Skorohod's Theorem Paper Unofficial Second Draft

1 Introduction and Preliminary Results

%fact about countably valued random variables converging in $\mathbb{P}$ converging almost surely

Theorem 1.1. *(Skorohod Representation Theorem)* Let E be a complete and separable metric space. Let $\{X_n\}_{n=1}^{\infty}$ and X be E-valued r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \Rightarrow X$. Then there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and r.v.'s $\left\{\widetilde{X}_n\right\}_{n=1}^{\infty}$ and $\widetilde{X}$ on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ such that,

(i) $\widetilde{X}_n \overset{D}{=} X_n$
(ii) $\widetilde{X} \overset{D}{=} X$
(iii) $\widetilde{X}_n \to \widetilde{X}$ $\widetilde{\mathbb{P}}$-a.s.
Source: M647 Lecture 1 (revised) Theorem 1.4.5.
%REWRITE THOEREM WITH CONSISTENT NOTATION

Proposition 1.2.. If $X = c$ $\mathbb{P}$-a.s., for some $c \in E$, then $X_n \Rightarrow X \implies X_n \xrightarrow{\mathbb{P}} X$.
%REVISE AND CITE SOURCE

Note: The proposition is in the more general setting of metric-spaced valued random variables, whereas the analogous statement of this proposition metioned in Billingsley is in the real-valued random variable setting. However, the proof (which we shall provide in full detail below) is pretty much the same as in this case.
%EDIT THIS NOTE TO ACCOUNT FOR PROOF NOT BEING MENTIONED

Proposition 1.3.
%PROPOSITION ABOUT CONVERGENCE IN DISTRIBUTION BEING PRESERVED FOR $\mathbb{P}$-a.s. CONTINUOUS FUNCTIONS

Proposition 1.4.
%CONSIDER INCLUDING THOEREM ON OPEN MAPPING COMPOSITION PRESERVING ALMOST EVERYWHERE CONVERGENCE
%ALSO INCLUDE MEASURE-PRESERVING PROPERTIES OF TRANSLATIONS
%INCLUDE GENERAL CASE OF THAT IN THIS PROPOSITION

Lemma 1.5. Given a probability space $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ such that $\widetilde{\Omega} := [0,1]$ and $\mathcal{B}([0,1]) \subset \widetilde{\Sigma}$, If $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$, $\widetilde{X} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ and $||\widetilde{X}_n - \widetilde{X}||$, for every $n \in \mathbb{N}$, is monotonically increasing on $\widetilde{\Omega}$, and $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{X}$, then $\widetilde{X}_n \xrightarrow{\text{pointwise}} \widetilde{X}$.
%GENERALIZE THIS LEMMA FOR ARBITRARY POSETS

Proof.

Given $\widetilde{\omega} \in \widetilde{\Omega}$ and $\epsilon > 0$, choose $N \geq 1$ sufficiently large such that for all $n \geq N$, we have $\widetilde{\mathbb{P}}\left[\,|\widetilde{X}_n - \widetilde{X}| \geq \epsilon\right] < \dfrac{\widetilde{\mathbb{P}}\left(\left[\widetilde{\omega}, 1\right)\right)}{2}$. Since $||\widetilde{X}_n - \widetilde{X}||$ is monotonically increasing, we find that if (towards contradiction), we have $||\widetilde{X}_n\left(\widetilde{\omega}\right) - \widetilde{X}\left(\widetilde{\omega}\right)|| \geq \epsilon$, or all $\widetilde{\Omega} \ni \widetilde{\omega}' \geq \widetilde{\omega}$, we have

$$||\widetilde{X}_n\left(\widetilde{\omega}'\right) - \widetilde{X}\left(\widetilde{\omega}'\right)|| \geq ||\widetilde{X}_n\left(\widetilde{\omega}\right) - \widetilde{X}\left(\widetilde{\omega}\right)|| \geq \epsilon,$$

and it follows that

$$\widetilde{\mathbb{P}}\left[\left[\widetilde{\omega}, 1\right)\right] \leq \widetilde{\mathbb{P}}\left[\,||\widetilde{X}_n - x|| \geq \epsilon\right] < \frac{\widetilde{\mathbb{P}}\left(\left[\widetilde{\omega}, 1\right)\right)}{2},$$

which is a contradiction. Then we conclude that $||\widetilde{X}_n\left(\widetilde{\omega}\right) - \widetilde{X}\left(\widetilde{\omega}\right)|| < \epsilon$. □
%LEMMA ABOUT INCREASING FUNCTION ON $[0,1]$ CONVERGING $\widetilde{\mathbb{P}}$-A.S. IF IT CONVERGES IN $\widetilde{\mathbb{P}}$.

%MOVE NEXT LEMMAS IN THE NEXT SECTION
Lemma 1.6.

(i) If $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ and $A \in \Sigma$. then $\mathbb{1}_A X_n \xrightarrow{\mathbb{P}\text{-a.s.}} \mathbb{1}_A X$, as $n \to +\infty$.

(ii) If $X_n \xrightarrow{\mathbb{P}} X$ and $A \in \Sigma$. then $\mathbb{1}_A X_n \xrightarrow{\mathbb{P}} \mathbb{1}_A X$, as $n \to +\infty$.

(iii) If $X_n \Rightarrow X$ and $A \in \Sigma$. then $\mathbb{1}_A X_n \Rightarrow \mathbb{1}_A X$, as $n \to +\infty$.

%NOTE THAT PART (III) OF THE THOEREM IS WRONG

Outline of proof.
%DRAW FROM PREVIOUS OUTLINES TO MAKE NEW OUTLINE

Proposition 1.7. If $\{A_i\}_{i\in\mathbb{N}} \subset \Sigma$ is a partition of Ω, then $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ iff $\mathbb{1}_{A_i}X_n \xrightarrow{\mathbb{P}\text{-a.s.}} \mathbb{1}_{A_i}X$, for each $i \in \mathbb{N}$ as $n \to +\infty$.
%REVISE THIS PROPOSITION TO INCLUDE CONVERGENCE IN MEASURE

2 Conditional Probability, Probability Trees, And Convergence in Distribution

%MAKE NOTE ABOUT CONDITIONAL PROBABILITY DISTRIBUTIONS DEFINED BY BAYE'S RULE IN CONTRAST TO CONDITIONAL EXPECTATION

Theorem 2.1. Let $A \in \Sigma$, suppose $\mathbb{P}[A] > 0$, and note that $(\Omega, \Sigma, \mathbb{P}[\,\cdot\,|A])$ is a probability space. There exists an surjective bounded operator
$(-)_A : \mathcal{L}^1(\mathbb{P}; \mathcal{X}) \to \mathcal{L}^1(\mathbb{P}[\,\cdot\,|A]; \mathcal{X}) \subset \mathcal{L}^1(\mathbb{P}; \mathcal{X})$ such that given $X \in \mathcal{L}^0(\mathbb{P}; \mathcal{X})$, we have

$$\int_B X_A d\mathbb{P}[\,\cdot\,|A] = \int_B \mathbb{1}_A X d\mathbb{P} = \mathbb{E}[\mathbb{1}_{A\cap B}X],$$

for all $B \in \Sigma$ defined by

$$X_A := \mathbb{P}[A] \cdot X.$$

Outline of Proof. Note that $\nu := \int_{(-)} \mathbb{1}_A X d\mathbb{P}$ is a measure on (Ω, Σ) such that $\nu \ll \mathbb{P}[\,\cdot\,|A]$. Then our conclusion immediately follows by the Radon Nikodym Theorem.
%USE THAT ARGUMENT TO TALK ABOUT THE FUNCTION THAT EXISTS AND IS AN OPERATOR

Observe that for all $B \in \Sigma$, we have

$$\int_B X_A d\mathbb{P}[\,\cdot\,|A] = \int \mathbb{P}[A] \cdot \left(\sum_{x\in\mathrm{supp}(X)} [\mathbb{1}_{B\cap\{X=x\}}x] \right) d\mathbb{P}[\,\cdot\,|A]$$

$$
\begin{aligned}
&= \sum_{x \in \mathrm{supp}(X)} \left[\mathbb{P}[A] \cdot \left(\int \mathbb{1}_{B \cap \{X = x\}} d\mathbb{P}[\,\cdot\,|A] \right) \cdot x \right] \\
&= \sum_{x \in \mathrm{supp}(X)} [\mathbb{P}[A] \cdot \mathbb{P}[B \cap \{X = x\} | A] \cdot x] \\
&= \sum_{x \in \mathrm{supp}(X)} \mathbb{P}[A \cap (B \cap \{X = x\})] \cdot x \\
&= \sum_{x \in \mathrm{supp}(X)} \left[\left(\int \mathbb{1}_{A \cap B \cap \{X = x\}} d\mathbb{P} \right) \cdot x \right] \\
&= \int \sum_{x \in \mathrm{supp}(X)} [\mathbb{1}_{A \cap B \cap \{X = x\}} x] d\mathbb{P} \\
&= \int \sum_{x \in \mathrm{supp}(X)} [\mathbb{1}_A \mathbb{1}_B \mathbb{1}_{X = x} x] d\mathbb{P} \\
&= \int_B \mathbb{1}_A X d\mathbb{P}. \qquad \square
\end{aligned}
$$

%CONSIDER DELETING THIS THEOREM AND PROOF

Proposition 2.2. Suppose $\mathbb{P}[A] > 0$ and $\{W_n\}_{n \in \mathbb{N}} \in \mathcal{L}^0(\mathbb{P}[\,\cdot\,|A]; \mathcal{X})$, $W \in \mathcal{L}^0(\mathbb{P}[\,\cdot\,|A]; \mathcal{X})$ such that $W_n \Rightarrow W$ in $\mathbb{P}$-distribution as $n \to +\infty$. Then $W_n \Rightarrow W$ in $\mathbb{P}[\,\cdot\,|A]$-distribution as $n \to +\infty$.

