# M800 Roger Temam 4/15 Report

## 1 Infinite Dimensional Stochastic Processes

**Definition 1.1.** Let $(Z, \mathcal{Z})$ be a measurable space. A family $\{X_i\}_{i\in I}$ of $Z$-valued r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **stochastic process**. For $\{i_1, \dots, i_m\} \subset I$, the measures defined on the product space $Z^m$ given by

$$\mu(X_{i_1}, \dots, X_{i_m})(S) = \mathbb{P}[(X_{i_1}, \dots, X_{i_m}) \in S]$$

are called the **finite dimensional distributions (f.d.d.'s)** of $\{X_i\}_{i\in I}$ (also called the **marginals**).
*Source: M647 Lecture 1 (revised), Definition 1.1.5*

*Remark:* In the SPDE context, we look at stochastic processes $\{X(t)\}_{t\geq 0}$ with values in Banach/Hilbert spaces.

**Definition 1.2.** Let $\mathcal{U}$ be a Hilbert space. A $\mathcal{U}$-valued stochastic process $(L(t))_{t\geq 0}$ is called a **Levy process** if,

*(i)* $L(0) = 0$
*(ii)* $L$ *has independent increments:* $\forall 0 \leq t_1 < t_2 < \dots < t_k$ the r.v.'s $L(t_k) - L(t_1), \dots, L(t_k) - L(t_{k-1})$ are independent.
*(iii)* $L$ *has stationary increments:* $L(t) - L(s) \sim L(t-s), \ \forall t \geq s \geq 0$.
*(iv) Stochastic continuity:* $\forall t_0 \geq 0$, $L(t) \to L(t_0)$ in probability as $t \to t_0$, i.e. $\forall \epsilon > 0$,

$$\lim_{t\to t_0} \mathbb{P}[||L(t) - L(t_0)||_{\mathcal{U}} > \epsilon] = 0.$$

*Source: M647 Lecture 2, Definition 2.2.1*

**Theorem 1.3.** Every Levy process $L$ has a cadlag modification, i.e. there is a $\mathcal{U}$-valued Levy process $\widetilde{L}$ such that

*(i)* $\widetilde{L}$ has cadlag sample paths a.s.
*(ii)* $\forall t \geq 0 \ \mathbb{P}\left[\widetilde{L}(t) = L(t)\right] = 1$.
*Source: M647 Lecture 2 (revised), Theorem 2.2.3*

*Proof.* For the sequence $\left\{t + n^{-1}\right\}_{n\in\mathbb{N}}$, we find that $L\left(t + n^{-1}\right) \xrightarrow{\mathbb{P}} L(t)$, and it follows that $L\left(t + n^{-1}\right) \xrightarrow{\text{a.s.}} L(t)$, since we find that the series

$$\overline{L}(t) := L(t+1) + \sum_{n=1}^{\infty} \left[ L\left(t + (n+1)^{-1}\right) - L\left(t + n^{-1}\right) \right] \quad (1.1)$$

converges in $\mathbb{P}$, so $\overline{L}(t)$ exists a.s., since $\left\{L\left(t + (n+1)^{-1}\right) - L\left(t + n^{-1}\right)\right\}_{n\in\mathbb{N}}$ is a sequence of independent random variables. Note moreover, that almost surely we have

$$(\forall t \in \mathbb{Q}_+)\left(\overline{L}(t) \text{ exists}\right)$$

Define

$$\widetilde{L}(t) := \begin{cases} 0 & \text{if } t = 0, \\ \overline{L}(t) & \text{if } t \in \mathbb{Q}_+ \setminus \{0\} \text{ and } \overline{L}(t) \text{ exists,} \\ L^*(t) \in \bigcap_{t'\in\mathbb{Q}_+,\, t<t'} \overline{\overline{L}([t,t'])} & \text{if } t \in \mathbb{R}_+ \setminus \mathbb{Q}_+ \text{ and such an } L^*(t) \text{ exists.} \\ L(t) & \text{otherwise.} \end{cases} \quad (1.2)$$

It shall then suffice to verify *(i)* and *(ii)* of the theorem, since the fact that $\widetilde{L}$ is a Levy process follows immediately by property *(ii)*.

