# M800 Roger Temam 4/4/23 Report

## 1 Vector-Valued Measures

**Definition 1.1.** A function $F$ from a field $\mathcal{F}$ of subsets of a set $\Omega$ to a Banach space $X$ is called a **finitely additive vector measure**, or simply a **vector measure**, if whenever $E_1$ and $E_2$ are disjoint members of $\mathcal{F}$ then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$.

If, in addition, $F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$ in the norm topology of $X$ for all sequences $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint members of $\mathcal{F}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$, then $F$ is termed a **countably additive vector measure**, or simply, we say that $F$ is **countably additive**.
*Source: Diestel, Uhl Section I-1. Definition 1 (page 1-2)*

*Remark 1.1.* In this report, we talk about vector-valued measures in Banach spaces, but in a future draft, I hope to generalize this report to vector valued measures in a complete metric space $(X, d)$, i.e., a function $F : \mathcal{F} \to X$ such that if $E_1, E_2 \in \mathcal{F}$ are disjoint, then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$ and a countable additive vector measure has the property that if $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint members of $\mathcal{F}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$, then $\sum_{n=1}^{N} F(E_n) \to F\left(\bigcup_{n=1}^{\infty} E_n\right)$ in $d$ as $N \to +\infty$, i.e., we have

$$\lim_{N\to+\infty} d\left(\sum_{n=1}^{N} F(E_n), F\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = 0.$$

This situation is useful specifcally for random measures, which according to **Proposition 1.2** of the *4/8/23 Report (1-17-23 Update)*, can be defined (particularly in the case of $\mathbb{R}$-valued random measures) by a countably additive vector measure.$M : \mathcal{F} \to \mathfrak{L}^0(\mathbb{P}; \mathbb{R})$ in the complete metric space $d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}$ on the space $\mathfrak{L}^0(\mathbb{P}; \mathbb{R})$ defined by

$$d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(X, Y) := \mathbb{E}\left[\frac{|X - Y|}{1 + |X - Y|}\right].$$

Recall of course that for $\{X_n\} \subset \mathfrak{L}^0(\mathbb{P};\mathbb{R})$, we have $X_n \to X$, for $X \in \mathfrak{L}^0(\mathbb{P};\mathbb{R})$ in $\mathbb{P}$ iff $X_n \xrightarrow{d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}} X$ as $n \to +\infty$.

**Example 1.2.** *Source: Diestel, Uhl Example 2-3 (page 2)*

*(i)* Let $T : \mathfrak{L}^\infty[0,1] \to X$ be a continuous linear operator. For each Lebesgue measurable set $E \subset [0,1]$, define $F(E)$ to $T(\mathbb{1}_E)$ ($\mathbb{1}_E$ denotes the characteristic or indicator function of $E$). Then by the linearity of $T$, $F$ is seen to be a finitely additive vector measure which may--even in the case that $X$ is the real numbers--fail to be countably additive. The simplest such general example of a noncountably additive measure is provided by considering any Hahn-Banach extension to $\mathfrak{L}^\infty[0,1]$ of a point mass functional on $C[0,1]$.

*(ii)* Let $T : \mathfrak{L}^1[0,1] \to X$ be a continuous linear operator. Again define $F(E) = T(\mathbb{1}_E)$ for each Legesgue measurble set $E \subset [0,1]$. Then evidently $F$ is finitely additive. Morover, for each $E$, one has

$$||F(E)|| \le ||\mathbb{1}_E||_1 ||T|| = m(E)||T||,$$

hence we have

$$\lim_m \left|\left| F\left(\bigcup_{n=1}^{\infty} E_n\right) - \sum_{n=1}^{m} F(E_n) \right|\right| = \lim_m \left|\left| F\left(\bigcup_{n=m+1}^{\infty} E_n\right) \right|\right| = \lim_m m\left(\bigcup_{n=m+1}^{\infty} E_n\right)||T|| = 0.$$

**Definition 1.3.** Let $F : \mathfrak{F} \to X$ be a vector measure. The **variation** of $F$ is the extended nonnegative function $|F|$ whose value on a set $E \in \mathfrak{F}$ is given by

$$|F|(E) = \sup_\pi \sum_{A \in \pi} ||F(A)||,$$

where the supremum is taken over all partitions $\pi$ of $E$ into a finite number of pairwise disjoint members of $\mathfrak{F}$. If $|F|(\Omega) < +\infty$, then $F$ will be called a **measure of bounded variation**.

The **semivariation** of $F$ is the extended nonnegative function $||F||$ whose value on a set $E \in \mathfrak{F}$ is given by

$$||F||(E) = \sup\left\{|\varphi F|(E):\ \varphi \in X^*, ||\varphi|| \leq 1\right\},$$

where $|\varphi F|$ is the variation of the real-valued measure $\varphi F$. If $||F||(\Omega) < +\infty$, then $F$ will be called a **measure of bounded semivariation**, which we will also call a **bounded vector measure**.
*Source: Diestel, Uhl Section I-1. Definition 4 (page 2)*

Direct verifications show that the variation of $F$ is a monotone finitely additive function on $\mathcal{F}$, while the semivariation of $F$ is a monotone subadditive function on $\mathcal{F}$. Also it is easy to see that for each $E \in \mathcal{F}$ one has $||F||(E) \leq |F|(E)$.

**Example 1.4.** *Source: Diestel, Uhl Example 5-8 (page 2-3)*

*(i)* The measure discussed in **Example 3 (ii)** is a measure of bounded variation

*(ii)* Let $\Sigma$ be a $\sigma$-field of Lebesgue measurable subsets of $[0,1]$ and define $F: \Sigma \to L^\infty[0,1]$, by $F(E) = \mathbb{1}_E$. Note that $F$ is of bounded semivariation but not of bounded variation.

*(iii)* Let $T: L^\infty[0,1] \to X$ be a continuous linear operator and for a Lebesgue measurable set $E \subset [0,1]$, define $F(E) = T(\mathbb{1}_E)$. We find that $F$ is of bounded semivariation.

*(iv)* Although little can be said of measures of unbounded semivariation, it is worth noting that a vector measure (in fact, a real-valued measure) need not be of bounded semivariation. Indeed, if $\mathcal{F}$ is the field of subsets of $\mathbb{N}$, the positive integers, consisting of sets that are either finite or have finite complements, then the measure $F: \mathcal{F} \to \mathbb{R}$ defined by

$$\begin{aligned} F(E) &= \text{cardinality of } E, && \text{if } E \text{ is finite,} \\ &= -\text{cardinality of } \mathbb{N} \setminus E, && \text{if } \mathbb{N} \setminus E \text{ is finite,} \end{aligned}$$

produces an example of a real-valued measure with unbounded semivariation.

**Proposition 1.5.** A vector measure of bounded variation is countably additive if and only if its variation is also countably additive.
*Source: Diestel, Uhl Section I-1. Proposition 9 (page 3)*

*Outline of Proof.* Suppose $F$ is countably additive. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of pairwise disjoint members of $\mathcal{F}$ such that $\displaystyle\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$. We want to show that

$$|F|\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} |F|(E_n).$$

Given a partition $\pi$ of $\bigcup_{n=1}^{\infty} E_n$, note that

$$\pi_n := \{A \cap E_n : A \in \pi\}$$

is a partition of $E_n$, for each $n \geq 1$, and we find by countable additivity of $F$ that

$$\begin{aligned}\sum_{A\in\pi} ||F(A)|| &= \sum_{A\in\pi} \left|\left|F\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right)\right|\right| = \sum_{A\in\pi} \left|\left|F\left(\bigcup_{n=1}^{\infty} [A \cap E_n]\right)\right|\right| \\ &\leq \sum_{A\in\pi} \left|\left|\sum_{n=1}^{\infty} F(A \cap E_n)\right|\right| \leq \sum_{A\in\pi} \left[\sum_{n=1}^{\infty} ||F(A \cap E_n)||\right] = \sum_{n=1}^{\infty} \sum_{B\in\pi_n} ||F(B)|| \\ &\leq \sum_{n=1}^{\infty} |F|(E_n),\end{aligned}$$

and we have shown that

$$|F|\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} |F|(E_n). \quad (1.1)$$

To show the other inequality

$$\sum_{n=1}^{\infty} |F|(E_n) \leq |F|\left(\bigcup_{n=1}^{\infty} E_n\right). \quad (1.2)$$

we prove the following claim:

*Claim.* $|F|$ is finitely additive.