Outline of proof.

$$
\begin{aligned}
\lim_{n \to +\infty} \int f d\mu_{W_n|A} &= \lim_{n \to +\infty} \int f(W_n) d\mathbb{P}[\,\cdot\,|A] = \lim_{n \to +\infty} \frac{1}{\mathbb{P}[A]} \int_A f(W_n) d\mathbb{P} \\
&= \frac{1}{\mathbb{P}[A]} \lim_{n \to +\infty} \int_A f d\mu_{W_n} = \frac{1}{\mathbb{P}[A]} \int_A f d\mu_W = \frac{1}{\mathbb{P}[A]} \int_A f(W) d\mathbb{P} \\
&= \int f(W) d\mathbb{P}[\,\cdot\,|A] = \int f d\mu_{W|A}. \quad \square
\end{aligned}
$$

3 Important Approximation Properties of Distributions And Convergence of Random Variables

Proposition 3.1. Given a probability space $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right)$ Let $\{A_i\}_{i \in \mathbb{N}} \subset \Sigma$, $\left\{\widetilde{A}_i\right\}_{i \in \mathbb{N}} \subset \widetilde{\Sigma}$ be a such that:

(i) $\{A_n\}_{n\in\mathbb{N}}$ is pairwise disjoint and $\left\{\widetilde{A}_i\right\}_{i\in\mathbb{N}}$ is pairwise disjoint.

(ii) $\mathbb{P}\left[\bigcup_{i\in\mathbb{N}} A_i\right] = \widetilde{\mathbb{P}}\left[\bigcup_{i\in\mathbb{N}} \widetilde{A}_i\right] = 1.$

Then given $Y \in \mathcal{L}^0(X;\mathbb{P})$, $\widetilde{Y} \in \mathcal{L}^0\left(X;\widetilde{\mathbb{P}}\right)$, if $\mathbb{1}_{A_i}Y \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{Y}$, for every $i \in \mathbb{N}$, then $Y \overset{D}{=} \widetilde{Y}$.

Proof. If $\mathbb{1}_{A_i}Y \overset{D}{=} \mathbb{1}_{\widetilde{A}_i}\widetilde{Y}$, for every $n \in \mathbb{N}$. then given measurable $\mathrm{C} \subset X$, we have

$$\begin{aligned}\mathbb{P}\left[Y^{-1}(\mathrm{C})\right] &= \mathbb{P}\left[\bigcup_{i\in\mathbb{N}}\left(Y^{-1}(\mathrm{C})\cap A_i\right)\right] = \sum_{i\in\mathbb{N}}\mathbb{P}\left[Y^{-1}(\mathrm{C})\cap A_i\right] = \sum_{i\in\mathbb{N}}\mathbb{P}\left[(\mathbb{1}_{A_i}Y)^{-1}(\mathrm{C})\right] \\ &= \sum_{i\in\mathbb{N}}\widetilde{\mathbb{P}}\left[\left(\mathbb{1}_{\widetilde{A}_i}\widetilde{Y}\right)^{-1}(\mathrm{C})\right] = \sum_{i\in\mathbb{N}}\widetilde{\mathbb{P}}\left[\widetilde{Y}^{-1}(\mathrm{C})\cap\widetilde{A}_i\right] = \widetilde{Y}\left[\bigcup_{i\in\mathbb{N}}\left(\widetilde{Y}^{-1}(\mathrm{C})\cap\widetilde{A}_i\right)\right] \\ &= \widetilde{\mathbb{P}}\left[\widetilde{Y}^{-1}(\mathrm{C})\right].\end{aligned}$$

□

%MENTION THAT EACH $X_{n,k}, X_k$ HAS A COUNTABLE IMAGE WITH ISOLATED POINTS
%TALK ABOUT COUNTABLE OF RANDOM VARIABLES HAVING A JOINT SEPARARABLE SUBSPACE THAT CONTAINS ALL IMAGES IN A PREVIOUS THEOREM
%MOVE THIS TO SECTION 2

Theorem 3.2. For every sequence $\{X_n\}_{n\in\mathbb{N}}$ of X-valued random variables and X-valued random variable X, for each $n \in \mathbb{N}$ there exists a series such that

$X_n = \sum_{k=0}^{\infty} X_{n,k}$, $X = \sum_{k=0}^{\infty} X_k$ of countably-valued random variables such that $||X_{n,k}||, ||X_k|| \leq 2^{-k}$, for $n, k \geq 1$, and satisfies the following three properties:

(i) $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $n \to +\infty$ if and only if $X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} X_k$ as $n \to +\infty$, for each $k \geq 0$.

(ii) $X_n \xrightarrow{\mathbb{P}} X$ as $n \to +\infty$ if and only if $X_{n,k} \xrightarrow{\mathbb{P}} X_k$ as $n \to +\infty$, for each $k \geq 0$.

(iii) $X_n \Rightarrow X$ as $n \to +\infty$ if and only if $(X_{n,0}, \ldots, X_{n,k}) \Rightarrow (X_1, \ldots, X_k)$ as $n \to +\infty$, for each $k \geq 0$.

(iv) It remains that way up to distribution equivalence, i.e., if we have a probability space $\left(\widetilde{\Omega}, \widetilde{\mathbb{P}}\right)$ and random variables $\left\{\widetilde{X}_{n,k}\right\}_{n\geq 1, k\geq 0}$ $\left\{\widetilde{X}_k\right\}_{k\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; X\right)$ such that

$\left(\widetilde{X}_{n,0}, \ldots, \widetilde{X}_{n,k}\right) \overset{D}{=} (X_{n,0}, \ldots, X_{n,k})$ and $\left(\widetilde{X}_{0}, \ldots, \widetilde{X}_{k}\right) \overset{D}{=} (X_0, \ldots, X_k)$ for each $k \geq 0,\ n \geq 1$, then $\widetilde{X}_n := \sum_{k=0}^{\infty} \widetilde{X}_{n,k}$ and $\widetilde{X} := \sum_{k=0}^{\infty} \widetilde{X}_k$ are well-defined and *(i)-(iii)* still hold. Moreover, we have $\widetilde{X}_n \overset{D}{=} X_n$ for each $n \geq 1$ and $\widetilde{X} \overset{D}{=} X$.

Outline of proof.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{X}$-Valued random variables and X be an $\mathcal{X}$-valued random variable. Set

$$\mathcal{X}_0 := \overline{\text{Span}_{\mathbb{C}}\left(X(\Omega) \cup \bigcup_{n\in\mathbb{N}} X_n(\Omega)\right)},$$

and note that $\mathcal{X}_0$ is an $\mathbb{P}$-a.s. separable subspace of $\mathcal{X}$ containing each image of X_n for all $n \geq 1$ and X , since the complex-rational span

$$\mathcal{S} := \text{Span}_{\mathbb{Q}[i]}\left(\{x_k\}_{k\in\mathbb{N}} \cup \bigcup_{n\in\mathbb{N}} \{x_{n,k}\}_{k\in\mathbb{N}}\right)$$

is the $\mathbb{Q}[i]$-linear combinations of countable union of $\mathbb{P}$-a.s. countable dense sets--i.e. $\{x_k\}_{k\in\mathbb{N}}$ of $X(\Omega)$ and $\{x_{n,j}\}_{j\in\mathbb{N}}$ of $X_n(\Omega)$ for each $n \geq 1$--is a countable $\mathbb{P}$-a.s. dense set of $\mathcal{X}_0$. Choose a countable basis $\{b_j\}_{j\in\mathbb{N}}$ of $\mathcal{X}_0$ such that $||b_j|| = 1$, for each $j \geq 1$, and note by the **Hahn-Banach Theorem** that for each $j_0 \geq 1$, there exists $\varphi_{b_{j_0}} \in \mathcal{X}^*$ such that

$$\varphi_{b_{j_0}}\left(\sum_{k=1}^{\infty} a_j b_j\right) = a_{j_0}, \text{ for every } \sum_{j=1}^{\infty} a_j b_j \in \mathcal{X}_0.$$

