuous function for some separable Banach space $\mathcal{Y}$, then $f(X_n) \Rightarrow f(X)$.
%PROPOSITION ABOUT CONVERGENCE IN DISTRIBUTION BEING PRESERVED FOR $\mathbb{P}$-a.s. CONTINUOUS FUNCTIONS
%PROVE WITHOUT SKORROHOD THEOREM

Proposition 1.4 *(scaling and translation invariance of the product Lebesgue measures)*.

Suppose $\bigotimes_{j=0}^{k} m$ is the product Lebesgue Measure, i.e., the measure on $\mathcal{B}\left(\mathbb{R}^{k+1}\right)$ generated by the premeasure M_0 on the collection of $k+1$-dimensional rectangles $I_0 \times \cdots \times I_k$ of open intervals $I_0, \ldots, I_k \subset \mathbb{R}^{k+1}$ defined by

$$M_0(I_0 \times \cdots \times I_k) := \prod_{j=0}^{k} m(I_j),$$

and $\phi : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is a bijective function defined by

$\phi(x_0, \ldots, x_k) := (a_0 x_0 + b_0, a_1 x_1 + b_1, \ldots, a_k x_k + b_k)$, for $a_0, \ldots, a_k \in \mathbb{R}^+, b_0, \ldots, b_k \in \mathbb{R}$.

Then for any $A \in \mathcal{B}\left(\mathbb{R}^{k+1}\right)$, we have

$$\left(\bigotimes_{j=0}^{k} m\right)(\phi(A)) = \left(\prod_{j=0}^{k} a_j\right) \cdot \left(\bigotimes_{j=0}^{k} m\right)(A).$$

%CONSIDER INCLUDING THOEREM ON OPEN MAPPING COMPOSITION PRESERVING ALMOST EVERYWHERE CONVERGENCE
%ALSO INCLUDE MEASURE-PRESERVING PROPERTIES OF TRANSLATIONS
%INCLUDE GENERAL CASE OF THAT IN THIS PROPOSITION
%CITE FOLLAND BUT

Lemma 1.5. Given a probability space $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ such that $\left(\widetilde{\Omega}, \preccurlyeq\right)$ is a poset and for the poset topology $\mathcal{T}(\preccurlyeq)$, we have $\mathcal{T}(\preccurlyeq) \subset \widetilde{\Sigma}$, if $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ and $||\widetilde{X}_n - \widetilde{X}||$, for every $n \in \mathbb{N}$, is monotonically increasing on $\widetilde{\Omega}$, and $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{X}$, then $\widetilde{X}_n \xrightarrow{\text{pointwise}} \widetilde{X}$.
%CITE MUNKRES OR WIKIPEDIA
%GENERALIZE THIS LEMMA FOR ARBITRARY POSETS

Proof. Given $\widetilde{\omega} \in \widetilde{\Omega}$ and $\epsilon > 0$, choose $N \geq 1$ sufficiently large such that for all $n \geq N$, we have $\widetilde{\mathbb{P}}\left[||\widetilde{X}_n - \widetilde{X}|| \geq \epsilon\right] < \dfrac{\widetilde{\mathbb{P}}\left(\left[\widetilde{\omega}, +\infty\right)\right)}{2}$. Since $||\widetilde{X}_n - \widetilde{X}||$ is monotonically increasing, we find that if (towards contradiction) we have $||\widetilde{X}_n(\widetilde{\omega}) - \widetilde{X}(\widetilde{\omega})|| \geq \epsilon$, then for all $\widetilde{\Omega} \ni \widetilde{\omega}' \succcurlyeq \widetilde{\omega}$ and $n \geq N$, we have

$$||\widetilde{X}_n(\widetilde{\omega}') - \widetilde{X}(\widetilde{\omega}')|| \geq ||\widetilde{X}_n(\widetilde{\omega}) - \widetilde{X}(\widetilde{\omega})|| \geq \epsilon,$$

and it follows that

$$\widetilde{\mathbb{P}}\Big[\big[\widetilde{\omega}, +\infty\big)\Big] \leq \widetilde{\mathbb{P}}\Big[||\widetilde{X}_n - x|| \geq \epsilon\Big] < \frac{\widetilde{\mathbb{P}}\Big(\big[\widetilde{\omega}, +\infty\big)\Big)}{2},$$

which is a contradiction. Then we conclude that $||\widetilde{X}_n(\widetilde{\omega}) - \widetilde{X}(\widetilde{\omega})|| < \epsilon$. ◻
%WRITE DOWN PROOF

§ 2 Conditional Probability, Probability Trees, And Convergence in Distribution

%COPY AND PASTE PAST SECTION TWO

%MOVE NEXT LEMMAS IN THE NEXT SECTION
Lemma 2.1. Suppose $A \in \Sigma$. Then

(i) If $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$. then $\mathbb{1}_A X_n \xrightarrow{\mathbb{P}\text{-a.s.}} \mathbb{1}_A X$, as $n \to +\infty$.

(ii) If $X_n \xrightarrow{\mathbb{P}} X$. then $\mathbb{1}_A X_n \xrightarrow{\mathbb{P}} \mathbb{1}_A X$, as $n \to +\infty$.

Proof.

(i) Since $\mathbb{P}\Big[A \cap \big\{X_n \to X\big\}^c\Big] \leq \mathbb{P}\big[X_n \not\to X\big] \leq 0$, we have

$$\mathbb{P}\Big[A \cap \big\{X_n \to X\big\}\Big] = \mathbb{P}\Big[A \cap \big\{X_n \to X\big\}\Big] + \mathbb{P}\Big[A \cap \big\{X_n \to X\big\}^c\Big] = \mathbb{P}[A].$$

It follows that

$$\mathbb{P}\big[\mathbb{1}_A X_n \to \mathbb{1}_A X\big] = \mathbb{P}\Big[A \cap \big\{\mathbb{1}_A X_n \to \mathbb{1}_A X_n\big\}\Big] + \mathbb{P}\Big[A^c \cap \big\{\mathbb{1}_A X_n \to \mathbb{1}_A X_n\big\}\Big]$$

$$= \mathbb{P}\left[A \cap \left\{X_n \to X\right\}\right] + \mathbb{P}\left[A^c \cap \left\{0 \to 0\right\}\right]$$
$$= \mathbb{P}[A] + \mathbb{P}\left[A^c\right]$$
$$= 1.$$

(ii) Given a subsequence $\{\mathbb{1}_A X_{n_k}\}_{k\in\mathbb{N}}$ of $\{\mathbb{1}_A X_n\}_{n\in\mathbb{N}}$ we choose a further subsequence $\{X_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{X_{n_k}\}_{k\in\mathbb{N}}$ such that $X_{n_{k_j}} \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $j \to +\infty$, and it follows by (i) that we have $\mathbb{1}_A X_{n_{k_j}} \xrightarrow{\mathbb{P}\text{-a.s.}} \mathbb{1}_A X$ as $j \to +\infty$. □

Example 2.2. The analogue of **Lemma 2.1** for convergence in distribution does not hold. Let $(\Omega, \mathbb{P}) := \left(\{H, T\}, \frac{1}{2}\delta_{\{H\}} + \frac{1}{2}\delta_{\{T\}}\right)$ be the coin flip space, and let

$X_{2n-1} := \mathbb{1}_{\{H\}}, X_{2n} := \mathbb{1}_{\{T\}}$, and noting that $\mu_{\mathbb{1}_{\{H\}}} = \mu_{X_n} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, for each $n \in \mathbb{N}$, we find that $X_n \Rightarrow X$, for $X := \mathbb{1}_{\{H\}}$, as $n \to +\infty$. However, for $A := \{H\}$, we have

$$\mathbb{1}_A X_{2n-1} = \mathbb{1}_{\{H\}\cap\{H\}} = \mathbb{1}_{\{H\}},$$
$$\mathbb{1}_A X_{2n} = \mathbb{1}_{\{T\}\cap\{H\}} = \mathbb{1}_{\varnothing} = 0,$$

hence $\mu_{\mathbb{1}_A X_{2n-1}} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $\mu_{\mathbb{1}_A X_{2n}} = \delta_0$, and hence $\mathbb{1}_A X_n \not\Rightarrow \mathbb{1}_A X$ as $n \to +\infty$.
%COUNTEREXAMPLE FOR CONVERGENCE IN DISTRIBUTION

Definition 2.3. We call $\{A_i\}_{i\in I} \subset \Sigma$ a $\mathbb{P}$-a.s. partition if it is pairwise disjoint and $\mathbb{P}\left[\bigcup_{i\in I} A_i\right] = 1.$
%DEFINE $\mathbb{P}$-a.s. PARTITION AND RELATE TO BRANCHES OF A TREE
%COME UP WITH MORE GENERAL DEFINITION

Proposition 2.4. Suppose $\{A_i\}_{i\in I} \subset \Sigma$ is a countable $\mathbb{P}$-a.s. partition of Ω, Then

(i) $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ iff $\mathbb{1}_{A_i} X_n \xrightarrow{\mathbb{P}\text{-a.s.}} \mathbb{1}_{A_i} X$, for each $i \in I$ as $n \to +\infty$.

(ii) $X_n \xrightarrow{\mathbb{P}} X$ iff $\mathbb{1}_{A_i} X_n \xrightarrow{\mathbb{P}} \mathbb{1}_{A_i} X$, for each $i \in I$ as $n \to +\infty$.

(iii) If $\mathbb{1}_{A_i} X_n \Rightarrow \mathbb{1}_{A_i} X$, for each $i \in I$, then $X_n \Rightarrow X$ as $n \to +\infty$.

%REVISE THIS PROPOSITION TO INCLUDE CONVERGENCE IN MEASURE
%TALK ABOUT HOW PROPOSITION 2.4 LENDS WAY OF PROVING CONVERGENCE BY PROVING SUCH BEHAVIOR BRANCH-WISE IN A PROBABILITY TREE

Proposition 2.5. Given a probability space $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ Let $\{A_i\}_{i\in I} \subset \Sigma$, $\left\{\widetilde{A}_i\right\}_{i\in I} \subset \widetilde{\Sigma}$ be a countable $\mathbb{P}$-a.s. and $\widetilde{\mathbb{P}}$-a.s. partition respectively Then

(i) Given $Y \in \mathfrak{L}^0(\mathcal{X};\mathbb{P})$, $\widetilde{Y} \in \mathfrak{L}^0\left(\mathcal{X};\widetilde{\mathbb{P}}\right)$, if $\mathbb{1}_{A_i}Y \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{Y}$, for every $i \in I$, then $Y \overset{D}{=} \widetilde{Y}$.

(ii) If $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0\left(\mathcal{X};\widetilde{\mathbb{P}}\right)$ such that for every $i \in I$, we have $\mathbb{1}_{A_i}X_n \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{X}_n$, for each $n \in \mathbb{N}$, and $\mathbb{1}_{A_i}X \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{X}$, and $\left(\mathbb{1}_{\widetilde{A}_i}\widetilde{X}_n, \widetilde{\mathbb{P}}\right) \Rightarrow (\mathbb{1}_{A_i}X, \mathbb{P})$ as $n \to +\infty$, then $\widetilde{X}_n \Rightarrow X$.
%TALK ABOUT CONVERGENCE IN DISTRIBUTION NOTATION SOMEWHERE

Proof.

(i) If $\mathbb{1}_{A_i}Y \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{Y}$, for every $n \in \mathbb{N}$. then given measurable $\mathrm{C} \subset \mathcal{X}$, we have

$$\begin{aligned}\mathbb{P}\left[Y^{-1}(\mathrm{C})\right] &= \mathbb{P}\left[\bigcup_{i\in\mathbb{N}}\left(Y^{-1}(\mathrm{C}) \cap A_i\right)\right] = \sum_{i\in\mathbb{N}}\mathbb{P}\left[Y^{-1}(\mathrm{C}) \cap A_i\right] = \sum_{i\in\mathbb{N}}\mathbb{P}\left[(\mathbb{1}_{A_i}Y)^{-1}(\mathrm{C})\right] \\ &= \sum_{i\in\mathbb{N}}\widetilde{\mathbb{P}}\left[\left(\mathbb{1}_{\widetilde{A}_i}\widetilde{Y}\right)^{-1}(\mathrm{C})\right] = \sum_{i\in\mathbb{N}}\widetilde{\mathbb{P}}\left[\widetilde{Y}^{-1}(\mathrm{C}) \cap \widetilde{A}_i\right] = \widetilde{Y}\left[\bigcup_{i\in\mathbb{N}}\left(\widetilde{Y}^{-1}(\mathrm{C}) \cap \widetilde{A}_i\right)\right] \\ &= \widetilde{\mathbb{P}}\left[\widetilde{Y}^{-1}(\mathrm{C})\right].\end{aligned}$$

(ii)
%TAKE ARBITRARY CONTINUOUS FUNCTIONS AND SUM UP THEIR INTEGRALS

%TALK ABOUT THE NEXT THOEREM BEING A PARTIAL CONVERSE

Conjecture 2.6. Suppose $\{X_n\}_{n\in\mathbb{N}}, X$ have countably closed isolated distributions (See Definition 3.1) and $X_n \Rightarrow X$. Then there exists a family $\{A_{n,i}\}_{n\in\mathbb{N},i\in I} \cup \{A_i\}_{i\in I} \subset \Sigma$ such that $\{A_{n,i}\}_{i\in I}$, for each $n \in \mathbb{N}$, and $\{A_i\}_{i\in I}$ form a $\mathbb{P}$-a.s. partition of Ω, and $\mathbb{1}_{A_{n,i}}X_n \Rightarrow \mathbb{1}_{A_i}X$, for each $i \in I$. More specifically, $\{A_{n,i}\}_{n\in\mathbb{N},i\in I} \cup \{A_i\}_{i\in I}$ can be chosen such that for every $i \in I$,

we have $A_i = X^{-1}(\{x\})$, for some unique $x \in$ Supp(μ_X), and $\lim_{n \to +\infty} \mathbb{P}[A_{n,i}] = \mathbb{P}[A_i]$..
%CHOOSE THIS ONE, CALL THIS A CONJECTURE, AND MAKE A REMARK ON IT

We prove this proposition at the end of the last section, since defining and deriving the topological properties of countably closed distributions are needed to prove this proposition.

%POSSIBLY DELETE
Theorem 2.7. Let $A \in \Sigma$, suppose $\mathbb{P}[A] > 0$, and note that $(\Omega, \Sigma, \mathbb{P}[\cdot|A])$ is a probability space. There exists an surjective bounded operator
$(-)_A : \mathfrak{L}^1(\mathbb{P}; \mathcal{X}) \to \mathfrak{L}^1(\mathbb{P}[\cdot|A]; \mathcal{X}) \subset \mathfrak{L}^1(\mathbb{P}; \mathcal{X})$ such that given $X \in \mathfrak{L}^0(\mathbb{P}; \mathcal{X})$, we have

$$\int_B X_A d\mathbb{P}[\cdot|A] = \int_B \mathbb{1}_A X d\mathbb{P} = \mathbb{E}[\mathbb{1}_{A \cap B} X],$$

for all $B \in \Sigma$ defined by

$$X_A := \mathbb{P}[A] \cdot \mathbb{1}_A X,$$

which extends to a surjective bounded operator $(-)_A : \mathfrak{L}^0(\mathbb{P}; \mathcal{X}) \to \mathfrak{L}^0(\mathbb{P}[\cdot|A]; \mathcal{X})$.

Proposition 2.8. Suppose $\mathbb{P}[A] > 0$ and $\{W_n\}_{n \in \mathbb{N}} \in \mathfrak{L}^0(\mathbb{P}[\cdot|A]; \mathcal{X})$, $W \in \mathfrak{L}^0(\mathbb{P}[\cdot|A]; \mathcal{X})$ such that $(\mathbb{1}_A W_n, \mathbb{P}) \Rightarrow (\mathbb{1}_A W, \mathbb{P})$ as $n \to +\infty$. Then $(W_n, \mathbb{P}[\cdot|A]) \Rightarrow (W, \mathbb{P}[\cdot|A])$--and also and more importantly $(\mathbb{1}_A W_n, \mathbb{P}[\cdot|A])) \Rightarrow (\mathbb{1}_A W, \mathbb{P}[\cdot|A])$--as $n \to +\infty$.
EDIT HYPHENATED ADDITION

§ 3 Definition and Topological Properties of Countable Isolated Sets

%WRITE DEFINITION OF COUNTABLY CLOSED ISOLATED SETS AND DISTRIBUTION
%SHOW PROPERTIES OF DISTANCE BEING SPREAD OUT AND NO LIMIT POINTS

For this section, we shall assume that (E, d) is a metric space.

%CHANGE USE OF $\mathcal{X}$ TO E (ALSO TALK ABOUT WHEN DISTRIBUTION IS CLOSED)
Definition 3.1. We call a set $A \subset E$ **isolated** if every $a \in A$ is isolated in A. Additionally, given an E-valued random variable X, we say that X has an **isolated distribution** if Supp(μ_X) is isolated.

