M800 Roger Temam 4/22 Report

1 Levy-Khinchin Decomposition

Let W be a $\mathcal{U}$-valued Wiener process.

Proposition 1.1. $W(t)$ is $\mathcal{L}^2(\mathbb{P};\mathcal{U})$, for all, i.e. we have $\mathbb{E}||W(t)||_{\mathcal{U}}^2 < +\infty$.
Source: M647 Lecture 2 (revised), Theorem 2.2.4 (i).

Outline of Proof. We shall proceed with a similar proof to that of *Proposition 1.4* of the *4/15 Report*, nothing that for every $n \geq 1$, we find by the mean-zero and independent increments conditions that

$$\begin{aligned}\mathbb{E}||W(t)||_{\mathcal{U}}^2 &= \text{Var}(W(t)) = \sum_{j=1}^{n} \text{Var}\left(W\left(\frac{jt}{n}\right) - W\left(\frac{(j-1)t}{n}\right)\right) \\ &= \sum_{j=1}^{n} \text{Var}\mathbb{E}\left|\left|W\left(\frac{j \cdot t}{n}\right) - W\left(\frac{(j-1)t}{n}\right)\right|\right|_{\mathcal{U}}^2 \\ &= n\mathbb{E}\left|\left|W\left(\frac{t}{n}\right)\right|\right|_{\mathcal{U}}^2\end{aligned}$$

giving us

$$\mathbb{E}||W(t)||_{\mathcal{U}}^2 = +\infty \Longrightarrow \forall n \geq 1\left(\mathbb{E}\left|\left|W\left(\frac{t}{n}\right)\right|\right|_{\mathcal{U}}^2 = +\infty\right).$$

This will then lead to a similar contradiction of the stochastic continuity and $W(0) = 0$ conditions to the one given in the aforemented proof of *Proposition 1.4* of the *4/15 Report*. ◻

Proposition 1.2. the random vector $(\langle W(t_1), u_1\rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n\rangle_{\mathcal{U}})$ has the multivariate normal distribution on $\mathbb{R}^n$, with mean zero. That is, $\forall \Gamma \in \mathcal{B}\left(\mathbb{R}^n\right)$

$$\mathbb{P}[(\langle W(t_1), u_1\rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n\rangle_{\mathcal{U}}) \in \Gamma] = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \int_{\Gamma} e^{-1/2y^T\Sigma^{-1}y} dy \quad (1.1)$$

Here Σ is the $n \times n$ covariance matrix given by:

$\Sigma_{i,j} := \mathbb{E}[\langle W(t_i), u_i \rangle_{\mathcal{U}} \langle W(t_j), u_j \rangle_{\mathcal{U}}] \quad (1.2)$
Source: M647 Lecture 2 (revised), Theorem 2.2.4 (ii).

Outline of the proof. This result can be derived by applying the independent increments in time invariance property to each component $\langle W(t_i), u_i \rangle_{\mathcal{U}}$ to a uniform partition of increments for each $n \geq 1$, i.e.we get that

$$\langle W(t_i), u_i \rangle_{\mathcal{U}} = \sum_{k=1}^{m} \left\langle W\left(\frac{kt_i}{m}\right) - W\left(\frac{(k-1)t_i}{m}\right), u_i \right\rangle_{\mathcal{U}},$$

$$\left\langle W\left(\frac{kt_i}{m}\right) - W\left(\frac{(k-1)t_i}{m}\right), u_i \right\rangle_{\mathcal{U}} \sim \left\langle W\left(\frac{t_i}{m}\right), u_i \right\rangle_{\mathcal{U}} \text{ i.i.d.}$$

We then take $m \to +\infty$ and find by the Central Limit Theorem that $(\langle W(t_1), u_1 \rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n \rangle_{\mathcal{U}})$ is component-wise normal, which we know from the properties of Gaussian (i.e. multivariate normal) distributions is equivalent to a Gaussian distribution. We moreover know that such a Gaussian distribution is uniquely deterned by (1.1), with the covariance matrix specified in (1.2). □

Definition 1.3. Let $Q : \mathcal{U} \to \mathcal{U}$ be a bounded, linear, symmetric, positive, operator and of trace class on $\mathcal{U}$, then $Q \in L_1^+(\mathcal{U})$.

(Trace class: $\exists$ sequences $(a_k), (b_k) \subset \mathcal{U}$ such that $Qu = \sum_k \langle a_k, u \rangle b_k$, $\forall u \in U$ and such that $\sum_k ||a_k||_{\mathcal{U}} ||b_k||_{\mathcal{U}} < +\infty$).

Let $Q \in L_1^+(\mathcal{U})$. Then $\exists$ an orthonormal basis $(u_n)_{n=1}^{\infty}$ of U consisting of eigenvectors of Q with corresponding eigenvalues $\{\nu_n\}_{n=1}^{\infty}$. Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d) real-valued Brownian motions. Define W

$$(1.3) \qquad W(t) = \sum_{n=1}^{\infty} \sqrt{\nu_n} \beta_n(t) u_n.$$

Then, the sum in (1.3) converges in $L^2(\Omega; C([0,T],\mathcal{U}))$ $\forall T \geq 0$. The process W in (1.3) is a Levy process and its sample paths are continuous a.s. (Wiener process) For any Wiener process W, $\exists Q \in L_1^+(\mathcal{U})$, some i.i.d Brownian motion $\{\beta_n\}_{n=1}^{\infty}$ such that (1.3) holds.