*(i)* To show that $\widetilde{L}$ has cadlag sample paths a.s., we want to show that for all $t \geq 0$, we have

$$\lim_{s\to t^+} \widetilde{L}(s) = \widetilde{L}(t).$$

First, note that almost surely $\overline{L}(t)$ exists for all $t \in \mathbb{Q}_+ \setminus \{0\}$. Next, we want to verify the following claim:

***Claim.*** $\widetilde{L}|(\mathbb{Q}_+ \setminus \{0\})$ is cadlag a.s.

Given a decreasing sequence $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_+ \setminus \{0\}$ such that $q_k \searrow q$, for $q \in \mathbb{Q}_+ \setminus \{0\}$, we want to show that

$$\lim_{k \to +\infty} \widetilde{L}(q_k) = \widetilde{L}(q) \text{ a.s. } (1.3)$$

First, note that $q$ is of the form

$$q = \sum_{j=1}^{N} n_j^{-1}$$

for $n_1 \leq n_2 \leq \cdots \leq n_N$ and all $q_k$ is of the form

$$q_k = \sum_{j=1}^{N_k} n_{j,k}^{-1},$$

for $n_{1,k} \leq n_{2,k} \leq \cdots \leq n_{N_k,k}$. As $q_k \searrow q$, we find $n_{j,k} = n_j$ eventually for $1 \leq j \leq N$ as $k \to +\infty$ and $n_{j,k} \to +\infty$ for all $j > N$ (for whichever $j$ subindex exists for each $k$). As a result, for each $l \geq 1$, we find by recursive diagonalization of the sequences

$$\left\{ L\left( \sum_{j=1}^{\min(N_k,l)} n_{j,k}^{-1} \right) \right\}_{k \in \mathbb{N}}$$

that we have

$$\lim_{k \to +\infty} \overline{L}\left( \sum_{j=1}^{\min(N_k,l)} n_{j,k}^{-1} \right) = \overline{L}\left( \sum_{j=1}^{\min(N,l)} n_j^{-1} \right),$$

almost surely and it follows taking $l \to +\infty$ and applying further diagonalization that

$$\lim_{k \to +\infty} \widetilde{L}(q_k) = \lim_{k \to +\infty} \overline{L}(q_k) = \overline{L}(q) = \widetilde{L}(q)$$

almost surely, and $(1.3)$ is reached.

Now that we have proved the claim, we then want to verify that a cadlag restriction $\widetilde{L}|(\mathbb{Q}_+ \setminus \{0\})$ on the dense subset $\mathbb{Q}_+ \setminus \{0\} \subset \mathbb{R}_+$ has a Cadlag extension $L^*$ that almost surely agrees with $\widetilde{L}$. It can be shown (using a similar argument to the argument that a continuous function on a dense subset of a metric space extends to a continuous extension on the metric space) that the cadlag property of $\widetilde{L}|(\mathbb{Q}_+ \setminus \{0\})$ implies preservation of Cauchy decreasing sequences $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_+ \setminus \{0\}$, to the image, and hence for every $t \in \mathbb{R}_+$, we find that every decreasing sequence $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_+ \setminus \{0\}$ such that $q_k \searrow t$, we find $\left\{\widetilde{L}(q_k)\right\}_{k\in\mathbb{N}}$ is Caucy and hence converges to some uniquely determined limit $L^*(t)$.

To show that $L^*$ (in the almost sure scenario that $\widetilde{L}|(\mathbb{Q}_+ \setminus \{0\})$ is Cadlag where $L^*$ exists on all $\mathbb{R}_+$) agrees almost surely with $\widetilde{L}$. We find that the case where $t > 0$ is immediate, since almost surely $\overline{L}(t) = \widetilde{L}|(\mathbb{Q}_+ \setminus \{0\}) = L^*(t)$ exists if $t \in \mathbb{Q}_+ \setminus \{0\}$ and $\widetilde{L}(t) = L^*(t)$ by construction of  if $t \in \mathbb{R}_+ \setminus \mathbb{Q}_+$. We are then left with the case where $t = 0$, which is a more special case of part *(ii)* (which we prove next).