Given disjoint $E_1, E_2 \in \mathcal{F}$, note by $(1.1)$ (by setting $E_3 = E_4 = \cdots = 0$) we have $|F|(E_1 \cup E_2) \leq |F|(E_1) + |F|(E_2)$ and it remaions to show that

$|F|(E_1)+|F|(E_2) \leq |F|(E_1 \cup E_2)$. we find that for any partition $\pi_1, \pi_2$ of $E_1$ and $E_2$, we find that $\pi_1 \cup \pi_2$ is a partition of $E_1 \cup E_2$, we find that

$$\sum_{A_1 \in \pi_1} ||F(A_1)|| + \sum_{A_2 \in \pi_2} ||F(A_2)|| = \sum_{A \in \pi_1 \cup \pi_2} ||F(A)|| \leq |F|(E_1 \cup E_2),$$

and so

$$|F|(E_1)+|F|(E_2) = \sup_{\pi_1, \pi_2} \left( \sum_{A_1 \in \pi_1} ||F(A_1)|| + \sum_{A_2 \in \pi_2} ||F(A_2)|| \right) \leq |F|(E_1 \cup E_2),$$

Now we proceed to prove $(1.2)$. Note that

$$\begin{aligned} \sum_{n=1}^{\infty} |F|(E_n) &= \sup_{\{\pi_n\}_{n\in\mathbb{N}}} \sum_{n=1}^{\infty} \sum_{A \in \pi_n} ||F(A)|| \\ &= \sup\left\{ \sum_{k=1}^{\infty} ||F(A_k)|| : \{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F} \text{ is pairwise disjoint and } \bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} E_n \right\} \quad (1.3) \end{aligned}$$

and it shall suffice to show that for every pairwise disjoint $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} E_n$, we have

$$\sum_{k=1}^{\infty} ||F(A_k)|| \leq |F|\left( \bigcup_{n=1}^{\infty} E_n \right)$$

Given pairwise disjoint $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} E_n$, we note by the above *Claim* that $|F|$ is finitely additive and we repeatedly utilize finite additivity show show that for every for every $N \geq 1$, we have

$$\sum_{k=1}^{N} ||F(A_k)|| \leq \sum_{k=1}^{N} |F|(A_k) = |F|\left( \bigcup_{k=1}^{N} A_k \right) \leq |F|\left( \bigcup_{k=1}^{N} A_k \right) + |F|\left( \left( \bigcup_{n=1}^{\infty} E_n \right) \setminus \left( \bigcup_{k=1}^{N} A_k \right) \right)$$

$$= |F|\left(\bigcup_{n=1}^{\infty} E_n\right).$$

Next, suppose conversely that $|F|$ is countably additive. Then for a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint members of $\mathcal{F}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$, we want to show that

$$F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n).$$

Note by finite additivity that

$$F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{N} F(E_n) + F\left(\bigcup_{n=N+1}^{\infty} E_n\right)$$

and it shall suffice to show that

$$\lim_{N\to\infty} F\left(\bigcup_{n=N+1}^{\infty} E_n\right) = 0.$$

Since $|F|$ is countably subadditive, we have $|F|\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} |F|(E_n)$, hence it immediatly follows that

$$\lim_{N\to\infty} \left\|F\left(\bigcup_{n=N+1}^{\infty} E_n\right)\right\| \leq \lim_{N\to\infty} |F|\left(\bigcup_{n=N+1}^{\infty} E_n\right) = \lim_{N\to\infty} \sum_{n=N+1}^{\infty} |F|(E_n) = 0. \quad \square$$

*Remark 1.2.* Note that this proof is still technically an outline because $(1.3)$ is a statement that may be intuitive (since a sequence of partitions of disjoint sets can be viewed up to refinement as a sequence of disjoint sets, and vice versa) but nevertheless should be proven just to make sure.

**Proposition 1.6.** Let $F : \mathcal{F} \to \mathcal{X}$ be a vector measure. Then for $E \in \mathcal{F}$, one has

*(a)*

$$||F||(E) = \sup\left\{\left|\left|\sum_{A_n\in\pi} \epsilon_n F(A_n)\right|\right|\right\},$$

where the supremum is taken over all partitions $\pi$ of $E$ into finitely many disjoint members of $\mathcal{F}$ and a finite collections $\{\epsilon_n\}$ satisfying $|\epsilon_k| \le 1$; and

*(b)*

$$\begin{aligned}\sup\{||F(H)|| : E \supset H \in \mathcal{F}\} \le ||F||(E) \\ \le 4\sup\{||F(H)|| : E \supset H \in \mathcal{F}\}.\end{aligned}$$

Consequently a vector measure is of bounded semivariation on $\Omega$ if and only if its range is bounded in $\mathcal{X}$.
*Source: Diestel, Uhl Section I-1. Proposition 11 (page 4)*

In view of the above proposition, a vector measure of bounded semivariation will also be called a **bounded vector measure**.

*Proof.*

*(a)* Given a finite collections $\{\epsilon_n\}$ satisfying $|\epsilon_k| \le 1$, we have

$$\begin{aligned}\left|\left|\sum_{A_n\in\pi} \epsilon_n F(A_n)\right|\right| &= \sup\left\{\left|\varphi \sum_{A_n\in\pi} \epsilon_n F(A_n)\right| : \varphi \in \mathcal{X}^*, ||\varphi|| \le 1\right\} \\ &= \sup\left\{\left|\sum_{A_n\in\pi} \epsilon_n \varphi F(A_n)\right| : \varphi \in \mathcal{X}^*, ||\varphi|| \le 1\right\} \\ &\le \sup\left\{\sum_{A_n\in\pi} |\epsilon_n| \cdot |\varphi F(A_n)| : \varphi \in \mathcal{X}^*, ||\varphi|| \le 1\right\} \\ &\le \sup\left\{\sum_{A_n\in\pi} |\varphi F(A_n)| : \varphi \in \mathcal{X}^*, ||\varphi|| \le 1\right\} \\ &\le ||F||(E),\end{aligned}$$

which shows that

$$||F||(E) \geq \sup\left\{\left|\left|\sum_{A_n\in\pi} \epsilon_n F(A_n)\right|\right|\right\}$$

To show the reverse inequality, observe that given $\varphi \in \mathcal{X}^*$ such that $||\varphi|| \leq 1$ and a finite partition $\pi$ of $E$, we have

$$\begin{aligned}
\sum_{A\in\pi} |\varphi F(A)| &= \sum_{A\in\pi} \operatorname{sgn}(\varphi F(A))\varphi F(A) \\
&= \sum_{A\in\pi} \operatorname{sgn}(\varphi F(A))\varphi F(A) \\
&= \varphi\left(\sum_{A\in\pi} \operatorname{sgn}(\varphi F(A))F(A)\right) \\
&\leq ||\varphi||\left|\left|\sum_{A\in\pi} \operatorname{sgn}(\varphi F(A))F(A)\right|\right| \\
&\leq \left|\left|\sum_{A\in\pi} \operatorname{sgn}(\varphi F(A))F(A)\right|\right|,
\end{aligned}$$

which shows that

$$||F||(E) \leq \sup\left\{\left|\left|\sum_{A_n\in\pi} \epsilon_n F(A_n)\right|\right|\right\}.$$

*(b)* Using *part (a)*, note that given $H \subset E$, we find that

$$||F(H)|| = ||F(H) + 0 \cdot F(E \setminus H)|| \leq \sup\left\{\left|\left|\sum_{A_n\in\pi} \epsilon_n F(A_n)\right|\right|\right\} = ||F||(E).$$

To show $||F||(E) \leq 4\sup\{||F(H)|| : E \supset H \in \mathcal{F}\}$, take a partition $\pi$ of $E$ and $\varphi \in \mathcal{X}^*$ such that $||\varphi|| \leq 1$. Set

$$A^{\mathrm{Re},+} := \Big\{x \in E : x \in A, \text{ for some } A \in \pi \text{ such that } \mathrm{Re}(\varphi(F(A_n))) > 0\Big\},$$
$$A^{\mathrm{Re},-} := \Big\{x \in E : x \in A, \text{ for some } A \in \pi \text{ such that } \mathrm{Re}(\varphi(F(A_n))) < 0\Big\},$$
$$A^{\mathrm{Im},+} := \Big\{x \in E : x \in A, \text{ for some } A \in \pi \text{ such that } \mathrm{Im}(\varphi(F(A_n))) > 0\Big\},$$
$$A^{\mathrm{Im},-} := \Big\{x \in E : x \in A, \text{ for some } A \in \pi \text{ such that } \mathrm{Im}(\varphi(F(A_n))) > 0\Big\},$$

$$\pi_{A^{\mathrm{Re},+}} := \Big\{A \in \pi_E : A \subset A^{\mathrm{Re},+}\Big\},$$
$$\pi_{A^{\mathrm{Re},-}} := \Big\{A \in \pi_E : A \subset A^{\mathrm{Re},-}\Big\},$$
$$\pi_{A^{\mathrm{Im},+}} := \Big\{A \in \pi_E : A \subset A^{\mathrm{Im},+}\Big\},$$
$$\pi_{A^{\mathrm{Im},-}} := \Big\{A \in \pi_E : A \subset A^{\mathrm{Im},-}\Big\}.$$

We find that

$$\begin{aligned}
\sum_{A \in \pi} |\varphi F(A)| &= \sum_{A \in \pi} \mathrm{sgn}(\varphi F(A))\varphi F(A) \\
&= -\sum_{A \in \pi_{A^{\mathrm{Re},+}}} \varphi F(A) + \sum_{A \in \pi_{A^{\mathrm{Re},-}}} \varphi F(A) - i \sum_{A \in \pi_{A^{\mathrm{Im},+}}} \varphi F(A) + i \sum_{A \in \pi_{A^{\mathrm{Im},-}}} \varphi F(A) \\
&= -\varphi\left(\sum_{A \in \pi_{A^{\mathrm{Re},+}}} F(A)\right) + \varphi\left(\sum_{A \in \pi_{A^{\mathrm{Re},-}}} F(A)\right) - i\varphi\left(\sum_{A \in \pi_{A^{\mathrm{Im},+}}} F(A)\right) + i\varphi\left(\sum_{A \in \pi_{A^{\mathrm{Im},-}}} F(A)\right) \\
&= -\varphi F\left(A^{\mathrm{Re},+}\right) + \varphi F\left(A^{\mathrm{Re},-}\right) - i\varphi F\left(A^{\mathrm{Im},+}\right) + i\varphi F\left(A^{\mathrm{Im},-}\right) \\
&\le ||\varphi|| \cdot ||F\left(A^{\mathrm{Re},+}\right)|| + ||\varphi|| \cdot ||F\left(A^{\mathrm{Re},-}\right)|| + ||\varphi|| \cdot ||F\left(A^{\mathrm{Im},+}\right)|| + ||\varphi|| \cdot ||F\left(A^{\mathrm{Im},-}\right)|| \\
&\le ||F\left(A^{\mathrm{Re},+}\right)|| + ||F\left(A^{\mathrm{Re},-}\right)|| + ||F\left(A^{\mathrm{Im},+}\right)|| + ||F\left(A^{\mathrm{Im},-}\right)|| \\
&\le 4\sup\{||F(H)|| : E \supset H \in \mathcal{F}\},
\end{aligned}$$

which shows $||F||(E) \le 4\sup\{||F(H)|| : E \supset H \in \mathcal{F}\}$. □

Let $\mathcal{F}$ be a field of subsets of $\Omega$ and $F : \mathcal{F} \to X$ be a bounded vector measure If $f$ is a scalar-valued simple function on $\Omega$, i.e., $f := \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}$ where $\alpha_i$ are nonzero scalars and $E_1, \ldots, E_n$ are pairwise disjoint members of $\mathcal{F}$, define $T_F(f) = \sum_{i=1}^{n} \alpha_i F(E_i)$.

