M800 Roger Temam 4/21 Corrected 4/8 Report

1 Random Measures

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\mathcal{X}$ be a banach space.

Definition 1.1. . A **random measure** $\nu : \Omega \times \mathcal{E} \to \mathbb{R}$ is a function such that

(i) $\nu(\cdot, F)$ is $\mathbb{P}$-measurable for any fixed $F \in \mathcal{E}$.

(ii) $\nu(\omega, \cdot)$ is a (signed-)measure on $(E, \mathcal{E})$, for any fixed $\omega \in \Omega$.

Proposition 1.2. Let $M : \mathcal{E} \to \mathcal{L}^0(\Omega; \mathbb{R})$ be a vector-valued measure. $\nu_M : \Omega \times \mathcal{E} \to \mathbb{R}$ defined by $\nu_M(\omega, E) := M(E)(\omega)$ is a random measure. Conversely, for every random measure ν, $M_\nu : \mathcal{E} \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ defined by $M_\nu(E) := \nu(\cdot, E)$ is a random measure.

Remarks 1.1.

(i) The set of random measures and the set of $\mathcal{L}^0(\mathbb{P}; \mathbb{R})$-valued measures are in fact isomorphic in the category Set. Denote this isomorphism by $M_{(-)}$.

(ii) Note that $\mathcal{L}^0(\mathbb{P}; \mathbb{R})$ is a complete metric space, with a metric defined as follows

$$d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(X, Y) = \mathbb{E}\left[\frac{|X - Y|}{1 + |X - Y|}\right]. \quad (1.1)$$

Definition. We further define $\mathcal{L}^0(\nu; \mathbb{R})$ (the space of random ν-measurable functions) as the set of functions $Q : E \times \Omega \to \mathbb{R}$ such that the function $Q' : E \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ defined by $Q'(e) := Q(e, \cdot) : \Omega \to \mathbb{R}$ is $(E, \mathcal{E})$-measurable.
%possibly figure out less awkward way to define it

Furthermore, for a sequence $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^0(\nu; \mathbb{R})$, we write $Q_n \xrightarrow{\nu} Q$ if for every $\epsilon > 0$, we have

$$\nu(\cdot, \{|Q_n - Q| \geq \epsilon\}) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to +\infty \quad (1.2)$$

Proposition 1.3. Let $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^0(\nu; \mathbb{R})$ and $Q \in \mathcal{L}^0(\nu; \mathbb{R})$. The following are equivalent.

1. $Q_n \xrightarrow{\nu} Q$

2. For all $\epsilon_1 > 0$, we have $M_\nu(\{|Q_n - Q| \geq \epsilon_1\}) \xrightarrow{\mathbb{P}} 0$ as $n \to +\infty$, i.e., for all $\epsilon_2 > 0$, we have

$$\lim_{n \to +\infty} \mathbb{P}[|M_\nu(\{|Q_n - Q| \geq \epsilon_1\})| \geq \epsilon_2] = 0, \quad (1.3)$$

3. For all $\epsilon > 0$, we have

$$\lim_{n \to +\infty} d_{\mathcal{L}^0(\nu;\mathbb{R})}(M_\nu(\{|Q_n - Q| \geq \epsilon\}), 0) = 0. \quad (1.4)$$

4. For every subsequences $\{Q_{n_k}\}_{k\in\mathbb{N}}$, there exists a further subsequence $\{Q_{n_j}\}_{j\in\mathbb{N}}$ such that almost surely for $\omega \in \Omega$, we have

$$Q_{n_j}(\cdot, \omega) \to Q(\cdot, \omega) \text{ as } j \to +\infty \ \nu(\cdot, \omega)\text{-a.e.} \quad (1.5)$$

Remarks 1.2.

(i) In general, $\mathcal{L}^0(\nu;\mathbb{R})$ is a vector space with respect to the induced addition and scalar multiplication operations.

(ii) The above proposition motivates another concept (and even more notation to go with it!). For a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\nu;\mathbb{R})$, we state that $Q_n \xrightarrow{\nu\text{-a.e.}} Q$, for $Q \in \mathcal{L}^0(\nu;\mathbb{R})$ if almost surely for $\omega \in \Omega$, (2.1.5) holds.