*(ii)* Let $t \geq 0$. To show that $\mathbb{P}\left[\widetilde{L}(t) = L(t)\right] = 1$, note first by stochastic continuity that $L(s) \xrightarrow{\mathbb{P}} L(t)$ as $s \to t^+$. In the case where $t \in \mathbb{Q}_+$, we find by $(1.1)$ and $(1.2)$ that almost surely we have $L(t) = \overline{L}(t) = \widetilde{L}(t)$, since $L(t)$ and $\overline{L}(t)$ are both $\mathbb{P}$-limits of the sequence $\left\{L\left(t+n^{-1}\right)\right\}_{n\in\mathbb{N}}$. Next, in the case where $t \in \mathbb{R}_+ \setminus \mathbb{Q}_+$, we can choose $q_k \searrow t$ and note that since almost surely we have $\widetilde{L}(q_k) = L(q_k)$, for each $k \in \mathbb{N}$, and it follows from the fact that $\widetilde{L}(q_k) \xrightarrow{\text{a.s.}} \widetilde{L}(t)$ and $L(q_k) \xrightarrow{\mathbb{P}} L(t)$ that $\widetilde{L}(t)$ and $L(t)$ are both $\mathbb{P}$-limits of the sequence $\{L(q_k)\}$, and hence agree almost surely. $\square$

**Proposition 1.4.** Let $L$ a Levy process

*(i)* $\mathbb{E}||L_t||_{\mathcal{H}} < +\infty \ \forall t \geq 0$

*(ii)* If $L$ is a mean-zero Levy process, then $L$ is a martingale with respect to its natural filtration $\{\sigma(X_t)\}_{t\geq 0}$.
*Source: M647 Lecture 2 (revised), Proposition 2.5.2*

*Remark 1.1:* This proposition is a more general claim than that of the cited proposition. A mean-zero Levy Processes are $\mathfrak{L}^1(\mathbb{P};\mathcal{U})$ by definition, while part *(i)* claims that *all* Levy processes are $\mathfrak{L}^1(\mathbb{P};\mathcal{U})$.

*Proof.*

*(i)* In the case that $t = 0$, we have $L_t = 0$ and we are done. Suppose $t > 0$, and suppose towards contradiction otherwise, i.e., that

$$\mathbb{E}||L_t||_{\mathcal{U}} = +\infty.$$

Observe that for all $k \geq 1$, and $1 \leq l \leq k$ we have

$$L(t) = \sum_{l=1}^{k} L\left(\frac{lt}{k}\right) - L\left(\frac{(l-1)t}{k}\right),$$

and the random variables

$$L\left(\frac{lt}{k}\right) - L\left(\frac{(l-1)t}{k}\right) \sim L\left(\frac{t}{k}\right) \quad (1.4)$$

are i.i.d. for $1 \leq l \leq k$. Then for all $k \geq 1$, and $1 \leq l \leq k$, we have

$$\begin{aligned} k\mathbb{E}\left|\left|L\left(\frac{t}{k}\right)\right|\right|_{\mathcal{U}} &= \sum_{l=1}^{k} \mathbb{E}\left|\left|L\left(\frac{lt}{k}\right) - L\left(\frac{(l-1)t}{k}\right)\right|\right|_{\mathcal{U}} \geq \mathbb{E}\left|\left|\sum_{l=1}^{k} L\left(\frac{lt}{k}\right) - L\left(\frac{(l-1)t}{k}\right)\right|\right|_{\mathcal{U}} \\ &= \mathbb{E}||L(t)||_{\mathcal{U}} = +\infty \\ \implies \mathbb{E}\left|\left|L\left(\frac{t}{k}\right)\right|\right|_{\mathcal{U}} &= +\infty. \end{aligned}$$

We find that for all $k \geq 1$, we have for each $1 \leq l \leq k$

$$\mathbb{E}\left|\left|L\left(\frac{t}{k}\right)\right|\right|_{\mathcal{U}} = +\infty,$$

and it follows by $(1.4)$ that for each $j, k \geq 1$, we find that for $1 \leq l \leq j$

$$\mathbb{E}\left|\left|L\left(\frac{lt}{jk}\right) - L\left(\frac{(l-1)t}{jk}\right)\right|\right|_{\mathcal{U}} = \mathbb{E}\left|\left|L\left(\frac{t}{jk}\right)\right|\right|_{\mathcal{U}} = +\infty,$$

hence as $j \to +\infty$ we find by a corollary of the *Strong Law of Large Numbers Billingsley,* *page 284* that

$$j^{-1}\sum_{l=1}^{j}\left|\left|L\left(\frac{lt}{jk}\right)-L\left(\frac{(l-1)t}{jk}\right)\right|\right|_{\mathcal{U}} \xrightarrow{\text{a.s.}} \mathbb{E}\left|\left|L\left(\frac{t}{jk}\right)\right|\right|_{\mathcal{U}} = +\infty.$$

