M800 Roger Temam 4/22 Report

1 Levy-Khinchin Decomposition

Let W be a $\mathcal{U}$-valued Wiener process.

Proposition 1.1. $W(t)$ is $\mathcal{L}^2(\mathbb{P};\mathcal{U})$, for all, i.e. we have $\mathbb{E}||W(t)||_{\mathcal{U}}^2 < +\infty$.
Source: M647 Lecture 2 (revised), Theorem 2.2.4 (i).

Outline of Proof. We shall proceed with a similar proof to that of *Proposition 1.4* of the *4/15 Report*, nothing that for every $n \geq 1$, we find by the mean-zero and independent increments conditions that

$$\begin{aligned}\mathbb{E}||W(t)||_{\mathcal{U}}^2 = \text{Var}(W(t)) &= \sum_{j=1}^{n} \text{Var}\left(W\left(\frac{jt}{n}\right) - W\left(\frac{(j-1)t}{n}\right)\right) \\ &= \sum_{j=1}^{n} \text{Var}\mathbb{E}\left|\left|W\left(\frac{j\cdot t}{n}\right) - W\left(\frac{(j-1)t}{n}\right)\right|\right|_{\mathcal{U}}^2 \\ &= n\mathbb{E}\left|\left|W\left(\frac{t}{n}\right)\right|\right|_{\mathcal{U}}^2\end{aligned}$$

giving us

$$\mathbb{E}||W(t)||_{\mathcal{U}}^2 = +\infty \Longrightarrow \forall n \geq 1\left(\mathbb{E}\left|\left|W\left(\frac{t}{n}\right)\right|\right|_{\mathcal{U}}^2 = +\infty\right).$$

This will then lead to a similar contradiction of the stochastic continuity and $W(0) = 0$ conditions to the one given in the aforemented proof of *Proposition 1.4* of the *4/15 Report*. □

Proposition 1.2. the random vector $(\langle W(t_1), u_1\rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n\rangle_{\mathcal{U}})$ has the multivariate normal distribution on $\mathbb{R}^n$, with mean zero. That is, $\forall \Gamma \in \mathcal{B}(\mathbb{R}^n)$

$$\mathbb{P}[(\langle W(t_1), u_1\rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n\rangle_{\mathcal{U}}) \in \Gamma] = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \int_{\Gamma} e^{-1/2y^T\Sigma^{-1}y} dy \quad (1.1)$$

Here Σ is the $n \times n$ covariance matrix given by:

$\Sigma_{i,j} := \mathbb{E}[\langle W(t_i), u_i \rangle_{\mathcal{U}} \langle W(t_j), u_j \rangle_{\mathcal{U}}] \quad (1.2)$
Source: M647 Lecture 2 (revised), Theorem 2.2.4 (ii).

Outline of the proof. This result can be derived by applying the independent increments in time invariance property to each component $\langle W(t_i), u_i \rangle_{\mathcal{U}}$ to a uniform partition of increments for each $n \geq 1$, i.e.we get that

$$\langle W(t_i), u_i \rangle_{\mathcal{U}} = \sum_{k=1}^{m} \left\langle W\left(\frac{kt_i}{m}\right) - W\left(\frac{(k-1)t_i}{m}\right), u_i \right\rangle_{\mathcal{U}},$$

$$\left\langle W\left(\frac{kt_i}{m}\right) - W\left(\frac{(k-1)t_i}{m}\right), u_i \right\rangle_{\mathcal{U}} \sim \left\langle W\left(\frac{t_i}{m}\right), u_i \right\rangle_{\mathcal{U}} \text{ i.i.d.}$$

We then take $m \rightarrow +\infty$ and find by the **Central Limit Theorem** that $(\langle W(t_1), u_1 \rangle_{\mathcal{U}}, \ldots, \langle W(t_n), u_n \rangle_{\mathcal{U}})$ is component-wise normal, which we know from the properties of Gaussian (i.e. multivariate normal) distributions is equivalent to a Gaussian distribution. We moreover know that such a Gaussian distribution is uniquely deternined by (1.1), with the covariance matrix specified in (1.2). $\square$

Definition 1.3. Let $Q : \mathcal{U} \rightarrow \mathcal{U}$ be a bounded, linear, symmetric, positive, operator and of trace class on $\mathcal{U}$, then $Q \in L_1^+(\mathcal{U})$.