This Q is called the **covariance operator of** W.
Source: M647 Lecture 2 (revised), Definition 2.2.5.

In a later draft of this report, we will show that this covariance operator is well-defined.

Proposition 1.4. Let P be a $\mathcal{U}$-valued CPP with intensity measure μ. If $\mathbb{E}||P(t)||_{\mathcal{U}} < +\infty$ $\forall t$ then

$\mathbb{E}[P(t)] = t\int_{\mathcal{U}} y d\mu(y)$ holds $\forall t \geq 0$.

We define,

$\widehat{P}(t) := P(t) - \mathbb{E}[P(t)]$.

We call $\widehat{P}$ a **compensated compound Poisson process (CCPP)**.

Outline of Proof. To be done in a future draft. □

- CCPP is a mean-zero Levy process, with jump-discontinuities.

Source: M647 Lecture 2 (revised), Definition 2.5.1.

Levy-Khinchin Decomposition: A Levy-process is a sum of a deterministic linear growth term, a Wiener process, a compound Poisson process and a series of compensated compound Poisson process.

Let L be a $\mathcal{U}$-valued Levy process. Let $A \in \mathcal{B}(\mathcal{U})$ such that $0 \in \overline{A}$. Define,

$$\pi_A(t) := \sum_{s\in(0,t]} \mathbb{1}_A(\Delta L(s)) = \#\{s \in (0,t] : \Delta L(s) \in A\}$$

Fact: π_A is a Poisson process.

Let $\nu(A)$ denote the intensity of π_A:

$$\nu(A) = \frac{1}{t}\mathbb{E}[\pi_A(t)] = \mathbb{E}[\pi_A(1)]$$

Definition 1.5. We call ν the **Levy measure of** L.
Source: M647 Lecture 2 (revised), Definition 2.5.3.

Lemma 1.6. Let L be a Levy process with Levy measure ν. Then, for $A \in \mathcal{B}(\mathcal{U})$ with $0 \in \overline{A}$,

$$L_A(t) := \sum_{s\in(0,t]} \Delta L(s)\mathbb{1}_A(\Delta L(s))$$

is a compound Poisson process with Levy measure $\nu|_A$.
Source: M647 Lecture 2 (revised), Lemma 2.5.4.

Outline of Proof. To be done in a future draft. □

A Levy process is a sum of a deterministic linear growth term, a Wiener process, a compound
Poisson process and a series of compensated compound Poisson processes.

Theorem 1.7. Let L be a $\mathcal{U}$-valued Levy process with Levy measure ν. Given $\{r_n\}_{n=1}^{\infty}$ $r_n \downarrow 0$, define $A_0 := \{y \in \mathcal{U} : ||y||_{\mathcal{U}} \geq r_0\}$ and $A_n := \{y \in \mathcal{U} : r_{n+1} \leq ||y||_{\mathcal{U}} < r_n\}$. Then the following statements hold:

(1) The CPP $\{L_{A_n}\}_{n=0}^{\infty}$ given by $L_{A_n}(t) = \sum_{s\in(0,t]} \Delta L(s)\mathbb{1}_{A_n}(\Delta L(s))$ are independent.
(2) There $\exists a \in \mathcal{U}$ and a Wiener process W that is independent of $\{L_{A_n}\}_{n=0}^{\infty}$ such that

$$L(t) = at + W(t) + L_{A_0}(t) + \sum_{n=1}^{\infty} \widehat{L}_{A_n}(t) \quad (1.4)$$

and, with probability 1, the series on the right hand side converges uniformly on compact subsets of $[0, +\infty)$.
Source: M647 Lecture 2 (revised), Definition 3.1.1. (NOTE: I will submit my revised notes containing this defintiion to supplement a future draft of this report)

Outline of Proof. To prove the claim that $\sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)$ on compact subsets, we suppose towards contradiction that $\sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)$ goes not converge uniformly on compact subsets and get a contradiction of Stochastic continuity of L.

Next, to prove (1.4), we want to show that

$$\widetilde{W}(t) := L(t) - \left[L_{A_0}(t) + \sum_{n=1}^{\infty} \widehat{L}_{A_n}(t) \right] = at + W(t)$$

for some $a \in \mathcal{H}$ by taking $a := \mathbb{E}\left[\widetilde{W}(1)\right]$ and showing that $W(t) := \widetilde{W}(t) - at$ is a Wiener process. We do this as follows:

(i) Noting by noting by *Proposition 1.4* of the *4/15 Report* that $W(t)$ is integrable.

(ii) Using *Theorem 1.3* of the *4/15 Report* (i.e. that $L(t)$ has a Cadlag modification $\widetilde{L}(t)$) and the deduction of jump discontinuities from $L(t)$ to $\widetilde{W}(t)$ to show that $\widetilde{W}(t)$ has a.s. continuous paths,

(iii) Prove in a later draft of this report that $\mathbb{E}[L(t)] = \mathbb{E}[L(1)]t$, giving us

$$\mathbb{E}\left[\widetilde{W}(t) - at\right] = \mathbb{E}\left[\widetilde{W}(t)\right] - at = 0. \quad \square$$

Remark 1.1:

(i) This decomposition helps identify the nump and quadratic variation which is needed to apply the Ito formula.

(ii) Informally, $dL = adt + dW + d\pi + d\widehat{\pi}$, where π is a Poisson random measure (called the jump measure of L).

Sources:

Spring 2022-M647 Lecture 2-3

2 Ito's Lemma and Formula

I will add to my previous exposition of this section in a future draft.