Then almost surely for each $k \geq 1$, we have some $1 \leq l_k \leq j_k$ such that

$$\left|\left|L\left(\frac{l_k t}{j_k k}\right)-L\left(\frac{(l_k-1)t}{j_k k}\right)\right|\right|_{\mathcal{U}} \geq 1. \quad (1.5)$$

We then conclude that almost surely there exists $\frac{r_k t}{j_k k} \leq \frac{t}{k}$ such that $\frac{r_k t}{j_k k} \to 0$ as $k \to +\infty$, but the sequence

$$\left\{\left|\left|L\left(\frac{l_k t}{j_k k}\right)-L\left(\frac{(l_k-1)t}{j_k k}\right)\right|\right|_{\mathcal{U}}\right\}_{k\in\mathbb{N}}$$

isn't Cauchy by $(1.5)$, which contradicts the fact that $L\left(\frac{r_k t}{j_k k}\right) \xrightarrow{\mathbb{P}} L(0) = 0$ as $k \to +\infty$.

*(ii)* Note that we proved in part *(i)* that $L_t$ is $\mathfrak{L}^1(\mathbb{P};\mathcal{U})$, for all $t \geq 0$ and we use the mean-zero hypothesis to prove the remaining condition that $\mathbb{E}[L_t|\sigma(L_s)] = L_s$. Note that since

$$\mathbb{E}[L_t|\sigma(L_s)] = \mathbb{E}[L_t - L_s|\sigma(L_s)] + \mathbb{E}[L_s|\sigma(L_s)] = \mathbb{E}[L_t - L_s|\sigma(L_s)] + L_s$$

we find the desired condition to be equivalent to the condition that

$$\mathbb{E}[L_t - L_s|\sigma(L_s)] = 0, \quad (1.6)$$

and so it shall suffice to prove that. $\sigma(L_s)$-measurability $0$ is immediate. Observe that $L_t - L_s$ and $\mathbb{1}_G$, for $G \in \sigma(L_s)$ are independent. It follows that

$$\mathbb{E}[(L_t - L_s)\mathbb{1}_G] = \mathbb{E}[L_t - L_s]\cdot\mathbb{E}[\mathbb{1}_G] = 0 = \mathbb{E}[0\cdot\mathbb{1}_G],$$

since

$\mathbb{E}[L_t - L_s] = \mathbb{E}[L_{t-s}] = 0.$

and $(1.6)$ is satisfied. ◻

**Examples 1.5.**

**1.** A **Wiener process** is a $\mathcal{U}$-valued, integrable, mean-zero Levy process with a.s. continuous paths.
*Source: M647 Lecture 2 (revised), Definition 2.2.2*

**2.** Let $\mu$ be a finite Borel measure on $\mathcal{U}$ (Hilbert space) such that $\mu(\{0\}) = 0$. A **compound Poisson process (CPP) with intensity measure** $\mu$ is a Levy process $P$ with cadlag $\mathcal{U}$-value sample paths s.t.

$$\mathbb{P}[P(t) \in \Gamma] = e^{-\mu(\mathcal{U})t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^{*k}(\Gamma),\ \forall t \geq 0,\ \Gamma \in \mathcal{B}(\mathcal{U}).$$

Here, $\mu^{*j} = \mu * \mu * \cdots * \mu,\ \mu^{*0} := \delta_0 =$ probability measure concentrated at the point $0 \in \mathcal{U}$.

Hence, the ony difference between a Levy process and a Wiener process is the possible occurences of jumps (discontinuities).
*Source: M647 Lecture 2 (revised), Definition 2.4.1*

**Sources:**

*Spring 2022-M647 Lecture 1-2 (revised)*

*Probability and Measures* § *22*
*Billingsley*

## 2 Parametrization Techniques

*Notation 2.1:* Given a Hilbert space $\mathcal{U}$, $L_2^0(\mathcal{U})$--or simply $L_2^0$ when context is clear--refers to the (Hilbert) space of all Hilbert-Schmit operators.

Let $\Phi : (t, \omega, x) \mapsto \Phi(t, \omega, x)$ be a measurable mapping from

$(\Omega_T \times E, \mathcal{P}_T \times \mathcal{B}(E))$ into $\left(L_2^0(\mathcal{U}), \mathcal{B}\left(L_2^0(\mathcal{U})\right)\right)$,

Thus, in particular, for arbitrary $x \in E$, $\Phi(\cdot, \cdot, x)$ is a predictable $L_2^0$-valued process. Additionally, let $\mu$ be a finite positive meaure on $(E, \mathcal{E})$.