(Trace class: $\exists$ sequences $(a_k), (b_k) \subset \mathcal{U}$ such that $Qu = \sum_k \langle a_k, u \rangle b_k$, $\forall u \in U$ and such that $\sum_k ||a_k||_{\mathcal{U}} ||b_k||_{\mathcal{U}} < +\infty$).

Let $Q \in L_1^+(\mathcal{U})$. Then $\exists$ an orthonormal basis $(u_n)_{n=1}^{\infty}$ of U consisting of eigenvectors of Q with corresponding eigenvalues $\{\nu_n\}_{n=1}^{\infty}$. Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d) real-valued Brownian motions. Define W

$$(1.3) \qquad W(t) = \sum_{n=1}^{\infty} \sqrt{\nu_n} \beta_n(t) u_n.$$

Then, the sum in (1.3) converges in $L^2(\Omega; C([0,T], \mathcal{U}))$ $\forall T \geq 0$. The process W in (1.3) is a Levy process and its sample paths are continuous a.s. (Wiener process) For any Wiener process W, $\exists Q \in L_1^+(\mathcal{U})$, some i.i.d Brownian motion $\{\beta_n\}_{n=1}^{\infty}$ such that (1.3) holds.

This Q is called the **covariance operator of** W.
Source: M647 Lecture 2 (revised), Definition 2.2.5.

In a later draft of this report, we will show that this covariance operator is well-defined.

Proposition 1.4. Let P be a $\mathcal{U}$-valued CPP with intensity measure μ. If $\mathbb{E}||P(t)||_{\mathcal{U}} < +\infty$ $\forall t$ then

$\mathbb{E}[P(t)] = t\int_{\mathcal{U}} y d\mu(y)$ holds $\forall t \geq 0$.

We define,

$$\widehat{P}(t) := P(t) - \mathbb{E}[P(t)].$$

We call $\widehat{P}$ a **compensated compound Poisson process (CCPP)**.
Source: M647 Lecture 2 (revised), Definition 2.5.1.

Outline of Proof. Note that

$$P(t) := \sum_{k=1}^{N(t)} X_k,$$

where $\{X_k\}_{k=1}^{\infty}$ is the sequence waiting time between the change in iteration of $N(t)$, which we know are i.i.d. with the intensity measure μ as the distribution, i.e., we have $\mu_{X_k} = \mu$, for every $k \geq 1$. Since

$$\mathbb{E}[N(t)] = t, \ \mathbb{E}[X_1] = \int_{\mathcal{U}} y d\mu(y),$$

we find that

$$
\begin{aligned}
\mathbb{E}[P(t)] &= \mathbb{E}\left[\sum_{k=1}^{N(t)} X_k\right] \\
&= \int_{(0,+\infty)^{\mathbb{N}}\times\mathbb{N}} \sum_{k=1}^{n} x_k d\mu_{\left(\{X_k\}_{k=1}^{\infty}, N(t)\right)}\left(\{x_k\}_{k=1}^{\infty}, n\right) \\
&= \int_{(0,+\infty)^{\mathbb{N}}\times\mathbb{N}} \sum_{k=1}^{n} x_k d\left[\left(\bigotimes_{k=1}^{\infty}\mu_{X_k}\right)\otimes \mu_{N(t)}\right]\left(\{x_k\}_{k=1}^{\infty}, n\right) \\
&= \int_{\mathbb{N}} \int_{(0,+\infty)^{\mathbb{N}}} \sum_{k=1}^{n} x_k d\left(\bigotimes_{k=1}^{\infty}\mu_{X_k}\right)\left(\{x_k\}_{k=1}^{\infty}\right) d\mu_{N(t)}(n) \\
&= \int_{\mathbb{N}} \sum_{k=1}^{n} \left[\int_{(0,+\infty)^{\mathbb{N}}} x_k d\left(\bigotimes_{k=1}^{\infty}\mu_{X_k}\right)\left(\{x_k\}_{k=1}^{\infty}\right)\right] d\mu_{N(t)}(n) \\
&= \int_{\mathbb{N}} \sum_{k=1}^{n} \left[\int_{(0,+\infty)} x_k d\mu_{X_k}(x_k)\right] d\mu_{N(t)}(n) \\
&= \int_{\mathbb{N}} \sum_{k=1}^{n} \mathbb{E}[X_k] d\mu_{N(t)}(n) \\
&= \int_{\mathbb{N}} n\mathbb{E}[X_1] d\mu_{N(t)}(n) \\
&= \int_{\mathbb{N}} n d\mu_{N(t)}(n)\cdot \mathbb{E}[X_1] \\
&= \mathbb{E}[N(t)]\cdot\mathbb{E}[X_1] \\
&= t\int_{\mathcal{U}} y d\mu(y) \qquad \square
\end{aligned}
$$

- CCPP is a mean-zero Levy process, with jump-discontinuities.