Lemma 3.2. If $A = \{x_i\}_{1 \leq i < \#A+1}$ is a countable isolated set, then there exists

$\{\epsilon_i\}_{1\leq i<\#A+1} \subset (0,+\infty)$ such that $\{B_E(x_i,\epsilon_i)\}_{1\leq i<\#A+1}$ is pairwise disjoint.

Proof. For each $1 \leq i < \#A+1$, recursively choose $\epsilon_1 > 0$ such that $\overline{B_E(x_1,\epsilon_1)}$ contains x_1 but no other $x_i \in A$, then for $i+1$, noting that $d\left(x_{i+1}, \bigcup_{j=1}^{i} B_E(x_j,\epsilon_j)\right) > 0$, choose $0 < \epsilon_{i+1} \leq \frac{1}{2} d\left(x_{i+1}, \bigcup_{j=1}^{i} B_E(x_j,\epsilon_j)\right)$ sufficiently small such that $\overline{B_E(x_{i+1},\epsilon_{i+1})}$ contains x_{i+1} and no other $x_k \in A$.

We find that $\{B_E(x_i,\epsilon_i)\}_{1\leq i<\#A+1}$ is pairwise disjoint since given $1 \leq i_1 < i_2 < \#A+1$, we find for every $x \in B_E(x_{i_2},\epsilon_{i_2})$, we find by recursive definition that

$$d(x_{i_1},x) \geq d(x_{i_2},x_{i_1}) - d(x_{i_2},x) \geq d\left(x_{i_2}, \bigcup_{j=1}^{i_2-1} B_E(x_j,\epsilon_j)\right) - \epsilon_{i_2} \geq \frac{1}{2} d\left(x_{i_2}, \bigcup_{j=1}^{i_2-1} B_E(x_j,\epsilon_j)\right) \geq \epsilon_{i_2},$$

and we have $x \notin B_E(x_{i_1},\epsilon_{i_1})$. □

Proposition 3.3. $A \subset E$ is a set of closed isolated points if and only if A contains no limit points.

Proof. If $A \subset E$ is isolated, then given $x \in A$, there exists $\epsilon > 0$ such that $B_E(x,\epsilon) \cap (A \setminus \{x\})$ so for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset A \setminus \{x\}$, we have $d(x_n,x) \geq \epsilon$, and we conclude that x is not a limit point. Conversely and contrapositively, if A contains a limit point x, then there exists $\{x_n\}_{n\in\mathbb{N}} \subset A \setminus \{x\}$ such that $x_n \to x$ in E, and we conclude that x is not an isolated point since there does not exist an open neighborhood containing x but not some $x_N \in A \setminus \{x\}$ for some $N \in \mathbb{N}$ sufficiently large. □

%CONSIDER QUANITFYING ON ALL $x \in E$ AND NOT JUST $x \in E \setminus A$
Proposition 3.4. If $A \subset E$ is a countably compact isolated set, then for every $x \in E \setminus A$, we have $d(x,A) = d(x,y)$ for some $y \in A$. Moreover, we have $d(x,A) > 0$.

Outline of proof. For each $n \in \mathbb{N}$, choose $y_n \in A$ such that $d(x,y_n) = d(x,A) + n^{-1}$, and note that $\{y_n\}_{n\in\mathbb{N}}$ is a bounded sequence, and hence by sequential compactness there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$, we have $k \in \mathbb{N}$, such that $y_{n_k} \to y$ for some $y \in E$, and we have $y \in A$ since A is closed. and we have

$$d(x,y) = \lim_{k \to +\infty} d(x, y_{n_K}) = \lim_{k \to +\infty} \left[d(x,A) + n_k^{-1} \right] = d(x,A).$$

We then find $d(x,A) > 0$ since if $d(x,A) = 0$, then $y_n \to x$ as $n \to +\infty$, and $x \in A$, since A is closed, which contradicts $x \in E \setminus A$. $\square$
%USE COMPACT AND SEQUENTIAL COMPACTNESSN

%CONSIDER DELETING THIS PROOF
Previous Draft of Proof. If towards contradiction, $d(x,A) = 0$ for $x \in E \setminus A$, then we can choose $\{x_n\}_{n \in \mathbb{N}} \subset A$ such that $d(x_n, x) < n^{-1}$, and we find since $x_n \to x$ as $n \to +\infty$, we find that x is a limit point of A, and we have $x \in A$ since A is closed; however, this contradicts A being a set of isolated points by **Lemma 3.2.** $\square$

%REWORD THIS PROPOSITION AND PROOF; REWORD SO THAT HYPOTHESIS IS SECOND COUNTABILITY; OR AT LEAST TALK ABOUT POSSIBLE CONJECTURE THAT THIS IS EQUIVALENT
Corollary 3.5. If there exists a sequence $\{C_k\}_{k \geq 0}$ of compact sets such that

$$C_0 \subset C_1 \subset \cdots \subset C_k \subset \cdots \text{ and } \overline{\bigcup_{k=0}^{\infty} C_k} = E, \quad (3.1)$$

and If $A \subset E$ is a countably closed isolated set such that given $x \in E$, we have for $k \geq 0$ sufficiently large $A \cap \overline{B(x,\epsilon)} \subset C_k$ for $\epsilon > 0$ sufficiently small, then for every $x \in E \setminus A$, we have $d(x,A) = d(x,y)$ for some $y \in A$. Moreover, we have $d(x,A) > 0$.

Outline of proof. Given $x \in E$, $k \geq 1$ sufficiently large, and $\epsilon > 0$ sufficiently small so that $A \cap \overline{B(x,\epsilon)} \subset C_k$, we apply the argumentation in the proof of Proposition 3.4 to $A' := A \cap \overline{B(x,\epsilon)}$ and $x \in A'$, noting that A' is compact. $\square$

Corollary 3.6. If there exists sequence $\{C_k\}_{k \geq 0}$ of compact sets such that (3.1) holds and If $A \subset E$ is a countably compact isolated set such that given $x \in E$, we have for $k \geq 1$ sufficiently large $A \cap \overline{B(x,\epsilon)} \subset C_k$ for $\epsilon > 0$ sufficiently small (as hypothesized in **Corollary 3.5**), then given $c \in E$, there exists an enumeration $\{x_i\}_{1 \leq i < \#A+1}$ such that x_i is ordered from closest to c to furthest, i.e., $d(x, x_i)$ is increasing for every $1 \leq i < \#A + 1$.
%COROLLARY ABOUT BEING ABLE TO ENUMERATE COUNTABLY CLOSED ISOLATED SETS

Outline of proof. Recursively, for each $1 \leq i < \#A + 1$ choose $x_i \in A \setminus \{x_{i_0}\}_{1 \leq i_0 < i}$ such that

$d(c, x_i) = d(c, A \setminus \{x_{i_0}\}_{1 \le i_0 < i})$. $\square$

%CONSIDER ADJUSTING REQUIRED CONDITIONS OF "COUNTABLY COMPACT ISOLATED SETS" FOR THIS SECTION
%EXPLAIN WHY THE LEMMAS WERE DONE IN SECTION 3

%MENTION INTENT TO PROVE CONJECTURE LATER (IF NOT DONE SO NOW)
%PROVE CONJECTURE 2.6 IN THIS SECTION
Outline of Proof of Theorem 2.6.

§ 4 Fundamental Approximation Properties of Distributions And Convergence of Random Variables

%REPLACE THE PHRASE "COUNTABLY CLOSED IMAGE" WITH "COUNTABLY CLOSED ISOLATED DISTRIBUTION"
%WRITE STATEMENTS CONSIDER POSSIBILITY THAT DISTRIBUTIONS APPROXIMATING EVERYTHING MAY HAVE FINITE DISTRIBUTION AND NOT JUST COUNTABLE

Lemma 4.1. If X is separable, then there exists a sequence $\{C_k\}_{k \ge 0}$ of compact subsets of X such that (3.1) holds (with $E := X$ in this case).

such that for every $x \in X$, there exists a sequence $\{x_k\}_{k \ge 0}$ such that $x_k \in C_k$, for each $k \ge 0$ and $\sum_{k=0}^{\infty} x_k = x$ as $k \to +\infty$.

Proof. Choose a countable basis $\{b_j\}_{j \ge 0}$ of X such that $||b_j|| = 1$, for each $j \ge 0$, and note by the **Hahn-Banach Theorem** that for each $j_0 \ge 0$, there exists $\varphi_{b_{j_0}} \in X^*$ such that

$$\varphi_{b_{j_0}}\left(\sum_{j=0}^{\infty} a_j b_j\right) = a_{j_0}, \text{ for every } \sum_{j=0}^{\infty} a_j b_j \in X.$$

Define $(-)_k : X \to X$ by

%DOUBLE CHECK THIS DEFINITION
%CHANGE THIS DEFINTION TO MAKE THE DISTRIBUTION ISOLATED
$(x)_0 := 2^{-1}\lfloor 2\varphi_{b_0}(x) \rfloor b_0,$

$$(x)_{k+1} := \sum_{j=0}^{k} \left[\left(2^{-(k+j+2)} \lfloor 2^{k+j+2} \varphi_{b_j}(x) \rfloor - 2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(x) \rfloor \right) b_j \right] \quad (4.1)$$
$$+ 2^{-(2(k+1)+1)} \lfloor 2^{2(k+1)+1} \varphi_{b_{k+1}}(x) \rfloor b_{k+1},$$

%MAKE SURE THIS IS CORRECT

where $\lfloor a \rfloor := \lfloor \text{Re}(a) \rfloor + i \lfloor \text{Im}(a) \rfloor$, for all $a \in \mathbb{C}$. Define $\phi_k := \sum_{j=0}^{k} (-)_j$, note that the image of ϕ_k by construction is countably closed and isolated, and define $C_k := \phi_k(\mathcal{X}) \cap \overline{B_{\mathcal{X}}(0,k)}$, and note that C_k is compact since C_k is a closed and bounded subset contained in the finite dimensional subspace $\text{Span}(b_1, \ldots, b_k)$. Note by construction that $C_0 \subset C_1 \subset \cdots \subset C_k \subset \cdots$ and to verify that $\overline{\bigcup_{k=0}^{\infty} C_k} = \mathcal{X}$, Note by construction that for all $x \in \mathcal{X}$, we find that

$$\begin{aligned}
&\lim_{k \to +\infty} (\mathbb{1}_{\overline{B_{\mathcal{X}}(0,k)}}(\phi_k(x)) \phi_k(x)) \\
&= \lim_{k \to +\infty} \left(\mathbb{1}_{||y|| \le k}(\phi_k(x)) \sum_{j=0}^{k} (x)_j \right) \\
&= \lim_{k \to +\infty} (\mathbb{1}_{||y|| \le k}(\phi_k(x))) \\
&\quad \cdot \lim_{k \to +\infty} \left(2^{-1} \lfloor 2 \varphi_{b_0}(x) \rfloor b_0 + \sum_{j=1}^{k} \left[\sum_{m=0}^{k} \left[\left(2^{-(j+m+1)} \lfloor 2^{j+m+1} \varphi_{b_m}(x) \rfloor - 2^{-(j+m)} \lfloor 2^{j+m} \varphi_{b_m}(x) \rfloor \right) b_m \right] \right. \right. \\
&\qquad \left. \left. + 2^{-(2(j+1)+1)} \lfloor 2^{2(j+1)+1} \varphi_{b_{j+1}}(x) \rfloor b_{j+1} \right] \right) \\
&= \lim_{k \to +\infty} \sum_{j=0}^{k} \left[2^{-(j+k+1)} \lfloor 2^{j+k+1} \varphi_{b_j}(x) \rfloor b_j \right] \\
&= \sum_{j=0}^{\infty} \varphi_{b_j}(x) b_j \\
&= x.
\end{aligned}$$

□

%DELETE THIS VERSION OF THE LEMMA

%REDO LEMMA 4.1

Lemma 4.1. If $\mathcal{X}$ is separable, then there exists a sequence $\{C_k\}_{k \ge 0}$ of compact sets such that such that for every $x \in \mathcal{X}$, there exists a sequence $\{x_k\}_{k \ge 0}$ such that $x_k \in C_k$, for each $k \ge 0$ and $\sum_{k=0}^{\infty} x_k = x$ as $k \to +\infty$..

%MAKE A LEMMA ABOUT SEPARABLE BANACH SPACES BEING UNIFORMLY

APPROXIMATED BY COUNTABLE CLOMPACT ISOLATED SETS AND MOVE SOME OF PROOF OF THEOREM 4.2 TO PROOF OF THIS LEMMA
%PROVE THIS LEMMA AND MODEL SETS AFTER THIS ONE

Outline of proof. Choose a countable basis $\{b_j\}_{j\geq 0}$ of $\mathcal{X}$ such that $||b_j|| = 1$, for each $j \geq 0$, and note by the **Hahn-Banach Theorem** that for each $j_0 \geq 0$, there exists $\varphi_{b_{j_0}} \in \mathcal{X}^*$ such that

$$\varphi_{b_{j_0}}\left(\sum_{j=0}^{\infty} a_j b_j\right) = a_{j_0}, \text{ for every } \sum_{j=0}^{\infty} a_j b_j \in \mathcal{X}.$$

Define $(-)_k : \mathcal{X} \to \mathcal{X}$ by

%DOUBLE CHECK THIS DEFINITION
%CHANGE THIS DEFINTION TO MAKE THE DISTRIBUTION ISOLATED

$$(x)_0 := 2^{-1}\lfloor 2\varphi_{b_0}(x)\rfloor b_0,$$

$$(x)_{k+1} := \sum_{j=0}^{k}\left[\left(2^{-(k+j+2)}\lfloor 2^{k+j+2}\varphi_{b_j}(x)\rfloor - 2^{-(k+j+1)}\lfloor 2^{k+j+1}\varphi_{b_j}(x)\rfloor\right) b_j\right]$$
$$+2^{-(2(k+1)+1)}\lfloor 2^{2(k+1)+1}\varphi_{b_{k+1}}(x)\rfloor b_{k+1},$$

%MAKE SURE THIS IS CORRECT

where $\lfloor a\rfloor := \lfloor \mathrm{Re}(a)\rfloor + i\lfloor \mathrm{Im}(a)\rfloor$, for all $a \in \mathbb{C}$. Define $\phi_k := \sum_{j=0}^{k}(-)_j$, note that the image of ϕ_k by construction is countably closed and isolated, and define $C_k := \phi_k(\mathcal{X}) \cap \overline{B_{\mathcal{X}}(0,k)}$.
Noting by construction that
%EXPLAIN WHY THIS SET IS COMPACT

%REVISE THIS DERIVATION

$$\lim_{k\to+\infty}(\mathbb{1}_{\overline{B_{\mathcal{X}}(0,k)}}(\phi_k(x))\phi_k(x))$$
$$= \lim_{k\to+\infty}\left(\mathbb{1}_{||y||\leq k}(\phi_k(x))\sum_{j=0}^{k}(x)_j\right)$$
$$= \lim_{k\to+\infty}(\mathbb{1}_{||y||\leq k}(\phi_k(x)))$$
$$\cdot \lim_{k\to+\infty}\left(2^{-1}\lfloor 2\varphi_{b_0}(x)\rfloor b_0 + \sum_{j=1}^{k}\left[\sum_{m=0}^{k}\left[\left(2^{-(j+m+1)}\lfloor 2^{j+m+1}\varphi_{b_m}(x)\rfloor - 2^{-(j+m)}\lfloor 2^{j+m}\varphi_{b_m}(x)\rfloor\right) b_m\right]\right.\right.$$

$$+2^{-(2(j+1)+1)}\lfloor 2^{2(j+1)+1}\varphi_{b_{j+1}}(x)\rfloor b_{j+1}\Big]\Big)$$

$$= \lim_{k\to+\infty}\sum_{j=0}^{k}\Big[2^{-(j+k+1)}\lfloor 2^{j+k+1}\varphi_{b_j}(x)\rfloor b_j\Big]$$

$$= \sum_{j=0}^{\infty}\varphi_{b_j}(x)b_j$$

$$= x,$$

%REWRITE THEOREM AND EXPLAIN WHY COROLLARY 3.6 GETS TRIGGERED

Theorem 4.2. For every sequence $\{X_n\}_{n\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables and $\mathcal{X}$-valued random variable X, for each $n \in \mathbb{N}$ there exists a series such that

$X_n = \sum_{k=0}^{\infty} X_{n,k},\ X = \sum_{k=0}^{\infty} X_k$ of random variables with countably closed isolated distribution such that $||X_{n,k}||, ||X_k|| \le 2^{-k}$, for $n, k \ge 1$, and satisfies the following four properties:

(i) $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $n \to +\infty$ if and only if $X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} X_k$ as $n \to +\infty$, for each $k \ge 0$.

(ii) $X_n \xrightarrow{\mathbb{P}} X$ as $n \to +\infty$ if and only if $X_{n,k} \xrightarrow{\mathbb{P}} X_k$ as $n \to +\infty$, for each $k \ge 0$.