Define

%DOUBLE CHECK THIS DEFINITION

$$X_{n,0} := \sum_{j=1}^{\infty} 2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X_n) \rfloor b_j,$$

$$X_0 := \sum_{j=1}^{\infty} 2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X) \rfloor b_j,$$

$$X_{n,k+1} := \sum_{j=1}^{\infty} \left(2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right) b_j,$$

$$X_{k+1} := \sum_{j=1}^{\infty} \left(2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X) \rfloor \right) b_j,$$

where $\lfloor a \rfloor := \lfloor \mathrm{Re}(a) \rfloor + i \lfloor \mathrm{Im}(a) \rfloor$, for all $a \in \mathbb{C}$. Since

$$\left| 2^{-(j+1)} \lfloor 2^{j+1} a \rfloor - 2^{-j} \lfloor 2^j a \rfloor \right| \leq 2 \cdot 2^{-(j+1)} = 2^{-j},$$

$$a = 2^{-j} \lfloor 2^j a \rfloor + \sum_{k=1}^{\infty} \left[2^{-(j+k+1)} \lfloor 2^{m+j+1} a \rfloor - 2^{-(j+k)} \lfloor 2^{j+k} a \rfloor \right],$$

for all $j \geq 1$, note for $k_0 \geq 1$ that

$$||X_{n,k_0}|| \leq \sum_{j=1}^{\infty} \left| 2^{-(k_0+j+1)} \lfloor 2^{k_0+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k_0+j)} \lfloor 2^{k_0+j} \varphi_{b_j}(X_n) \rfloor \right| ||b_j|| \leq \sum_{j=1}^{\infty} 2^{-(k_0+j)} = 2^{-k_0},$$

$$\begin{aligned}
&\implies \left|\left| X_n - \sum_{k=1}^{k_0} X_{n,k} \right|\right| \\
&= \left|\left| \sum_{j=1}^{\infty} \left(\varphi_{b_j}(X_n) - \sum_{k=0}^{k_0} [\varphi_{b_j}(X_{n,k})] \right) b_j \right|\right| \\
&= \left|\left| \sum_{j=1}^{\infty} \left[2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X_n) \rfloor + \sum_{k=1}^{\infty} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right.\right.\right. \\
&\qquad \left.\left.\left. - \left(2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(X_n) \rfloor + \sum_{k=1}^{k_0} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right) \right] b_j \right|\right| \\
&= \left|\left| \sum_{j=1}^{\infty} \left[\sum_{k=k_0+1}^{\infty} \left[2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(X_n) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(X_n) \rfloor \right] \right] b_j \right|\right| \\
&\leq \sum_{j=1}^{\infty} \left[\sum_{k=k_0+1}^{\infty} 2^{-(j+k)} ||b_j|| \right] \\
&= 2^{-k_0},
\end{aligned}$$

which shows that $\sum_{k=0}^{k_0} X_{n,k} \to X_n$ uniformly as $k_0 \to +\infty$, for every $n \geq 1$, and by similar derivation we have $||X_{k_0}|| \leq 2^{-k_0}$ for all $k_0 \geq 1$ and $\sum_{k=0}^{k_0} X_k \to X$ uniformly as $k_0 \to +\infty$.

%INSTEAD TALK ABOUT $\mathcal{L}^\infty$-CONVERGENCE

%SIMPLIFY RESULTS USING DIAGONALIZATION

(i)

$\implies$ Suppose $X_n \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $n \to +\infty$. Then $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}\text{-a.s.}} \varphi_{b_j}(X)$ as $n \to +\infty$ for each $j \geq 1$, and $\mathbb{P}$-a.s. convergence of $X_{n,0}, X_{n,k+1}$ for each $k \geq 1$ follows.

$\impliedby$ Suppose conversely $X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} X_k$ as $n \to +\infty$ for each $k \geq 1$. Noting that $\sum_{k=0}^{N} X_{n,k} \xrightarrow{\mathbb{P}\text{-a.s.}} \sum_{k=0}^{N} X_k$, we find that given $\epsilon > 0$, we find $\mathbb{P}$-a.s. we can choose $N \geq 1$ sufficiently large such that $\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| \leq 2^{-N} < \epsilon/3$, for every $n \geq 1$ and $\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| < \epsilon/3$ eventually for $n \geq 1$, and we can show $\mathbb{P}$-a.s. convergence of $X_n = \sum_{k=0}^{\infty} X_{n,k}$ to $X = \sum_{k=0}^{\infty} X_k$ from there.

(ii)

$\implies$ Similar to proving *(i)*, we find the hypothesis implies $\varphi_{b_j}(X_n) \xrightarrow{\mathbb{P}} \varphi_{b_j}(X)$ as $n \to +\infty$ for each $j \geq 1$, and convergence in $\mathbb{P}$ of $X_{n,0}, X_{n,k+1}$ for each $k \geq 1$ follows.

$\impliedby$ Similar to proving *(i)*, we note $\sum_{k=0}^{N} X_{n,k} \xrightarrow{\mathbb{P}} \sum_{k=0}^{N} X_k$ and we can choose $N \geq 1$ sufficiently large such that $\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| \leq 2^{-N} < \epsilon/3$, for every $n \geq 1$, and since

$$\lim_{n \to +\infty} \mathbb{P}\left[\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3\right] = 0 \text{ as } n \to +\infty$$

and

$$||X_n - X|| \geq \epsilon \implies \left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3 \vee \left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right| \geq \epsilon/3 \vee \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| \geq \epsilon/3,$$

which in turn implies $\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3$, since $\left|\left|\sum_{k=N+1}^{\infty} X_{n,k}\right|\right|, \left|\left|\sum_{k=N+1}^{\infty} X_k\right|\right| < \epsilon/3$, we find $\mathbb{P}[||X_n - X|| \geq \epsilon] = O\left(\mathbb{P}\left[\left|\left|\sum_{k=0}^{N} X_{n,k} - \sum_{k=0}^{N} X_k\right|\right| \geq \epsilon/3\right]\right)$ our conclusion follows.

(iii)

$\Longrightarrow$ Note that the mapping

$$\phi_k : x \mapsto ((x)_0, \ \dots, (x)_k) \text{ where } (x)_0 := \sum_{j=1}^{\infty} 2^{-(j+1)} \lfloor 2^{j+1} \varphi_{b_j}(x) \rfloor b_j,$$

$$(x)_{k_0+1} := \sum_{j=1}^{\infty} (2^{-(k+j+1)} \lfloor 2^{k+j+1} \varphi_{b_j}(x) \rfloor - 2^{-(k+j)} \lfloor 2^{k+j} \varphi_{b_j}(x \rfloor) b_j, \ \forall 0 \leq k_0 < k \qquad (3.1)$$

is an almost everywhere continuous mapping from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), m_{\mathcal{B}(\mathcal{X})}) \to \left(\mathcal{X}^k, \mathcal{B}\left(\mathcal{X}^k\right), m_{\mathcal{B}\left(\mathcal{X}^k\right)}\right)$.

%EXPLAIN WHY THAT CONTINUITY IS

It immediately follows that $X_n \Rightarrow X$ implies $(X_{n,0}, \dots, X_{n,k}) \Rightarrow (X_0, \dots, X_k)$, for every $k \geq 0$
%MENTION THOEREM THAT STATES THIS

$\Longleftarrow$ We find for each $k \geq 0$ by hypothesis that for each $N \geq 1$, we have $\sum_{k=0}^{N} X_{n,k} \Rightarrow \sum_{k=0}^{N} X_k$ .Given $f \in C_b(\mathcal{X}; \mathbb{R})$, we find for each $N \geq 0$, we have

$$\lim_{n \to +\infty} \int f d\mu_{\sum_{k=0}^{N} X_{n,k}} = \int f d\mu_{\sum_{k=0}^{N} X_k}. \quad (3.2)$$

Moreover, we find since $\sum_{k=0}^{N} X_k \xrightarrow{u} X$ as $N \to +\infty$, we find by continuity of f that $f\left(\sum_{k=0}^{N} X_k\right) \xrightarrow{u} f(X)$ as $N \to +\infty$, Then for each $n \geq 1$, we can choose $N_n \geq 0$ such that $N_{n+1} > N_n$ and sufficiently large such that

$$\left| f\left(\sum_{k=0}^{N_n} X_k\right) - f(X) \right| \leq n^{-1}. \quad (3.3)$$