Let $B(\mathcal{F})$ denote the space of all scalar-valued functions on $\Omega$ that are uniform limits of simple functions modeled on $\mathcal{F}$.

**Proposition 1.7.** Let $\mathcal{F}$ be a field of subsets of the set $\Omega$ and let $F : \mathcal{F} \to \mathcal{X}$ be a bounded vector measure. then there is a unique continuous extension of $T_F$, still denoted by $T_F$, to $B(\mathcal{F})$.
*Source: Diestel, Uhl Section I-1. (page 6)*

*Proof.* It shall suffice to prove that $T_F$ is a continuous linear map from $B(\mathcal{F})$ (with the uniform norm $||f|| := \sup_{\omega \in \Omega} |f(\omega)|$) to $\mathcal{X}$, since it would immediately follow from from continuity and Cauchy Completion of the spaces (as well as the **Open Mapping Theorem** applied to $T_F(B(\mathcal{F}))$) that

$$T_F(f) := \lim_{n \to \infty} T_F(f_n),$$

where $f_n$ is a sequence of simple function such that $f_n \to f$ uniformly as $n \to \infty$, is a well-defined, uniquely determined continuous extension.

Observe for $f := \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}$, with $E_1, \ldots, E_N$ pairwise disjoint members of $\mathcal{F}$, we have

$$||T_F(f)|| = ||f|| \left|\left| \sum_{i=1}^{n} \frac{\alpha_i}{||f||} F(E_i) \right|\right| \leq ||f|| \cdot ||F||(\Omega),$$

proving continuity of $T_F$. □

**Definition 1.8.** Let $\mathcal{F}$ be a field of subsets of the set $\Omega$ and let $F : \mathcal{F} \to \mathcal{X}$ be a bounded vector measure. For each $f \in B(\mathcal{F})$, $\int f dF$ is defined by

$$\int f dF := T_F(f),$$

where $T_F$ is as above.
*Source: Diestel, Uhl Section I-1. Definition 12 (page 6)*

**Theorem 1.9.** Let $\mathcal{F}$ (resp. $\Sigma$) be a field (resp. $\sigma$-field) of subsets of the set $\Omega$. Suppose $\mu$ is an extended real-valued nonnegative finitely additive measure on $\Sigma$. Then there is a one-to-one linear correspondence between $L(B(\mathcal{F}); \mathcal{X})$ (resp. $L\left(\mathcal{L}^\infty(\mu); \mathcal{X}\right)$) and the space of all bounded vector measures $F : \mathcal{F} \to \mathcal{X}$ (resp. all bounded vector measures $F : \Sigma \to \mathcal{X}$ that vanish on $\mu$-null sets) defined by $F \leftrightarrow T_F$ if $T_F f = \int f dF$ for all $f \in B(\mathcal{F})$ (resp. $L_\infty(\mu)$). Moreover $||T_F|| = ||F||(\Omega)$.
*Source: Diestel, Uhl Section I-1. Theroem 13 (page 6)*

*Proof.* Let $M(\mathcal{F}, \mathcal{X})$ denote the space of all bounded vector measures (respectively, let $M(\mu, \mathcal{X})$ denote the space of all bounded vector measures that vanish on $\mu$-null sets). To show this, it shall suffice to show $T_{(-)} : M(\mu, \Omega) \to L(B(\mathcal{F}); \mathcal{X})$ (resp. $M(\mu, \Omega) \to L\left(\mathcal{L}^\infty(\mu); \mathcal{X}\right)$) is invertible. Define $F_{(-)} : L(B(\mathcal{F}); \mathcal{X}) \to M(\mu, \Omega)$ (resp. $F_{(-)} : L\left(\mathcal{L}^\infty(\mu); \mathcal{X}\right) \to M(\mu, \Omega)$) by

$F_T(A) := T(\mathbb{1}_A)$, for all $T \in L(B(\mathcal{F}); \mathcal{X})$ and $A \in \mathcal{F}$ (resp. $L\left(\mathcal{L}^\infty(\mu); \mathcal{X}\right)$ and $A \in \Sigma$)

First we show that $F_T$ is a bounded vector measure from $\mathcal{F} \to \mathcal{X}$ (resp. a bounded measure from $\Sigma \to \mathcal{X}$ that vanishes on $\mu$-null sets). It follows by linearity that $F_T$ is a vector measure. To show that $F_T$ is bounded, we find that given $\varphi \in \mathcal{X}^*$ and $||\varphi|| \leq 1$ and a finite partition $\pi$ of $\Omega$, we have

$$\begin{aligned}\sum_{A \in \pi} |\varphi F_T(A)| &= \sum_{A \in \pi} \operatorname{sgn}(\varphi F_T(A)) \varphi F_T(A) \\ &= \varphi\left(\sum_{A \in \pi} \operatorname{sgn}(\varphi F_T(A)) T(\mathbb{1}_A)\right) \\ &= \varphi T\left(\sum_{A \in \pi} \operatorname{sgn}(\varphi F_T(A)) \mathbb{1}_A\right) \\ &\leq ||T||, \qquad (1.4)\end{aligned}$$

and it follows that $||F_T||(\Omega) \leq ||T|| < +\infty$. Next, in the case where $T \in L\left(\mathcal{L}^\infty(\mu); \mathcal{X}\right)$, we want to show that $F_T$ vanishes on $\mu$-null sets. Given $A \in \Sigma$ such that $\mu(A) = 0$, we find that $\mathbb{1}_A = 0$ $\mu$-a.e., and it follows that

$F_T(A) = T(\mathbb{1}_A) = T(0) = 0.$

Now we show that $F_{(-)}$ is the inverse of $T_{(-)}$. To show that $T_{(-)} \circ F_{(-)} = \text{id}_{L(B(\mathcal{F});X)}$ (resp. $T_{(-)} \circ F_{(-)} = \text{id}_{L(\mathfrak{L}^{\infty}(\mu);X)}$) shall suffice by density to show that

$$[(T_{(-)} \circ F_{(-)})(T)](f) = T(f),$$

for every simple function $f := \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i} \in B(\mathcal{F})$ (resp. $\in \mathfrak{L}^{\infty}(\mu)$), (with $E_1, \ldots, E_N$ pairwise disjoint members of $\mathcal{F}$). Given such a simple function $f$ and $T \in L(B(\mathcal{F}); X)$ (resp. $T \in L(\mathfrak{L}^{\infty}(\mu); X)$), we find that

$$[(T_{(-)} \circ F_{(-)})(T)](f) = [T_{(-)}(F_T)](f) = \sum_{i=1}^{n} \alpha_i F_T(E_i) = \sum_{i=1}^{n} \alpha_i T(\mathbb{1}_{E_i}) = T(f),$$

To show that $F_{(-)} \circ T_{(-)} = \text{id}_{M(\mathcal{F},X)}$ (resp. $T_{(-)} \circ F_{(-)} = \text{id}_{M(\mu,X)}$), observe that for all $F \in M(\mathcal{F}, X)$ and $E \in \mathcal{F}$ (resp. $F \in M(\mu, X)$ and $E \in \Sigma$), we have

$$[(F_{(-)} \circ T_{(-)})(F)](E) = [F_{(-)}(T_F)](E) = T_F(\mathbb{1}_E) = F(E).$$

Finally, we show that $||T_F|| = ||F||(\Omega)$. By $(1.4)$, we find that $||F||(\Omega) \leq ||T_F||$ and it remains to show that $||T_F|| \leq ||F||(\Omega)$, we find that for $f := \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i} \in B(\mathcal{F})$ (resp. $\in \mathfrak{L}^{\infty}(\mu)$) such that $||f|| \leq 1$, we find that $E_1, \ldots, E_n$ is a partition of $\Omega$ and $|\alpha_i| \leq 1$ for all $1 \leq i \leq n$, and it follows by **Proposition 1.6** *(a)* that

$$||T_F(f)|| = \left|\left|\sum_{i=1}^{n} \alpha_i F(E_i)\right|\right| \leq \sup\left\{\left|\left|\sum_{A_n \in \pi} \epsilon_n F(A_n)\right|\right|\right\} = ||F||(\Omega). \quad \square$$

**Definition 1.10.**

*(i)* Let $\mathcal{F}$ be a field of subsets of the set $\Omega$ and let $F : \mathcal{F} \to X$ be a vector measure. $F$ is said to be **strongly additive** whenever given a sequence $\{E_n\}_{n \in \mathbb{N}}$ of pairwise disjoint members of $\mathcal{F}$, the series $\sum_{n=1}^{\infty} F(E_n)$ converges in norm.