(iii) $X_n \Rightarrow X$ as $n \to +\infty$ if and only if $(X_{n,0},\ \dots, X_{n,k}) \Rightarrow (X_1,\ \dots, X_k)$ as $n \to +\infty$, for each $k \ge 0$.

(iv) It remains that way up to distribution equivalence, i.e., if we have a probability space $\left(\widetilde{\Omega}, \widetilde{\mathbb{P}}\right)$ and random variables $\left\{\widetilde{X}_{n,k}\right\}_{n\ge1,k\ge0} \left\{\widetilde{X}_k\right\}_{k\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that $\left(\widetilde{X}_{n,0},\ \dots, \widetilde{X}_{n,k}\right) \overset{D}{=} (X_{n,0},\ \dots, X_{n,k})$ and $\left(\widetilde{X}_0,\ \dots, \widetilde{X}_k\right) \overset{D}{=} (X_0,\ \dots, X_k)$ for each $k \ge 0,\ n \ge 1$, then $\widetilde{X}_n := \sum_{k=0}^{\infty}\widetilde{X}_{n,k}$ and $\widetilde{X} := \sum_{k=0}^{\infty}\widetilde{X}_k$ are well-defined and *(i)*-*(iii)* still hold. Moreover, we have $\widetilde{X}_n \overset{D}{=} X_n$ for each $n \ge 1$ and $\widetilde{X} \overset{D}{=} X$.

Outline of proof.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{X}$-Valued random variables and X be an $\mathcal{X}$-valued random variable. Set

$$\mathcal{X}_0 := \overline{\text{Span}_{\mathbb{C}}\left(X(\Omega) \cup \bigcup_{n\in\mathbb{N}} X_n(\Omega)\right)},$$

and note that $\mathcal{X}_0$ is an $\mathbb{P}$-a.s. separable subspace of $\mathcal{X}$ containing each image of X_n for all $n \geq 1$ and X , since the complex-rational span

$$\mathcal{S} := \text{Span}_{\mathbb{Q}[i]}\left(\{x_k\}_{k\in\mathbb{N}} \cup \bigcup_{n\in\mathbb{N}} \{x_{n,k}\}_{k\in\mathbb{N}}\right)$$

is the $\mathbb{Q}[i]$-linear combinations of countable union of $\mathbb{P}$-a.s. countable dense sets--i.e. $\{x_k\}_{k\in\mathbb{N}}$ of $X(\Omega)$ and $\{x_{n,j}\}_{j\in\mathbb{N}}$ of $X_n(\Omega)$ for each $n \geq 1$--is a countable $\mathbb{P}$-a.s. dense set of $\mathcal{X}_0$. Choose a countable basis $\{b_j\}_{j\in\mathbb{N}}$ of $\mathcal{X}_0$ such that $||b_j|| = 1$, for each $j \geq 1$, and note by the **Hahn-Banach Theorem** that for each $j_0 \geq 1$, there exists $\varphi_{b_{j_0}} \in \mathcal{X}^*$ such that

$$\varphi_{b_{j_0}}\left(\sum_{k=1}^{\infty} a_j b_j\right) = a_{j_0}, \text{ for every } \sum_{j=1}^{\infty} a_j b_j \in \mathcal{X}_0.$$

%SHORTEN THE ABOVE EXPLANATION

Define $(-)_k : \mathcal{X} \to \mathcal{X}$ for $k \geq 0$ as in (4.1) and define

%DOUBLE CHECK THIS DEFINITION
%CHANGE THIS DEFINTION TO MAKE THE DISTRIBUTION ISOLATED
$$X_{n,k} := (X_n)_k, \quad X_k := (X)_k, \quad (4.2)$$

%MAKE DIFFERENT DERIVATION
Since

$$\left|2^{-(j+1)}\lfloor 2^{j+1}a\rfloor - 2^{-j}\lfloor 2^j a\rfloor\right| \leq 2 \cdot 2^{-(j+1)} = 2^{-j},$$
$$a = 2^{-j}\lfloor 2^j a\rfloor + \sum_{k=1}^{\infty}\left[2^{-(j+k+1)}\lfloor 2^{m+j+1}a\rfloor - 2^{-(j+k)}\lfloor 2^{j+k}a\rfloor\right],$$

for all $j \geq 1$, note for $k_0 \geq 1$ that

$$||X_{n,k_0}|| \leq \sum_{j=1}^{\infty}\left|2^{-(k_0+j+1)}\lfloor 2^{k_0+j+1}\varphi_{b_j}(X_n)\rfloor - 2^{-(k_0+j)}\lfloor 2^{k_0+j}\varphi_{b_j}(X_n)\rfloor\right| ||b_j|| \leq \sum_{j=1}^{\infty} 2^{-(k_0+j)} = 2^{-k_0},$$
$$\implies \left|\left|X_n - \sum_{k=1}^{k_0} X_{n,k}\right|\right|$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{\infty} \left(\varphi_{b_j}(X_n) - \sum_{k=0}^{k_0} [\varphi_{b_j}(X_{n,k})] \right) b_j \right\| \\
&= \left\| \sum_{j=1}^{\infty} \left[2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X_n) \rfloor + \sum_{k=1}^{\infty} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right. \right. \\
&\qquad \left. \left. - \left(2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X_n) \rfloor + \sum_{k=1}^{k_0} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right) \right] b_j \right\| \\
&= \left\| \sum_{j=1}^{\infty} \left[\sum_{k=k_0+1}^{\infty} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right] b_j \right\| \\
&\leq \sum_{j=1}^{\infty} \left[\sum_{k=k_0+1}^{\infty} 2^{-(j+k)} ||b_j|| \right] \\
&= 2^{-k_0},
\end{aligned}
$$

which shows that $\sum_{k=0}^{k_0} X_{n,k} \to X_n$ pointwise as $k_0 \to +\infty$, for every $n \geq 1$, and by similar derivation we have $||X_{k_0}|| \leq 2^{-k_0}$ for all $k_0 \geq 1$ and $\sum_{k=0}^{k_0} X_k \to X$ pointwise as $k_0 \to +\infty$.

%INSTEAD TALK ABOUT $\mathcal{L}^\infty$-CONVERGENCE

%SIMPLIFY RESULTS USING DIAGONALIZATION

(i)

$\Longrightarrow$ Suppose $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $n \to +\infty$. Then $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}\text{-a.s.}} \varphi_{b_j}(X)$ as $n \to +\infty$ for each $j \geq 1$, and $\mathbb{P}$-a.s. convergence of $X_{n,0}, X_{n,k+1}$ for each $k \geq 1$ follows.

%SHOW THAT $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}\text{-a.s.}} \varphi_{b_j}(X)$ AS $n \to +\infty$ FOR $j \geq 0$ AND PROVE RESULT FROM THERE

$\Longleftarrow$ Suppose conversely $X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} X_k$ as $n \to +\infty$ for each $k \geq 1$. Noting that $\sum_{k=0}^{N} X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} \sum_{k=0}^{N} X_k$, we find that given $\epsilon > 0$, we find $\mathbb{P}$-a.s. for every $N \geq 1$ we we have

$\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| < \epsilon/3$ eventually for $n \geq 1$, and eventually for $N \geq 1$ we have $\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| < \epsilon/3$ and we can show $\mathbb{P}$-a.s. convergence of $X_n = \sum_{k=0}^{\infty} X_{n,k}$ to $X = \sum_{k=0}^{\infty} X_k$ from there.

(ii)

$\Longrightarrow$ Similar to proving *(i)*, we find the hypothesis implies $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}} \varphi_{b_j}(X)$ as $n \to +\infty$ for each $j \geq 1$, and convergence in $\mathbb{P}$ of $X_{n,0}, X_{n,k+1}$ for each $k \geq 1$ follows.

%FIND SUBSEQUENCE CONVERGING POINTWISE USING (I) AND DIAGONALIZATION AND HYPOTHESES
%IF FIRST STRATEGY DOESN'T WORK MAKE SIMILAR CHANGE TO (I) AND SHOW THAT $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}} \varphi_{b_j}(X)$ AS $n \to +\infty$ FOR $j \geq 0$ AND PROVE RESULT FROM THERE

$\Longleftarrow$ Similar to proving *(i)*, we note $\sum_{k=0}^{N} X_{n,k} \xrightarrow{\mathbb{P}} \sum_{k=0}^{N} X_k$ and we can choose $N \geq 1$ sufficiently large such that $\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| < \epsilon/3$, for every $n \geq 1$, and since

$$\lim_{n \to +\infty} \mathbb{P}\left[\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3\right] = 0 \text{ as } n \to +\infty$$

and

$$||X_n - X|| \geq \epsilon \Longrightarrow \left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3 \vee \left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right| \geq \epsilon/3 \vee \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| \geq \epsilon/3,$$

which in turn implies $\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3$, since

$\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_{k}\right|\right| < \epsilon/3$, we find

$\mathbb{P}[||X_n - X|| \geq \epsilon] = O\left(\mathbb{P}\left[\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_{k}\right|\right| \geq \epsilon/3\right]\right)$ our conclusion follows.

(iii)

%CLARIFY THE A.S. CONTINUOUS MAPPING AND THE MEASURE THAT WE'RE LOOKING AT

$\Longrightarrow$ Note that the mapping

$\phi_k : x \mapsto ((x)_0, \dots, (x)_k)$ where $(x)_0 := \sum_{j=1}^{\infty} 2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(x) \rfloor b_j$,

$$(x)_{k_0+1} := \sum_{j=1}^{\infty} (2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(x) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(x \rfloor) b_j, \ \forall 0 \leq k_0 < k \qquad (4.1)$$

is an almost everywhere continuous mapping from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), m_{\mathcal{B}(\mathcal{X})}) \to \left(\mathcal{X}^k, \mathcal{B}\left(\mathcal{X}^k\right), m_{\mathcal{B}\left(\mathcal{X}^k\right)}\right)$.

%EXPLAIN WHY THAT CONTINUITY IS

%EXPLAIN THE CONTINUITY DIFFERENTLY

It immediately follows that $X_n \Rightarrow X$ implies $(X_{n,0}, \dots, X_{n,k}) \Rightarrow (X_0, \dots, X_k)$, for every $k \geq 0$

%MENTION THOEREM THAT STATES THIS

%USE EITHER DOMINATED CONVERGENCE THEOREM OR EGOROV'S THEOREM INSTEAD OF UNIFORM CONVERGENCE

$\Longleftarrow$ We find for each $k \geq 0$ by hypothesis that for each $N \geq 1$, we have $\sum_{k=0}^{N} X_{n,k} \Rightarrow \sum_{k=0}^{N} X_k$

.Given $f \in C_b(\mathcal{X}; \mathbb{R})$, we find for each $N \geq 0$, we have

$$\lim_{n \to +\infty} \int f d\mu_{\sum_{k=0}^{N} X_{n,k}} = \int f d\mu_{\sum_{k=0}^{N} X_k}. \qquad (4.2)$$

Moreover, we find since $\sum_{k=0}^{N} X_k \xrightarrow{u} X$ as $N \to +\infty$, we find by continuity of f that $f\left(\sum_{k=0}^{N} X_k\right) \xrightarrow{u} f(X)$ as $N \to +\infty$, Then for each $n \geq 1$, we can choose $N_n \geq 0$ such that $N_{n+1} > N_n$ and sufficiently large such that

$$\left| f\left(\sum_{k=0}^{N_n} X_k\right) - f(X) \right| \leq n^{-1}. \quad (4.3)$$

We find that

$$\begin{aligned}
&\lim_{n \to +\infty} \left| \int f d\mu_{X_n} - \int f d\mu_X \right| \\
&\leq \lim_{n \to +\infty} \left(\left| \int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} \right| + \left| \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k} \right| \right. \\
&\quad \left. + \left| \int f d\mu_{\sum_{k=0}^{N} X_k} - \int f d\mu_X \right| \right) \\
&\leq \lim_{n \to +\infty} \left(\left| \int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} \right| \right) + \lim_{n \to +\infty} \left(\left| \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k} \right| \right) \\
&\quad + \lim_{n \to +\infty} \mathbb{E}\left[\left| f\left(\sum_{k=0}^{N_n} X_k\right) - f(X) \right| \right].
\end{aligned}$$

By (4.2) and diagonalization we have

$$\left| \int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} \right|, \left| \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k} \right| \to 0 \text{ as } n \to +\infty,$$

and by (3.3) we have $\mathbb{E}\left[\left| f\left(\sum_{k=0}^{N_n} X_k\right) - f(X) \right| \right] = O\left(n^{-1}\right)$ as $n \to +\infty$, and we conclude that $\lim_{n \to +\infty} \left| \int f d\mu_{X_n} - \int f d\mu_X \right| = 0$, and our conclusion has been reached.

%PROOF OF (I)-(III) FOLLOWS BY THE SAME DERIVATION AND USE SEQUENCE OF BASIS COEFFICIENT FUNCTIONALS TO SHOW WELL-DEFINEDNESS AND EQUIVALENCE IN DISTRIBUTION

(iv) Suppose we have random variables $\left\{\widetilde{X}_{n,k}\right\}_{n\geq1,k\geq0}$, $\left\{\widetilde{X}_k\right\}_{k\geq0} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)$ such that $\left(\widetilde{X}_{n,0},\ \dots,\widetilde{X}_{n,k}\right) \overset{D}{=} (X_{n,0},\ \dots,X_{n,k})$ and $\left(\widetilde{X}_0,\ \dots,\widetilde{X}_k\right) \overset{D}{=} (X_0,\ \dots,X_k)$ for each $k\geq 0,\ n\geq 1$. It follows immediately by this hypothesis that $\widetilde{\mathbb{P}}$-a.s., we have $||\widetilde{X}_{n,k}||, ||\widetilde{X}_k|| \leq 2^{-k}$ for $n,k\geq 1$, and hence well-definedness of $\widetilde{X}_n := \sum_{k=0}^{\infty}\widetilde{X}_{n,k}$ for every $n\geq 1$ and $\widetilde{X} := \sum_{k=0}^{\infty}\widetilde{X}_k$ immediately follow.

To prove $\implies$ of *(i)-(iii)*, we note that we can explicitly construct $\left\{\widetilde{X}_{n,k}\right\}_{n\geq1,k\geq0}$, $\left\{\widetilde{X}_k\right\}_{k\geq0} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)$ such that *(i)-(iii)* hold by using the $m_{\mathfrak{B}(\mathcal{X})}$-almost everywhere continuous mappings $(-)_k:\mathcal{X}\to\mathcal{X}$ defined as in (4.1), i.e., given $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right),\ \widetilde{X}\in\mathfrak{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right),$, we can set

$$\overline{\widetilde{X}}_{n,k} := \left(\widetilde{X}_n\right)_k,\ \overline{\widetilde{X}}_k := \left(\widetilde{X}\right)_k,$$

and we find by construction of $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}$, $\widetilde{X}$ that $\widetilde{\mathbb{P}}$-a.s., we have $\widetilde{X}_{n,k} = \overline{\widetilde{X}}_{n,k}$ and $\widetilde{X}_k = \overline{\widetilde{X}}_k$ for $n\geq 1,\ k\geq 0$. Conditions *(i)-(iii)* working for $\left\{\widetilde{X}_{n,k}\right\}_{n\geq1,k\geq0}$, $\left\{\widetilde{X}_k\right\}_{k\geq0}$, $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}$, $\widetilde{X}$ follows immediately from $\widetilde{\mathbb{P}}$-a.s. equality of $\left\{\widetilde{X}_{n,k}\right\}_{n\geq1,k\geq0}$, $\left\{\widetilde{X}_k\right\}_{k\geq0}$ to the explicit construction $\left\{\overline{\widetilde{X}}_{n,k}\right\}_{n\geq1,k\geq0}$, $\left\{\overline{\widetilde{X}}_k\right\}_{k\geq0}$, hence the proof proceeds identically to the original proofs of *(i)-(iii)*.