Definition 1.4. We further define $\mathcal{L}^0(\nu;\mathbb{R})$ (the space of random ν-measurable functions) as the set of functions $Q : E \times \Omega \to \mathbb{R}$ such that the function $Q' : E \to \mathcal{L}^0(\mathbb{P};\mathbb{R})$ defined by $Q'(e) := Q(e, \cdot) : \Omega \to \mathbb{R}$ is $(E, \mathcal{E})$-measurable.

Furthermore, for a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\nu;\mathbb{R})$, we write $Q_n \xrightarrow{\nu} Q$ if for every $\epsilon > 0$, we have

$$\nu(\cdot, \{|Q_n - Q| \geq \epsilon\}) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to +\infty \quad (1.6)$$

Definition 1.5. The random integral of $Q \in \mathcal{L}^0(\nu;\mathbb{R})$ (if it exists) with respect to a random measure ν is defined as follows:

(i) For simple functions of the form $S(\omega,s) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_k}(\omega)\cdot \mathbb{1}_{F_k}(s)$, for $A_1, \ldots, A_N \in \Sigma$, $x_1, \ldots, x_k \in \mathbb{R}$, and $F_1, \ldots, F_k \in \mathcal{E}$, we have

$$\int S(\cdot,s)d\nu(\cdot,s) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_k}(\cdot)\nu(\cdot,F_k).$$

(ii) In general, for Q such that

$$\int |Q(\omega,e)|d|\nu|(\omega,e) < +\infty, \quad (1.7)$$

a.s. for $\omega \in \Omega$, we define $\int Q(\cdot,s)d\nu(\cdot,s)$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega;\mathbb{R})$ (if it exists) of random integrals of a sequence of simple functions S_n such that $|S_n| \le |Q|$ a.s. and $S_n \xrightarrow{\nu} Q$, i.e. we have

$$\int S_n d\nu \xrightarrow{\mathbb{P}} Y. \quad (1.8)$$

Any Q such that (1.7) holds and $Y \in \mathcal{L}^0(\Omega;\mathbb{R})$ exists such that (1.8) is satisfied for some sequence $\{S_n\}_{n\in\mathbb{N}}$ of simple functions and we call ν-integrable, and the set of such functions we call $\mathcal{L}^1(\nu;\mathbb{R})$.

Remarks 1.3:

(i) $\mathcal{L}^1(\nu;\mathbb{R})$, like $\mathcal{L}^0(\nu;\mathbb{R})$, is a vector space.

(ii) Random integrals of random functions with respect to random integrals are a more general case of a stochastic Integral, as we shall soon see in the next section. For now, let's provide some examples:

Examples 1.6.

(i) First, we can define $\nu(\omega,E) := m(E)$, where $\mathcal{E} := \mathcal{B}(\mathbb{R}_+)$ and m is the Lebesgue measure. The integral of a random function Q over this random measure is simply the Borel extension of the Riemann integral over a stochastic process, i.e., we have

$$\int_{(a,b)} Q dm = \int_a^b Q dt$$

(ii) Next, we can define $\nu(\omega, E) := \mu_{X(\omega)}(E)$, where $(X_t)_{t \in \mathbb{R}_+}$ is a cadlag process and for a cadlag function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, we define μ_γ to be the Borel extension of the (signed-)premeasure

$$\mu_\gamma(a, b) := \gamma(b) - \gamma(a).$$

The random integral over this random measure is in turn the Borel extension of the Ito integral over X, i.e., we have

$$\int_{(a,b)} Q d\mu_X = \int_a^b Q dX.$$

(iii) Next, we have a **Poisson Random measure** with respect to $(E, \mathcal{A}, \mu)$, for some measure space with σ-finite measure μ. The Poisson random measure with inensity measure μ is a random measure $(\omega, A) \mapsto N_A(\omega)$ such that

(a) $\forall A \in \mathcal{A}$, N_A is a Poisson random variable with rate $\mu(A)$.

(b) If sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$ don't intersect then the corresponding random variables from (a) are mutually independent

The poisson random measure gives rise to integration with respect to the Poisson random measure on some $Q \in \mathcal{L}^0(N; \mathbb{R})$. In practice, since Poisson distributions are discrete, and hence integrals over a Poisson random measure end up being series, as so

$$\int_A Q dN = \sum_{x \in A} Q(\cdot, x) N(\cdot, \{x\}) \qquad (1.9)$$

As will be mentioned in future reports, there is a very intimate relationship between Poisson random measures (over Borel σ-algebras in particular). and Levy processes, and so integration over poisson processes will often occur.