*(ii)* A family $\left\{F_\tau : \mathcal{F} \to X | \tau \in A\right\}$ of strongly additive vector measures is said to be

**uniformly strongly additive** whenever for any sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint members of $\mathcal{F}$, then

$$\lim_m \left|\left|\sum_{n=m}^{\infty} F_\tau(E_n)\right|\right| = 0,$$

uniformly in $\tau \in T$.
*Source: Diestel, Uhl Section I-1. Definition 14 (page 7)*

Of course, countably additive vector measures on sigma-fields are strongly additive. It is also important to realize that in the definition of strong additivity the convergence of the series $\sum_{n=1}^{\infty} F(E_n)$ is unconditional in norm (since every subseries also converges).

A wide but by no means exhaustive class of strongly additive vector measures is furnished by:

**Proposition 1.11.** If $F : \mathcal{F} \to X$ is a vector measure of bounded variation, then $F$ is strongly additive.
*Source: Diestel, Uhl Section I-1. Proposition 15 (page 7)*

*Proof.* Note that $\{E_n\}_{n\in\mathbb{N}}$ is a partition of $\Omega$, so $\{E_1, \ldots, E_n\}$ is a partition of $\Omega_n := \bigcup_{k=1}^{n} E_k$.
We find that

$$\sum_{k=1}^{n} ||F(E_n)|| \leq |F|(\Omega_n) \leq |F|(\Omega) < +\infty. \quad \square$$

Let $(\Omega, \Sigma, \mu)$ be a finite measure.

**Definition 1.12.** A function $f : \Omega \to X$ is called **simple** if there exists $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \mathbb{1}_{E_i}$. A function $f : \Omega \to X$ is called **$\mu$-measurable** if there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ with $\lim_n ||f_n - f|| = 0$ **$\mu$-almost everywhere**. A function $f : \Omega \to X$ is called **weakly $\mu$-measurable** if for each $\varphi \in X^*$ the numerical function $\varphi f$ is **$\mu$-meausrable**. More generally, if $\Gamma \subset X^*$ and $\varphi f$ is measurable for each $\varphi \in \Gamma$, then $f$ is called **$\Gamma$-measurable**. If $f : \Omega \to X^*$ is $X$-measurable (when $X$ is

identified with its image under the natural imbedding of $\mathcal{X}$ into $\mathcal{X}^{**}$), then $f$ is called **weak*-measurable**.
*Source: Diestel, Uhl Section II-1. Definition 1 (page 41)*

In the litereature, the terms **strong measurability** and **scalar measurability** are often used to describe $\mu$-measruability and weak $\mu$-measurability, respectively. Sometimes reference to the measure $\mu$ will be suppressed when there is no chance of ambiguity.

**Theorem 1.13.** *(Pettis Measurability Theorem)* A function $f : \Omega \to \mathcal{X}$ is measurable if and only if

*(i)* $f$ is $\mu$-essentially separably valued, i.e., there exists $E \in \Sigma$ with $\mu(E) = 0$ and such that $f[\Omega \setminus E]$ is a (norm) separable subset of $\mathcal{X}$, and

*(ii)* $f$ is weakly $\mu$-measurable.
*Source: Diestel, Uhl Section II-1. Theorem 2 (page 42)*

*Outline of proof.*

$\Longrightarrow$ Suppose $f : \Omega \to \mathcal{X}$ is measurable. Choose $f_n := \sum_{i=1}^{N_n} x_{n,i} \mathbb{1}_{E_{n,i}}$ such that $\lim_{n\to\infty} ||f_n - f|| = 0$ $\mu$-a.e., i.e., we find that for

$$E := \Big\{\omega \in \Omega : f_n(\omega) \not\to f(\omega) \text{ in norm}\Big\},$$

we have $\mu(E) = 0$, and $f[\Omega \setminus E]$ is a norm separable subset of $\mathcal{X}$, since

$$A := \{x_{n,i} : n \geq 1,\ 1 \leq i \leq N_n\}$$

is a countable set such that for every $f(\omega) \in f[\Omega \setminus E]$, we have $f_n(\omega) = x_{n,q} \in A$, for some $1 \leq q \leq N_n$ and $f_n(\omega) \to f(\omega)$ in norm. We have shown that condition *(i)* is satisfied. Since for every $\varphi \in \mathcal{X}^*$, we have $\varphi f_n = \sum_{i=1}^{N_n} \varphi(x_{n,i}) \mathbb{1}_{E_{n,i}}$, and hence is $\mu$-measurable, and it follows that $\varphi f$ is $\mu$-measurable since $\varphi f_n \xrightarrow{\mu\text{-a.e.}} \varphi f$ as $n \to \infty$, and we conclude that condition *(ii)* that $f$ is $\mu$-measurable is met.

$\Longleftarrow$ Suppose conversely that conditions *(i)* and *(ii)* are met. Choose $E \in \mathcal{E}$ such that $\mu(E) = 0$ and $f[\Omega \setminus E]$ is separable. Choose countable $\{x_k\}$ dense in $f[\Omega \setminus E]$, and since

$$||x|| := \sup\left\{\varphi(x) : ||\varphi|| = 1,\ \varphi \in \mathcal{X}^*\right\},$$

we can, for every $j, k \in \mathbb{N}$, choose $\left\{\varphi_{j,k}^{(n)}\right\}_{n\in\mathbb{N}}$ such that $||\varphi_{j,k}^{(n)}|| = 1$ and

$$||x_j - x_k|| - \varphi_{j,k}^{(n)}(x_j - x_k) \le n^{-1}.$$

Next for every $n \ge 1$, choose the largest $\epsilon_n > 0$ such that

$\varphi_{j,k}^{(n)}(x_j - x_k) \ge 2\epsilon_n$, for every $1 \le j \ne k \le n$.

Set

$$A_{j,k}^{(n)} := \left\{x : |\varphi_{j,k}^{(n)}(x - x_k)| < \epsilon_n\right\} = \left(\varphi_{j,k}^{(n)}\right)^{-1}\left(\left(\varphi_{j,k}^{(n)}(x_k) - \epsilon_n, \varphi_{j,k}^{(n)}(x_k) + \epsilon_n\right)\right).$$

Finally, for every $n \in \mathbb{N}$, define $f_n : \Omega \to \mathcal{X}$ by

$$f_n := \sum_{k=1}^{n} (\max_{1\le j\le n} \mathbb{1}_{f^{-1}\left(A_{j,k}^{(n)}\right)}) \cdot x_k,$$

and observe first that $f_n$ is measurable (and more specifically a simple function) since

$$f^{-1}\left(A_{j,k}^{(n)}\right) = \left(\varphi_{j,k}^{(n)} f\right)^{-1}\left(\left(\varphi_{j,k}^{(n)}(x_k) - \epsilon_n, \varphi_{j,k}^{(n)}(x_k) + \epsilon_n\right)\right),$$

which are measurable sets by the hypothesis that $f$ is weakly $\mu$-measurable.

From the fact that $f_n$ is valued by $\varphi_{j,k}^{(n)}$, which approximates (after possibly needed tweeks to the values of $f_n$ in a later draft) $||f(\omega) - x_k||$, which will in turn use the dense subset $\{x_k\}$ to approximate the value of $f(\omega)$, we will then get the desired result that $f_n \to f$ pointwise on $\Omega \setminus E$ as $n \to +\infty$. $\square$

**Corollary 1.14.**

*(i)* A function $f : \Omega \to X$ is $\mu$-measurable if and only if $f$ is the $\mu$-almost everywhere uniform limit of a sequence of countably valued $\mu$-measurable functions.

*(ii)* A $\mu$-essentially separably valued function $f : \Omega \to X$ is $\mu$-measurable if there exists a norming set $\Gamma \subset X^*$ such that the numerical function $\varphi f$ is $\mu$-measurable for each $\varphi \in \Gamma$. (Recall $\Gamma \subset X^*$ is norming if

$$||x|| = \sup\{|\varphi x| \,/\, ||\varphi|| : \varphi \in \Gamma\}$$

for each $x \in X$.)
*Source: Diestel, Uhl Section II-1. Corollary 3-4 (page 42-43)*

*Outline of proof.*

(*i*) $\Longleftarrow$ follows from the fact that $\mu$-a.e. uniform convergence implies $\mu$-a.e. pointwise convergence.To prove $\Longrightarrow$ , it follows by **Theorem 1.13** we can choose $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable. Next, choose dense $\{x_k\} \subset f(\Omega \setminus E)$. To prove $\Longrightarrow$ , it follows by **Theorem 1.13** we can choose $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable. Next, choose dense $\{x_k\} \subset f(\Omega \setminus E)$. Choose a normalized basis $\{b_k\} \subset \{x_k\}$ (i.e., a basis such that $||b_k|| = 1$, for every $k \in \mathbb{N}$) of span($f(\Omega \setminus E)$) and note that

$$\text{span}_{\mathbb{Q}[i]}(\mathcal{B}) := \left\{ \sum_{k=1}^{n} q_k b_k : b_1, \ \dots, b_n \in \mathcal{B}, q_1, \ \dots, q_n \in \mathbb{Q}[i] \right\}$$

is a countable dense subset of $f(\Omega \setminus E)$. Define $f_n : \Omega \to X$ as follows

$$f_n(\omega) = \sum_{k=1}^{\infty} \frac{\lfloor 10^{k+n} \text{Re}(a_k(\omega)) \rfloor + i \lfloor 10^{k+n} \text{Im}(a_k(\omega)) \rfloor}{10^{k+n}} \cdot b_k,$$

where $a_k(\omega) \in \mathbb{C}$ is the (possibly zero) $k$th coefficient of the value of $f(\omega)$ projected to $b_k$, i.e., the values such that

$$f(\omega) = \sum_{k=1}^{\infty} a_k(\omega) \cdot b_k.$$

We note that $f_n$ is well-defined since $\{b_k\}$ is a basis and hence the values of $a_k(\omega)$ that determine the values of $f_n$ are uniquely determined. We moreover find that $f_n$ are countably valued, and we can show that $f_n$ is measurable using **Theorem 1.13**, i.e., we find that $f_n(\Omega)$ is separable since it is countably valued as a subset of $\text{span}_{\mathbb{Q}[i]}(\mathcal{B})$, and $\mu$-measurable, since given $\varphi \in X^*$, we find

$$\varphi f_n(\omega) = \sum_{k=1}^{\infty} \frac{\lfloor 10^{k+n}\text{Re}(a_k(\omega))\rfloor + i\lfloor 10^{k+n}\text{Im}(a_k(\omega))\rfloor}{10^{k+n}} \cdot \varphi(b_k),$$

which is a measurable function from $\Omega$ to $\mathbb{R}$ since it involves "measurable operations" (I shall give a more rigorous argument in the next draft).