%MAKE SURE TO TALK ABOUT $\mathfrak{L}^\infty$ CONVERGENCE IN PLACE OF UNIFORM CONVERGENCE TO SHOW THAT IT IS IDENTICAL

Finally, to show that $\widetilde{X}_n \stackrel{D}{=} X_n$ for each $n \geq 1$ and $\widetilde{X} \stackrel{D}{=} X$, note that since by hypotheses that $\left(\widetilde{X}_{n,0}, \ldots, \widetilde{X}_{n,N}\right) \stackrel{D}{=} (X_{n,0}, \ldots, X_{n,N})$ and $\left(\widetilde{X}_0, \ldots, \widetilde{X}_N\right) \stackrel{D}{=} (X_0, \ldots, X_k)$ for each $N \geq 0,\ n \geq 1$, we find for each $N \geq 0$ we have $\sum_{k=0}^{N} \widetilde{X}_{n,k} \stackrel{D}{=} \sum_{k=0}^{N} X_{n,k}$ for each $n \geq 1$ and $\sum_{k=0}^{N} \widetilde{X}_k \stackrel{D}{=} \sum_{k=0}^{N} X_k$. Since $\mathcal{L}^\infty$-convergence implies convergence in distribution, we have $\sum_{k=0}^{N} X_{n,k} \Rightarrow X_n, \sum_{k=0}^{N} \widetilde{X}_{n,k} \Rightarrow \widetilde{X}_n$, for each $n \geq 1$ and $\sum_{k=0}^{N} X_k \Rightarrow X, \sum_{k=0}^{N} \widetilde{X}_k \Rightarrow \widetilde{X}$ as $N \to +\infty$, we find $\widetilde{X}_n \stackrel{D}{=} X_n$ for each $n \geq 1$ and $\widetilde{X} \stackrel{D}{=} X$ follows by uniqueness of distribution limits up to distribution equivalence. □

%TALK ABOUT HOW RANDOM VARIABLES CAN THEN BE SEEN AS MAPPING TO AN INFINITE TREE WITH COUNTABLY MANY BRANCHES

§ 5 Skorohod's Theorem For Random Variables With Countably Closed Isolated Distributions

First, we prove Skorohod's Theorem in the following special case below:

Theorem 5.1. *Skorohod's Representation Theorem* holds such that $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, if $\{X_n\}_{n \in \mathbb{N}}, X$ are $\mathcal{X}$-valued random variables with countably closed isolated distributions.

To prove this, we first prove the following two lemmas:

Lemma 5.2. *Skorohod's Representation Theorem* holds such that $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, if $\{X_n\}_{n \in \mathbb{N}}$ are $\mathcal{X}$-valued random variables with countably closed isolated distributions, and X is $\mathbb{P}$-a.s. constant.

Proof. Suppose $\{X_n\}_{n \in \mathbb{N}}$ are $\mathcal{X}$-valued random variables with a countable image of isolated points, and $X = c$ $\mathbb{P}$-a.s. for some $c \in \mathcal{X}$. For each $n \in \mathbb{N}$, we can by **Corollary 3.6** enumerate $\mathrm{Supp}(\mu_{X_n}) := \{x_{n,i}\}_{1 \leq i < \#\mathrm{Supp}(X_n)+1}$ such that $x_{n,i}$ is ordered from closest to c to furthest, i.e., for every $i_0 \in \mathbb{N}$, we have

$$\min\{||x_{n,i} - c|| : i_0 \leq i\} = ||x_{n,i_0} - c||, \quad (5.1)$$

which we can do since Supp(μ_{X_n}) is isolated. Then for every $n \in \mathbb{N}$, define $q_{n,i}$ recursively for $0 \leq i < \#\text{Supp}(\mu_{X_n}) + 1$ $q_{n,i}$ by $q_{n,0} := 0$ and $q_{n,i+1} := q_{n,i} + \mathbb{P}[X_n = x_{n,i+1}]$. For each $n \in \mathbb{N}$, define $\widetilde{X}_n : \widetilde{\Omega} \to \mathcal{X}$ by $\widetilde{X}_n\left(\widetilde{\omega}\right) := x_{n,i}$, for $\widetilde{\omega} \in [q_{n,i-1}, q_{n,i})$, and set $\widetilde{X} : \widetilde{\Omega} \to \mathcal{X}$ equal to the constant c. We find by construction that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, and $X \overset{D}{=} \widetilde{X}$, and it remains to show that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$ and $n \to +\infty$.

To show that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$ (and more generally that $\widetilde{X}_n \xrightarrow{\text{pointwise}} \widetilde{X}$) as $n \to +\infty$, note by **Proposition 1.2** that since $\widetilde{X}_n \Rightarrow \widetilde{X}$ and $\widetilde{X} = c$, we find that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{X}$, as $n \to +\infty$. By (5.1), we find for all $\widetilde{\omega}, \widetilde{\omega}' \in \widetilde{\Omega}$ such that $\widetilde{\omega} \leq \widetilde{\omega}'$, we find

$$||\widetilde{X}_n\left(\widetilde{\omega}\right) - \widetilde{X}\left(\widetilde{\omega}\right)|| = ||\widetilde{X}_n\left(\widetilde{\omega}\right) - c|| \leq ||\widetilde{X}_n\left(\widetilde{\omega}'\right) - c|| = ||\widetilde{X}_n\left(\widetilde{\omega}'\right) - \widetilde{X}\left(\widetilde{\omega}'\right)||,$$

which shows that $||\widetilde{X}_n - \widetilde{X}||$ is monotonically increasing on $\widetilde{\Omega}$, and our conclusion immediately follows by **Lemma 1.5**. □

Lemma 5.3. if $\{X_n\}_{n\in\mathbb{N}}, X$ are $\mathcal{X}$-valued random variables with countably closed isolated distributions, there exists $\{\widehat{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$, $\widehat{X} \in \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$ for $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mathbb{P}}) := ([0, 1), \mathcal{B}([0, 1)), m_{[0,1)})$, such that $X_n \overset{D}{=} \widehat{X}_n$, for each $n \in \mathbb{N}$, $\widehat{X} \overset{D}{=} X$, and $\widehat{X}_n \xrightarrow{\widehat{\mathbb{P}}} \widehat{X}$ as $n \to +\infty$.

Proof. Enumerate $\text{Supp}(\mu_X) = \{x_j\}_{1\leq j<\#\text{Supp}(\mu_X)+1}$ in order from greatest to least of $\mathbb{P}[X = x_j]$. Recursively choose N_i, n_i, for each $0 \leq i < \#\text{Supp}(\mu_X) + 1$, as follows. Set $N_0, n_0 := 0$. Then for $i + 1$, choose $N_{i+1} > N_i$ sufficiently large such that

$$\sum_{j_0=N_{i+1}+1}^{\#\text{Supp}(\mu_X)} \mathbb{P}[X = x_{j_0}] < \frac{1}{2}(i+1)^{-1}, \quad (5.2)$$

and then choose $n_{i+1} > n_I$ sufficiently large so that for $\epsilon_{i+1} > 0$ sufficiently small such that

$\epsilon_{i+1} \le (i+1)^{-1}$ and $\{B_{\mathcal{X}}(x_j, \epsilon_{i+1})\}_{1\le j\le \min\left(\#\mathrm{Supp}(\mu_X), N_{i+1}\right)}$ is pairwise disjoint, and $B_{\mathcal{X}}(x_j, \epsilon_{i+1})$, for each $1 \le j \le \min\left(\#\mathrm{Supp}(\mu_X), N_{i+1}\right)$, is a μ_X-continuity set (since X has a countably closed isolated distribution), and for $n \ge n_{i+1}$, we have

$$|\mathbb{P}[X_n \in \ B_{\mathcal{X}}(x_j, \epsilon_{i+1})] - \mathbb{P}[X \in \ B_{\mathcal{X}}(x_j, \epsilon_{i+1})]| < (i+1)^{-1}2^{-(i+j+2)}. \quad (5.3)$$

For every $i \ge 0$ $n_i < n \le n_{i+1}$ and $1 \le j < \#\mathrm{Supp}(\mu_X) + 1$, enumerate

$$\mathrm{Supp}(\mu_{X_n}) \cap B_{\mathcal{X}}(x_j, \epsilon_i) := \{x[n,i,j,k]\}_{1\le k<\#\left(\mathrm{Supp}(\mu_{X_n})\cap B_{\mathcal{X}}(x_j,\epsilon_i)\right)+1},$$

for $1 \le j \le \min\left(\#\mathrm{Supp}(\mu_X), N_i\right)$, (5.4)

$$\mathrm{Supp}(\mu_{X_n}) \setminus \left[\bigcup_{j=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} B_{\mathcal{X}}(x_j, \epsilon_i)\right] := \{x[n,i,j,k]\}_{1\le k<\#\left(\mathrm{Supp}(\mu_{X_n})\setminus\left[\bigcup_{j=1}^{\min\left(\#\mathrm{Supp}(\mu_X),N_i\right)} B_{\mathcal{X}}(x_j,\epsilon_i)\right]\right)+1},$$

for

$j = N_i + 1,$ (5.5)

then define $q[n,i,j,k]$ recursively for $i \ge 0$, $n_i < n \le n_{i+1}$, $1 \le j \le \min\left(\#\mathrm{Supp}(\mu_X), N_i\right)$, and $k \ge 0$ as follows:

$$q[n,i,1,0] := 0,$$

$$q[n,i,j+1,0] := \begin{cases} q[n,i,j,0] + \mathbb{P}[X_n \in B_{\mathcal{X}}(x_j, \epsilon_i)], & j < N_i, \\ \mathbb{P}\left[X_n \in \displaystyle\bigcup_{j_0=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} B_{\mathcal{X}}(x_{j_0}, \epsilon_i)\right] & j = N_i, \end{cases} \quad (5.6)$$

$$q[n,i,j,k+1] := q[n,i,1,k] + \mathbb{P}[X_n = x[n,i,j,k+1]]. \quad (5.7)$$

Then for each $0 \le j < \mathrm{Supp}(\mu_X) + 1$, define q_j recursively by $q_0 := 0$ and $q_{j+1} := q_j + \mathbb{P}[X = x_{j+1}]$.

Next, define $\{\widehat{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\widehat{\mathbb{P}}; \mathcal{X})$, $\widehat{X} \in \mathcal{L}^0(\widehat{\mathbb{P}}; \mathcal{X})$ by $\widehat{X}_n(\widehat{\omega}) = x[n,i,j,k]$, for some $i \ge 0$, $n_i < n \le n_{i+1}$, $1 \le j \le \min\left(\#\mathrm{Supp}(\mu_X), N_i\right)$, and $k \ge 1$ such that $\widehat{\omega} \in [q[n,i,j,k-1], q[n,i,j,k])$ and $\widehat{X}(\widehat{\omega}) = x_i$, for $1 \le i < \#\mathrm{Supp}(\mu_X) + 1$ such that

$\widehat{\omega} \in [q_{i-1}, q_i)$.

First, we show that for each $n \in \mathbb{N}$, we have $\widehat{X}_n \overset{D}{=} X_n$, and $\widehat{X} \overset{D}{=} X$. Since $\{X_n\}_{n \in \mathbb{N}}, X$ have countable images, it shall suffice that for each $n \in \mathbb{N}$ and $x_n \in \text{Supp}(\mu_{X_n})$, we have $\widehat{\mathbb{P}}[\widehat{X}_n = x_n] = \mathbb{P}[X_n = x_n]$, and for each $x \in \text{Supp}(\mu_X)$, we have $\widehat{\mathbb{P}}[\widehat{X} = x] = \mathbb{P}[X = x]$. To show $\widehat{\mathbb{P}}[\widehat{X}_n = x_n] = \mathbb{P}[X_n = x_n]$, given $n \in \mathbb{N}$ and $x_n \in \text{Supp}(\mu_{X_n})$, let $i_0 \geq 0$ be the highest $i \geq 0$ such that $n_i < n$. We then have two cases.

Case 1. Suppose $x_n \in \text{Supp}(\mu_{X_n}) \cap B_X(x_{j_0}, \epsilon_{i_0})$, for some $1 \leq j_0 \leq \min\Big(\#\text{Supp}(\mu_X), N_{i_0}\Big)$ such that $j_0 \leq N_{i_0}$. Then by (5.4), we have $x_n = x[n, i_0, j_0, k_0]$, for some $1 \leq k_0 < \#\Big(\text{Supp}(\mu_{X_n}) \cap B_X(x_{j_0}, \epsilon_{i_0})\Big) + 1$, hence by (5.6) in the case where $j := j_0 - 1 < N_{i_0}$, and by (5.7) we have $\widehat{X}_n^{-1}(\{x[n, i_0, j_0, k_0]\}) = [q[n, i_0, j_0, k_0 - 1], q[n, i_0, j_0, k_0])$, so we conclude that

$$\begin{aligned}
\widehat{\mathbb{P}}[\widehat{X}_n = x_n] = \widehat{\mathbb{P}}[\widehat{X}_n &= x[n, i_0, j_0, k_0]] \\
&= m_{[0,1)}([q[n, i_0, j_0, k_0 - 1], q[n, i_0, j_0, k_0])) \\
&= \mathbb{P}[X_n = x[n, i_0, j_0, k_0]] \\
&= \mathbb{P}[X_n = x_n]. \qquad (5.8)
\end{aligned}$$

Case 2. Suppose $x_n \in \text{Supp}(\mu_{X_n}) \setminus \left[\bigcup_{j=1}^{\min\left(\#\text{Supp}(\mu_X), N_i\right)} B_X(x_j, \epsilon_i)\right]$. Then by (5.5), we have $x_n = x[n, i_0, j_0, k_0]$ for $j_0 := N_{i_0} + 1$ and some $1 \leq k_0 < \#\left(\text{Supp}(\mu_{X_n}) \setminus \left[\bigcup_{j=1}^{\min\left(\#\text{Supp}(\mu_X), N_i\right)} B_X(x_j, \epsilon_i)\right]\right) + 1$, and it follows by (5.6) in the case where $j := j_0 - 1 = N_{i_0}$ and by (5.7) we have $\widehat{X}_n^{-1}(\{x[n, i_0, j_0, k_0]\}) = [q[n, i_0, j_0, k_0 - 1], q[n, i_0, j_0, k_0])$, so we conclude that similarly (5.8) holds.

Next, noting that for each $x \in \text{Supp}(\mu_X)$, we have $x = x_{i_0}$, for some $1 \leq i_0 < \text{Supp}(\mu_X) + 1$, and it immediately follows by the fact that $\widehat{X}^{-1}(\{x\}) = [q_{i_0 - 1}, q_{i_0})$ that we have

$$\widehat{\mathbb{P}}[\widehat{X} = x] = m_{[0,1)}([q_{i_0-1}, q_{i_0})) = \mathbb{P}[X = x_{i_0}] = \mathbb{P}[X = x].$$

Finally, we show that $\widehat{X}_n \xrightarrow{\widehat{\mathbb{P}}} \widehat{X}$ as so: First, observe that for all $i \geq 1$, $n_i < n \leq n_{i+1}$, since for all $j_0 \geq 1$, we have by (5.3)

$$\begin{aligned}
\mathbb{P}[X = x_{j_0}] - i^{-1}2^{-(i+j_0+2)} &= \mathbb{P}[X \in B_X(x_{j_0}, \epsilon_i)] - i^{-1}2^{-(i+j_0+2)} \\
&\leq \mathbb{P}[X_n \in B_X(x_{j_0}, \epsilon_i)] \\
&\leq \mathbb{P}[X \in B_X(x_{j_0}, \epsilon_i)] + i^{-1}2^{-(i+j_0+2)} \\
&= \mathbb{P}[X = x_{j_0}] + i^{-1}2^{-(i+j_0+2)}, \qquad (5.9)
\end{aligned}$$

we find for $j_0 \geq 1$, $i_0 \geq j_0$, and $n \geq n_i$, we have by (5.9)

$$\begin{aligned}
q_{j_0} - i_0^{-1}2^{-(j_0+2)} &\leq \sum_{j=1}^{j_0} \mathbb{P}[X = x_j] - \sum_{j=1}^{j_0} i_0^{-1}2^{-(j_0+j+2)} \\
&\leq \sum_{j=1}^{j_0} \left[\mathbb{P}[X = x_j] - i_0^{-1}2^{-(i_0+j+2)}\right] \\
&\leq \sum_{j=1}^{j_0} \mathbb{P}[X_n \in B_X(x_j, \epsilon_{i_0})] = q[n, i_0, j_0, 0] \\
&\leq \sum_{j=1}^{j_0} \left[\mathbb{P}[X = x_j] + i_0^{-1}2^{-(i_0+j+2)}\right] \\
&\leq \sum_{j=1}^{j_0} \mathbb{P}[X = x_j] + \sum_{j=1}^{j_0} i_0^{-1}2^{-(j_0+j+2)} \\
&\leq q_{j_0} + i_0^{-1}2^{-(j_0+2)}
\end{aligned}$$

and it follows that

$$\begin{aligned}
[q_{j_0-1}, q_{j_0}) \cap [q[n, i_0, j_0 - 1, 0], q[n, i_0, j_0, 0]) &\supset \left[q_{j_0-1} - i_0^{-1}2^{-(j_0-1+2)}, q_{j_0} + i_0^{-1}2^{-(j_0+2)}\right) \\
&\supset \left[q_{j_0-1} - i_0^{-1}2^{-(j_0+2)}, q_{j_0} + i_0^{-1}2^{-(j_0+2)}\right). \qquad (5.10)
\end{aligned}$$