Notation 1.1: For a cadlag function X we denote,

(i) The left limit of X at t by $X_{t^-} = X(t-) = \lim_{\epsilon\to 0^+} X(t-\epsilon)$,
(ii) And the size of jumps Δ by $\Delta X(t) = X(t) - X(t-)$.

Levy-Khinchin Decomposition: A Levy-process is a sum of a deterministic linear growth term, a Wiener process, a compound Poisson process and a series of compensated compound Poisson process.

Let L be a $\mathcal{U}$-valued Levy process. Let $A \in \mathcal{B}(\mathcal{U})$ such that $0 \in \overline{A}$. Define,

$$\pi_A(t) := \sum_{s\in(0,t]} \mathbb{1}_A(\Delta L(s)) = \#\{s \in (0,t] : \Delta L(s) \in A\}$$

Fact: π_A is a Poisson process. We shall prove this in our proof of **Lemma 1.6**.

Let $\nu(A)$ denote the intensity of π_A:

$$\nu(A) = \frac{1}{t}\mathbb{E}[\pi_A(t)] = \mathbb{E}[\pi_A(1)]$$

Definition 1.5. We call ν the **Levy measure of** L.
Source: M647 Lecture 2 (revised), Definition 2.5.3.

Lemma 1.6. Let L be a Levy process with Levy measure ν. Then, for $A \in \mathcal{B}(\mathcal{U})$ with $0 \in \overline{A}$,

$$L_A(t) := \sum_{s\in(0,t]} \Delta L(s)\mathbb{1}_A(\Delta L(s))$$

is a compound Poisson process with Levy measure $\nu|_A$.
Source: M647 Lecture 2 (revised), Lemma 2.5.4.

Outline of Proof. First, we want to show that π_A is a Poisson process. To show this, let

$$I_{n,k} := \left(\frac{kt}{n}, \frac{(k+1)t}{n}\right] (n \geq 1,\ 0 \leq k < n),$$

and note that for every $s \in (0,t]$ we find that

$$\mathbb{P}[\exists(s \in I_{n,k})(\Delta(L(s) \in A)] = \mathbb{P}[\exists(s \in I_{n,j})(\Delta(L(s) \in A)]$$

and, so the random variables $\mathbb{1}_{\exists(s\in I_{n,k})(\Delta(L(s)\in A)}$, for every $0 \leq k < n$ are i.i.d., hence

$$\sigma_{n,A}(t) := \#\{1 \leq k \leq n-1 : \exists(s \in I_{n,k})(\Delta(L(s) \in A)\}$$

has a binomial distribution. It follows that

$$\mathbb{P}[\pi_A(t) = j] \approx \mathbb{P}[\sigma_{n,A}(t) = j]$$

and hence $\pi_A(t)$ is the limit of a sequence of binomial distributions, and hence is a Poisson distribution.

Next, we show that the definition provided in *Example 1.5-2* of the *4/15 report*, i.e. we have

$$\mathbb{P}[L_A(t) \in \Gamma] = e^{-\nu(A)t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \nu^{*k}(\Gamma), \ \forall t \geq 0, \ \Gamma \in \mathcal{B}(\mathcal{U}) \cap \mathcal{P}(A), \quad (1.2)$$

by verifying an equivalent definition of a compound Poisson process (that I hope to talk about and verify as equivalent in **Appendix B** of my **M647 notes**) by identifying the i.i.d. random variables $\{X_n\}_{n=1}^{\infty}$ of the jump discontinuities of L that are in A, i.e. $X_n := \Delta L(T_n)$, where T_n is the nth (stopping-)time where L has a jump discontinuity. In order to do this, we need to do the following:

(i) Show that L almost surely has at most countably many jump discontinuities, or else we can formulate a contradiction of stochastic continuity.