We find that

$$\begin{aligned}
&\lim_{n \to +\infty} \left| \int f d\mu_{X_n} - \int f d\mu_X \right| \\
&\leq \lim_{n \to +\infty} \left(\left| \int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} \right| + \left| \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k} \right| \right. \\
&\qquad \left. + \left| \int f d\mu_{\sum_{k=0}^{N} X_k} - \int f d\mu_X \right| \right) \\
&\leq \lim_{n \to +\infty} \left(\left| \int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} \right| \right) + \lim_{n \to +\infty} \left(\left| \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k} \right| \right) \\
&\qquad + \lim_{n \to +\infty} \mathbb{E}\left[\left| f\left(\sum_{k=0}^{N_n} X_k\right) - f(X) \right| \right].
\end{aligned}$$

By (3.2) and diagonalization we have

$$\left|\int f d\mu_{X_n} - \int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}}\right|, \left|\int f d\mu_{\sum_{k=0}^{N_n} X_{n,k}} - \int f d\mu_{\sum_{k=0}^{N_n} X_k}\right| \to 0 \text{ as } n \to +\infty,$$

and by (3.3) we have $\mathbb{E}\left[\left|f\left(\sum_{k=0}^{N_n} X_k\right) - f(X)\right|\right] = O\left(n^{-1}\right)$ as $n \to +\infty$, and we conclude that $\lim_{n \to +\infty}\left|\int f d\mu_{X_n} - \int f d\mu_X\right| = 0$, and our conclusion has been reached.

(iv) Suppose we have random variables $\left\{\widetilde{X}_{n,k}\right\}_{n\geq 1, k\geq 0}, \left\{\widetilde{X}_k\right\}_{k\geq 0} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,k})$ and $\left(\widetilde{X}_0, \dots, \widetilde{X}_k\right) \overset{D}{=} (X_0, \dots, X_k)$ for each $k \geq 0,\ n \geq 1$. It follows immediately by this hypothesis that $\widetilde{\mathbb{P}}$-a.s., we have $||\widetilde{X}_{n,k}||, ||\widetilde{X}_k|| \leq 2^{-k}$ for $n, k \geq 1$, and hence well-definedness of $\widetilde{X}_n := \sum_{k=0}^{\infty} \widetilde{X}_{n,k}$ for every $n \geq 1$ and $\widetilde{X} := \sum_{k=0}^{\infty} \widetilde{X}_k$ immediately follow.

To prove $\implies$ of *(i)*-*(iii)*, we note that we can explicitly construct $\left\{\widetilde{X}_{n,k}\right\}_{n\geq 1, k\geq 0}, \left\{\widetilde{X}_k\right\}_{k\geq 0} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that *(i)*-*(iii)* hold by using the $m_{\mathfrak{B}(\mathcal{X})}$-almost everywhere continuous mappings $(-)_k : \mathcal{X} \to \mathcal{X}$ defined as in (3.1), i.e., given $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right),\ \widetilde{X} \in \mathfrak{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$, , we can set

$$\overline{\widetilde{X}}_{n,k} := \left(\widetilde{X}_n\right)_k,\ \overline{\widetilde{X}}_k := \left(\widetilde{X}\right)_k,$$

and we find by construction of $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}, \widetilde{X}$ that $\widetilde{\mathbb{P}}$-a.s., we have $\widetilde{X}_{n,k} = \overline{\widetilde{X}}_{n,k}$ and $\widetilde{X}_k = \overline{\widetilde{X}}_k$ for $n \geq 1,\ k \geq 0$. Conditions *(i)*-*(iii)* working for $\left\{\widetilde{X}_{n,k}\right\}_{n\geq 1, k\geq 0}, \left\{\widetilde{X}_k\right\}_{k\geq 0}, \left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}}, \widetilde{X}$ follows

immediately from $\widetilde{\mathbb{P}}$-a.s. equality of $\left\{\widetilde{X}_{n,k}\right\}_{n\geq 1, k\geq 0}$, $\left\{\widetilde{X}_{k}\right\}_{k\geq 0}$ to the explicit construction $\left\{\overline{\widetilde{X}}_{n,k}\right\}_{n\geq 1, k\geq 0}$, $\left\{\overline{\widetilde{X}}_{k}\right\}_{k\geq 0}$, hence the proof proceeds identically to the original proofs of *(i)*-*(iii)*.
%MAKE SURE TO TALK ABOUT $\mathcal{L}^{\infty}$ CONVERGENCE IN PLACE OF UNIFORM CONVERGENCE TO SHOW THAT IT IS IDENTICAL

Finally, to show that $\widetilde{X}_n \overset{D}{=} X_n$ for each $n \geq 1$ and $\widetilde{X} \overset{D}{=} X$, note that since by hypotheses that $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,N}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,N})$ and $\left(\widetilde{X}_{0}, \dots, \widetilde{X}_{N}\right) \overset{D}{=} (X_0, \dots, X_k)$ for each $N \geq 0,\ n \geq 1$, we find for each $N \geq 0$ we have $\sum_{k=0}^{N} \widetilde{X}_{n,k} \overset{D}{=} \sum_{k=0}^{N} X_{n,k}$ for each $n \geq 1$ and $\sum_{k=0}^{N} \widetilde{X}_{k} \overset{D}{=} \sum_{k=0}^{N} X_{k}$. Since $\mathcal{L}^{\infty}$-convergence implies convergence in distribution, we have $\sum_{k=0}^{N} X_{n,k} \Rightarrow X_n, \sum_{k=0}^{N} \widetilde{X}_{n,k} \Rightarrow \widetilde{X}_n$, for each $n \geq 1$ and $\sum_{k=0}^{N} X_{k} \Rightarrow X, \sum_{k=0}^{N} \widetilde{X}_{k} \Rightarrow \widetilde{X}$ as $N \to +\infty$, we find $\widetilde{X}_n \overset{D}{=} X_n$ for each $n \geq 1$ and $\widetilde{X} \overset{D}{=} X$ follows by uniqueness of distribution limits up to distribution equivalence. □

%proposition 1.G of 4/29 Report
%thoerem 1.1

4 Skorohod's Theorem For Random Variables With a Countable Image of Isolated Points

%FINISH THE STEPS OF THE ROUGH DRAFT PROOF AND NOTE WHAT'S WRONG

Theorem 4.1. *Skorohod's Representation Theorem* holds such that $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, if $\{X_n\}_{n\in\mathbb{N}}, X$ are $\mathcal{X}$-valued random variables with a countable image of isolated points.

Lemma 4.2. *Skorohod's Representation Theorem* holds such that $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := ([0,1), \mathcal{B}([0,1)), m_{[0,1)})$, if $\{X_n\}_{n\in\mathbb{N}}$ are $\mathcal{X}$-valued random variables with a countable image of isolated points, and X is $\mathbb{P}$-a.s. constant.

Outline of proof.

Suppose $\{X_n\}_{n\in\mathbb{N}}$ are $\mathcal{X}$-valued random variables with a countable image of isolated points, and $X = c$ $\mathbb{P}$-a.s. for some $c \in \mathcal{X}$. For each $n \in \mathbb{N}$, enumerate Supp$(\mu_{X_n}) := \{x_{n,i}\}_{1\le i<\#\text{Supp}(X_n)+1}$ such that $x_{n,i}$ is ordered from closest to x to furthest, i.e., for every $i_0 \in \mathbb{N}$, we have

$$\min\{||x_{n,i} - c|| : i_0 \le i\} = ||x_{n,i_0} - c||, \quad (4.1)$$

%POSSIBLY NUMBER THIS

which we can do since Supp(μ_{X_n}) is isolated. Then for every $n \in \mathbb{N}$, define $q_{n,i}$ recursively for $0 \le i < \#\text{Supp}(\mu_{X_n}) + 1$ $q_{n,i}$ by $q_{n,0} := 0$ and $q_{n,i+1} := q_{n,i} + \mathbb{P}[X_n = x_{n,i+1}]$. For each $n \in \mathbb{N}$, define $\widetilde{X}_n : \widetilde{\Omega} \to \mathcal{X}$ by $\widetilde{X}_n\left(\widetilde{\omega}\right) := x_{n,i}$, for $\widetilde{\omega} \in [q_{n,i-1}, q_{n,i})$, and set $\widetilde{X} : \widetilde{\Omega} \to \mathcal{X}$ equal to the constant c. We find by construction that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, and $X \overset{D}{=} \widetilde{X}$, and it remains to show that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$ and $n \to +\infty$.