Then for $j_0 \geq 1$, $i_0 \geq j_0$, and $n \geq n_{i_0}$, we have by (5.10)

$$\widehat{\mathbb{P}}[\{\widehat{X} = x_{j_0}\} \setminus \{\widehat{X}_n \in B_X(x_{j_0}, \epsilon_{i_0})\}] = m_{[0,1)}([q_{j_0-1}, q_{j_0}) \setminus [q[n, i_0, j_0 - 1, 0], q[n, i_0, j_0, 0]))$$

$$
\begin{aligned}
&= m_{[0,1)}[q_{j_0-1}, q_{j_0}) - m_{[0,1)}([q_{j_0-1}, q_{j_0}) \cap [q[n, i_0, j_0-1, 0], q[n, i_0, j_0, 0])) \\
&\leq m_{[0,1)}[q_{j_0-1}, q_{j_0}) - m_{[0,1)}\left(\left[q_{j_0-1} - i_0^{-1}2^{-(j_0+2)}, q_{j_0} + i_0^{-1}2^{-(j_0+2)}\right)\right) \\
&= i_0^{-1}2^{-(j_0+2-1)} = i_0^{-1}2^{-(j_0+1)}.
\end{aligned} \tag{5.11}
$$

We then conclude that given $\epsilon, \epsilon' > 0$, take $i \geq 1$ sufficiently large such that $i^{-1} \leq \epsilon$, then take $i_0 \geq i$ sufficiently large such that $i_0 \geq N_i$ and $i_0^{-1} \leq \epsilon'$ observe that that for all $n \geq n_{i_0}$, by (5.2) and (5.11), and the fact that

$$
\epsilon_{i_0} \leq i_0^{-1} \leq \epsilon',
$$

we find that

$$
\begin{aligned}
&\widehat{\mathbb{P}}[||\widehat{X}_n - \widehat{X}|| \geq \epsilon'] \leq \widehat{\mathbb{P}}[||\widehat{X}_n - \widehat{X}|| \geq \epsilon_{i_0}] \\
&= \widehat{\mathbb{P}}\left[\bigcup_{j_0=1}^{\#\mathrm{Supp}(\mu_X)} [\{\widehat{X} = x_{j_0}\} \cap \{||\widehat{X}_n - \widehat{X}|| \geq \epsilon_{i_0}\}]\right] \\
&= \sum_{j_0=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} \widehat{\mathbb{P}}[\{\widehat{X} = x_{j_0}\} \cap \{||\widehat{X}_n - x_{j_0}|| \geq \epsilon_{i_0}\}] + \sum_{j_0=N_i+1}^{\#\mathrm{Supp}(\mu_X)} \widehat{\mathbb{P}}[\{\widehat{X} = x_{j_0}\} \cap \{||\widehat{X}_n - x_{j_0}|| \geq \epsilon_{i_0}\}] \\
&\leq \sum_{j_0=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} \widehat{\mathbb{P}}[\{\widehat{X} = x_{j_0}\} \cap \{||\widehat{X}_n - x_{j_0}|| \geq \epsilon_{i_0}\}] + \sum_{j_0=N_i+1}^{\#\mathrm{Supp}(\mu_X)} \widehat{\mathbb{P}}[\widehat{X} = x_{j_0}] \\
&= \sum_{j_0=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} \widehat{\mathbb{P}}[\{\widehat{X} = x_{j_0}\} \setminus \{\widehat{X}_n \in B_X(x_{j_0}, \epsilon_{i_0})\}] + \sum_{j_0=N_i+1}^{\#\mathrm{Supp}(\mu_X)} \widehat{\mathbb{P}}[\widehat{X} = x_{j_0}] \\
&\leq \sum_{j_0=1}^{\min\left(\#\mathrm{Supp}(\mu_X), N_i\right)} \left[i_0^{-1}2^{-(j_0+1)}\right] + \frac{1}{2}i^{-1} \\
&\leq \frac{1}{2}i^{-1}\sum_{j_0=1}^{\infty}\left[2^{-j_0}\right] + \frac{1}{2}i^{-1} \\
&\leq \epsilon,
\end{aligned}
$$

which completes the proof. □

Remark 5.1.

(i) Now we are ready to prove **Theorem 5.1**. Before we do so, for any probability space $(\overline{\Omega}, \overline{\Sigma}, \overline{\mathbb{P}})$, and $\{Y_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\overline{\mathbb{P}}; \mathcal{X})$, $Y \in \mathcal{L}^0(\overline{\mathbb{P}}; \mathcal{X})$, for $y \in$ Supp(μ_Y) and have $\mathbb{1}_{Y=y}Y_n \Rightarrow \mathbb{1}_{Y=y}Y$ as $n \to +\infty$, we can set

$$(Y)_y := \mathbb{1}_{Y=y}Y,\ \overline{\mathbb{P}}_y := \overline{\mathbb{P}}[\,\cdot\,|Y = y]. \qquad (5.12)$$

(ii) We prove this theorem first by using **Lemma 5.3** to choose $\{\widehat{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$, $\widehat{X} \in \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$ for $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mathbb{P}}) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, such that $X_n \overset{D}{=} \widehat{X}_n$, for each $n \in \mathbb{N}$, $\widehat{X} \overset{D}{=} X$, and $\widehat{X}_n \xrightarrow{\widehat{\mathbb{P}}} \widehat{X}$ as $n \to +\infty$. We're then able (by **Lemma 2.1** *(ii)*) to split such sequences into branches of events where $\widehat{X} = x$, for any $x \in$ Supp(μ_X), and use **Lemma 5.2** and choose $\{Y_n(x)\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(m_{[0,1)}; \mathcal{X})$, $Y(x) \in \mathcal{L}^0(m_{[0,1)}; \mathcal{X})$ such that $(Y_n(x), m_{[0,1)}) \overset{D}{=} ((X_n)_x, \widehat{\mathbb{P}}_x)$, for each $n \in \mathbb{N}$ and $(Y(x), m_{[0,1)}) \overset{D}{=} ((X)_x, \widehat{\mathbb{P}}_x)$ such that $Y_n(x) \xrightarrow{m_{[0,1)}\text{-a.s.}} Y(x)$. Then we can combine these functions so that we have $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}, \widetilde{X}$ constructed so that $\left(\widetilde{X}_n\right)_x \overset{D}{=} Y_n(x_j)$, for each $n \in \mathbb{N}$, and $\left(\widetilde{X}\right)_x \overset{D}{=} Y(x)$, and the overall distribution determined by the probability tree with $\widetilde{X} = x$ as each event branch so that we have $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, and $\widetilde{X} \overset{D}{=} X$, and we have $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$.

(iii) We shall elaborate on what on an intuitive and theoretical level is meant by "probability tree" and "event branch" in § 2 in the next draft.

Proof of Theorem 5,1.

Suppose $\{X_n\}_{n\in\mathbb{N}}, X$ are $\mathcal{X}$-valued random variables with countably closed isolated distributions. By **Lemma 5.3**, we can choose $\{\widehat{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$, $\widehat{X} \in \mathcal{L}^0(\mathcal{X}; \widehat{\mathbb{P}})$ for $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mathbb{P}}) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, such that $X_n \overset{D}{=} \widehat{X}_n$, for each $n \in \mathbb{N}$, $\widehat{X} \overset{D}{=} X$, and $\widehat{X}_n \xrightarrow{\widehat{\mathbb{P}}} \widehat{X}$ as $n \to +\infty$. Without loss of generality, we can then suppose $\{X_n\}_{n\in\mathbb{N}}, X$ are (like $\{\widehat{X}_n\}_{n\in\mathbb{N}}, \widehat{X}$) such that $X_n \xrightarrow{\mathbb{P}} X$ as $n \to +\infty$, and more specifically, we let $\{(X_n)_x\}_{n\in\mathbb{N}}, (X)_x, \mathbb{P}_x$ be defined as in (5.12). By **Lemma 2.1** *(ii)* and **Proposition 2.8**, we find that for every $x \in$ Supp(μ_X) we have $((X_n)_x, \mathbb{P}_x) \Rightarrow ((X)_x\mathbb{P}_x)$ as $n \to +\infty$ (since $(X_n)_x \xrightarrow{\mathbb{P}} (X)_x$ as $n \to +\infty$). We can then choose

$\{Y_n(x)\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(m_{[0,1)};\mathcal{X})$, $Y(x) \in \mathcal{L}^0(m_{[0,1)};\mathcal{X})$ such that $(Y_n(x), m_{[0,1)}) \overset{D}{=} ((X_n)_x, \mathbb{P}_x)$, for each $n \in \mathbb{N}$ and $(Y(x), m_{[0,1)}) \overset{D}{=} ((X)_x, \mathbb{P}_x)$ such that $Y_n(x) \xrightarrow{m_{[0,1)}\text{-a.s.}} Y(x)$.

Enumerate $\text{Supp}(\mu_X) := \{x_i\}_{1\le i<\#\text{Supp}(\mu_X)+1}$; recursively define $q_0 := 0$, $q_i := q_{i-1} + \mathbb{P}[X = x_{i-1}]$, for every $1 \le i < \#\text{Supp}(\mu_X) + 1$. Then define $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)$, $\widetilde{X} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)$ by

$$\widetilde{X}_n\left(\widetilde{\omega}\right) := \sum_{i=1}^{\#\text{Supp}(\mu_X)} \mathbb{1}_{[q_{i-1},q_i)}\left(\widetilde{\omega}\right) \cdot \left[Y(x_i) \circ \left(\mathbb{P}[X = x_i]^{-1} \cdot \left(\widetilde{\omega} - q_{i-1}\right)\right)\right], \text{ for every } n \in \mathbb{N},$$

$$\text{and } \widetilde{X}\left(\widetilde{\omega}\right) := \sum_{i=1}^{\#\text{Supp}(\mu_X)} \mathbb{1}_{[q_{i-1},q_i)}\left(\widetilde{\omega}\right) \cdot \left[Y(x_i) \circ \left(\mathbb{P}[X = x_i]^{-1} \cdot \left(\widetilde{\omega} - q_{i-1}\right)\right)\right], \tag{5.13}$$

for every $\widetilde{\omega} \in \widetilde{\Omega}$, and it remains to show that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, $\widetilde{X} \overset{D}{=} X$, and $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$.

To show that $\widetilde{X}_n \overset{D}{=} X_n$, given $n \in \mathbb{N}$, and $\widetilde{X} \overset{D}{=} X$, since $\left\{X^{-1}(\{x_i\})\right\}_{1\le i<\#\text{Supp}(\mu_X)+1} \subset \Sigma$, $\left\{\widetilde{X}^{-1}(\{x_i\})\right\}_{1\le i<\#\text{Supp}(\mu_X)+1} \subset \widetilde{\Sigma}$ are partitions such that

$$\mathbb{P}\left[\bigcup_{i=1}^{\#\text{Supp}(\mu_X)} X^{-1}(\{x_i\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{i=1}^{\#\text{Supp}(\mu_X)} \widetilde{X}^{-1}(\{x_i\})\right] = 1,$$

it shall suffice by **Proposition 2.5** *(i)* to show that $\left(\widetilde{X}_n\right)_{x_i} \overset{D}{=} (X_n)_{x_i}$, for every $1 \le i < \#\text{Supp}(\mu_X) + 1$. Given $1 \le i < \#\text{Supp}(\mu_X) + 1$, we find by construction of $\widetilde{X}$ given in (5.13), we find for every $1 \le i < \#\text{Supp}(\mu_X) + 1$ that $\widetilde{X}^{-1}(\{x_i\}) = [q_{i-1}, q_i) \setminus N_i$ for some null $N_i \in \widetilde{\Sigma}$, since for all $x \in \text{Supp}(\mu_X)$, we have $(X)_x = x$ $\mathbb{P}_x$-a.s., which further implies $Y(x) = x$, $m_{[0,1)}$-a.s. Then we have

$$\left(\widetilde{X}_n\right)_{x_i} = \mathbb{1}_{[q_i,q_{i-1})} \cdot \left[Y_n(x_i) \circ \left(\mathbb{P}[X = x_i]^{-1} \cdot ((\,\cdot\,) - q_{i-1}) \right) \right],$$
$$\left(\widetilde{X}\right)_{x_i} = \mathbb{1}_{[q_i,q_{i-1})} \cdot \left[Y(x_i) \circ \left(\mathbb{P}[X = x_i]^{-1} \cdot ((\,\cdot\,) - q_{i-1}) \right) \right], \qquad (5.14)$$

hence given $\mathrm{C} \in \mathcal{B}(\mathcal{X})$ we find by **Proposition 1.4** that

$$\begin{aligned}
\widetilde{\mathbb{P}}\left[\left(\widetilde{X}_n\right)_{x_i} \in \mathrm{C} \right] &= m_{[0,1)}\left(\left[(\mathbb{P}[X = x_i] \cdot Y_n(x_i))^{-1}(\mathrm{C}) + q_{i-1} \right] \cap [q_{i-1}, q_i) \right) \\
&= \mathbb{P}[X = x_i] \cdot m_{[0,1)}\left(Y_n(x_i)^{-1}(\mathrm{C}) \right) \\
&= \mathbb{P}[X = x_i] \cdot m_{[0,1)}\left(Y_n(x_i)^{-1}(\mathrm{C}) \right) \\
&= \mathbb{P}[X = x_i] \cdot \mathbb{P}_{x_i}[(X_n)_{x_i} \in \mathrm{C}] \\
&= \mathbb{P}[X = x_i] \cdot \frac{\mathbb{P}[(X_n)_{x_i} \in \mathrm{C}]}{\mathbb{P}[X = x_i]} \\
&= \mathbb{P}[(X_n)_{x_i} \in \mathrm{C}], \qquad (5.15)
\end{aligned}$$

so our desired conclusion of $\left(\widetilde{X}_n\right)_{x_i} \overset{D}{=} (X_n)_{x_i}$, and subsequently $\left(\widetilde{X}\right)_{x_i} \overset{D}{=} (X)_{x_i}$,.by similar derivation to (5.14) and (5.15), is met.

Finally, to show $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$, it shall suffice by **Proposition 2.4** *(i)* to show that

$$\left(\widetilde{X}_n\right)_{x_i} = \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})} \widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})} \widetilde{X} = \left(\widetilde{X}\right)_{x_i},$$

for each $1 \leq i < \#\mathrm{Supp}(\mu_X) + 1$, as $n \to +\infty$. Given $1 \leq i < \#\mathrm{Supp}(\mu_X) + 1$, we find by (5.13) and the open-mapping property (which I'll cite in a future draft) of $\left(\mathbb{P}[X = x_i]^{-1} \cdot ((\,\cdot\,) - q_{i-1}) \right) : [q_{i-1}, q_i) \to [0, 1)$ to (5.14), we find that $\left(\widetilde{X}_n\right)_{x_i} \to \left(\widetilde{X}\right)_{x_i}$ $\widetilde{\mathbb{P}}$-a.s., on $[q_{i-1}, q_i)$ as $n \to +\infty$, and it immediately follows from the fact that $\left(\widetilde{X}_n\right)_{x_i} = \left(\widetilde{X}\right)_{x_i} = 0$ on $\widetilde{\Omega} \setminus [q_{i-1}, q_i)$ that the conclusion of $\left(\widetilde{X}_n\right)_{x_i} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}\right)_{x_i}$ is reached.

□

Next, we prove the following thoerem that further generalizes **Theorem 5.1**:

Theorem 5.4. *Skorohod's Representation Theorem* holds such that

$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{k+1}, \mathcal{B}\left([0,1)^{k+1}\right), \bigotimes_{j=0}^{k} m_{[0,1)}\right)$ for $k \geq 0$ if $\{(X_{n,0}, \dots, X_{n,k})\}_{n\in\mathbb{N}}, (X_0, \dots X_k)$ are $k+1$-dimensional $\mathcal{X}$-valued random variables with countably closed isolated distributions.

Proof. In proving this partial verision of *Skorohod's Representation Theorem*, we end up proving the more general **Lemma 5.5**, which implies this result by using the following inductive argument:

The base case immediately follows by **Theorem 5.1**, since $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) = ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$ in that situation. In the inductive step, where $\{(X_{n,0}, \dots, X_{n,k+1})\}_{n\in\mathbb{N}}, (X_0, \dots X_{k+1})$ are $k+2$-dimensional $\mathcal{X}$-valued random variables with countable images, we find by the inductive hypothesis that for $\overrightarrow{X}_n := (X_{n,0}, \dots, X_{n,k})$ for each $n \in \mathbb{N}$, $\overrightarrow{X} := (X_0, \dots, X_k)$, there exists, for each $n \in \mathbb{N}$, some

$$\left\{\overline{\overrightarrow{X}}_n\right\}_{n\in\mathbb{N}} := \{(\overline{X}_{n,0}, \dots, \overline{X}_{n,k})\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right),$$
$$\overline{\overrightarrow{X}} := (\overline{X}_0, \dots, \overline{X}_k) \in \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right),$$

such that we have $\overrightarrow{X}_n \overset{D}{=} \overline{\overrightarrow{X}}_n$, for each $n \in \mathbb{N}$, $\overrightarrow{X} \overset{D}{=} \overline{\overrightarrow{X}}$, and $\overline{\overrightarrow{X}}_n \xrightarrow{\bigotimes_{j=0}^{k} m_{[0,1)}\text{-a.s.}} \overline{\overrightarrow{X}}$ as $n \to +\infty$. By **Lemma 5.5**, we can define $\left\{\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right\}_{n\in\mathbb{N}}, \left\{\widetilde{X}_0, \dots, \widetilde{X}_k\right\} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ as in (5.15), and choose $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k+1} m_{[0,1)}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\bigotimes_{j=0}^{k+1} m_{[0,1)}; \mathcal{X}\right)$, such that we have $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,k+1})$, for each $n \in \mathbb{N}$, $\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right) \overset{D}{=} (X_0, \dots, X_{k+1})$, and $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$ as $n \to +\infty$, satisfying the conclusion of *Skorohod's Representation Thoerem* for $\{(X_{n,0}, \dots, X_{n,k+1})\}_{n\in\mathbb{N}}, (X_0, \dots X_{k+1})$. $\square$

Essentially, one can think of **Theorem 5.4** as more of a checkpoint, and less of a useful result in and of itself, whereas in terms of further building machinery for our proof of Skorohod's Thoerem, **Lemma 5.5** is the true result that we're seeking out. As we'll see once we use it in our proof of **Corollary 6.1**.