We find that $f_n \xrightarrow{u} f$ as $n \to +\infty$, since

$$\begin{aligned} ||f_n - f|| &\le \sum_{k=1}^{\infty} \left| \frac{\lfloor 10^{k+n+1}\text{Re}(a_k(\omega))\rfloor + i\lfloor 10^{k+n+1}\text{Im}(a_k(\omega))\rfloor}{10^{k+n+1}} - a_k(\omega) \right| ||b_k|| \\ &= \sum_{k=1}^{\infty} 10^{-(k+n)} < 10^{-n}. \end{aligned}$$

*(ii)* Suppose a $\mu$-essentially separably valued function $f : \Omega \to X$ and there exists a norming set $\Gamma \subset X^*$ such that the numerical function $\varphi f$ is $\mu$-measurable for each $\varphi \in \Gamma$. We can choose $E \in \mathcal{E}$ such that $\mu(E) = 0$ and $f[\Omega \setminus E]$ is separable, choose countable $\{x_k\}$ dense in $f[\Omega \setminus E]$, and then for every $j, k \in \mathbb{N}$, choose $\left\{\varphi_{j,k}^{(n)}\right\}_{n\in\mathbb{N}} \subset \Gamma$ exactly as in the $\Longleftarrow$ part of the proof of **Theorem 1.13**, which in turn allows us to proceed exactly the same way as that proof in constructing a sequence of $\mu$-measurable simple functions $f_n$ such that $f_n \to f$. □

**Example 1.15.** *Source: Diestel, Uhl Section II-1. Corollary 5-7 (page 43-44)*

*(i)* Let $\{e_t : t \in [0, 1]\}$ be an orthonormal basis for the nonseparable Hilbert space $\ell^2([0, 1])$. Define $f : [0, 1] \to \ell^2([0, 1])$ by $f(t) = e_t$. This is a weakly measurable function that is not measurable.

*(ii) (Sierpinski)* Let $\{r_n\}_{n\in\mathbb{N}}$ be the sequence of Rademacher function on $[0, 1]$. i.e., for $t \in [0, 1]$, $r_n(t) = \text{sign}\left(\sin\left(2^n \pi t\right)\right)$. Define $f : [0, 1] \to \ell^\infty$ by

$f(t) = ((r_1(t)+1)/2, (r_2(r)+1)/2, \ldots)$. We find $f$ is weak* measurable but not weakly measurable.

*(iii) (Hagler)* Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subintervals of $[0,1]$ such that: (i) $A_1 = [0,1]$; (ii) Each $A_n$ is a nonempty subinterval of $[0,1]$; (iii) $\lim_n \mu(A_n) = 0$ where $\mu$ is the Lebesgue measure; (iv) $A_n = A_{2n} \cup A_{2n+1}$, for all $n$, and (v) $A_n \cap A_j = \varnothing$, for each apir $m$ and $j$ with $m \neq j$ and $2^i \leq m,\ j \leq 2^{i+1}-1$ (bisect the interval and keep bisecting). Define $f:[0,1] \to \ell^\infty$ by $f(t) = (\mathbb{1}_{A_n}(t))_{n\in\mathbb{N}}$ for $t \in [0,1]$. We find that $f$ is (nontrivially) weakly measurable.

**Sources:**

*Vector Measures Section I-1., II-1.*
*Diestel, Uhl*

## 2 The Radon-Nikodym Property and Conditional Expectation

**Example 2.1.** *Source: Diestel, Uhl Section III-1. Example 1, 1', 2, 2' (page 60)*

*(i) The failure of the Radon-Nikodym Theorem for a $c_0$-valued measure.* Let $\Omega = [0,1]$ and let $\mu$ be a Lebesgue measure on $\Sigma$, the $\sigma$-field of Lebesgue measurable subsets of $[0,1]$. Define a measure $G:\Sigma \to c_0$ by

$$G(E) = \left(\int_E \sin\left(2^n \pi t\right) d\mu(t)\right)_{n\in\mathbb{N}}$$

We find using some calculations done in *Diestel-Uhl III-1-Example 1 (page 60)* that $G(E)$ has no Radon-Nikodym derivative.

*(ii) The failure of the Riesz Representation Theorem for an operator $T:L_1[0,1] \to c_0$.* Let $(\Omega, \Sigma, \mu)$ be as in part (i) of this example. Define $T:L_1(\mu) \to c_0$ by

$$Tf = \left(\int_{[0,1]} f(t)\sin\left(2^n \pi t\right) d\mu(t)\right)_{n\in\mathbb{N}}.$$

According to the *Riemann-Lebesgue lemma*, $T$ is a $c_0$-valued linear operator $L_1(\mu)$. However, we find using some calculations done in *Diestel-Uhl III-1-Example 1' (page 60)* there is no $g \in L_\infty(\mu, c_0)$ such that

$$Tf = \int_\Omega fg d\mu \text{ for all } f \in L_1(\mu).$$

*(iii)* An $L_1(\mu)$-valued measure with no Radon-Nikodym derivative. Suppose $(\Omega, \Sigma, \mu)$ is a finite measure space without atoms. Define $G : \Sigma \to L_1(\mu)$ by $G(E) = \mathbb{1}_E$. Then $G$ is countably additive, $\mu$-continuous and is of bounded variation; in fact, $|G|(E) = \mu(E)$ for each $E \in \Sigma$. However, we find using somme calculations done in *Diestel-Uhl III-1-Example 2 (page 61)* that $G(E)$ has no Radon-Nikodym derivative.

*(iv)* An $L_1(\mu)$-valued measure with no Radon-Nikodym derivative. Suppose $(\Omega, \Sigma, \mu)$ is a finite measure space without atoms. Let $T$ be the identity operator on $L_1(\mu)$. We find using some calculations done in *Diestel-Uhl III-1-Example 2' (page 61)* there is no $g \in L_\infty(\mu, c_0)$ such that

$$Tf = \int_\Omega fg d\mu \text{ for all } f \in L_1(\mu).$$

Let $(\Omega, \Sigma, \mu)$ be a finite nonnegative real-valued measure, and $X$ is a Banach space.

**Definition 2.2.** We state that a vector measure $F$ is **$\mu$-continuous** if

$$\lim_{\mu(E) \to 0} F(E),$$

i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta \implies ||F(E)|| < \epsilon$. We write $F \ll \mu$.
*Source: Diestel, Uhl Section I-2. Definiton 3 (page 11)*

*Remark 2.1.* Note that $F \ll \mu$ does not necessarily mean that $F$ vanishes on $\mu$-null sets unless both $F$ and $\mu$ are countably additive and defined on a $\sigma$-field. We shall discuss this distinction on the next draft.

Now assume for the remainder of this section that $\mu$ is countably additive.

**Definition 2.3.**

*(ii)* A Banach space $X$ has the **Radon-Nikodym property** with respect to $(\Omega, \Sigma, \mu)$ if for each $\mu$-continuous vector measure $G : \Sigma \to X$ of bounded variation there exists $g \in \mathcal{L}^1(\mu, X)$

such that $G(E) = \int_E g d\mu$ , for all $E \in \Sigma$.   (2.1)

*(iii)* A Banach space $X$ has the **Radon-Nikodym property** if $X$ has the Radon-Nikodym property with respect to every finite measure space.

*(iv)* A bounded linear operator $T: L_1(\mu) \to X$ is **Reisz representable** (or simply **representable**) if there exists $g \in \mathcal{L}^\infty(\mu, X)$ such that

$Tf = \int_\Omega f g d\mu$ for all $f \in \mathcal{L}^1(\mu)$   (2.2)

*Source: Diestel, Uhl Section III-1. Definiton 3 (page 61)*

According to previous examples, the space $c_0$ does not have the Radon-Nikodym property and $\mathcal{L}^1(\mu)$ does not have the Radon-Nikodym property when $\mu$ has no atoms. More generally, if $\mu$ is not purely atomic, then $\Omega$ contains a subset $\Omega_0$ such that $\mu|_{\Omega_0}$, we obtain an example of an $\mathcal{L}^1(\mu)$-valued measure without a derivative. Therefore, whenever $\mu$ is not purely atomic, the space $\mathcal{L}^1(\mu)$ does not have the Radon-Nikodym property. On the other hand, if $(\Omega, \Sigma, \mu)$ is purely atomic, then every Banach space has the Radon Nikodym property with respect to $(\Omega, \Sigma, \mu)$.

The fundamental connection between representable operators on $\mathcal{L}^1(\mu)$ and vector measures with Radon-Nikodym derivatives is contained in the following straightforward lemma.

