M800 Roger Temam 4/8 Report

1 Random Measures

Due to a lot of roadblocks that came up with proving some fundamental results that will take a fair amount of time to patch up, and I don't want the work done on this section to hold up much-needed work I need to do on future sections that you definitely find more important, I've decided to call off the draft of this section until the future.

To make up, I made a decently well-detailed update of the next section where I outline the proof of the Ito Isometry in its most general form.

2 Stochastic Integration

For the two following defintions, let $\mathcal{X}$ be a Banach space, and let P be an $\mathcal{X}$-valued stochastic process.

Definition 2.1. The (Riemann) Integral of a (Banach-valued) stochastic process P over time on the interval (a, b) is defined as follows:

(i) For simple processes of the form $P := S(\omega, t) = \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(\omega)\mathbb{1}_{(t_k, t_{k-1})}(t)] \cdot x_k$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $\quad x_1, \ldots, x_k \in \mathcal{X}$, and $t_0 := a < t_1 < \cdots < t_N := b$ $(1 \le k \le N)$, we have

$$\int_a^b S dt := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(t_k - t_{k-1})] \cdot x_k.$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_{\mathcal{X}} dt < +\infty, \quad (2.1)$$

a.s. for $\omega \in \Omega$, we define $\int_a^b P dt$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; \mathcal{X})$ (if it exists) of stochastic integrals of sequences of simple processes $\{S_n\}_{n \in \mathbb{N}}$ such that $||S_n||_{\mathcal{X}} \le ||P||_{\mathcal{X}}$ $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dt \xrightarrow{\mathbb{P}} Y. \quad (2.2)$$

Any process such that (2.2) exists we call **(Riemann) integrable over** (a, b).
Note: I haven't really found a great source that corresponds well to this specific definition, but I have a source for a **Definition 2.6**, which more or less generalizes both this definition and a later definition; I'll look for a good source in a future draft.

For the next definition, let X be real-valued cadlag process.

Definition 2.2. A **stochastic (Stiltjes) Integral (or an ito integral) of a (Banach-valued) stochastic process P with respect to X on the interval (a, b)** is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $x_1, \ldots, x_N \in \mathcal{X}$, and $t_0 := a < t_1 < \cdots < t_N := b$ ($1 \leq k \leq N$), we have

$$\int_a^b S dX := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(X(t_k) - X(t_{k-1}))] \cdot x_k.$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_{\mathcal{X}} d|\mu_{X(\omega)}|(\omega, t) < +\infty, \quad (2.3)$$

a.s. for $\omega \in \Omega$, where $\mu_{X(\omega)}$ is given in **Examples 1.3 (ii)**, we define $\int_a^b P dX$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; \mathcal{X})$ (if it exists) of stochastic integrals of sequences of simple processes $\{S_n\}_{n \in \mathbb{N}}$ such that $||S_n||_{\mathcal{X}} \leq ||P||_{\mathcal{X}}$ for all $n \in \mathbb{N}$ and $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dX \xrightarrow{\mathbb{P}} Y. \quad (2.4)$$

Any function such that (2.4) exists, we call **Ito integrable** with respect to X over (a, b).
Note: I haven't really found a great source that corresponds well to this specific definition, but I have a source for a **Definition 2.6**, which more or less generalizes both this definition and a later definition; I'll look for a good source in a future draft.

Remark 2.1.

(i) Note that **Definition 2.2** actually generalizes **Definition 2.1** in the sense that Definition 2.1 is a special case of **Definition 2.2** in the case where X is the deterministic idenity function $t \mapsto t$ on $\mathbb{R}_+$.