Lemma 5.5. Suppose given $k \geq 0$, we have that $\{(X_{n,0}, \dots, X_{n,k})\}_{n\in\mathbb{N}}, (X_0, \dots, X_{k+1})$ are $k+2$ dimensional $\mathcal{X}$-valued random vectors with countably closed isolated distributions such that $(X_{n,0}, \dots, X_{n,k+1}) \Rightarrow (X_0, \dots, X_{k+1})$ as $n \to +\infty$ and there exists $\{\overline{X}_{n,0}, \dots, \overline{X}_{n,k}\}_{n\in\mathbb{N}}, \{\overline{X}_0, \dots, \overline{X}_k\} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right)$ such that for $\vec{X}_n := (X_{n,0}, \dots, X_{n,k})$, $\overrightarrow{\overrightarrow{X}}_n := (\overline{X}_{n,0}, \dots, \overline{X}_{n,k})$, for each $n \in \mathbb{N}$, and $\vec{X} := (X_0, \dots, X_k)$, $\overrightarrow{\overrightarrow{X}} := (\overline{X}_0, \dots, \overline{X}_k)$, we have $\vec{X}_n \overset{D}{=} \overrightarrow{\overrightarrow{X}}_n$, for each $n \in \mathbb{N}$, $\vec{X} \overset{D}{=} \overrightarrow{\overrightarrow{X}}$, and $\overrightarrow{\overrightarrow{X}}_n \xrightarrow{\bigotimes_{j=0}^{k} m\text{-a.s.}} \overrightarrow{\overrightarrow{X}}$ as $n \to +\infty$. Define $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{k+2}, \mathcal{B}\left([0,1)^{k+2}\right), \bigotimes_{j=0}^{k+1} m_{[0,1)}\right)$, and define $\left\{\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right\}_{n\in\mathbb{N}}, \left\{\widetilde{X}_0, \dots, \widetilde{X}_k\right\} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ by

$$\widetilde{X}_{n,j}\left(\widetilde{\omega}\right) := (\overline{X}_{n,j} \circ (\pi_0, \dots, \pi_k))\left(\widetilde{\omega}\right), \ \widetilde{X}_j\left(\widetilde{\omega}\right) := (\overline{X}_j \circ (\pi_0, \dots, \pi_k))\left(\widetilde{\omega}\right), \quad (5.16)$$

for each $n \in \mathbb{N}, 0 \leq j \leq k$. Then there exists $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that:

(i) for each $n \in \mathbb{N}$, we have $(X_{n,0}, \dots, X_{n,k+1}) \overset{D}{=} \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)$, and $(X_0, \dots, X_{k+1}) \overset{D}{=} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$.

(ii) $\widetilde{X}_{n,k+1} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}_{k+1}$ (and hence $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$) as $n \to +\infty$.

Remark 5.2.

(i) While the proof of this lemma is extremely long and technichal, the proof of this lemma is fundamentally the same strategy as the proof of **Theorem 5.1**. We first take the case where X_{k+1} is $\mathbb{P}$-a.s. constant, and prove it using similar methods that mirror **Lemma 5.2**, though in doing so, we also utilize another probability tree trick similar to our proof of **Theorem 5.1** where we break down the distribution of $X_{n,k}$ into outcome branches where $\vec{X}_n = \vec{x}_n$, for each $\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})$, and then define $\widetilde{X}_{n,k}$ in (5.20) in a way that mirrors the piecewise definition of $\widetilde{X}_{n,k}$ in (5.13) in the proof of **Theorem 5.1**.

After that, we then define $\widetilde{X}_{n,k}$ in (5.25) via a probability tree scheme with $X_{k+1} = x$, for each $x \in$ Supp$(\mu_{X_{k+1}})$, in a way that once again mirrors the piecewise definition of $\widetilde{X}_{n,k}$ in (5.13) in the proof of **Theorem 5.1**.

(ii) Note that somewhere along the course of this proof, we define ϕ_i, for each $1 \leq i <$ #Supp$(\mu_{X_{k+1}}) + 1$, as so in (5.26). Note that ϕ_i is analogous to the factor of $\left(\mathbb{P}[X = x_i]^{-1} \cdot ((\cdot) - q_{i-1})\right)$ in (5.13) in the proof of **Theorem 5.1**.

We find that ϕ_i is surjective, and hence right invertible, and in particular the restriction $\phi_i|\pi_{k+1}^{-1}[q_{i-1}, q_i) : \pi_{k+1}^{-1}[q_{i-1}, q_i) \to [0, 1)^{k+2}$ is fully invertible. We abuse notation and let ϕ_i^{-1} be a right inverse of ϕ_i^{-1} (which is a full inverse of $\phi_i|\pi_{k+1}^{-1}[q_{i-1}, q_i)$), and note that by (5.16), we have

$$\overset{\approx}{X}_n = \overset{\Rightarrow}{X}_n \circ (\pi_0, \ldots, \pi_k) \circ \phi_i^{-1} = \overset{\Rightarrow}{X}_n \circ (\pi_0, \ldots, \pi_k) \circ \phi_i^{-1} \circ \phi_i, \quad (5.17)$$

which justifies the derivation of (5.28). However, this derivation needs to be better-written and in a future-draft, I'll improve the notation of this derivation.

(iii) As also mentioned in *Remark 5.1 (iii)*, we shall elaborate on what on an intuitive and theoretical level is meant by "probability tree" and "event branch" in $\S\, 2$ in the next draft.

Proof of Lemma 5.5.

Note by **Theorem 5.1** that for $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mathbb{P}}) := ([0, 1), \mathcal{B}([0, 1)), m_{[0,1)})$, we can choose $\{(\widehat{X}_{n,0}, \ldots, \widehat{X}_{n,k+1})\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widehat{\mathbb{P}}; \mathcal{X}^{k+2}\right), (\widehat{X}_0, \ldots, \widehat{X}_{k+1}) \in \mathcal{L}^0\left(\widehat{\mathbb{P}}; \mathcal{X}^{k+2}\right)$ such that we have $(\widehat{X}_{n,0}, \ldots, \widehat{X}_{n,k+1}) \overset{D}{=} (X_{n,0}, \ldots, X_{n,k+1})$, for each $n \in \mathbb{N}$, $(\widehat{X}_0, \ldots, \widehat{X}_{k+1}) \overset{D}{=} (X_0, \ldots, X_{k+1})$, and $(\widehat{X}_{n,0}, \ldots, \widehat{X}_{n,k+1}) \xrightarrow{\widehat{\mathbb{P}}\text{-a.s.}} (\widehat{X}_0, \ldots, \widehat{X}_{k+1})$ as $n \to +\infty$, and so by **Lemma 2.1** *(i)*, we find that

$$\mathbb{1}_A \widehat{X}_{n,k+1} \Rightarrow \mathbb{1}_A \widehat{X}_{k+1} \text{ for every } A \in \widehat{\Sigma}, \qquad (5.18)$$

and for $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$, it shall suffice to show that

$(\widehat{X}_{n,0}, \ \dots, \widehat{X}_{n,k+1}) \overset{D}{=} \left(\widetilde{X}_{n,0}, \ \dots, \widetilde{X}_{n,k+1}\right)$, for each $n \in \mathbb{N}$,
$(\widehat{X}_{0}, \ \dots, \widehat{X}_{k+1}) \overset{D}{=} \left(\widetilde{X}_{0}, \ \dots, \widetilde{X}_{k+1}\right)$, and property *(ii)* of the above lemma holds. Then without loss of generality, we can suppose $(\Omega, \Sigma, \mathbb{P}) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$ and $(X_{n,0}, \ \dots, X_{n,k+1}) \xrightarrow{\mathbb{P}\text{-a.s.}} (X_0, \ \dots, X_{k+1})$ and hence we can assume (5.18) holds for $\{X_{n,k+1}\}_{n\in\mathbb{N}}, X_{k+1},$ and any $A \in \Sigma$.

First, we prove the following claim:

Claim. **Lemma 5.5** (the lemma above) holds in the special case that $X_{k+1} = c$ $\mathbb{P}$-a.s., for some $c \in \mathcal{X}$.

Proof. For every $\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})$, we shall use the notation given in (5.12) and for any probability space $(\overline{\Omega}, \overline{\Sigma}, \overline{\mathbb{P}})$, set

$$(X_{n,k+1})_{\vec{x}_n} := \mathbb{1}_{\vec{X}_n = \vec{x}_n} X_{n,k+1}, \ \overline{\mathbb{P}}_{\vec{x}_n} := \overline{\mathbb{P}}[\,\cdot\,|\vec{X}_n = \vec{x}_n].$$

For each $n \in \mathbb{N}$, enumerate $\text{Supp}(\mu_{X_{n,k+1}}) := \{x_{n,i}\}_{1\le i<\#\text{Supp}(\mu_{X_{n,k+1}})+1}$ such that $x_{n,i}$ is ordered from closest to c to furthest, i.e., for every $i_0 \in \mathbb{N}$, we have

$$\min\{||x_{n,i} - c|| : i_0 \le i\} = ||x_{n,i_0} - c||, \quad (5.19)$$

which we can do by **Corollary 3.6** since $\text{Supp}(\mu_{X_{n,k+1}})$ is isolated.

Then for every $n \in \mathbb{N}$, $\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})$, and $0 \le i < \#\text{Supp}(\mu_{X_{n,k+1}}) + 1$ define $q_{n,i}[\vec{x}_n]$ recursively by $q_{n,0}[\vec{x}_n] := 0$, $q_{n,i+1} := q_{n,i}[\vec{x}_n] + \mathbb{P}_{\vec{x}_n}[(X_{n,k+1})_{\vec{x}_n} = x_{n,i+1}]$. Then for each $n \in \mathbb{N}$ and $0 \le i < \#\text{Supp}(\mu_{X_{n,k+1}}) + 1$, define $Y_n(\vec{x}_n) \in \mathcal{L}^0(m_{[0,1)}; \mathcal{X})$ by $Y_n(\vec{x}_n)(\widehat{\omega}) := x_{n,i}$, for $\widehat{\omega} \in [q_{n,i-1}[\vec{x}_n], q_{n,i}[\vec{x}_n])$. Next, for every $n \in \mathbb{N}$, define
$\vec{\widetilde{X}}_n := \left(\widetilde{X}_{n,0}, \ \dots, \widetilde{X}_{n,k}\right), \vec{\widetilde{X}} := \left(\widetilde{X}_{0}, \ \dots, \widetilde{X}_{k}\right)$, and define
$\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ by

$$\widetilde{X}_{n,k+1} := \sum_{\vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n})} \mathbb{1}_{\widetilde{\vec{X}}_n = \vec{x}_n} [Y_n(\vec{x}_n) \circ \pi_{k+1}], \quad (5.20)$$

and $\widetilde{X}_{k+1} := c$. We shall next prove properties *(i)* and *(ii)* hold for $\left\{\left(\widetilde{X}_{n,0}, \ldots, \widetilde{X}_{n,k+1}\right)\right\}_{n \in \mathbb{N}}, \left(\widetilde{X}_0, \ldots, \widetilde{X}_{k+1}\right)$.

To show property *(i)* holds, since $\left\{\vec{X}_n^{-1}(\{\vec{x}_n\})\right\}_{\vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n})} \in \Sigma$, $\left\{\widetilde{\vec{X}}_n^{-1}(\{\vec{x}_n\})\right\}_{\vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n})} \subset \widetilde{\Sigma}$

are partitions such that

$$\mathbb{P}\left[\bigcup_{\vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n})} \vec{X}_n^{-1}(\{\vec{x}_n\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{\vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n})} \widetilde{\vec{X}}_n^{-1}(\{\vec{x}_n\})\right] = 1,$$

it shall suffice by **Proposition 2.5** *(i)* to show that given $n \in \mathbb{N}$, we have $\left(\widetilde{X}_{n,0}, \ldots, \widetilde{X}_{n,k+1}\right) \stackrel{D}{=} (X_{n,0}, \ldots, X_{n,k+1})$, by showing that

$$\mathbb{1}_{\widetilde{\vec{X}}_n = \vec{x}_n}\left(\widetilde{\vec{X}}_n, \widetilde{X}_{n,k+1}\right) \stackrel{D}{=} \mathbb{1}_{\vec{X}_n = \vec{x}_n}(\vec{X}_n, X_{n,k+1}), \text{ for every } \vec{x}_n \in \mathrm{Supp}(\mu_{\vec{X}_n}). \quad (5.21)$$

Note first that for every $\mathrm{C} \in \mathcal{B}(\mathcal{X})$, we have

$$\begin{aligned} m_{[0,1)}\left(Y_n(\vec{x}_n)^{-1}(\mathrm{C})\right) &= m_{[0,1)}\left(\bigcup_{x_{n,i} \in \mathrm{Supp}(\mu_{X_{n,k+1}}) \cap \mathrm{C}} \{Y_n(\vec{x}_n) = x_{n,i}\}\right) \\ &= \sum_{x_{n,i} \in \mathrm{Supp}(\mu_{X_{n,k+1}}) \cap \mathrm{C}} m_{[0,1)}(\{Y_n(\vec{x}_n) = x_{n,i}\}) \\ &= \sum_{x_{n,i} \in \mathrm{Supp}(\mu_{X_{n,k+1}}) \cap \mathrm{C}} m_{[0,1)}([q_{n,i-1}[\vec{x}_n], q_{n,i}[\vec{x}_n])) \\ &= \sum_{x_{n,i} \in \mathrm{Supp}(\mu_{X_{n,k+1}}) \cap \mathrm{C}} \mathbb{P}_{\vec{x}_n}[(X_{n,k+1})_{\vec{x}_n} = x_{n,i}] \end{aligned}$$

$$= \mathbb{P}_{\vec{x}_n}\left[\bigcup_{x_{n,i}\in \mathrm{Supp}(\mu_{X_{n,k+1}})\cap \mathrm{C}} \{(X_{n,k+1})_{\vec{x}_n} = x_{n,i}\}\right]$$
$$= \mathbb{P}_{\vec{x}_n}[(X_{n,k+1})_{\vec{x}_n} \in \mathrm{C}], \tag{5.22}$$

which shows that $(Y_n(\vec{x}_n), m_{[0,1)}) \overset{D}{=} ((X_{n,k+1})_{\vec{x}_n}, \mathbb{P}_{\vec{x}_n})$. Additionally noting by hypothesis we have $\overrightarrow{\overrightarrow{X}}_n \overset{D}{=} \vec{X}_n$, it follows that for every $\mathrm{C}_0, \ldots, \mathrm{C}_{k+1} \in \mathcal{B}(\mathcal{X})$, we find by (5.20) and (5.22) that

$$\widetilde{\mathbb{P}}\left[\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\left(\widetilde{\vec{X}}_n, \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] = \widetilde{\mathbb{P}}\left[\left\{\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\widetilde{\vec{X}}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \left\{\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$
$$= \widetilde{\mathbb{P}}\left[\left\{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \left\{\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$
$$= \mathbb{1}_{\vec{x}_n\in\prod_{j=0}^{k}\mathrm{C}_j}\widetilde{\mathbb{P}}\left[\left\{\widetilde{\vec{X}}_n = \vec{x}_n\right\} \cap \left\{\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$
$$= \mathbb{1}_{\vec{x}_n\in\prod_{j=0}^{k}\mathrm{C}_j}\widetilde{\mathbb{P}}\left[\left\{\overrightarrow{\overrightarrow{X}}_n \circ (\pi_0, \ldots, \pi_k) = \vec{x}_n\right\} \cap \{Y_n(\vec{x}_n)\circ \pi_{k+1} \in \mathrm{C}_{k+1}\}\right]$$