**Lemma 2.4.** *Source: Diestel, Uhl Section III-1. Lemma 4 (page 61)*

Let $T: \mathcal{L}^1(\mu) \to X$ be a bounded linear operator. For $E \in \Sigma$, define $G(E)$ by

$G(E) = T(\mathbb{1}_E)$.

Then $T$ is representable if and only if there exists $g \in \mathcal{L}^\infty(\mu, X)$ such that

$G(E) = \int_E g d\mu$

for all $E \in \Sigma$. In this case, the function $g \in \mathcal{L}^\infty(\mu, X)$ is such that

$$T(f) = \int_{\Omega} fg d\mu$$

for all $f \in L^1(\mu)$. Moreover, $||g||_{\infty} = ||T||$.
*Source: Diestel, Uhl Section III-1. Lemma 3 (page 62)*

*Outline of Proof.* If $T$ is representable, then there exists $g \in \mathcal{L}^{\infty}(\mu, X)$ such that $(2.2)$ holds, for all $E \in \Sigma$, we have $\mathbb{1}_E \in \mathcal{L}^1(\mu)$ and we find that

$$G(E) = T(\mathbb{1}_E) = \int \mathbb{1}_E g d\mu = \int_E g d\mu.$$

If conversely there exists $g \in \mathcal{L}^{\infty}(\mu, X)$ such that $G(E) = \displaystyle\int_E g d\mu$, we find by linearity of $T$ that $(2.2)$ holds for simple functions and $(2.2)$ holding more generally immediately follows by the **Dominated Convergence Theorem** (for vector valued functions). □

**Theorem 2.5.** Let $X$ be a Banach space and $(\Omega, \Sigma, \mu)$ be a finite measure space. Then $X$ has the Radon-Nikodym property with respect to $(\Omega, \Sigma, \mu)$ if and only if each $T \in L\left(\mathcal{L}^1(\mu); X\right)$ is representable.
*Source: Diestel, Uhl Section III-1. Theorem 5 (page 63)*

*Outline of proof.*

$\Longrightarrow$ We find that given $T \in L\left(L^1(\mu); X\right)$, we find that $F_T : \Sigma \to X$ is a $\mu$-continuous vector-measure, and it follows by the Radon Nikodym property that there exists $g \in \mathcal{L}^{\infty}(\mu, X)$ such that $(2.1)$ holds, and it follows by **Lemma 2.4** that $T$ is representable.

$\Longleftarrow$ Given a $\mu$-continuous vector measure $G : \Sigma \to X$ of bounded variation, we find that $T_G \in L\left(\mathcal{L}^1(\mu); X\right)$, hence $T_G$ is representable, and it follows by **Lemma 2.3** that $(2.1)$ holds, and we conclude that $X$ has the Radon-Nikodym property with respect to $(\Omega, \Sigma, \mu)$. □

**Theorem 2.6.** If $X$ has a boundedly complete Schauder basis $\{x_n\}_{n \in \mathbb{N}}$, then $X$ has the Radon-Nikodym property.
*Source: Diestel, Uhl Section III-1. Theorem 6 (page 64)*

*Outline of proof.* It shall suffice to prove that $X$ is representable. Choose a boundedly complete Schauder basis $\{x_n\}_{n \in \mathbb{N}}$, and define $x_n^* \in X^*$ by

$$x_n^*\left(\sum_{k\in\mathbb{N}}\alpha_k x_k\right) = \alpha_n$$

Let $T \in L\big(\mathfrak{L}^1(\mu); X\big)$. Given $f \in \mathfrak{L}^1(\mu)$, we find by the (scalar) Riesz Representation Theorem, we find that for every $k \in \mathbb{N}$, there exists $g_k \in \mathfrak{L}^\infty(\mu)$ such that

$$x^*T(f) = \int_\Omega f g_k d\mu.$$

We define $g : \Omega \to X$ such that

$$g(\omega) := \sum_{k\in\mathbb{N}} g_k(\omega)x_k,$$

and find that $g \in \mathfrak{L}^\infty(\mu; X)$. Then for each $E \in \Sigma$, we find that

$$T(\mathbb{1}_E) = \sum_{k\in\mathbb{N}} x_k^* T(\mathbb{1}_E) = \sum_{k\in\mathbb{N}}\left(\int_E g_k d\mu\right)x_k = \int_E g d\mu,$$

and our conclusion follows by **Lemma 2.4**. ◻

**Theorem 2.7.** A Banach space $X$ has the Radon-Nikodym property with respect to $(\Omega, \Sigma, \mu)$ if and only if every bounded linear operator $T : \mathfrak{L}^1(\mu) \to X$ admits a factorization $T = LS$.

where $L : \ell_1 \to X$ and $S : \mathfrak{L}^1(\mu) \to \ell_1$ are continuous linear operators.

In this case, for each $\epsilon > 0$, $L, S$ can be chosen such that $||S|| \leq ||T|| + \epsilon$ and $||L|| \leq 1$.
*Source: Diestel, Uhl Section III-1. Theorem 8 (page 66)*

We shall prove this in the next draft.

**Definition 2.8.** Let $\mathfrak{B}$ be a sub-$\sigma$-field of $\Sigma$ and $f \in \mathfrak{L}^1(\mu, X)$. An element $g$ of $\mathfrak{L}^1(\mu, X)$ is called **the conditional expectation of f** relative to $\mathfrak{B}$ if $g$ is $\mathfrak{B}$-measurable and

$$\int_E g d\mu = \int_E f d\mu \text{ for all } E \in \mathfrak{B}. \quad (2.3)$$

In this case $g$ is denoted by $\mathbb{E}[f|\mathcal{B}]$.
*Source: Diestel, Uhl Section IV-1. Definition 1 (page 121-122)*

It is clear that $\mathbb{E}[f|\mathcal{B}]$ is uniquely defined whenever it is defined. It is not clear that $\mathbb{E}[f|\mathcal{B}]$ is defined for all $f \in \mathcal{L}^1(\mu, \mathcal{X})$ (and for that matter any $\mathcal{X}$ whether it has the Radon-Nikodym property or not). Of course, if $\mathcal{X}$ has the Radon-Nikodym property, then it's more obvious to show that $\mathbb{E}[f|\mathcal{B}]$ exists, which we'll do for the following lemma:

**Lemma 2.9.** If $\mathcal{X} := \mathbb{C}$, then $\mathbb{E}[f|\mathcal{B}]$ exists.
*Source: Diestel, Uhl Section IV-1. Lemma 3 (page 122)*

*Proof.* Given $f \in \mathcal{L}^1(\mu, \mathcal{X})$, note that the measure $\nu := \int_{(-)} f d\mu : \mathcal{B} \to \mathcal{X}$ is absolutely continuous with respect to the restriction measure $\mu|\mathcal{B} : \mathcal{B} \to [0, +\infty)$, i.e., for every $A \in \mathcal{B}$, we have

$$(\mu|\mathcal{B})(A) = 0 \Longrightarrow \int_A f d\mu = 0.$$

Since $\mathcal{X}$ has the Randon-Nikodym property (as per the **Radon-Nikodym Theorem**), we have some $g \in \mathcal{L}^1(\mu|\mathcal{B}, \mathcal{X})$ such that

$$\int_A f d\mu = \nu(A) = \int_A g d\mu, \text{ for all } E \in \mathcal{B},$$

which satisfies (2.3), and we conclude that $g = \mathbb{E}[f|\mathcal{B}]$. $\square$

**Proposition 2.10.** $\mathbb{E}[\,\cdot\,|\mathcal{B}]$ is a contractive (and hence bounded) linear operator on the set of all $f \in \mathcal{L}^1(\mu, \mathcal{X})$ such that $\mathbb{E}[f|\mathcal{B}]$ exists.

More generally, we have the following properties for $f, g \in \mathcal{L}^1(\mu, \mathcal{X})$, $a \in \mathbb{C}$, and $p \geq 1$ if $\mathbb{E}[f|\mathcal{B}]$ and $\mathbb{E}[g|\mathcal{B}]$ exist:

*(i)* $\mathbb{E}[\mathbb{E}[f|\mathcal{B}]|\mathcal{B}] = \mathbb{E}[f|\mathcal{B}]$, and more generally $\mathbb{E}[f|\mathcal{B}] = f$, if $f$ is $\mathcal{B}$-measurable.
*(ii)* $\mathbb{E}[0|\mathcal{B}] = 0$ and $\mathbb{E}[f + ag|\mathcal{B}] = \mathbb{E}[f|\mathcal{B}] + a\mathbb{E}[g|\mathcal{B}]$.
*(iii)* $\mathbb{E}[f|\mathcal{B}]$ is $\mu$-a.e. uniquely determined.
*(iv)* For $A \in \Sigma$, we find $\mathbb{E}[\mathbb{1}_A|\mathcal{B}]$ (often called the **conditional probability** with notation $\mathbb{P}[A|\mathcal{B}] := \mathbb{E}[\mathbb{1}_A|\mathcal{B}]$) exists and $0 \leq \mathbb{E}[\mathbb{1}_A|\mathcal{B}] \leq 1$--and more generally $0 \leq \mathbb{E}[\mathbb{1}_A|\mathcal{B}] \leq \mathbb{1}_A$ $\mu$-a.e.

*(v)* If $f$ is a countably valued simple function of the form $f := \sum_{k=1}^{\infty} x_k \mathbb{1}_{A_k}$ for a pairwise disjoint $\{A_k\} \subset \Sigma$, we have

$$\mathbb{E}[f|\mathcal{B}] = \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}]x_k. \quad (2.4)$$

*(vi)* $||\mathbb{E}[f|\mathcal{B}]||_p \leq ||f||_p$.