(ii) Note that in both preceding definitions, we work specifically with simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on S), $x_1, \ldots, x_N \in X$, and $t_0 := a < t_1 < \cdots < t_N := b$ $(1 \leq k \leq N)$, as opposed to simple processes in general, which are more generally of the form $\sum_{k=1}^{N} X_k(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, where $X_k \in \mathcal{F}_{t_{k-1}}$. In a later draft, we shall define Riemann integrability of such processes in the more conventional sense, and show this, we shall talk about the classic notions of $\mathcal{L}^1$ convergence for stochastic processes, and how it generalizes $\mathcal{L}^1$ convergence for deterministic functions defined over an interval $[0, T]$ or $\mathbb{R}_+$.

(iii) Note that **Definition 2.1** and **Definition 2.2** have not been generalized by the previous section (unfortunately), since the previous section deals with real-valued integrable functions over a random measure and this section deals with Banach-valued Riemann integration over a stochastic process, which is a random measure with additional filtration structure. Hopefully in later reports, we shall talk about integration of random measures in a Banach valued setting, as well as stochastic Lebesgue integration.

(iv) Note that **Definition 2.1** and **Definition 2.2** as presented are incomplete in the sense that it only accounts for limits of one specific simple function. In the case of pure Riemann integration (without the introduction of random measures over the analogous measurable space), this does not exactly work as ideal. In a later draft, we shall remedy this problem by providing a more rigorous definition of Riemann integrability of a stochastic process.

Example 2.4. The most common example of such a stochastic Stiltjes Integral is the one with respect to the Wiener process W, i.e. $\int_0^T P_t dW_t$. There are lots of nice properties involving this integral, and variants of that integral, such as the Ito Isometry, which we shall now prove. *Note:* As before, I haven't really found a great source that corresponds well to this specific example. And as before, I'll look for a good source in a future draft.

Propositon 2.5. *(Ito Isometry; Hilbert Space Version)* Let $W: [0, T] \times \Omega \to \mathbb{R}$ denote the canonical real-valued Weiner process defined up to time $T > 0$, and let $X: [0, T] \times \Omega \to \mathcal{H}$ be an a.s. $L^2(0, T)$ stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then

$$\mathbb{E}\left(\left|\left|\int_0^T X_t dW_t\right|\right|_{\mathcal{H}}^2\right) = \mathbb{E}\left[\int_0^T ||X_t||_{\mathcal{H}}^2 dt\right]. \quad (2.6)$$

Source: Ito Isometry (from Wikipedia link here)

Remark 2.2.

(i) The proposition in the cited wikipedia article tells us that

$$\mathbb{E}\left(\left(\int_0^T X_t dW_t\right)^2\right) = \mathbb{E}\left[\int_0^T (X_t)^2 dt\right],$$

for an adapted *real-valued* process X_t. Noting that

$$||a||_{\mathcal{H}}^2 = a^2$$

for $a \in \mathcal{H}$ in the special case where the Hilbert space $\mathcal{H}$ is one dimensional (i.e. we have $\mathcal{H} \cong \mathbb{R}$), we find that the proposition cited is (more or less) this proposition in the single dimensional case. The proof moreover is also very similar.

(ii) It's worth noting that the proof of this proposition relies on simple functions of the elegant form

$$S := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}} \cdot \mathbb{1}_{(t_{k-1}, t_k)}$$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on S),
$x_1, \ldots, x_N \in X$, and $t_0 := a < t_1 < \cdots < t_N := b$ ($1 \le k \le N$).

The previous section tries to define random measures, which is in some ways (though not completely in other ways) a generalization of the Stiltjes Integral. So far I've only defined random measures in the real-value case. The hope is that this framework of stochastic processes can be talked about more generally in the context of (Bochner/Pettis) integration of a Hilbert-valued (and maybe even Banach-valued function) function over a random measure, which in turn allows us to talk about SDE's in a more general framework.

(iii) To expand on a specific instance where the idea of *part (ii)* gives us a pretty elegant (and potentially useful) result, it's worth noting that I'm pretty sure the Ito-Isometry also holds for the situation of processes adapted to Levy Processes that are integrated with respect to Levy Processes. I shall verify this claim in a future draft.