To show that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$ (and more generally that $\widetilde{X}_n \xrightarrow{\text{pointwise}} \widetilde{X}$) as $n \to +\infty$, note by **Proposition 1.2** that since $\widetilde{X}_n \Rightarrow \widetilde{X}$ and $\widetilde{X} = c$, we find that $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}} \widetilde{X}$, as $n \to +\infty$. By (4.1), we find for all $\widetilde{\omega}, \widetilde{\omega}' \in \widetilde{\Omega}$ such that $\widetilde{\omega} \le \widetilde{\omega}'$, we find

$$||\widetilde{X}_n\left(\widetilde{\omega}\right) - \widetilde{X}\left(\widetilde{\omega}\right)|| = ||\widetilde{X}_n\left(\widetilde{\omega}\right) - c|| \le ||\widetilde{X}_n\left(\widetilde{\omega}'\right) - c|| = ||\widetilde{X}_n\left(\widetilde{\omega}'\right) - \widetilde{X}\left(\widetilde{\omega}'\right)||,$$

which shows that $||\widetilde{X}_n - \widetilde{X}||$ is monotonic on $\widetilde{\Omega}$, and our conclusion immediately follows by **Lemma 1.5**. ◻

Outline of proof of Theorem 4,1.

Suppose $\{X_n\}_{n\in\mathbb{N}}, X$ are $\mathcal{X}$-valued random variables with a countable image of isolated points. For every $x \in$ Supp(μ_X), set

$$X_n\left[X^{-1}(\{x\})\right] := \mathbb{1}_{X=x}X_n,\ X\left[X^{-1}(\{x\})\right] := \mathbb{1}_{X=x}X,\ \mathbb{P}_{X^{-1}(\{x\})} := \mathbb{P}[\,\cdot\,|X = x].$$

By Lemma 1.6, we find for every $x \in$ Supp(μ_X) that $X_n\left[X^{-1}(\{x\})\right] \Rightarrow X\left[X^{-1}(\{x\})\right]$, and it

follows by Proposition 2.2 that we have
$\left(X_n\left[X^{-1}(\{x\})\right], \mathbb{P}_{X^{-1}(\{x\})}\right) \Rightarrow \left(X\left[X^{-1}(\{x\})\right], \mathbb{P}_{X^{-1}(\{x\})}\right)$. Since $X\left[X^{-1}(\{x\})\right] = x$ $\mathbb{P}_{X^{-1}(\{x\})}$-a.s., by Lemma 4.2 we can then choose

$$\left\{\widehat{X}_n\left[X^{-1}(\{x\})\right]\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(m_{[0,1)};\mathcal{X}),\ \widehat{X}\left[X^{-1}(\{x\})\right] \in \mathcal{L}^0(m_{[0,1)};\mathcal{X}),$$

such that

$$\begin{aligned}
&\left(X_n\left[X^{-1}(\{x\})\right], \mathbb{P}_{X^{-1}(\{x\})}\right) \overset{D}{=} \left(\widehat{X}_n\left[X^{-1}(\{x\})\right], m_{[0,1)}\right), \text{ for every } n \in \mathbb{N},\\
&\left(X\left[X^{-1}(\{x\})\right], \mathbb{P}_{X^{-1}(\{x\})}\right) \overset{D}{=} \left(\widehat{X}_n\left[X^{-1}(\{x\})\right], m_{[0,1)}\right), \text{ and}\\
&\widehat{X}_n\left[X^{-1}(\{x\})\right] \xrightarrow{m_{[0,1)}\text{-a.s.}} \widehat{X}\left[X^{-1}(\{x\})\right] \text{ as } n \to +\infty.
\end{aligned} \tag{4.4}$$

Enumerate $\mathrm{Supp}(\mu_X) := \{x_i\}_{1\le i<\#\mathrm{Supp}(\mu_X)+1}$; recursively define $q_0 := 0$, $q_{i+1} := q_i + \mathbb{P}[X = x_i]$. Then for every $1 \le i < \#\mathrm{Supp}(\mu_X) + 1$, define $\left\{\widetilde{X}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0(m_{[0,1)};\mathcal{X}), \widetilde{X} \in \mathcal{L}^0(m_{[0,1)};\mathcal{X})$ by

$$\begin{aligned}
&\widetilde{X}_n\left(\widetilde{\omega}\right) := \sum_{i=1}^{\#\mathrm{Supp}(\mu_X)} \mathbb{1}_{[q_i,q_{i-1})}\left(\widetilde{\omega}\right) \cdot \left[\widehat{X}_n\left[X^{-1}(\{x_i\})\right]\left(\mathbb{P}[X = x_i]^{-1} \cdot \left(\widetilde{\omega} - q_{i-1}\right)\right)\right], \text{ for every } n \in \mathbb{N},\\
&\text{and } \widetilde{X}\left(\widetilde{\omega}\right) := \sum_{i=1}^{\#\mathrm{Supp}(\mu_X)} \mathbb{1}_{[q_i,q_{i-1})}\left(\widetilde{\omega}\right) \cdot \left[\widehat{X}\left[X^{-1}(\{x_i\})\right]\left(\mathbb{P}[X = x_i]^{-1} \cdot \left(\widetilde{\omega} - q_{i-1}\right)\right)\right],
\end{aligned} \tag{4.5}$$

for every $\widetilde{\omega} \in \widetilde{\Omega}$, and it remains to show that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, $\widetilde{X} \overset{D}{=} X$, and $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$.

To show that $\widetilde{X}_n \overset{D}{=} X_n$, for each $n \in \mathbb{N}$, and $\widetilde{X} \overset{D}{=} X$, we set

$$\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] := \mathbb{1}_{\widetilde{X}=x_i}\widetilde{X}_n,\ \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right] := \mathbb{1}_{\widetilde{X}=x_i}\widetilde{X},$$

and since $\left\{X^{-1}(\{x_i\})\right\}_{1\leq i<\#\text{Supp}(\mu_X)+1} \in \Sigma$, $\left\{\widetilde{X}^{-1}(\{x_i\})\right\}_{1\leq i<\#\text{Supp}(\mu_X)+1} \in \widetilde{\Sigma}$ are partitions such that

$$\mathbb{P}\left[\bigcup_{i=1}^{\#\text{Supp}(\mu_X)} X^{-1}(\{x_i\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{i=1}^{\#\text{Supp}(\mu_X)} \widetilde{X}^{-1}(\{x_i\})\right] = 1,$$

it shall suffice by Proposition 3.1 to show that $\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \overset{D}{=} X_n\left[X^{-1}(\{x_i\})\right]$, for every $i \in \mathbb{N}$. By construction of $\widetilde{X}$ given in (4.5) for every $i \in \mathbb{N}$, we have $\widetilde{X}^{-1}(\{x_i\}) = [q_{i-1}, q_i)$, since we have $X\left[X^{-1}(\{x_i\})\right] = x_i\ \mathbb{P}_{X^{-1}(\{x_i\})}$-a.s., which further implies $\widehat{X}\left[X^{-1}(\{x_i\})\right] = x_i\ \widetilde{\mathbb{P}}$-a.s. $\widehat{X}\left[X^{-1}(\{x_i\})\right] = x_i\ m_{[0,1)}$-a.s. Then we have

$$\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] = \mathbb{1}_{[q_i,q_{i-1})}\cdot\left[\widehat{X}_n\left[X^{-1}(\{x_i\})\right]\circ\left(\mathbb{P}[X = x_i]^{-1}\cdot((\,\cdot\,) - q_{i-1})\right)\right],\ \text{for every } n \in \mathbb{N},$$

$$\text{and } \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right] = \mathbb{1}_{[q_i,q_{i-1})}\cdot\left[\widehat{X}\left[X^{-1}(\{x_i\})\right]\circ\left(\mathbb{P}[X = x_i]^{-1}\cdot((\,\cdot\,) - q_{i-1})\right)\right], \tag{4.6}$$

hence by (4.4) we find given $\mathrm{C} \in \mathcal{B}(\mathcal{X})$ we have
%CONSIDER CITING PROPOSITION 1.4

$$\begin{aligned}\widetilde{\mathbb{P}}\left[\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \in \mathrm{C}\right] &= m_{[0,1)}\left(\left[\mathbb{P}[X = x_i]\cdot\left(\widehat{X}_n\left[X^{-1}(\{x_i\})\right]^{-1}(\mathrm{C})\right) + q_{i-1}\right] \cap [q_{i-1}, q_i)\right)\\ &= \mathbb{P}[X = x_i]\cdot m_{[0,1)}\left(\left(\widehat{X}_n\left[X^{-1}(\{x_i\})\right]^{-1}(\mathrm{C})\right)\right)\\ &= \mathbb{P}[X = x_i]\cdot m_{[0,1)}\left(\left(\widehat{X}_n\left[X^{-1}(\{x_i\})\right]^{-1}(\mathrm{C})\right)\right)\\ &= \mathbb{P}[X = x_i]\cdot \mathbb{P}\left[X_n\left[X^{-1}(\{x_i\})\right] \in \mathrm{C} | X = x_i\right]\\ &= \mathbb{P}\left[X_n\left[X^{-1}(\{x_i\})\right] \in \mathrm{C}\right],\end{aligned} \tag{4.7}$$

so our desired conclusion of $\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \overset{D}{=} X_n\left[X^{-1}(\{x_i\})\right]$, for every $n \in \mathbb{N}$, and subsequently $\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \overset{D}{=} X_n\left[X^{-1}(\{x_i\})\right]$.by similar derivation to (4.7), is met.