*(vii)* $\{f_n\} \subset \mathcal{L}^p(\mu, X)$ is a sequence such that $\mathbb{E}[f_n|\mathcal{B}]$ exists, for each $n \in \mathbb{N}$, and $f_n \xrightarrow{\mathcal{L}^p(\mu,X)} f$ as $n \to +\infty$, then $\mathbb{E}[f|\mathcal{B}]$ exists and $\mathbb{E}[f_n|\mathcal{B}] \xrightarrow{\mathcal{L}^p(\mu,X)} \mathbb{E}[f|\mathcal{B}]$ as $n \to +\infty$

*Source: Diestel, Uhl Section IV-1. Lemma 3-Theorem 4 (page 122-123)*

*Remark 2.2.* Note that there is a slight abuse of notation going on with the expected-value notation going on for the proofs of *parts (i)* and *(ii)*, since we do not prove that $\mathbb{E}[f|\mathcal{B}]$ is $\mu$-a.e. uniquely determined until *part (iii)* (we actually utilize *parts (i)* and *(ii)* to prove uniqueness), so to say something like "$\mathbb{E}[f|\mathcal{B}] = \mathbb{E}[g|\mathcal{B}]$" is to say that if there is some $h_f$ and $h_g$ such that that $(2.3)$ holds for $f$ and $g$ respectively, then it also holds for $g$ and $f$ respectively. When writing "$\mathbb{E}[f|\mathcal{B}]$" in any conditional probability equation, you can substitute it for some $h$ such that $(2.3)$ holds for $f$. Whether or not that function is $\mu$-a.e. uniquely determined doesn't matter for the proof of *parts (iv)* and *(v)*, though it does matter for the proof of *part (vi)* and *(vii)* (which uses *(vi)*), since we need the fact that the conditional probability of countably simple functions, can be expressed in terms of $(2.4)$, <u>no matter what the choice of the function</u>, to show that $||\mathbb{E}[f|\mathcal{B}]||_p \leq ||f||_p$.

*Proof.* Note that $\mathbb{E}[\,\cdot\,|\mathcal{B}]$ being a contractive linear operator follows immediately from properties *(ii)*, *(iii)*, *(vi)* and *(vii)* showing that $\mathbb{E}[\,\cdot\,|\mathcal{B}]$ is a well-defined contractive linear operator on the set of all $f \in \mathcal{L}^1(\mu, X)$ such that $\mathbb{E}[f|\mathcal{B}]$ exists.

*(i)* Note that when $f$ is $\mathcal{B}$-measurable, we find $(2.3)$ holds trivially (since we have $\int_E f d\mu = \int_E f d\mu$), and $\mathbb{E}[\mathbb{E}[f|\mathcal{B}]|\mathcal{B}]] = \mathbb{E}[f|\mathcal{B}]$ immediately follows since $\mathbb{E}[f|\mathcal{B}]$ is $\mathcal{B}$-measurable.

*(ii)* $\mathbb{E}[0|\mathcal{B}] = 0$ follows immediately from *part (i)*, since $0$ is $\mathcal{B}$-measurable, and observe that $h := \mathbb{E}[f|\mathcal{B}] + a\mathbb{E}[g|\mathcal{B}]$ is a $\mathcal{B}$-measurable $\mathcal{L}^1(\mu, X)$-function such that for every $E \in \mathcal{B}$, we have

$$\int_E f + ag d\mu = \int_E f d\mu + a\int_E g d\mu = \int_E \mathbb{E}[f|\mathcal{B}]d\mu + a\int_E \mathbb{E}[g|\mathcal{B}]d\mu = \int_E h d\mu.$$

*(iii)* First, we prove that $\mathbb{E}[0|\mathcal{B}]$ is $\mu$-a.e. uniquely determined. Given $h$ such that $(2.3)$ holds for $f = 0$, we find that

$$\int_E (h-0)d\mu = \int_E hd\mu = \int_E 0d\mu = 0 \ \forall E \in \mathcal{B} \Longrightarrow h = 0 \ \mu\text{-a.e.}$$

*(See Folland § 2.2 Proposition 2.16 (page 51))*

Next, given $f \in \mathcal{L}^1(\mathcal{X}, \mu)$, and given $h_1, h_2 \in \mathcal{L}^1(\mathcal{X}, \mu)$ such that $h_1$ and $h_2$ are $\mathcal{B}$-measurable and $(2.3)$ holds for $f$, we find by part (ii) that

$$\mathbb{E}[0|\mathcal{B}] = \mathbb{E}[f|\mathcal{B}] - \mathbb{E}[f|\mathcal{B}] = h_1 - h_2,$$

and it immediately follows from the fact that $\mathbb{E}[0|\mathcal{B}] = 0$ is $\mu$-a.e. uniquely determined that $h_1 = h_2$ $\mu$-a.e.

*(iv)* Existence of $\mathbb{E}[\mathbb{1}_A|\mathcal{B}]$ follows immediately from **Lemma 2.9**, and it remains to show that $0 \le \mathbb{E}[\mathbb{1}_A|\mathcal{B}] \le \mathbb{1}_A$ (and hence $0 \le \mathbb{E}[\mathbb{1}_A|\mathcal{B}] \le 1$) $\mu$-a.e.. Given $A \in \Sigma$, first observe that we have

$$\int_{\mathbb{E}[f|\mathcal{B}]<0} \mathbb{E}[\mathbb{1}_A|\mathcal{B}]d\mu \le 0,$$

$$\int_{\mathbb{E}[f|\mathcal{B}]<0} \mathbb{E}[\mathbb{1}_A|\mathcal{B}]d\mu = \mu(A \cap \{\mathbb{E}[\mathbb{1}_A|\mathcal{B}] < 0\}) \ge 0,$$

and it follows that $\int_{\mathbb{E}[f|\mathcal{B}]<0} \mathbb{E}[\mathbb{1}_A|\mathcal{B}]d\mu = 0$, so $\mathbb{E}[\mathbb{1}_A|\mathcal{B}] \ge 0$ $\mu$-a.e. We additionally find that if $\mu(\mathbb{E}[\mathbb{1}_A|\mathcal{B}] > \mathbb{1}_A) > 0$, then

$$\begin{aligned}\mu(\mathbb{E}[\mathbb{1}_A|\mathcal{B}] > \mathbb{1}_A\}) = \int_{\mathbb{E}[\mathbb{1}_A|\mathcal{B}]>\mathbb{1}_A} \mathbb{1}_A d\mu &= \int_{\mathbb{E}[\mathbb{1}_A|\mathcal{B}]>\mathbb{1}_A} \mathbb{E}[\mathbb{1}_A|\mathcal{B}]d\mu \\ &> \int_{\mathbb{E}[\mathbb{1}_A|\mathcal{B}]>\mathbb{1}_A} \mathbb{1}_A d\mu = \mu(\mathbb{E}[\mathbb{1}_A|\mathcal{B}] > \mathbb{1}_A),\end{aligned}$$

and we reach a contradiction. Therefore, we find $\mathbb{E}[\mathbb{1}_A|\mathcal{B}] \le \mathbb{1}_A$ $\mu$-a.e. and our conclusion is reached.

*(v)* Note that

$$h := \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}]x_k,$$

is well-defined and $\mathcal{B}$-measurable, since we find by *part (iv)* that $\mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}]$ exists and $\mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}] \leq \mathbb{1}_A$, resulting in pointwise convergence of $\sum_{k=1}^{\infty} \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}]x_k$. Next, observe that for every $E \in \mathcal{B}$, we have

$$\int_E f d\mu = \sum_{k=1}^{\infty} \left[ \left( \int_E \mathbb{1}_{A_k} d\mu \right) x_k \right] = \sum_{k=1}^{\infty} \left[ \left( \int_E \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}] d\mu \right) x_k \right] = \int_E h d\mu,$$

therefore $(2.3)$ holds, and we conclude that $h = \mathbb{E}[f|\mathcal{B}]$.

*(vi)* First, Suppose $f$ is a countably-valued simple function, i.e., of the form $f := \sum_{k=1}^{\infty} x_k \mathbb{1}_{A_k}$ for a pairwise disjoint $\{A_k\} \subset \Sigma$. We find that since *part (iv)* shows us that
$0 \leq \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}] \leq \mathbb{1}_{A_k}( \leq 1)$ $\mu$-a.e., we have

$$\begin{aligned}
||\mathbb{E}[f|\mathcal{B}]||_p &= \left( \int \sum_{k=1}^{\infty} \left[ \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}]^p ||x_k||^p \right] d\mu \right)^{1/p} \\
&\leq \left( \int \sum_{k=1}^{\infty} \left[ \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}] ||x_k||^p \right] d\mu \right)^{1/p} \\
&= \left( \sum_{k=1}^{\infty} \left[ \left( \int \mathbb{E}[\mathbb{1}_{A_k}|\mathcal{B}] d\mu \right) ||x_k||^p \right] \right)^{1/p} \\
&= \left( \sum_{k=1}^{\infty} \left[ \left( \int \mathbb{1}_{A_k} d\mu \right) ||x_k||^p \right] \right)^{1/p} \\
&= \left( \int \sum_{k=1}^{\infty} \left[ \mathbb{1}_{A_k}^p ||x_k||^p \right] d\mu \right)^{1/p} \\
&= ||f||_p,
\end{aligned}$$