(ii) Show that $\{X_n\}_{n=1}^{\infty}$ is i.i.d. It suffices to show that $L^{T_n}(t) := L(T_n + t) - L(T_n)$ have the same distribution, since doing so leads to the result that

$$X_1 = \Delta L^{T_1}(0) \sim \Delta L^{T_n}(0) = X_n. \ \square$$

A Levy process is a sum of a deterministic linear growth term, a Wiener process, a compound
Poisson process and a series of compensated compound Poisson processes.

Theorem 1.7. Let L be a $\mathcal{U}$-valued Levy process with Levy measure ν. Given $\{r_n\}_{n=1}^{\infty}$ $r_n \downarrow 0$, define $A_0 := \{y \in \mathcal{U} : ||y||_{\mathcal{U}} \geq r_0\}$ and $A_n := \{y \in \mathcal{U} : r_{n+1} \leq ||y||_{\mathcal{U}} < r_n\}$. Then the following statements hold:

(1) The CPP $\{L_{A_n}\}_{n=0}^{\infty}$ given by $L_{A_n}(t) = \sum_{s \in (0,t]} \Delta L(s) \mathbb{1}_{A_n}(\Delta L(s))$ are independent.

(2) There $\exists a \in \mathcal{U}$ and a Wiener process W that is independent of $\{L_{A_n}\}_{n=0}^{\infty}$ such that

$$L(t) = at + W(t) + L_{A_0}(t) + \sum_{n=1}^{\infty} \widehat{L}_{A_n}(t) \quad (1.4)$$

and, with probability 1, the series on the right hand side converges uniformly on compact subsets of $[0, +\infty)$.
Source: M647 Lecture 3 (revised), Definition 3.1.1. (NOTE: I will submit my revised notes containing this defintiion to supplement a future draft of this report)

Outline of Proof. To prove the claim that $\sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)$ on compact subsets, we suppose towards contradiction that $\sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)$ goes not converge uniformly on compact subsets and get a contradiction of Stochastic continuity of L.

Next, to prove (1.4), we want to show that

$$\widetilde{W}(t) := L(t) - \left[L_{A_0}(t) + \sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)\right] = at + W(t)$$

for some $a \in \mathcal{H}$ by taking $a := \mathbb{E}\left[\widetilde{W}(1)\right]$ and showing that $W(t) := \widetilde{W}(t) - at$ is a Wiener process. We do this as follows:

(i) Noting by noting by *Proposition 1.4* of the *4/15 Report* that $W(t)$ is integrable.

(ii) Using *Theorem 1.3* of the *4/15 Report* (i.e. that $L(t)$ has a Cadlag modification $\widetilde{L}(t)$) and the deduction of jump discontinuities from $L(t)$ to $\widetilde{W}(t)$ to show that $\widetilde{W}(t)$ has a.s. continuous paths,

(iii) Prove in a later draft of this report that $\mathbb{E}[L(t)] = \mathbb{E}[L(1)]t$, giving us

$$\mathbb{E}\left[\widetilde{W}(t) - at\right] = \mathbb{E}\left[\widetilde{W}(t)\right] - at = 0. \quad \square$$

Remark 1.2:

(i) This decomposition helps identify the nump and quadratic variation which is needed to apply the Ito formula.

(ii) Informally, $dL = adt + dW + d\pi + d\widehat{\pi}$, where π is a Poisson random measure (called the

jump measure of L).

Sources:

Spring 2022-M647 Lecture 2-3

2 Ito's Lemma and Formula

Assume that Φ is an L_2^0-valued process stochatically integrable in $[0,T]$, φ is an $\mathcal{H}$-valued predictable process Bochner integrable on $[0,T]$, $\mathbb{P}$-a.s., and $X(0)$ is an $\mathcal{F}_0$-measurable $\mathcal{H}$-valued random variable.