Lemma 1.7. Suppose $Q \in \mathcal{L}^0(\nu; \mathbb{R})$

(i) If there exists a sequence $Q_n \in \mathcal{L}^1(\nu; \mathbb{R})$ such that

$$\int |Q_n - Q| d\nu \xrightarrow{\mathbb{P}} 0,$$

then $\int Q_n d\nu \xrightarrow{\mathbb{P}} \int Q d\nu$.

(ii) If ν is finite and $Q \in \mathcal{L}^1(\nu; X)$, and $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(\nu;\mathbb{R})$ is a sequence of simple functions such that $|S_n| \leq |Q|$ and $S_n \xrightarrow{\nu} Q$, then

$$\int S_n d\nu \xrightarrow{\mathbb{P}} \int Q d\nu$$

Proof.

(i) Assume without loss of generality that ν is a positive measure (since $\nu = \nu_+ - \nu_-$). Our conclusion immediately follows from the fact that for fixed $\omega \in \Omega$, we have

$$\left| \int Q_n(\omega, s) d\nu(\omega, s) - \int Q(\omega, s) d\nu(\omega, s) \right| \leq \int |Q_n(\omega, s) - Q(\omega, s)| d\nu(\omega, s)$$

(ii) Given a subsequence $\{S_{n_k}\}$, we find almost surely for fixed $\omega \in \Omega$ that there exists a further subsequence $\{S_{n_{k_j}}\}$ such that $S_{n_{k_j}} \xrightarrow{\nu\text{-a.e.}} Q$ as $j \to +\infty$. Since it follows by hypothesis that $|Q - S_{n_{k_j}}| \leq 2|Q|$ we find by the **Dominated Convergence Thoerem** that

$$\int |S_{n_{k_j}} - Q| d\nu \xrightarrow{\text{a.s.}} 0 \text{ as } j \to +\infty,$$

and we have shown that $\int |S_n - Q| d\nu \xrightarrow{\mathbb{P}} 0$. Our conclusion immediately follows by part *(i)*.
□

Sources:

Stochastic Differential Equations and Diffusion Processes 2nd ed. Chapter I § 8
Ikeda, Watanabe

Lévy Processes and Stochastic Calculus § 2.3.1- § 2.3.2
Applebaum

2 Stochastic Integration

Definition 2.1. A (Riemann) Integral of a stochastic process P over time on the interval (a, b) is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_k, t_{k-1})}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $x_1, \dots, x_k \in X$, and $t_0 := a < t_1 <\dots< t_N := b$ ($1 \le k \le N$), we have

$$\int_a^b S dt := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}} \cdot (t_k - t_{k-1}).$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_X dt < +\infty, \quad (2.1)$$

a.s. for $\omega \in \Omega$, we define $\int_a^b P dt$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; X)$ (if it exists) of stochastic integrals of sequences of simple S_n processes such that $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dt \xrightarrow{\mathbb{P}} Y. \quad (2.2)$$

Any process such that (2.2) exists we call **(Riemann) integrable over** (a, b).

Let X be real-valued cadlag process.

Definition 2.2. A **stochastic (Stiltjes) Integral (or an ito integral)** of a stochastic process P with respect to X on the interval (a, b) is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_k, t_{k-1})}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $x_1, \dots, x_k \in X$, and $t_0 := a < t_1 <\dots< t_N := b$ ($1 \le k \le N$), we have

$$\int_a^b SdX := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}} \cdot (X(t_k) - X(t_{k-1})).$$

(ii) For any process P such that

$$\int_a^b ||P(\omega,t)||_X d|\mu_{X(\omega)}|(\omega,t) < +\infty, \quad (2.3)$$

a.s. for $\omega \in \Omega$, where $\mu_{X(\omega)}$ is given in **Examples 1.3 (ii)**, we define $\int_a^b PdX$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; \mathcal{X})$ (if it exists) of stochastic integrals of sequences of simple S_n processes such that $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dX \xrightarrow{\mathbb{P}} Y. \quad (2.4)$$

Any function such that (2.4) exists, we call **Ito integrable** with respect to X over (a,b).

Example 2.4. The most common example of such a stochastic Stiltjes Integral is the one with respect to the Wiener process W, i.e. $\int_0^T P_t dW_t$. There are lots of nice properties involving this integral, and variants of that integral, such as the Ito Isometry, which we shall now prove.