Proof. Let

$$S(t) = \sum_{j=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_k, t_{k-1})} a_k \quad \left(t_0 := 0 < t_1 < \cdots < t_N := T,\ a_k \in \mathcal{H},\ A_{t_{k-1}} \in \mathcal{F}^W_{t_{k-1}}\right)$$

be a simple function. Since $\mathbb{1}_{A_{t_{k-1}}},\ W_{t_k} - W_{t_{k-1}}\ (1 \leq j \leq N)$ are independent and $W_{t_j} - W_{t_{j-1}}, W_{t_k} - W_{t_{k-1}}\ (j \neq k)$ are independent, we have

$$\begin{aligned}
\mathbb{E}\left(\left| \left| \int_0^T S(t) dW_t \right| \right|^2_{\mathcal{H}} \right) &= \mathbb{E}\left(\left\langle \int_0^T S(t) dW_t, \int_0^T S(t) dW_t \right\rangle_{\mathcal{H}} \right) \\
&= \mathbb{E}\left(\left\langle \sum_{j=1}^{N} [\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] a_j, \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})] a_k \right\rangle_{\mathcal{H}} \right) \\
&= \mathbb{E}\left(\sum_{k=1}^{N} \sum_{j=1}^{N} ([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \langle a_j, a_k \rangle_{\mathcal{H}} \right) \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}\left(\left[\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})^2 \right] \right) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}\left[(W_{t_j} - W_{t_{j-1}})^2 \right] \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}[W_{t_j} - W_{t_{j-1}}] \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \mathbb{E}[W_{t_k} - W_{t_{k-1}}]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}](t_j - t_{j-1}) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \cdot 0 \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \cdot 0 \\
&= \mathbb{E}\left[\sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} (t_j - t_{j-1}) \right] \\
&= \mathbb{E}\left[\int_0^T \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} dt \right] \\
&= \mathbb{E}\left[\int_0^T \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \right. \\
&\qquad \left. + \sum_{j \neq k}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_k, t_{k-1})} dt \right] \\
&= \mathbb{E}\left[\int_0^T \sum_{k=1}^{N} \sum_{j=1}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} dt \right] \\
&= \mathbb{E}\left[\int_0^T \left\langle \sum_{j=1}^{N} a_j \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})}, \sum_{k=1}^{N} a_k \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_k, t_{k-1})} \right\rangle_{\mathcal{H}} dt \right] \\
&= \mathbb{E}\left[\int_0^T ||S(t)||_{\mathcal{H}}^2 dt \right].
\end{aligned}
$$

Then for any integrable process X_t such that $\int_0^T X_t dW_t$ is $L^2(\mathbb{P}; \mathcal{H})$, we can choose a sequence of simple processes $S_n(t)$ such that almost surely we have

$$
\lim_{n \to +\infty} S_n(t) = X_t, \ \lim_{n \to +\infty} \int_0^T S_n(t) dt = \int_0^T X_t dt, \ \lim_{n \to +\infty} \int_0^T S_n(t) dW_t = \int_0^T X_t dW_t, \quad (2.6)
$$

and then satisfy (2.6) by passing the limit using the **Dominated Convergence Theorem.** □

Now let X be an $\mathcal{X}$-valued stochastic process, and P, be an operator-valued, i.e. $L(\mathcal{X})$-valued, stochastic process. We will now define stochastic integration under this setting:

Definition 2.6. A **stochastic (Stiltjes) Integral (or an ito integral) of a (Operator-valued) stochastic process P with respect to X on the interval (a, b)** is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(\omega)\mathbb{1}_{(t_{k-1}, t_k)}(t)] \cdot \Phi_k$, for $A_{t_{k-1}} \in \mathcal{F}_{t_k}$ (where $\mathcal{F}_t$ is the filtration on P), $\Phi_1, \ldots, \Phi_N \in L(\mathcal{X})$, and $t_0 := a < t_1 < \cdots < t_N := b$ ($1 \leq k \leq N$), we have

$$\int_a^b SdX := \sum_{k=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \Phi_k(X(t_k) - X(t_{k-1})).$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_{L(\mathcal{X})} d||\mu_{X(\omega)}||(\omega, t) < +\infty, \quad (2.7)$$

a.s. for $\omega \in \Omega$, where $\mu_{X(\omega)}$ is given in **Examples 1.3 (ii)**, we define $\int_a^b PdX$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; L(\mathcal{X}))$ (if it exists) of stochastic integrals of sequences of simple S_n processes such that $||S_n||_{L(\mathcal{X})} \leq ||P||_{L(\mathcal{X})}$ and $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dX \xrightarrow{\mathbb{P}} Y. \quad (2.8)$$

Any function such that (2.8) exists, we call **Ito integrable** with respect to X over (a, b).
Source: Lototsky, Rozovskey (page 128-131) and Spring 2022-M647 Lecture 3 (revised) Definition 3.2.3

Remark 2.3.

(i) Note that **Definition 2.4** is a generalization of the definition of *A643 Spring 2021 Lecture 4/M647 Spring 2022 Lecture 3* that it came from, as well as the *Da Prato, Zabczyk* § *4.2 (page 90-96)* source that it came from, in the sense that the integration is over any stochastic process X as opposed to simply a Wiener process W.

(ii) Note furthermore that **Definition 2.4** is a generalization of **Definition 2.2** (as well as **Definition 2.1**), in the sense that the integration **Definition 2.2** can be imbedded into the instance of **Definition 2.4** where $\mathcal{X} := \mathbb{R}$, using the canonical isometry $\mathbb{R} \to \mathcal{L}(\mathbb{R})$ defined by $a \mapsto T_a$, where $T_a : \mathbb{R} \to \mathbb{R}$ is defined by $T_a(x) = ax$, hence integrating over real-valued

stochastic processes can be looked at as integrating over $\mathcal{L}(\mathbb{R})$-valued stochastic processes.

(iii) As a generalization of **Definition 2.2**, the ideas and caveats mentioned in *Remark 2.1 (ii)-(iv)* also (more or less) apply to **Definition 2.4**, although integrating vector-valued functions over random measures have not been established yet, and neither has integrating operator-valued functions over vector valued measures (not even in the deterministic setting!)

(iv) In a later draft, I shall provide some concrete examples of the kind of integration mentioned in **Definition 2.4**.

(v) For this next version of Ito's Isometry (i.e. **Proposition 2.7**), we need the $L(\mathcal{U}, \mathcal{H})$-valued process P on $(0, +\infty)$ to additionally be almost surely **Hilbert-Schmidt** operator-valued on $(0, T)$. In other words, for any $t \in (0, T)$, we assume that almost surely for $\omega \in \Omega$, we have

$$||P_t(\omega)||_{L_2(\mathcal{U},\mathcal{H})} := \sqrt{\sum_{i \in I} ||P_t(\omega)e_i||^2_{\mathcal{H}}} < +\infty,$$

where $\{e_i : i \in I\}$ is an orthonormal basis of $\mathcal{H}$. We shall moreover let $L_2(\mathcal{U}, \mathcal{H})$ denote the set of Hilbert-Schmidt opeators, call $|| \cdot ||_{L_2(\mathcal{U},\mathcal{H})}$ the **Hilbert-Schmidt norm**, and note that $L_2(\mathcal{U}, \mathcal{H})$ is a Hilbert Space on the inner-product

$$\langle A, B \rangle_{L_2(\mathcal{U},\mathcal{H})} := \sum_{i \in I} \langle Ae_i, Be_i \rangle.$$

We finally define an a.s. Hilbert-Schmidt operator-valued process to to be **X-admissible on $[0, T]$ if**

$$\int_0^T \langle Q_X P_{t t} P_t \rangle_{L_2(\mathcal{U},\mathcal{H})} dt < +\infty.$$

We shall talk about Hilbert-Schmidt Operators in more detail in a future report.
Source: Hilbert–Schmidt Operator (from Wikipedia link here), *Lototsky, Rozovskey (Page 130)*, and *Da Prato, Zabczyk (page 90-91)*

We shall now state and outline the proof of the operator-valued version of the Ito Isometry.