$$= \mathbb{1}_{\vec{x}_n\in\prod_{j=0}^{k}\mathrm{C}_j}\left[\left(\bigotimes_{j=0}^{k} m_{[0,1)}\right)\left(\overrightarrow{\overrightarrow{X}}_n^{-1}(\{\vec{x}_n\})\right) \cdot m_{[0,1)}\left(Y_n(\vec{x}_n)^{-1}(\mathrm{C}_{k+1})\right)\right]$$
$$= \mathbb{1}_{\vec{x}_n\in\prod_{j=0}^{k}\mathrm{C}_j}\mathbb{P}[\vec{X}_n = \vec{x}_n] \cdot \mathbb{P}_{\vec{x}_n}[(X_{n,k+1})_{\vec{x}_n} \in \mathrm{C}_{k+1}]$$
$$= \mathbb{1}_{\vec{x}_n\in\prod_{j=0}^{k}\mathrm{C}_j}\mathbb{P}[\{\vec{X}_n = \vec{x}_n\} \cap \{(X_{n,k+1})_{\vec{x}_n} \in \mathrm{C}_{k+1}\}]$$
$$= \mathbb{P}\left[\left\{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \{\mathbb{1}_{\vec{X}_n=\vec{x}_n} X_{n,k+1} \in \mathrm{C}_{k+1}\}\right]$$
$$= \mathbb{P}\left[\left\{\mathbb{1}_{\vec{X}_n=\vec{x}_n}\vec{X}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \{\mathbb{1}_{\vec{X}_n=\vec{x}_n} X_{n,k+1} \in \mathrm{C}_{k+1}\}\right]$$
$$= \mathbb{P}\left[\mathbb{1}_{\vec{X}_n=\vec{x}_n}(\vec{X}_n, X_{n,k+1}) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right],$$

which shows that (5.21) holds, and we conclude that $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,k+1})$. Since $X_{k+1} = c$ $\mathbb{P}$-a.s., and $\widetilde{X}_{k+1} = c$, and

$$\mu_{\overset{\widetilde{\rightrightarrows}}{X}} = \widetilde{\mathbb{P}}\left[\overset{\widetilde{\rightrightarrows}}{X}^{-1}(\cdot)\right] = \left[\widetilde{\mathbb{P}} \circ (\pi_0, \dots, \pi_k)^{-1}\right]\left(\overset{\rightrightarrows}{X}^{-1}(\cdot)\right) = \left(\bigotimes_{j=0}^{k} m_{[0,1)}\right)\left(\overset{\rightrightarrows}{X}^{-1}(\cdot)\right) = \mu_{\overset{\rightrightarrows}{X}},$$

we find by hypothesis that $\overset{\widetilde{\rightrightarrows}}{X} \overset{D}{=} \overset{\rightrightarrows}{X} \overset{D}{=} \vec{X}$, hence we have

$$\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right) \overset{D}{=} \left(\overset{\widetilde{\rightrightarrows}}{X}, c\right) \overset{D}{=} (\vec{X}, c) \overset{D}{=} (X_0, \dots, X_{k+1}),$$

so we conclude that property *(i)* is met.

To prove property *(ii)*, we shall define an order $\preccurlyeq$ so that,the argument is similar to the proof of **Lemma 5.2**, by verifying that that $||\widetilde{X}_n - \widetilde{X}||$ is monotonically increasing on some poset $\preccurlyeq$ on $\widetilde{\Omega}$ such that $\mathcal{T}(\preccurlyeq) \subset \widetilde{\Sigma}$, and then let our conclusion follow by **Lemma 1.5**. To do this, take $\widetilde{\omega}, \widetilde{\omega}' \in \widetilde{\Omega}$. Define

$$\widetilde{\omega} \preccurlyeq \widetilde{\omega}' \iff ||\widetilde{X}_{n,k+1}\left(\widetilde{\omega}\right) - c|| \le ||\widetilde{X}_{n,k+1}\left(\widetilde{\omega}'\right) - c||. \quad (5.23)$$

Note that it is a poset such that $\mathcal{T}(\preccurlyeq) \subset \widetilde{\Sigma}$, since we find that given $\widetilde{\omega} \in \widetilde{\Omega}$, we find by (5.19) and (5.23), that

$$\left[\widetilde{\omega}, +\infty\right) = ||\widetilde{X}_{n,k+1} - c||^{-1}\left(\left[||\widetilde{X}_{n,k+1}\left(\widetilde{\omega}\right) - c||, +\infty\right)\right),$$
$$\left(-\infty, \widetilde{\omega}\right] = ||\widetilde{X}_{n,k+1} - c||^{-1}\left(\left[0, ||\widetilde{X}_{n,k+1}\left(\widetilde{\omega}\right) - c||\right]\right),$$

and it follows that $\left[\widetilde{\omega}, +\infty\right), \left(-\infty, \widetilde{\omega}\right] \in \widetilde{\Sigma}$, completing the proof of the claim. □

Next, we'll prove the above lemma in the general case by defining $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}}, \widetilde{X}_{k+1}$ in similar fashion to the proof of **Theorem 5.1**. For every $x \in \text{Supp}(\mu_{X_{k+1}})$, we shall once again use the notation given in (5.12) and for any probability space $(\overline{\Omega}, \overline{\Sigma}, \overline{\mathbb{P}})$ set

$$(X_{n,k+1})_x := \mathbb{1}_{X_{k+1}=x} X_{n,k+1},\ (X_{k+1})_x := \mathbb{1}_{X_{k+1}=x} X_{k+1},\ \overline{\mathbb{P}}_x := \overline{\mathbb{P}}[\,\cdot\,|X_{k+1} = x].$$

We find since (5.18) holds for $\{X_{n,k+1}\}_{n\in\mathbb{N}}, X_{k+1},$ for every $x \in \mathrm{Supp}(\mu_{X_{k+1}})$ that $(X_{n,k+1})_x \Rightarrow (X_{k+1})_x$, and it follows by **Proposition 2.8** that we have $((X_{n,k+1})_x, \mathbb{P}_x) \Rightarrow ((X_{k+1})_x, \mathbb{P}_x)$ as $n \to +\infty$. Since $(X_{n,k+1})_x = x$ $\mathbb{P}_x$-a.s., we find that the previously proven *Claim* shows that the special case of the above lemma holds for $\{(X_{n,k+1})_x\}_{n\in\mathbb{N}}, (X_{k+1})_x$ as random variables of $(\Omega, \Sigma, \mathbb{P}_x)$. Setting $\overset{\rightrightarrows}{X}_n := \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right)$, we can then choose $\left\{\widetilde{X}_{n,k+1}(x)\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right),\ \widetilde{X}_{k+1}(x) \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that

$$((\overrightarrow{X}_n, (X_{n,k+1})_x), \mathbb{P}_x) \overset{D}{=} \left(\left(\overset{\rightrightarrows}{X}_n, \widetilde{X}_{n,k+1}(x)\right), \widetilde{\mathbb{P}}\right) \text{ for every } n \in \mathbb{N},$$

$$((\overrightarrow{X}, (X_{k+1})_x), \mathbb{P}_x) \overset{D}{=} \left(\left(\overset{\rightrightarrows}{X}, \widetilde{X}_{k+1}(x)\right), \widetilde{\mathbb{P}}\right),$$

and $\widetilde{X}_{n,k+1}(x) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}_{k+1}(x)$,

$$\left(\text{and more generally } \left(\overset{\rightrightarrows}{X}_n, \widetilde{X}_{n,k+1}(x)\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\overset{\rightrightarrows}{X}, \widetilde{X}_{k+1}(x)\right)\right), \text{ as } n \to +\infty. \quad (5.24)$$

As before, we enumerate $\mathrm{Supp}(\mu_{X_{k+1}}) := \{x_i\}_{1\le i<\#\mathrm{Supp}(\mu_{X_{k+1}})+1}$; and define $q_0 := 0$, $q_{i+1} := q_i + \mathbb{P}[X_{k+1} = x_i]$. Then define $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ by

$$\widetilde{X}_{n,k+1} := \sum_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})} \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1}, q_i)} \left[\widetilde{X}_{n,k+1}(x_i) \circ \phi_i\right], \text{ for every } n \in \mathbb{N}, \text{ and}$$

$$\widetilde{X}_{k+1} := \sum_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})} \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1}, q_i)} \left[\widetilde{X}_{k+1}(x_i) \circ \phi_i\right], \quad (5.25)$$

where

$$\phi_i := \left(\pi_0, \dots, \pi_k, \max\left(\min\left(\mathbb{P}[X_{k+1} = x_i]^{-1} \cdot (\pi_{k+1} - q_{i-1}), 1\right), 0\right)\right). \quad (5.26)$$

It then remains to show that properties *(i)* and *(ii)* holds for $\left\{\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)\right\}_{n\in\mathbb{N}}$, $\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$ (in the general case).

To show that property *(i)* holds, we shall prove $(X_{n,0}, \dots, X_{n,k+1}) \overset{D}{=} \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)$, given $n \in \mathbb{N}$, and let $(X_0, \dots, X_{k+1}) \overset{D}{=} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$ follow by similarity. Since $\left\{X_{k+1}^{-1}(\{x_i\})\right\}_{1\le i<\#\mathrm{Supp}(\mu_{X_{k+1}})+1} \in \Sigma$, $\left\{\widetilde{X}_{k+1}^{-1}(\{x_i\})\right\}_{1\le i<\#\mathrm{Supp}(\mu_{X_{k+1}})+1} \subset \widetilde{\Sigma}$ are partitions such that

$$\mathbb{P}\left[\bigcup_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})+1} X_{k+1}^{-1}(\{x_i\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})+1} \widetilde{X}_{k+1}^{-1}(\{x_i\})\right] = 1,$$

it shall suffice by **Proposition 2.5** *(i)* to show that

$$\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots \widetilde{X}_{n,k+1}\right) \overset{D}{=} \mathbb{1}_{X_{k+1}=x_i}(X_{n,0}, \dots, X_{n,k+1}), \text{ for every } 1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1, \quad (5.27)$$

By construction of $\widetilde{X}_{k+1}$ given in (5.25) we find given $1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1$, we have $\widetilde{X}_{k+1}^{-1}(\{x_i\}) = \pi_{k+1}^{-1}[q_{i-1}, q_i)$, since we have $(X_{k+1})_{x_i} = x_i$ $\mathbb{P}_{x_i}$-a.s., which further implies $\widetilde{X}_{k+1}(x_i) = x_i$ $\widetilde{\mathbb{P}}$-a.s. Then we have

$$\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) = \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\overset{\rightrightarrows}{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i) \circ \phi_i\right),$$
$$\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right) = \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\overset{\rightrightarrows}{\widetilde{X}}, \widetilde{X}_{k+1}(x_i) \circ \phi_i\right).$$

hence, given $C_0, \dots, C_{k+1} \in \mathcal{B}(\mathcal{X})$, we find by (5.18) (explained in greater detail in *Remark 5.2 (ii)*) that

$$\left\{\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1} C_j\right\}$$

$$= \left[\mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\circ\phi_i\right)\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)$$

$$= \left(\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\circ\phi_i\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1},q_i)\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \left(\left[\left(\overrightarrow{\widetilde{X}}_n \circ \phi_i^{-1}\circ\phi_i, \widetilde{X}_{n,k+1}(x_i)\circ\phi_i\right)\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1},q_i)\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \left(\left[\left(\overrightarrow{\widetilde{X}}_n \circ \phi_i^{-1}, \widetilde{X}_{n,k+1}(x_i)\right)\circ\phi_i\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1},q_i)\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \phi_i^{-1}\left(\left(\overrightarrow{\widetilde{X}}_n \circ \phi_i^{-1}, \widetilde{X}_{n,k+1}(x_i)\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \phi_i^{-1}\left(\bigcap_{j=0}^{k}\left[\left(\overline{X}_{n,j}\circ(\pi_0,\ \dots,\pi_{k+1})\circ\phi_i^{-1}\right)^{-1}(\mathrm{C}_j)\right] \cap \widetilde{X}_{n,k+1}(x_i)^{-1}(\mathrm{C}_{k+1})\right)$$
$$\cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \phi_i^{-1}\left(\bigcap_{j=0}^{k}\left[(\overline{X}_{n,j}\circ(\pi_0,\ \dots,\pi_{k+1}))^{-1}(\mathrm{C}_j)\right] \cap \widetilde{X}_{n,k+1}(x_i)^{-1}(\mathrm{C}_{k+1})\right)$$
$$\cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)$$

$$= \phi_i^{-1}\left(\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right). \tag{5.28}$$

It follows by (5.24) and (5.28) and **Proposition 1.4** that

$$\widetilde{\mathbb{P}}\left[\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0},\ \dots \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1}\mathrm{C}_j\right]$$

$$= \widetilde{\mathbb{P}}\left[\phi_i^{-1}\left(\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\right)\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cup \mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)\right]$$

$$= \widetilde{\mathbb{P}}\left[\phi_i^{-1}\left(\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\right)\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right] + \widetilde{\mathbb{P}}\left[\mathbb{1}_{\vec{0}\in\prod_{j=0}^{k+1}\mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1},q_i)^c\right)\right]$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \widetilde{\mathbb{P}}\left[\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot \widetilde{\mathbb{P}}\left[\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right]$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \widetilde{\mathbb{P}}\left[\left(\overrightarrow{\widetilde{X}}_n, \widetilde{X}_{n,k+1}(x_i)\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot (1 - m_{[0,1)}[q_{i-1}, q_i))$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \mathbb{P}_{x_i}\left[(\overrightarrow{X}_n, (X_{n,k+1})_{x_i}) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot (1 - \mathbb{P}[X_{n,k+1} = x_i])$$

$$= \mathbb{P}\left[\left\{(\overrightarrow{X}_n, (X_{n,k+1})_{x_i}) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right\} \cap \{X_{k+1} = x_i\}\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot \mathbb{P}[X_{n,k+1} \neq x_i]$$

$$= \mathbb{P}\left[\left(\left\{(\overrightarrow{X}_n, (X_{n,k+1})_{x_i}) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right\} \cap \{X_{k+1} = x_i\}\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \{X_{n,k+1} \neq x_i\}\right]$$

$$= \mathbb{P}\left[\mathbb{1}_{X_{k+1}=x_i}(\overrightarrow{X}_n, (X_{n,k+1})_{x_i}) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right],$$

and (5.27) immediately follows.

Finally, to prove property *(ii)*, it shall suffice by **Proposition 2.4** *(i)* to show that

$$\left(\widetilde{X}_{n,k+1}\right)_{x_i} = \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})} \widetilde{X}_{n,k+1} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})} \widetilde{X}_{k+1} = \left(\widetilde{X}_{k+1}\right)_{x_i},$$

for each $1 \leq i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1$, as $n \to +\infty$. Given $1 \leq i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1$, we note that $\widetilde{X}^{-1}(\{x_i\}) = \pi_{k+1}^{-1}[q_{i-1}, q_i)$ and we find by (5.25) and the open-mapping property of $\phi_i | \pi_{k+1}^{-1}[q_{i-1}, q_i) : \pi_{k+1}^{-1}[q_{i-1}, q_i) \to [0,1)^{k+2}$ (which I'll cite in a future draft) that

$$\left(\widetilde{X}_{n,k+1}\right)_{x_i} = \widetilde{X}_{n,k+1}(x_i) \circ \phi_i \to \widetilde{X}_{k+1}(x_i) \circ \phi_i = \left(\widetilde{X}_{k+1}\right)_{x_i}, \ \widetilde{\mathbb{P}}\text{-a.s., on } \pi_{k+1}^{-1}[q_{i-1}, q_i) \text{ as } n \to +\infty,$$

and it immediately follows from the fact that $\left(\widetilde{X}_{n,k+1}\right)_{x_i} = \left(\widetilde{X}_{k+1}\right)_{x_i} = 0$ on $\widetilde{\Omega} \setminus \pi_{k+1}^{-1}[q_{i-1}, q_i)$ that the conclusion of $\left(\widetilde{X}_n\right)_{x_i} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}\right)_{x_i}$ is reached. □

§ 6 Generalizing Skorohod's Thoerem for Arbitrary Random Variables

Let $\left(\ell^{\infty}(\mathcal{X}), ||\cdot||_{\infty}\right)$ and $\left(\ell^{1}(\mathcal{X}), ||\cdot||_{1}\right)$ be the banach spaces defined by

$$\ell^{\infty}(\mathcal{X}) := \left\{(x_j)_{j=0}^{\infty} \in \mathcal{X}^{\mathbb{N}\cup\{0\}} : ||x_j|| \le m \text{ for all } j \ge 0, \text{ for some } m \ge 0\right\},$$

$$\ell^{1}(\mathcal{X}) := \left\{(x_j)_{j=0}^{\infty} \in \mathcal{X}^{\mathbb{N}\cup\{0\}} : \sum_{k=0}^{\infty} ||x_k|| < +\infty\right\},$$

with norms $||\cdot||_{\infty}, ||\cdot||_{1}$ defined by

$$||(x_j)_{j=0}^{\infty}||_{\infty} := \inf\left\{m \ge 0 : ||x_j|| \le m \text{ for all } j \ge 0\right\},$$

$$||(x_j)_{j=0}^{\infty}||_{1} := \sum_{j=0}^{\infty} ||x_j||.$$

Corollary 6.1. Suppose $\left\{(X_{n,j})_{j=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right), (X_j)_{j=0}^{\infty} \in \mathfrak{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right)$, such that $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$ and $X_{n,j}, X_j$ for each $n \ge 1, j \ge 0$ has a countably closed isolated distribution. Then *Skorohod's Representation Thoerem* holds such that

$$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{\mathbb{N}\cup\{0\}}, \mathcal{B}\left([0,1)^{\mathbb{N}\cup\{0\}}\right), \bigotimes_{k=0}^{\infty} m_{[0,1)}\right).$$

Remark 6.1.