Propositon 2.5. *(Ito Isometry)* Let $W : [0,T] \times \Omega \to \mathcal{H}$ denote the canonical real-valued Weiner process defined up to time $T > 0$, and let $X : [0,T] \times \Omega \to \mathcal{H}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then

$$\mathbb{E}\left(\left| \left| \int_0^T X_t dW_t \right| \right|_{\mathcal{H}}^2 \right) = \mathbb{E}\left[\int_0^T ||X_t||_{\mathcal{H}}^2 dt \right]. \quad (2.5)$$

Proof. Let

$$S(t) = \sum_{j=1}^{N} a_j \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \quad \left(t_0 := 0 < t_1 < \cdots < t_N := T,\ a_j \in \mathcal{H},\ A_{t_{j-1}} \in \mathcal{F}_{t_{j-1}}^W \right)$$

be a simple function. Since $\mathbb{1}_{A_{t_{j-1}}},\ W_{t_j} - W_{t_{j-1}}\ (1 \le j \le N)$ are independent and

$W_{t_j} - W_{t_{j-1}}, W_{t_k} - W_{t_{k-1}}$ $(j \neq k)$ are independent, we have

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\int_0^T S(t)dW_t\right|\right|_{\mathcal{H}}^2\right) &= \mathbb{E}\left(\left\langle\int_0^T S(t)dW_t, \int_0^T S(t)dW_t\right\rangle_{\mathcal{H}}\right) \\
&= \mathbb{E}\left(\left\langle\sum_{j=1}^N a_j \mathbb{1}_{A_{t_{j-1}}}(W_{t_j} - W_{t_{j-1}}), \sum_{k=1}^N a_k \mathbb{1}_{A_{t_{k-1}}}(W_{t_k} - W_{t_{k-1}})\right\rangle_{\mathcal{H}}\right) \\
&= \mathbb{E}\left(\sum_{k=1}^N \sum_{j=1}^N ([\mathbb{1}_{A_{t_{j-1}}}(W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}}(W_{t_k} - W_{t_{k-1}})]) \langle a_j, a_k\rangle_{\mathcal{H}}\right) \\
&= \sum_{k=1}^N \sum_{j=1}^N \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}}(W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}}(W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{E}\left(\left[\mathbb{1}_{A_{t_{j-1}}}(W_{t_j} - W_{t_{j-1}})^2\right]\right) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}}(W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}}(W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}\left[(W_{t_j} - W_{t_{j-1}})^2\right] \\
&\quad + \sum_{j \neq k} \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}[W_{t_j} - W_{t_{j-1}}] \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \mathbb{E}[W_{t_k} - W_{t_{k-1}}] \\
&= \sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}](t_j - t_{j-1}) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \cdot 0 \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \cdot 0 \\
&= \mathbb{E}\left[\sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}}(t_j - t_{j-1})\right] \\
&= \mathbb{E}\left[\int_0^T \sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} dt\right] \\
&= \mathbb{E}\left[\int_0^T \sum_{j=1}^N \langle a_j, a_j\rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\neq k}^{N} \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j,t_{j-1})} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_k,t_{k-1})} dt \Bigg] \\
&= \mathbb{E}\left[\int_0^T \sum_{k=1}^{N}\sum_{j=1}^{N} \langle a_j, a_k\rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j,t_{j-1})} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j,t_{j-1})} dt\right] \\
&= \mathbb{E}\left[\int_0^T \left\langle \sum_{j=1}^{N} a_j \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j,t_{j-1})}, \sum_{k=1}^{N} a_k \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_k,t_{k-1})} \right\rangle_{\mathcal{H}} dt\right] \\
&= \mathbb{E}\left[\int_0^T ||S(t)||_{\mathcal{H}}^2 dt\right].
\end{aligned}
$$

Then for any integrable process X_t such that $\int_0^T X_t dW_t$ is $L^2(\mathbb{P};\mathcal{H})$, we can choose a sequence of simple processes $S_n(t)$ such that almost surely we have

$$
\lim_{n\to+\infty} S_n(t) = X_t,\ \lim_{n\to+\infty}\int_0^T S_n(t)dt = \int_0^T X_t dt,\ \lim_{n\to+\infty}\int_0^T S_n(t)dW_t = \int_0^T X_t dW_t,
$$

and then satisfy (2.5) by passing the limit using the **Dominated Convergence Theorem**. $\square$

Sources:

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