**Theorem 2.1.** *(Stochastic Fubini Theorem)* Assume that:

$$\int_E ||\Phi(\cdot, \cdot, x)||_T \mu(dx) < +\infty.$$

Then almost surely we have

$$\int_E \left[\int_0^T \Phi(t,x)dW(t)\right]d\mu(x) = \int_0^T \left[\int_E \Phi(t,x)d\mu(x)\right)]dW(t).$$

*Source: Da Prato, Zabczyk, Theorem 4.18 (page 109)*

We shall prove a more general analogue (at least, hopefully more general, refer to *remark 2.2 (ii)*) of this thoerem (that I've formulated myself) that we shall call the "Fubini Thoerem for random measures". Basically, this theorem asserts that for random measures $\nu$ and $\eta$ on $(E, \mathcal{E})$ and $(G, \mathcal{G})$, respectively on the product random measure $\xi := \nu \otimes \eta$ on $(E \times G, \mathcal{E} \otimes \mathcal{G})$ (defined analogously by the random measure generated by the premeasure $\xi(A \times B) := \nu(A)\eta(B)$, for $A \in \mathcal{E}$, $G \in \mathcal{G}$), we have that the integration done in different order almost surely agrees, for any Banach Space $\mathcal{X}$-valued random $\xi$-measurable function $\Phi$.

**Theorem 2.2.** *(Fubini Theorem for random measures)* Suppose that $\Phi \in \mathcal{L}^1(\xi; \mathcal{X})$, i.e., that $\Phi$ is a $\xi$-random measurable function such that

$$\int_{E \times G} ||\Phi||_{\mathcal{X}} d\xi < +\infty \text{ a.s.}$$

Then almost surely we have

$$\int_{E \times G} \Phi d\xi = \int_G \left[\int_E \Phi(e,g)d\nu(e)\right]d\eta(g) = \int_E \left[\int_G \Phi(e,g)d\eta(g)\right]d\nu(e). \quad (2.1)$$

*Outline of the Proof.* First take $\Phi$ to be a simple function of the form

$$\sum_{j=1}^{N} x_j \mathbb{1}_{A_j \times B_j} \quad (2.2)$$

and observe that

$$\int_{E\times G} \Phi d\xi = \sum_{j=1}^{N} x_j \xi(A_j \times B_j) = \sum_{j=1}^{N} x_j \nu(A_j)\eta(B_j) = \sum_{j=1}^{N} x_j \int_E \mathbb{1}_{A_j}(e) d\nu(e) \int_G \mathbb{1}_{B_j}(g) d\eta(g)$$

and $(2.1)$ is met by linearity of (Bochner)-integration over random measures.

The next part is met by utilizing an analogous "monotone class lemma" to pass $\Phi$ over a $\mathbb{P}$-limit of of simple functions of the form of $(2.2)$, and then invoke **Lemma 1.6** *(ii)* of my 4/8 notes to get $\mathbb{P}$-convergence, then choose the appropriate subsequence to get the desired almost sure result. □

*Remark 2.2.*

*(i)* Note that the proof of **Thoerem 2.2** I've outlined relies on a pretty different strategy than the one done in the proof of the previously-mentioned Stochastic Fubini Theorem in *Da Prato, Zabczyk*, where there, a Borel-Cantelli Lemma, Markov inequality, and Ito-isometry argument is used to get the result. In my argument, I relate it to the arguments done in *Folland*, and claim analogous measure theoretic constructions and principles hold for random measures. (which in the future will require a more in-depth report)

*(ii)* As a disclaimer, the proof of **Thoerem 2.2** is not as general as I would like, i.e., it doesn't quite generalize **Theorem 2.1**, the issue being that $W(t)$ (and hence the "random measure" $\mu_W$) in that theorem is $\mathcal{H}$-valued (which can be done, but I have not done so in my notes) and the kernel is $L_2^0$-valued, whereas both random measures are $\mathbb{R}$-valued and the kernel is $\mathcal{H}$-valued. In either a later edit of this report or a later report, I plan to further generalize this theorem accordingly.

**Sources:**

*Stochastic equations in infinite dimensions* § *4.6*
*Da Prato, Zabczyk*

*Real Analysis, Modern Techniques* § *2.5*