Lemma 2.1. The following process

$$X(t) = X(0) + \int_0^t \varphi(s)ds + \int_0^t \Phi(s)dW(s),\ t \in [0,T], \quad (2.1)$$

is well-defined.
Source: Da Prato, Zabczyk, Theorem 4.16 (page 105)
Proof. By definition of Bochner and stochastic integrability, respectively, we find that $\int_0^t \varphi(s)ds$ and $\int_0^t \Phi(s)dW(s)$ exist as a stochastic process, i.e. a $\mathcal{F}_t$-measurable random variable for every $t \in [0,T]$, since $\varphi(s), \Phi(s), W(s)$ are $\mathcal{F}_t$-measurable for all $s \in [0,t]$, so $\int_0^t \varphi(s)ds$ and $\int_0^t \Phi(s)dW(s)$ (as a limit of sums of $\mathcal{F}_t$-measurable functions) are $\mathcal{F}_t$-measurable. It immediately follows that $X(t)$ exists and is $\mathcal{F}_t$-measurable for every $t \in [0,T]$.
□

Theorem 2.2. *(Ito's Formula)* Let X be a process satisfying (2.1), and assume that the function $F : [0,T] \times \mathcal{H} \to \mathbb{R}$ and its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0,T] \times \mathcal{H}$. Under these conditions, $\mathbb{P}$-a.s., for all $t \in [0,T]$, we have

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) &+ \int_0^t \langle F_x(s, X(s)), \Phi(s)dW(s)\rangle \\ &+ \int_0^t [F_t(s, X(s)) + \langle F_x(s, X(s)), \varphi(s)\rangle] + \frac{1}{2}\mathrm{Tr}\Big[F_{xx}(s, X(s))\big(\Phi(s)Q^{1/2}\big)\big(\Phi(s)Q^{1/2}\big)^*\Big]ds, \end{aligned}$$

where Q is the covariance operator of W.

Source: Da Prato, Zabczyk, Theorem 4.17 (page 105)
%SEE WHAT HAPPENS WHEN F is a vector-valued function

In this draft, we shall focus on the case where $X(t) := B_t$, where B_t is a finite n-dimensional Brownian motion (i.e., we have $\mathcal{H} = \mathbb{R}^n$ for some $n \geq 1$). In a later draft, we will dry to generalize this proof to the more general case given in the theorem.

Remark 2.1: Recall that the **quadratic covariation** $[X, Y]_t$ is defined as

$$[X, Y]_t = \lim_{||P|| \to 0} \sum_{k=1}^{n} [(X(t_k) - X(t_{k-1}))(Y(t_k) - Y(t_{k-1}))] \text{ ,}$$

$$(P := \{t_0 := 0 < t_1 < \cdots < t_{n_p - 1} < t_{n_P - 1} := t\},\ ||P|| := \min\{t_{k_i} - t_{k_{i-1}} : 1 \leq i \leq p\})$$

and the **quadratic variation** is defined as

$$[X]_t := [X, X]_t$$

Remark 2.2: Note that this proof outlines utilize **stochastic differentials**, in particular dt (noting that stochastic differentials generalize deterministic differentials) and dB_t. In a future report (hopefully the upcoming 7/22 report), I will define stochastic differentials (and more generally what I call "random differentials")

Outline of proofs in real-valued case (i.e. when $n = 1$). It shall suffice to show using stochastic differentials that

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial_2 f}{\partial^2 B_t}\right) dt + \frac{\partial f}{\partial B_t} dB_t. \quad (2.1)$$

We shall proceed as explained in [these notes](). By **Multivariable Taylor's Theorem**, we have

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial_2 f}{\partial^2 t}(dt)^2 + \frac{\partial_2 f}{\partial t \partial B_t} dt dB_t + \frac{1}{2}\frac{\partial_2 f}{\partial^2 B_t}(dB_t)^2 + \cdots$$

We claim that $\mathbb{P}$-a.s., we have

$$(dB_t)^2 = dt,$$

or equivalently, for all t, we have $[B_s]_t = t$.

Observe that the specific finite sequence $P_n = \left\{0 < \frac{t}{n} < \frac{2t}{n} < \cdots < \frac{(n-1)t}{n} < t\right\}$ makes up a partition such that $||P_n|| \to 0$ as $n \to +\infty$, and it follows by the SLLN that we $\mathbb{P}$-a.s. have

$$
\begin{aligned}
[B_s]_t &= \lim_{n \to +\infty} \sum_{k=1}^{n} \left(B\left(\frac{kt}{n}\right) - B\left(\frac{(k-1)t}{n}\right)\right)^2 \\
&= \lim_{n \to +\infty} n \cdot \frac{1}{n} \sum_{k=1}^{n} \left(B\left(\frac{kt}{n}\right) - B\left(\frac{(k-1)t}{n}\right)\right)^2 \\
&= \lim_{n \to +\infty} n\mathbb{E}\left[\left(B\left(\frac{t}{n}\right) - B(0)\right)^2\right] \\
&= \lim_{n \to +\infty} n\text{Var}\left(B\left(\frac{t}{n}\right)\right) \\
&= \lim_{n \to +\infty} n\left(\frac{1}{n} \cdot \text{Var}(B_t)\right) \\
&= t
\end{aligned}
$$