%CONSIDER CITING PROPSITION 1.4 IN THIS PARAGRAPH INSTEAD OF "THE OPEN

MAPPING PROPERTY"

Finally, to show $\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}$, it shall suffice by Proposition 1.5 to show that

$$\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] = \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})}\widetilde{X}_n \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \mathbb{1}_{\widetilde{X}^{-1}(\{x_i\})}\widetilde{X}_n = \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right],$$

for each $1 \leq i < \#\text{Supp}(\mu_X) + 1$, as $n \to +\infty$. Given $1 \leq i < \#\text{Supp}(\mu_X) + 1$, we find by (4.4) and the open-mapping property of $\left(\mathbb{P}[X = x_i]^{-1} \cdot ((\,\cdot\,) - q_{i-1})\right) : [q_{i-1}, q_i) \to [0, 1)$ to (4.6), we find that $\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \to \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right]$ $\widetilde{\mathbb{P}}$-a.s. on $[q_{i-1}, q_i)$ as $n \to +\infty$,and it immediately follows from the fact that $\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] = \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right] = 0$ on $\widetilde{\Omega} \setminus [q_{i-1}, q_i)$ that the conclusion of$\widetilde{X}_n\left[\widetilde{X}^{-1}(\{x_i\})\right] \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}\left[\widetilde{X}^{-1}(\{x_i\})\right]$ is reached. □

%EXPLAIN WHAT'S MEANT BY THIS
%TRIGGER LEMMA 4.2 AND A PROBABILITY TREE ARGUMENT TO GET POINTWISE CONVERGENCE OVERALL

Theorem 4.3. *Skorohod's Representation Theorem* holds such that $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0, 1)^{k+1}, \mathcal{B}\left([0, 1)^{k+1}\right), \bigotimes_{j=0}^{k} m_{[0,1)}\right)$ for $k \geq 0$ if $\{(X_{n,0}, \ \ldots, X_{n,k})\}_{n \in \mathbb{N}}, (X_0, \ \ldots X_k)$ are $k + 1$-dimensional $\mathcal{X}$-valued random variables with a countable image of isolated points.

Proof. In proving this partial verision of *Skorohod's Representation Theorem*, we end up proving the more general **Lemma 3.4**, which implies this result by using the following inductive argument:

The base case immediately follows by **Theorem 4.1**, since $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) = ([0, 1), \mathcal{B}([0, 1)), m_{[0,1)})$ in that situation. In the inductive step, where $\{(X_{n,0}, \ \ldots, X_{n,k+1})\}_{n \in \mathbb{N}}, (X_0, \ \ldots X_{k+1})$ are $k + 2$-dimensional $\mathcal{X}$-valued random variables with countable images, we find by the inductive hypothesis that for $\vec{X}_n := (X_{n,0}, \ \ldots, X_{n,k})$ for each $n \in \mathbb{N}$, $\vec{X} := (X_0, \ \ldots, X_k)$, there exists, for each $n \in \mathbb{N}$, some

$$\left\{\overline{\vec{X}}_n\right\}_{n \in \mathbb{N}} := \{(\overline{X}_{n,0}, \ \ldots, \overline{X}_{n,k})\}_{n \in \mathbb{N}} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right),$$

$\overset{\Rightarrow}{X} := (\overline{X}_0, \dots, \overline{X}_k) \in \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right)$,

such that we have $\vec{X}_n \overset{D}{=} \overset{\Rightarrow}{X}_n$, for each $n \in \mathbb{N}$, $\vec{X} \overset{D}{=} \overset{\Rightarrow}{X}$, and $\overset{\Rightarrow}{X}_n \xrightarrow{\bigotimes_{j=0}^{k} m_{[0,1)}\text{-a.s.}} \overset{\Rightarrow}{X}$ as $n \to +\infty$. By **Lemma 4.4**, we can define $\left\{\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right)\right\}_{n\in\mathbb{N}}, \left\{\widetilde{X}_0, \dots, \widetilde{X}_k\right\} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ as in (3.1), and choose $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k+1} m_{[0,1)}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\bigotimes_{j=0}^{k+1} m_{[0,1)}; \mathcal{X}\right)$, such that we have $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,k+1})$, for each $n \in \mathbb{N}$, $\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right) \overset{D}{=} (X_0, \dots, X_{k+1})$, and $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$ as $n \to +\infty$, satisfying the conclusion of *Skorohod's Representation Thoerem* for $\{(X_{n,0}, \dots, X_{n,k+1})\}_{n\in\mathbb{N}}, (X_0, \dots X_{k+1})$. $\square$

Lemma 4.4. Suppose given $k \geq 0$, we have that $\{(X_{n,0}, \dots, X_{n,k+1})\}_{n\in\mathbb{N}}, (X_0, \dots, X_{k+1})$ are $k+1$ dimensional $\mathcal{X}$-valued random vectors with a countable image of isolated points such that $(X_{n,0}, \dots, X_{n,k+1}) \Rightarrow (X_0, \dots, X_{k+1})$ as $n \to +\infty$ and there exists $\{\overline{X}_{n,0}, \dots, \overline{X}_{n,k}\}_{n\in\mathbb{N}}, \{\overline{X}_0, \dots, \overline{X}_k\} \subset \mathcal{L}^0\left(\bigotimes_{j=0}^{k} m_{[0,1)}; \mathcal{X}\right)$ such that for $\vec{X}_n := (X_{n,0}, \dots, X_{n,k})$, $\overset{\Rightarrow}{X}_n := (\overline{X}_{n,0}, \dots, \overline{X}_{n,k})$, for each $n \in \mathbb{N}$, and $\vec{X} := (X_0, \dots, X_k)$, $\overset{\Rightarrow}{X} := (\overline{X}_0, \dots, \overline{X}_k)$, we have $\vec{X}_n \overset{D}{=} \overset{\Rightarrow}{X}_n$, for each $n \in \mathbb{N}$, $\vec{X} \overset{D}{=} \overset{\Rightarrow}{X}$, and $\overset{\Rightarrow}{X}_n \xrightarrow{\bigotimes_{j=0}^{k} m\text{-a.s.}} \overset{\Rightarrow}{X}$ as $n \to +\infty$. Define $\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{k+2}, \mathcal{B}\left([0,1)^{k+2}\right), \bigotimes_{j=0}^{k+1} m_{[0,1)}\right)$, and define $\left\{\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right)\right\}_{n\in\mathbb{N}}, \left\{\widetilde{X}_0, \dots, \widetilde{X}_k\right\} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ by

$$\widetilde{X}_{n,j}\left(\widetilde{\omega}\right) := (\overline{X}_{n,j} \circ (\pi_0, \dots, \pi_k))\left(\widetilde{\omega}\right),\ \widetilde{X}_j\left(\widetilde{\omega}\right) := (\overline{X}_j \circ (\pi_0, \dots, \pi_k))\left(\widetilde{\omega}\right), \quad (3.1)$$

for each $n \in \mathbb{N}, 0 \leq j \leq k$. Then there exists $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ such that:

(i) for each $n \in \mathbb{N}$, we have $(X_{n,0}, \dots, X_{n,k+1}) \overset{D}{=} \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)$, and

$(X_0, \dots, X_{k+1}) \overset{D}{=} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$.

(ii) $\widetilde{X}_{n,k+1} \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widetilde{X}_{k+1}$ (and hence $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$) as $n \to +\infty$.

Official outline of proof.

%MAKE THIS PART OF PROOF INTO ITS OWN CLAIM

First, we shall prove this lemma in the special case that $X_{k+1} = c$ $\mathbb{P}$-a.s. for some $c \in \mathcal{X}$. For every $\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})$, set

$$X_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right] := \mathbb{1}_{\vec{X}_n=\vec{x}} X_{n,k+1},\ X\left[\vec{X}^{-1}(\{\vec{x}_n\})\right] := \mathbb{1}_{\vec{X}=\vec{x}} X,\ \mathbb{P}_{\vec{X}_n^{-1}(\{x\})} := \mathbb{P}[\,\cdot\,|\vec{X}_n = \vec{x}_n].$$

%FIGURE OUT CENTRAL PLACE TO DEFINE THIS GENERAL NOTATION

For each $n \in \mathbb{N}$, enumerate $\text{Supp}(\mu_{X_{n,k+1}}) := \{x_{n,i}\}_{1 \le i < \#\text{Supp}(\mu_{X_{n,k+1}})+1}$ such that $x_{n,i}$ is ordered from closest to c to furthest, i.e., for every $i_0 \in \mathbb{N}$, we have

$$\min\{||x_{n,i} - c|| : i_0 \le i\} = ||x_{n,i_0} - c||, \quad (4.1)$$

%POSSIBLY NUMBER THIS

which we can do since $\text{Supp}(\mu_{X_{n,k+1}})$ is isolated.