and we've shown $||\mathbb{E}[f|\mathcal{B}]||_p \leq ||f||_p$ holds in the special case where $f$ is a countably valued simple function. Next, choose a sequence $\{f_n\}_{n\in\mathbb{N}}$ countably valued simple function $f_n := \sum_{k=1}^{\infty} x_{n,k}\mathbb{1}_{A_{n,k}}$function such that $||f_n||_p \leq ||f||_p$ with $\mu(A_{n,k}) > 0$ converging essentially uniformly to $f$. For each $f_n$. We find by *part (iv)* and the desired property holding for coutably valued simple functions (which include $f_n - f_m$ for $n, m \in \mathbb{N}$) that for $h_{n,m} := \mathbb{E}[f_n|\mathcal{B}] - \mathbb{E}[f_m|\mathcal{B}]$ we have

$$h_{n,m} = \mathbb{E}[f_n - f_m|\mathcal{B}],\ ||h_{n,m}||_p \leq ||f_n - f_m||_p,\ n, m \in \mathbb{N},$$

hence $\{\mathbb{E}[f_n|\mathcal{B}]\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{L}^p(\mu, X)$, and it follows that $g := \lim_{n\to+\infty} \mathbb{E}[f_n|\mathcal{B}]$ exists in $\mathcal{L}^p(\mu, X)$, and we conclude that

$$\int f d\mu = \lim_{n\to+\infty} \int f_n d\mu = \lim_{n\to+\infty} \int \mathbb{E}[f_n|\mathcal{B}]d\mu = \int g d\mu,$$

hence $g = \mathbb{E}[f|\mathcal{B}]$ we conclude that

$$||\mathbb{E}[f|\mathcal{B}]||_p = \lim_{n\to+\infty} ||\mathbb{E}[f_n|\mathcal{B}]||_p \leq \lim_{n\to+\infty} ||f_n||_p = ||f||_p.$$

*(vii)* $\{f_n\} \subset \mathcal{L}^p(\mu, X)$ is a sequence such that $\mathbb{E}[f_n|\mathcal{B}]$ exists, for each $n \in \mathbb{N}$, and $f_n \xrightarrow{\mathcal{L}^p(\mu,X)} f$ as $n \to +\infty$. Note by *(ii)* and *(vi)*, we find that for $n, m \in \mathbb{N}$

$$||\mathbb{E}[f_n|\mathcal{B}] - \mathbb{E}[f_m|\mathcal{B}]||_p = ||\mathbb{E}[f_n - f_m|\mathcal{B}]||_p \leq ||f_n - f_m||_p,$$

and it immediately follows that $\{\mathbb{E}[f_n|\mathcal{B}]\}$ is Cauchy in $\mathcal{L}^p(X, \mu)$, hence $g := \lim_{n\to+\infty} \mathbb{E}[f_n|\mathcal{B}]$ exists in $\mathcal{L}^p(X, \mu)$ and hence exists in $\mathcal{L}^1(X, \mu)$ (as does $f$ as a limit of $\{f_n\}$ in $\mathcal{L}^1(X, \mu)$), is $\mathcal{G}$-measurable as a $\mathcal{L}^0(X, \mu)$-limit of $\mathcal{G}$-measurable functions and we find that for all $E \in \mathcal{B}$

$$\int_E g d\mu = \lim_{n\to+\infty} \int_E \mathbb{E}[f_n|\mathcal{B}]d\mu = \lim_{n\to+\infty} \int_E f_n d\mu = \int_E f d\mu,$$

and we've shown that $g$ satisfies the definition of $\mathbb{E}[f|\mathcal{B}]$. $\square$

**Example 2.11.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of the disjoint sets in $\Sigma$ with $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\mu(A_n) > 0$ for all $n \in \mathbb{N}$. Let $B$ be the $\sigma$-field of all unions of members of $(A_n)_{n\in\mathbb{N}}$. If $f \in \mathfrak{L}^1(\mu; X)$ a straightforward check verifies the equality

$$\mathbb{E}[f|B] = \sum_{n=1}^{\infty} \frac{\int_{A_n} f d\mu}{\mu(A_n)} \mathbb{1}_{A_n}.$$

*Source: Diestel, Uhl Section IV-1. Example 2 (page 122)*

**Theorem 2.12.** Let $\mathcal{B}$ be a $\sigma$-subfield of $\Sigma$. Then $\mathbb{E}[f|\mathcal{B}]$ exists for every $f \in \mathfrak{L}^1(\mu, X)$. In fact if $f \in \mathfrak{L}^p(\mu, X)$ (($1 \leq p < \infty$), then

$$||\mathbb{E}[f|\mathcal{B}]||_p \leq ||f||_p. \quad (2.5)$$

Consequently, $\mathbb{E}[\cdot|\mathcal{B}]$ is a linear contractive projection on $\mathfrak{L}^p(\mu, X)$, $1 \leq p < \infty$.
*Source: Diestel, Uhl Section IV-1. Theorem 4 (page 123)*

*Proof.* **Proposition 2.10** *(v)* establishes existence for simple functions of the form countably-valued simple function, i.e., of the form $f := \sum_{k=1}^{\infty} x_k \mathbb{1}_{A_k}$ for a pairwise disjoint $\{A_k\} \subset \Sigma$. For any $f \in \mathfrak{L}^1(\mu, X)$, we can choose a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{L}^1(\mu, X)$ of simple functions such that $f_n \xrightarrow{\mathfrak{L}^p(\mu,X)} f$ as $n \to +\infty$, and **Proposition 2.10** *(vii)* then establishes existence of $\mathbb{E}[f|\mathcal{B}]$. Finally, **Proposition 2.10** *(vi)* immediately establishes (2.5) and hence that $\mathbb{E}[\cdot|\mathcal{B}]$ is contractive, and **Proposition 2.10** *(i)-(iii),* and *(vii)* establish that it's a linear projective mapping. □

**Sources:**


*Real Analysis, Modern Techniques, § 2.2*
*Folland*

*Vector Measures Section I-2. III-1., IV-1.*
*Diestel, Uhl*

## 3 Definitions of SDE's and solutions (in Hilbert Spaces)

We will consider the following liner equation

$$\left.\begin{array}{l} dX(t) = [AX(t) + f(t)]dt + BdW(t), \\ X(0) = \xi, \end{array}\right\} \quad (3.1)$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{U} \to \mathcal{H}$ are linear operators, $f$ is an $\mathcal{H}$-valued stochastic process.

We will assume that the deterministic Cauchy problem

$$u'(t) = Au(t),\ u(0) = x \in \mathcal{H}$$

is uniformly well-posed (see *Da Prato, Zabczyk Appendix A (page )*) and that $B$ is bounded, that is, we have the following:

*Hypothesis 3.1.*

*(i)* $A$ generates a strongly continuous semigroup $S(\cdot)$ in $\mathcal{H}$,
*(ii)* $B \in L(\mathcal{U}; \mathcal{H})$.
*Source: Da Prato, Zabczyk § 5.1.1 Hypothesis 5.1 (page 122)*

It is also natural to require the following:

*Hypothesis 3.2.*

*(i)* $f$ is a predictable process with Bochener integrable trajectories on arbitrary finite interval $[0, T]$ ,
*(ii)* $\xi$ is $\mathcal{F}_0$-measurable.
*Source: Da Prato, Zabczyk § 5.1.1 Hypothesis 5.1 (page 122)*

*Remark 3.1.* We can assume, without any restriction, that $\mathcal{U} = \mathcal{H}$. However in some applications, for example the wave or delay equations, it is convenient to have $B$ different from identity

An $\mathcal{H}$-valued predictable process $X(t),\ t \in [0, T],$ is said to be a **strong solution** to $(3.1)$ if $X$ takes values in $D(A)$, $\mathbb{P}_T$-a.s.,

$$\int_0^T |AX(s)|ds = x + \int_0^t [AX(s) + f(s)]ds + BW(s), \quad \mathbb{P}\text{-a.s.}$$

This definition is meaningful only if $BW$ is a $\mathcal{U}$-valued process and therefore requires that $\mathrm{Tr}\left(BQB^*\right) < +\infty$. Note that a strong solution should necessarily have a continuous modification.

An $\mathcal{H}$-valued predictable process $X(t),\ t \in [0,T],$ is said to be a **weak solution** of $(3.1)$ if the trajectories of $X(\cdot)$ are $\mathbb{P}$-a.s. Bochner integrable and if for all $\xi \in D\left(A^*\right)$ and all $t \in [0,T]$, we have

$$\langle X(t), \xi\rangle = \langle x, \xi\rangle + \int_0^t \left[\langle X(s), A^*\xi\rangle + \langle f(s), \xi\rangle\right]ds + \langle BW(t), \xi\rangle, \ \mathbb{P}\text{-a.s.}$$

*Source: Da Prato, Zabczyk § 5.1.1 Hypothesis 5.1 (page 122)*

This definition is meaningful for cylindrical Wiener process because the scalar process $\langle BW(t), \xi\rangle,\ t \in [0,T]$, are well-defined random processes (see *Da Prato, Zabczyk § 5.1.1*).

**Theorem 3.1.** Assume Hypothesis 3.1, 3.2, and $(3.2)$. Then equation $(3.1)$ has exactly one weak solution which is given by the formula

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)BdW(s), \quad t \in [0,T] \quad (3.4)$$

*Source: Da Prato, Zabczyk § 5.2 Theorem 5.4 (page 125)*

**Theorem 3.3.** Assume that

(i) $Q^{1/2}(H) \subset D(A)$ and $AQ^{1/2}$ is a Hilbert-Schmidt operator,
(ii) $x \in D(A)$ and $f \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A))$.

Then problem $(3.1)$ has a strong solution.
*Source: Da Prato, Zabczyk § 5.6 Theorem 5.38 (page 156)*

**Sources:**

*Stochastic Equations in Infinite Dimensions § 5.1, § 5.2, § 5.6*
*Da Prato, Zabczyk*