Noting that $dB_t = (dt)^{1/2}$, we have

$$
df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial_2 f}{\partial^2 B_t}\right)dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial_2 f}{\partial^2 t}(dt)^2 + \frac{\partial_2 f}{\partial t \partial B_t} dt dB_t + \cdots
$$

and

$$
(dt)^2, dt dB_t, \ldots = o(dt),
$$

and so

$$
\frac{1}{2}\frac{\partial_2 f}{\partial^2 t}(dt)^2, \frac{\partial_2 f}{\partial t \partial B_t} dt dB_t, \ldots \to 0
$$

and (2.1) is reached. □

Outline of proofs in the multidimensional case. It shall suffice to show using differentials that

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{k=1}^{m}\left[\frac{\partial_2 f}{\partial^2 B_t^k}\right]\right)dt + \sum_{k=1}^{m}\frac{\partial f}{\partial B_t^k}dB_t^k \quad (2.2)$$

Note by multidimensional Taylor's Theorem, we have

$$df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{m}\left[\frac{\partial f}{\partial B_t^i}dB_t^i\right] + \frac{1}{2}\frac{\partial_2 f}{\partial^2 t}(dt)^2 + \sum_{i=1}^{m}\left[\frac{\partial_2 f}{\partial t \partial B_t^i}dtdB_t^i\right] + \sum_{i=1}^{m}\left[\frac{1}{2}\frac{\partial_2 f}{\partial^2 B_t^i}\left(dB_t^i\right)^2\right]$$
$$+\sum_{j\neq k}\left[\frac{\partial_2 f}{\partial B_t^i \partial B_t^j}dB_t^i dB_t^j\right] + \cdots$$

Using the facts that $\left(dB_t^k\right)^2 = dt$ and $dtdB_t^k = o(dt)$ that we discussed in the one dimensional case, we get

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{k=1}^{m}\left[\frac{\partial_2 f}{\partial^2 B_t^i}\right]\right)dt + \sum_{i=1}^{m}\left[\frac{\partial f}{\partial B_t^i}dB_t^i\right] + \sum_{i\neq j}\left[\frac{\partial_2 f}{\partial B_t^i \partial B_t^j}dB_t^i dB_t^j\right] + \cdots$$

and it remains to verify (2.2) by showing that for $i \neq j$, we $\mathbb{P}$-a.s. have

$$dB_t^i dB_t^j = 0,$$

i.e., for all t, we have $\left[B_s^i, B_s^j\right]_t = 0$. Observe that $\mathbb{P}$-a.s., we have

$$\left[B_s^i, B_s^j\right]_t = \lim_{n\to+\infty}\sum_{k=1}^{n}\left[\left(B^i\left(\frac{kt}{n}\right) - B^i\left(\frac{(k-1)t}{n}\right)\right)\left(B^k\left(\frac{kt}{n}\right) - B^k\left(\frac{(k-1)t}{n}\right)\right)\right],$$

since B^i has almost sure continuous paths, and variation (of any order) has $\mathbb{P}$-a.s. a well-defined limit. Using the **Central Limit Theorem**, we note that

$$\sum_{i=1}^{n}\left[\left(B^i\left(\frac{kt}{n}\right) - B^i\left(\frac{(k-1)t}{n}\right)\right)\left(B^k\left(\frac{kt}{n}\right) - B^k\left(\frac{(k-1)t}{n}\right)\right)\right] \approx N\left(0, \frac{t^2}{n}\right)$$

as n gets large; therefore, since $\lim_{n \to +\infty} N\left(0, \frac{t^2}{n}\right) = 0$ in $\mathbb{P}$, we conclude that $\left[B_s^i, B_s^j\right]_t = 0$.

□

Sources:

Spring 2022-M647 Lecture 4

Stochastic equations in infinite dimensions § *4.5*
Da Prato, Zabczyk

Brownian Motion Notes (linked here)