Then for every $n \in \mathbb{N}$, $\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})$, and $0 \le i < \#\text{Supp}(\mu_{X_{n,k+1}}) + 1$ define $q_{n,i}[\vec{x}_n]$ recursively by $q_{n,0}[\vec{x}_n] := 0$, $q_{n,i+1} := q_{n,i}[\vec{x}_n] + \mathbb{P}_{\vec{X}_n^{-1}(\{\vec{x}_n\})}\left[X_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right] = x_{n,i+1}\right]$.

Then for each $n \in \mathbb{N}$ and $0 \le i < \#\text{Supp}(\mu_{X_{n,k+1}}) + 1$, define $\widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right] \in \mathcal{L}^0(m_{[0,1)}; \mathcal{X})$ by $\widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right]\left(\widetilde{\omega}\right) := x_{n,i}$, for $\widetilde{\omega} \in [q_{n,i-1}[\vec{x}_n], q_{n,i}[\vec{x}_n])$. Next, for every $n \in \mathbb{N}$, define $\widetilde{\vec{X}}_n := \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k}\right)$, $\widetilde{\vec{X}} := \left(\widetilde{X}_0, \dots, \widetilde{X}_k\right)$, and define $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$, $\widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}}; \mathcal{X}\right)$ by

$$\widetilde{X}_{n,k+1} := \sum_{\vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n})} \mathbb{1}_{\widetilde{\vec{X}}_n = \vec{x}_n}\left[\widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right] \circ \pi_{k+1}\right],$$

%POSSIBLY NUMBER THIS DEFINITION

and $\widetilde{X}_{k+1} := c$. We shall next prove *(i)* and *(ii)* hold for $\left\{\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)\right\}_{n\in\mathbb{N}}, \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$.

To show (i), since $\left\{\vec{X}_n^{-1}(\{\vec{x}_n\})\right\}_{\vec{x}_n\in\text{Supp}(\mu_{\vec{X}_n})} \in \Sigma$, $\left\{\widetilde{\vec{X}}_n^{-1}(\{\vec{x}_n\})\right\}_{\vec{x}_n\in\text{Supp}(\mu_{\vec{X}_n})} \subset \widetilde{\Sigma}$ are partitions such that

$$\mathbb{P}\left[\bigcup_{\vec{x}_n\in\text{Supp}(\mu_{\vec{X}_n})} \vec{X}_n^{-1}(\{\vec{x}_n\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{\vec{x}_n\in\text{Supp}(\mu_{\vec{X}_n})} \widetilde{\vec{X}}_n^{-1}(\{\vec{x}_n\})\right] = 1,$$

%ALSO MAKE AND CITE PROPOSITION ABOUT THE PRODUCT BOREL SET OF A BANACH SPACE

it shall suffice by Proposition 3.1 to show that given $n \in \mathbb{N}$, we have $\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) \overset{D}{=} (X_{n,0}, \dots, X_{n,k+1})$, by showing that

$$\mathbb{1}_{\widetilde{\vec{X}}_n=\vec{x}_n}\left(\widetilde{\vec{X}}_n, \widetilde{X}_{n,k+1}\right) \overset{D}{=} \mathbb{1}_{\vec{X}_n=\vec{x}_n}(\vec{X}_n, X_{n,k+1}),\ \text{for every } \vec{x}_n \in \text{Supp}(\mu_{\vec{X}_n}). \quad (4.2)$$

Note first that for every $\mathrm{C} \in \mathcal{B}(\mathcal{X})$, we have

$$\begin{aligned} m_{[0,1)}\left(\widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right]^{-1}(\mathrm{C})\right) &= m_{[0,1)}\left(\bigcup_{x_{n,i}\in\text{Supp}(\mu_{X_{n,k+1}})\cap\mathrm{C}} \widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right]^{-1}(\{x_{n,i}\})\right) \\ &= \sum_{x_{n,i}\in\text{Supp}(\mu_{X_{n,k+1}})\cap\mathrm{C}} m_{[0,1)}\left(\widehat{X}_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right]^{-1}(\{x_{n,i}\})\right) \\ &= \sum_{x_{n,i}\in\text{Supp}(\mu_{X_{n,k+1}})\cap\mathrm{C}} m_{[0,1)}([q_{n,i-1}[\vec{x}_n], q_{n,i}[\vec{x}_n]]) \\ &= \sum_{x_{n,i}\in\text{Supp}(\mu_{X_{n,k+1}})\cap\mathrm{C}} \mathbb{P}_{\vec{X}_n^{-1}(\{\vec{x}_n\})}\left[X_{n,k+1}\left[\vec{X}_n^{-1}(\{\vec{x}_n\})\right] = x_{n,i}\right] \end{aligned}$$

$$= \mathbb{P}_{\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})}\left[\bigcup_{x_{n,i} \in \mathrm{Supp}(\mu_{X_{n,k+1}}) \cap \mathrm{C}} \left\{X_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right] = x_{n,i}\right\}\right]$$

$$= \mathbb{P}_{\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})}\left[X_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right] \in \mathrm{C}\right],$$

which shows that $\left(\widehat{X}_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right], m_{[0,1)}\right) \overset{D}{=} \left(X_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right], \mathbb{P}_{\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})}\right)$.

Additionally noting by hypothesis we have $\overrightarrow{\overrightarrow{X}}_n \overset{D}{=} \overrightarrow{X}_n$, it follows that for every $\mathrm{C}_0, \ \dots, \mathrm{C}_{k+1} \in \mathcal{B}(\mathcal{X})$, we have

$$\widetilde{\mathbb{P}}\left[\mathbb{1}_{\widetilde{\overrightarrow{X}}_n = \vec{x}_n}\left(\widetilde{\overrightarrow{X}}_n, \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] = \widetilde{\mathbb{P}}\left[\left\{\mathbb{1}_{\widetilde{\overrightarrow{X}}_n = \vec{x}_n} \widetilde{\overrightarrow{X}}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \left\{\mathbb{1}_{\widetilde{\overrightarrow{X}}_n = \vec{x}_n} \widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$

$$= \widetilde{\mathbb{P}}\left[\left\{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \left\{\mathbb{1}_{\widetilde{\overrightarrow{X}}_n = \vec{x}_n} \widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$

$$= \mathbb{1}_{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j} \widetilde{\mathbb{P}}\left[\left\{\widetilde{\overrightarrow{X}}_n = \vec{x}_n\right\} \cap \left\{\mathbb{1}_{\widetilde{\overrightarrow{X}}_n = \vec{x}_n} \widetilde{X}_{n,k+1} \in \mathrm{C}_{k+1}\right\}\right]$$

$$= \mathbb{1}_{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j} \widetilde{\mathbb{P}}\left[\left\{\overrightarrow{\overrightarrow{X}}_n \circ (\pi_0, \ \dots, \pi_k) = \vec{x}_n\right\} \cap \left\{\widehat{X}_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right] \circ \pi_{k+1} \in \mathrm{C}_{k+1}\right\}\right]$$

$$= \mathbb{1}_{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j}\left[\left(\bigotimes_{j=0}^{k} m_{[0,1)}\right)\left(\overrightarrow{\overrightarrow{X}}_n^{-1}(\{\vec{x}_n\})\right) \cdot m_{[0,1)}\left(\widehat{X}_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right]^{-1}(\mathrm{C}_{k+1})\right)\right]$$

$$= \mathbb{1}_{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j} \mathbb{P}[\overrightarrow{X}_n = \vec{x}_n] \cdot \mathbb{P}_{\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})}\left[X_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right] \in \mathrm{C}_{k+1}\right]$$

$$= \mathbb{1}_{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j} \mathbb{P}\left[\{\overrightarrow{X}_n = \vec{x}_n\} \cap \left\{X_{n,k+1}\left[\overrightarrow{X}_n^{-1}(\{\vec{x}_n\})\right] \in \mathrm{C}_{k+1}\right\}\right]$$

$$= \mathbb{P}\left[\left\{\vec{x}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \{\mathbb{1}_{\overrightarrow{X}_n = \vec{x}_n} X_{n,k+1} \in \mathrm{C}_{k+1}\}\right]$$