Proposition 2.7. *(Ito Isometry; Operator Version):* For an $\mathcal{H}$-valued wiener process W and an a.s. $L_2(\mathcal{U}, \mathcal{H})$-valued process X that is Wiener-adapted to W and W-admissible on $[0, T]$, we have

$$\mathbb{E}\left[\left|\left|\int_0^T X_t dW_t\right|\right|_{\mathcal{H}}^2\right] = \mathbb{E}\int_0^T ||X_t Q^{1/2}||_{L_2(\mathcal{U},\mathcal{H})}^2 dt,$$

$$= \mathbb{E}\int_0^T ||\Psi(s)||_{L_2(\mathcal{U}_0,\mathcal{H})}^2 dt, \qquad (2.9)$$

where Q is the covariance operator of W and $\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$.
Source: Spring 2022-M647 Lecture 3 (revised) Proposition 3.2.4

Remark 2.4.

(i) Note that **Proposition 2.7** is a generalization of **Proposition 2.5**, observing that $L_2(\mathcal{H}, \mathbb{R}) \cong \mathcal{H}^* \cong \mathcal{H}$ and naturally, the proof goes through a similar derivation in the case of simple functions, as well as a similar convergence theorem/localization trick in the general case. The main difference is the fact that the dimensions The main difference is that the infinite dimensions cause there to be additional steps in the derivation, involving utilizing the covariance operator and interactions with the real-valued Brownian motion terms $\{\beta_n\}_{n\in\mathbb{N}}$, with both themselves and other terms.

(ii) On the subject of convergence thoerems/localization tricks, there is still work to be done on showing that the limit works in the way it does, either through usig a convergence theorem or localization. In this draft, I rather handwavily give reference to the Dominated Convergence Theorem, which definitely holds in the Bochner integration setting (though I have not yet verified this) and hence holds in the desired stochastic sense. But it might be more intuitively and flavorfully satisfying to verify the argument using the more general functional analytic trick of localization. Maybe I'll use this in a future draft.

Outline of proof. Let $\Psi(t)$ be a simple process of the form

$$\Psi(t) := \sum_{k=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_{k-1},t_k)} \Phi_k$$

Let Q be the covariance operator with eigenvalue coefficients $\{v_n\}_{n\in\mathbb{N}}$ of an orthonormal eigenbasis $(u_n)_{n\in\mathbb{N}}$, i.e., we have

$$W(t) := \sum_{n=1}^{\infty} \sqrt{v_n}\beta_n(t)u_n$$

for pairwise independent sequence of real-valued Brownian motions $\{\beta_n(t)\}_{n\in\mathbb{N}}$, and it follows that

$$\mathbb{E}\left(\left|\left|\int_0^T \Psi(t)dW_t\right|\right|_{\mathcal{H}}^2\right) = \mathbb{E}\left(\left|\left|\sum_{k=1}^{N} \mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sum_{n=1}^{\infty}\sqrt{\nu_n}\beta_n(t_k)u_n - \sum_{n=1}^{\infty}\sqrt{\nu_n}\beta_n(t_{k-1})u_n\right)\right|\right|_{\mathcal{H}}^2\right)$$

$$= \mathbb{E}\left(\left|\left|\sum_{k=1}^{N}\sum_{n=1}^{\infty} \mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sqrt{\nu_n}(\beta_n(t_k)-\beta_n(t_{k-1}))u_n\right)\right|\right|_{\mathcal{H}}^2\right)$$