(i) Note that any $\mathbf{X} \in \mathfrak{L}^0\left(\mathbb{P}; \ell^{\infty}(\mathcal{X})\right)$ can be expressed as $\mathbf{X} = (X_j)_{j=0}^{\infty}$, for $\{X_j\}_{j\ge0} \subset \mathfrak{L}^0(\mathbb{P}; \mathcal{X})$, such that almost surely for $\omega \in \Omega$, we have $||X_j(\omega)|| \le m$, for all $j \ge 0$, for some $m \ge 0$, or equivalently, we have $M \in \mathfrak{L}^0(\mathbb{P}; [0, +\infty))$, such that almost surely for $\omega \in \Omega$, we have $||X_j(\omega)|| \le M(\omega)$. To show this, we can define $X_j := \pi_j \circ \mathbf{X}$, and observe that if $\mathbf{X} \in \mathfrak{L}^0\left(\mathbb{P}; \ell^{\infty}(\mathcal{X})\right)$ we have $\pi_j(\mathbf{X}(\omega)) \in \ell^{\infty}(\mathcal{X})$ for any $\omega \in \Omega$.

(ii) Note that given any probability space $(\overline{\Omega}, \overline{\Sigma}, \overline{\mathbb{P}})$ and any $\left\{(\overline{Y}_{n,j})_{j=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0\left(\mathbb{P}; \ell^{\infty}(\mathcal{X})\right), (\overline{Y}_j)_{j=0}^{\infty} \in \mathfrak{L}^0\left(\mathbb{P}; \ell^{\infty}(\mathcal{X})\right)$, we have $(\overline{Y}_{n,j})_{j=0}^{\infty} \xrightarrow{\overline{\mathbb{P}}\text{-a.s.}} (\overline{Y}_j)_{j=0}^{\infty}$ as $n \to +\infty$ iff $\overline{Y}_{n,j} \xrightarrow{\overline{\mathbb{P}}\text{-a.s.}} \overline{Y}_j$ as $n \to +\infty$ for each $j \ge 0$, since for

$(y_{n,j})_{j=0}^{\infty} \in \mathcal{X}^{\mathbb{N}\cup\{0\}}$, $(y_j)_{j=0}^{\infty} \in \mathcal{X}^{\mathbb{N}\cup\{0\}}$, we have $(y_{n,j})_{j=0}^{\infty} \to (y_j)_{j=0}^{\infty}$ as $n \to +\infty$ iff $y_{n,j} \to y_j$ for
each $j \geq 0$, hence the event $(\overline{Y}_{n,j})_{j=0}^{\infty} \to (\overline{Y}_j)_{j=0}^{\infty}$ occurs precisely when $\overline{Y}_{n,j} \to \overline{Y}_j$, for each $j \geq 0$.

It can also be shown (though it takes some work that I possibly plan to provide in the next draft) that we have $(\overline{Y}_{n,j})_{j=0}^{\infty} \xrightarrow{\overline{\mathbb{P}}} (\overline{Y}_j)_{j=0}^{\infty}$ as $n \to +\infty$ iff $\overline{Y}_{n,j} \xrightarrow{\overline{\mathbb{P}}} \overline{Y}_j$ as $n \to +\infty$ for each $j \geq 0$.

(iii) It's a well-known functional analysis result that $\ell^1(\mathcal{X}) \subset \ell^\infty(\mathcal{X})$. It follows that $\mathcal{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right) \subset \mathcal{L}^0\left(\mathbb{P}; \ell^\infty(\mathcal{X})\right)$, hence what was mentioned in parts *(i)* and *(ii)* of this remark for $\mathcal{L}^0\left(\mathbb{P}; \ell^\infty(\mathcal{X})\right)$ also applies to $\mathcal{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right)$.

(iv) Another important fact to bear in mind with **Corollary 6.1** is the fact that condition *(i)* of **Theorem 4.2** is logically equivalent to $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$. In the next draft, we'll most-likely write this equivalence as its own proposition and fully prove it, but for now we'll outline the proof in this remark, starting with proving $\implies$ (which is pretty straightforward) in the next paragraph.

If $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$, then $(X_{n,0}, \dots, X_{n,k}) \Rightarrow (X_1, \dots, X_k)$ as $n \to +\infty$, for each $k \geq 0$, since $(\pi_0, \dots, \pi_k)$ is a continuous mapping,
$(X_{n,0}, \dots, X_{n,k}) = (\pi_0, \dots, \pi_k) \circ (X_{n,j})_{j=0}^{\infty}$ for each $n \in \mathbb{N}$, and
$(X_0, \dots, X_k) = (\pi_0, \dots, \pi_k) \circ (X_j)_{j=0}^{\infty}$.

Worth noting that all we need to prove **Corollary 6.1** is the $\implies$ side, since this side allows us to utilize **Theorem 4.2** from the hypothesis of $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$. We can then prove the converse in the next part of this remark, utilizing the conclusions of **Corollary 6.1** to prove the result (which unfortunately, it turns out we need to do so), since assuming condition *(i)* of **Theorem 4.2** is all we need to prove **Corollary 6.1** (another thing that I'll clairify in the next part of this remark)

(v) To prove the converse $\impliedby$ side of the fact that condition *(i)* of **Theorem 4.2** is logically equivalent to $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$, we first set

$$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{\mathbb{N}\cup\{0\}}, \mathcal{B}\left([0,1)^{\mathbb{N}\cup\{0\}}\right), \bigotimes_{k=0}^{\infty} m_{[0,1)}\right),$$

Condition *(i)* of **Theorem 4.2** allows us to use **Theorem 5.1** and **Lemma 5.5**, as done in the proof of **Corollary 6.1**, to define $\left\{\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}};\ell^1(\mathcal{X})\right), \left(\widetilde{X}_j\right)_{j=0}^{\infty} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}};\ell^1(\mathcal{X})\right)$ by (6.4). The proof of **Corollary 6.1** then establishes that $(X_{n,j})_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}$, for each $n \in \mathbb{N}$, $(X_j)_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$, and $\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$. as $n \to +\infty$, and $(X_{n,j})_{j=0}^{\infty} \Rightarrow (X_j)_{j=0}^{\infty}$ as $n \to +\infty$ immediately follows.

Proof of Corollary 6.1.

For every $k \geq 0$, we shall use **Theorem 5.1** and repeatedly **Lemma 5.5** to recursively choose for $k = 0$ $\{[X_{n,0}]_0\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(m_{[0,1)};\mathcal{X}), [X_0]_0 \in \mathcal{L}^0(m_{[0,1)};\mathcal{X})$ such that $[X_{n,0}]_0 \overset{D}{=} X_{n,0}$, for each $n \in \mathbb{N}$, $[X_0]_0 \overset{D}{=} X_0$, and $[X_{n,0}]_0 \xrightarrow{m_{[0,1)}\text{-a.s.}} [X_0]_0$. Then for $k+1$, choose $\{[X_{n,0}]_{k+1}, \ldots, [X_{n,k+1}]_{k+1}\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k+1} m_{[0,1)};\mathcal{X}\right)$ such that (analogous to (5.16) and the statement of **Lemma 5.5**) the following properties hold:

(a) The property

$$[X_{n,j}]_{k+1} := [X_{n,j}]_k \circ (\pi_0, \ldots, \pi_k),$$
$$[X_j]_{k+1} := [X_j]_k \circ (\pi_0, \ldots, \pi_k), \qquad (6.1)$$

holds for each $n \in \mathbb{N}, 0 \leq j \leq k$.

(b) For each $n \in \mathbb{N}$, we have $(X_{n,0}, \ldots, X_{n,k+1}) \overset{D}{=} ([X_{n,0}]_{k+1}, \ldots, [X_{n,k+1}]_{k+1})$, and $(X_0, \ldots, X_{k+1}) \overset{D}{=} ([X_0]_{k+1}, \ldots, [X_{k+1}]_{k+1})$.

(c) $([X_{n,0}]_{k+1}, \ldots, [X_{n,k+1}]_{k+1}) \xrightarrow{\bigotimes_{j=0}^{k+1} m_{[0,1)}\text{-a.s.}} ([X_0]_{k+1}, \ldots, [X_{k+1}]_{k+1})$, as $n \to +\infty$.

We shall prove that $\left\{\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \ell^\infty\left(\mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)\right), \left(\widetilde{X}_j\right)_{j=0}^{\infty} \in \ell^\infty\left(\mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)\right)$ defined, for each $\widetilde{\omega} \in [0,1)^{\mathbb{N}\cup\{0\}}$, by

$$\widetilde{X}_{n,k}(\widetilde{\omega}) := ([X_{n,k}]_k \circ (\pi_0, \dots, \pi_k))(\widetilde{\omega}),\ \widetilde{X}_k(\widetilde{\omega}) := ([X_k]_k \circ (\pi_0, \dots, \pi_k))(\widetilde{\omega}), \quad (6.2)$$

for each $n \in \mathbb{N}, k \geq 0$ is such that $(X_{n,j})_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}$, for each $n \in \mathbb{N}$, $(X_j)_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$. and $\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$. as $n \to +\infty$.

Given $n \in \mathbb{N}$, $C_1, \dots, C_m \in \mathcal{B}(\mathcal{X})$, and any ordered pair $k_1 < k_2 < \dots < k_m \in \mathbb{N} \cup \{0\}$, by property *(a)* for any $k < k'$, we find using (6.2), followed by by repeated use of (6.1), we have

$$\widetilde{X}_{n,k} = [X_{n,k}]_k \circ (\pi_0, \dots, \pi_k) = [X_{n,k}]_{k+1} \circ (\pi_0, \dots, \pi_{k+1}) = \dots = [X_{n,k}]_{k'} \circ (\pi_0, \dots, \pi_{k'}),$$

hence we have

$$\begin{aligned}
\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k_m}\right) &= ([X_{n,0}]_0 \circ \pi_0, [X_{n,1}]_1 \circ (\pi_0, \pi_1), \dots, [X_{n,k_m}]_{k_m} \circ (\pi_0, \dots, \pi_{k_m})) \\
&= ([X_{n,0}]_{k_m} \circ (\pi_0, \dots, \pi_{k_m}), [X_{n,1}]_{k_m} \circ (\pi_0, \dots, \pi_{k_m}), \dots, [X_{n,k_m}]_{k_m} \circ (\pi_0, \dots, \pi_{k_m})) \\
&= ([X_{n,0}]_{k_m}, \dots, [X_{n,k_m}]_{k_m}) \circ (\pi_0, \dots, \pi_{k_m}). \qquad (6.3)
\end{aligned}$$

Since we have $([X_{n,0}]_{k_m}, \dots, [X_{n,k_m}]_{k_m}) \overset{D}{=} (X_{n,0}, \dots, X_{n,k_m})$ by property *(b)* of $\{[X_{n,0}]_{k+1}, \dots, [X_{n,k+1}]_{k+1}\}_{n \in \mathbb{N}}$, we find by (6.3) that

$$\begin{aligned}
\mu_{\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}}\left(\bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right) &= \widetilde{\mathbb{P}}\left[\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty} \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right] \\
&= \widetilde{\mathbb{P}}\left[\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k_m}\right) \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right] \\
&= \widetilde{\mathbb{P}}\left[([X_{n,0}]_{k_m}, \dots, [X_{n,k_m}]_{k_m}) \circ (\pi_0, \dots, \pi_{k_m}) \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right] \\
&= \bigotimes_{j=0}^{k_m} m_{[0,1)}\left(([X_{n,0}]_{k_m}, \dots, [X_{n,k_m}]_{k_m}) \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right) \\
&= \mathbb{P}\left[(X_{n,0}, \dots, X_{n,k_m}) \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(C_i)\right]
\end{aligned}$$

$$= \mathbb{P}\left[(X_{n,j})_{j=0}^{\infty} \in \bigcap_{i=1}^{m} \pi_{k_i}^{-1}(\mathrm{C}_i) \right]$$
$$= \mu_{(X_{n,j})_{j=0}^{\infty}}\left(\bigcap_{i=1}^{m} \pi_{k_i}^{-1}(\mathrm{C}_i) \right),$$

and our conclusion of $(X_{n,j})_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty}$ is reached, and $(X_j)_{j=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$ follows from similarity.

To prove that $\left(\widetilde{X}_{n,j}\right)_{j=0}^{\infty} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_j\right)_{j=0}^{\infty}$ as $n \to +\infty$, it shall suffice by *Remark 6.1 (ii)* to prove that $\widetilde{X}_{n,k} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}_k$ as $n \to +\infty$, for each $k \geq 0$. Given $k \geq 0$, note by property *(c)*, and the fact that $[X_{n,0}]_0 \xrightarrow{m_{[0,1)}\text{-a.s.}} [X_0]_0$ as $n \to +\infty$, that we have $[X_{n,k}]_k \xrightarrow{\bigotimes_{j=0}^{k} m_{[0,1)}\text{-a.s.}} [X_{n,k}]_k$, as $n \to +\infty$. We then conclude by (6.2) and the fact that $(\pi_0, \dots, \pi_k)$ is an open map (reasoning that I'll elaborate in the next draft) that we have

$$\widetilde{X}_{n,k} = [X_{n,k}]_k \circ (\pi_0, \dots, \pi_k) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} [X_k]_k \circ (\pi_0, \dots, \pi_k) = \widetilde{X}_k \text{ as } n \to +\infty,$$

and our conclusion has been reached. □

Corollary 6.2. For any $\{X_n\}_{n=1}^{\infty} \subset \mathcal{L}^0(\mathbb{P}; \mathcal{X})$ such that for some $X \in \mathcal{L}^0(\mathbb{P}; \mathcal{X})$ such that $X_n \Rightarrow X$ as $n \to +\infty$, *Skorohod's Representation Thoerem* holds such that

$$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{\mathbb{N}\cup\{0\}}, \mathcal{B}\left([0,1)^{\mathbb{N}\cup\{0\}}\right), \bigotimes_{k=0}^{\infty} m_{[0,1)}\right).$$

Outline of proof.

By **Theorem 4.2**, we can choose the series $X_n = \sum_{k=0}^{\infty} X_{n,k}$, for each $n \in \mathbb{N}$, and $X = \sum_{k=0}^{\infty} X_k$ of random variables with countably closed isolated distribution such that $||X_{n,k}||, ||X_k|| \leq 2^{-k}$, for $n, k \geq 1$, and satisfies properties *(i)-(iv)* of the theorem. Noting that $\left\{(X_{n,k})_{k=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right), (X_j)_{j=0}^{\infty} \in \mathcal{L}^0\left(\mathbb{P}; \ell^1(\mathcal{X})\right)$), we find that since $X_n \Rightarrow X$ as $n \to +\infty$ by hypothesis, it follows by property *(iii)* of **Theorem 4.2** and *Remark 6.1 (v)* that

$(X_{n,k})_{k=0}^{\infty} \Rightarrow (X_k)_{k=0}^{\infty}$ as $n \to +\infty$. Then by **Corollary 6.1**, we can then choose $\left\{\left(\widetilde{X}_{n,k}\right)_{k=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \ell^{\infty}\left(\mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)\right), \left(\widetilde{X}_k\right)_{k=0}^{\infty} \in \ell^{\infty}\left(\mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)\right)$ such that $(X_{n,k})_{k=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_{n,k}\right)_{k=0}^{\infty}$, for each $n \in \mathbb{N}$, $(X_k)_{k=0}^{\infty} \overset{D}{=} \left(\widetilde{X}_k\right)_{k=0}^{\infty}$, and $\left(\widetilde{X}_{n,k}\right)_{k=0}^{\infty} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_k\right)_{k=0}^{\infty}$, as $n \to +\infty$. Therefore, by property *(iv)* of **Theorem 4.2**, we find that $\widetilde{X}_n := \sum_{k=0}^{\infty} \widetilde{X}_{n,k}$ and $\widetilde{X} := \sum_{k=0}^{\infty} \widetilde{X}_k$ are well-defined such that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \geq 1$, $\widetilde{X} \overset{D}{=} X$, and property *(i)* of **Theorem 4.2** still hold; so $\widetilde{X}_{n,k} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}_k$ as $n \to +\infty$, for every $k \geq 0$, and we conclude that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$ as $n \to +\infty$. □