$$= \mathbb{E}\left(\left\langle\sum_{k=1}^{N}\sum_{n=1}^{\infty} \mathbb{1}_{A_{t_{j-1}}}\Phi_j\left(\sqrt{\nu_m}(\beta_m(t_j)-\beta_m(t_{j-1}))u_m\right), \sum_{k=1}^{N}\sum_{n=1}^{\infty} \mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sqrt{\nu_n}(\beta_n(t_k)-\beta_n(t_{k-1}))u_n\right)\right\rangle_{\mathcal{H}}\right)$$

$$= \mathbb{E}\left(\sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right)$$

$$= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right]$$

Since $\mathbb{1}_{A_{t_{k-1}}}$, $W_{t_k} - W_{t_{k-1}}$ $(1 \le k \le N)$ are independent (and hence $\mathbb{1}_{A_{t_{k-1}}}$ and $\beta_n(t_k) - \beta_n(t_{k-1})$ are independent for all $n \ge 1$), $W_{t_j} - W_{t_{j-1}}, W_{t_k} - W_{t_{k-1}}$ $(j \neq k)$ are independent (and hence $\beta_n(t_j) - \beta_n(t_{j-1})$ and $\beta_n(t_k) - \beta_n(t_{k-1})$ are independent for all $n \ge 1$), and β_n, β_m $(m \neq n)$ are independent we find in the situation that $j \neq k$ or $m \neq n$, we have

$$\mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right] = 0,$$

and in the situation where $j = k$ and $m = n$ we have

$$\mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{A_{t_{k-1}}} v_n(\beta_n(t_k)-\beta_n(t_{k-1}))^2||\Phi_k u_n||_{\mathcal{H}}^2\right]$$

$$= ||\sqrt{v_n}\Phi_k u_n||_{\mathcal{H}}^2 \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}]\mathbb{E}\left[(\beta_n(t_k)-\beta_n(t_{k-1}))^2\right]$$

$$= ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}](t_k - t_{k-1}).$$

Since it can be shown (which we shall do in a later report about compact operators) that

$$||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})} = \sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}$$

follows that

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\int_0^T \Psi(t)dW_t\right|\right|^2_{\mathcal{H}}\right) &= \sum_{k=1}^{N}\sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}](t_k - t_{k-1}) \\
&= \mathbb{E}\left[\sum_{k=1}^{N}\sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{1}_{A_{t_{k-1}}}(t_k - t_{k-1})\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{N}\int_0^T \sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)dt\right] \\
&= \mathbb{E}\left[\int_0^T \sum_{k=1}^{N}\left[||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)dt\right]\right],
\end{aligned}$$

and through functional analysis techniques on compact operators (in particular taking advantage of the fact that $L_2(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert Space, for any two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2,$) , we can further show that

$$\sum_{k=1}^{N}\left[||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)\right] = ||Q^{1/2}\Psi(t)||^2_{L_2(\mathcal{U},\mathcal{H})},$$

and condition (2.9) is met for simple functions.

Then for any integrable process X_t such that $\int_0^T X_t dW_t$ is $L^1(\mathbb{P}; L_2(\mathcal{U}, \mathcal{H}))$, we proceed in similar fashion to the general case of the outline of the proof of **Propositon 2.5,** where we choose a sequence of simple processes $\Psi_n(t)$ satisfying an analogous condition to (2.6), and then show that (2.9) is satisfied by passing the limit using the **Dominated Convergence Theorem**. □

Sources:

Spring 2022-M647 Lecture 3 (revised)

Stochastic Partial Differential Equations § 3.3
Lototsky, Rozovskey

Stochastic equations in infinite dimensions § 4.2
Da Prato, Zabczyk

Ito Isometry (from Wikipedia link here)

Hilbert–Schmidt Operator (from Wikipedia link here)