$$= \mathbb{P}\left[\left\{\mathbb{1}_{\overrightarrow{X}_n = \vec{x}_n} \overrightarrow{X}_n \in \prod_{j=0}^{k} \mathrm{C}_j\right\} \cap \{\mathbb{1}_{\overrightarrow{X}_n = \vec{x}_n} X_{n,k+1} \in \mathrm{C}_{k+1}\}\right]$$

$$= \mathbb{P}\left[\mathbb{1}_{\vec{X}_n = \vec{x}_n} (\vec{X}_n, X_{n,k+1}) \in \prod_{j=0}^{k+1} \mathrm{C}_j \right],$$

which shows that (4.2) holds, and we conclude that $\left(\widetilde{X}_{n,0}, \ \dots, \widetilde{X}_{n,k+1}\right) \overset{D}{=} (X_{n,0}, \ \dots, X_{n,k+1})$. Since $X_{k+1} = c$ $\mathbb{P}$-a.s., and $\widetilde{X}_{k+1} = c$, and

$$\mu_{\tilde{\vec{X}}} = \widetilde{\mathbb{P}}\left[\tilde{\vec{X}}^{-1}(\,\cdot\,)\right] = \left[\widetilde{\mathbb{P}} \circ (\pi_0, \ \dots, \pi_k)^{-1}\right]\left(\vec{\vec{X}}^{-1}(\,\cdot\,)\right) = \left(\bigotimes_{j=0}^{k} m_{[0,1)}\right)\left(\vec{\vec{X}}^{-1}(\,\cdot\,)\right) = \mu_{\vec{\vec{X}}},$$

we find by hypothesis that $\tilde{\vec{X}} \overset{D}{=} \vec{X} \overset{D}{=} \vec{X}$, hence we have

$$\left(\widetilde{X}_0, \ \dots, \widetilde{X}_{k+1}\right) \overset{D}{=} \left(\tilde{\vec{X}}, c\right) \overset{D}{=} (\vec{X}, c) \overset{D}{=} (X_0, \ \dots, X_{k+1}),$$

so we conclude that (i) is met.

To prove (ii)
%USE SIMILAR STRATEGY TO LEMMA 4.1

Next, we'll prove this thoerem in the general case by defining $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}}, \widetilde{X}_{k+1}$ in similar fashion to the proof of Theorem 4.2. For every $x \in$ Supp$(\mu_{X_{k+1}})$, set

$$X_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right] := \mathbb{1}_{X_{k+1}=x} X_{n,k+1}, \ X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right] := \mathbb{1}_{X_{k+1}=x} X_{k+1},$$
$$\mathbb{P}_{X_{k+1}^{-1}(\{x\})} := \mathbb{P}[\,\cdot\,| X_{k+1} = x].$$

By Lemma 1.6, we find for every $x \in$ Supp$(\mu_{X_{k+1}})$ that $X_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right] \Rightarrow X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right]$, and it follows by Proposition 2.2 that we have $\left(X_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right], \mathbb{P}_{X_{k+1}^{-1}(\{x\})}\right) \Rightarrow \left(X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right], \mathbb{P}_{X_{k+1}^{-1}(\{x\})}\right)$ as $n \to +\infty$. Since $X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right] = x \ \mathbb{P}_{X_{k+1}^{-1}(\{x\})}$-a.s., we find that the previously proven special case holds for $\left\{X_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right\}_{n\in\mathbb{N}}, X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right]$ as random variables of $(\Omega, \Sigma, \mathbb{P}_{X_{k+1}^{-1}(\{x\})})$. Setting $\tilde{\vec{X}}_n := \left(\widetilde{X}_{n,0}, \ \dots, \widetilde{X}_{n,k}\right)$, we can then choose

$$\left\{\widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right),\ \widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x\})\right] \in \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right),$$

such that

$$\left(\left(\overrightarrow{X}_n, X_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right), \mathbb{P}_{X_{k+1}^{-1}(\{x\})}\right) \overset{D}{=} \left(\left(\overrightarrow{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right), \widetilde{\mathbb{P}}\right) \text{ for every } n\in\mathbb{N},$$
$$\left(\left(\overrightarrow{X}, X_{k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right), \mathbb{P}_{X_{k+1}^{-1}(\{x\})}\right) \overset{D}{=} \left(\left(\overrightarrow{\widetilde{X}}, \widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right), \widetilde{\mathbb{P}}\right),$$
$$\text{and } \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right] \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \widehat{X}\left[X^{-1}(\{x\})\right],$$
$$\left(\text{and more generally } \left(\overrightarrow{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right) \xrightarrow{\widetilde{\mathbb{P}}\text{-a.s.}} \left(\overrightarrow{\widetilde{X}}, \widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x\})\right]\right)\right), \text{ as } n\to+\infty. \quad (4.4)$$

As before, we enumerate $\operatorname{Supp}(\mu_{X_{k+1}}) := \{x_i\}_{1\le i<\#\operatorname{Supp}(\mu_{X_{k+1}})+1}$; and define $q_0 := 0$, $q_{i+1} := q_i + \mathbb{P}[X_{k+1} = x_i]$. Then define $\left\{\widetilde{X}_{n,k+1}\right\}_{n\in\mathbb{N}} \subset \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right), \widetilde{X}_{k+1} \in \mathcal{L}^0\left(\widetilde{\mathbb{P}};\mathcal{X}\right)$ by

%REFER TO THE PREVIOUS PROOF

%FIGURE OUT HOW TO DEFINE THESE FUNCTIONS
%FIND LETTER INDEXED BY x_i FOR TRANSLATION FUNCTION

$$\widetilde{X}_{n,k+1} := \sum_{i=1}^{\#\operatorname{Supp}(\mu_{X_{k+1}})} \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left[\widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\circ\phi_i\right], \text{ for every } n\in\mathbb{N}, \text{ and}$$
$$\widetilde{X}_{k+1} := \sum_{i=1}^{\#\operatorname{Supp}(\mu_{X_{k+1}})} \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left[\widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\circ\phi_i\right], \quad (4.5)$$

where

$$\phi_i := \left(\pi_0,\ \ldots,\pi_k,\ \min\left(\mathbb{P}[X_{k+1}=x_i]^{-1}\cdot(\pi_{k+1}-q_{i-1}),1\right)\right).$$

It remains to show that *(i)* and *(ii)* holds for $\left\{\left(\widetilde{X}_{n,0},\ \ldots,\widetilde{X}_{n,k+1}\right)\right\}_{n\in\mathbb{N}}$, $\left(\widetilde{X}_0,\ \ldots,\widetilde{X}_{k+1}\right)$ (in the general case).

To show that *(i)* (in the general case) holds, we shall prove

$(X_{n,0}, \dots, X_{n,k+1}) \overset{D}{=} \left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right)$, given $n \in \mathbb{N}$, and let $(X_0, \dots, X_{k+1}) \overset{D}{=} \left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right)$ follow by similarity. Since $\left\{X_{k+1}^{-1}(\{x_i\})\right\}_{1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}})+1} \in \Sigma$, $\left\{\widetilde{X}_{k+1}^{-1}(\{x_i\})\right\}_{1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}})+1} \subset \widetilde{\Sigma}$ are partitions such that

$$\mathbb{P}\left[\bigcup_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})+1} X_{k+1}^{-1}(\{x_i\})\right] = \widetilde{\mathbb{P}}\left[\bigcup_{i=1}^{\#\mathrm{Supp}(\mu_{X_{k+1}})+1} \widetilde{X}_{k+1}^{-1}(\{x_i\})\right] = 1,$$

it shall suffice by Proposition 3.1 to show that

$$\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots \widetilde{X}_{n,k+1}\right) \overset{D}{=} \mathbb{1}_{X_{k+1}=x_i}(X_{n,0}, \dots, X_{n,k+1}), \text{ for every } 1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1, \quad (4.6)$$

By construction of $\widetilde{X}_{k+1}$ given in (4.5) we find given $1 \le i < \#\mathrm{Supp}(\mu_{X_{k+1}}) + 1$, we have $\widetilde{X}_{k+1}^{-1}(\{x_i\}) = \pi_{k+1}^{-1}[q_{i-1}, q_i)$, since we have $X_{k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] = x_i \ \mathbb{P}_{X_{k+1}^{-1}(\{x_i\})}$-a.s., which further implies $\widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] = x_i \ \widetilde{\mathbb{P}}$-a.s. Then we have

$$\begin{aligned}
&\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots, \widetilde{X}_{n,k+1}\right) = \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\vec{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] \circ \phi_i\right), \\
&\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_0, \dots, \widetilde{X}_{k+1}\right) = \mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\vec{\widetilde{X}}, \widehat{X}_{k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] \circ \phi_i\right).
\end{aligned} \qquad (4.7)$$

Next, given $\mathrm{C}_0, \dots, \mathrm{C}_{k+1} \in \mathcal{B}(\mathcal{X})$, we find by (4.7) that

$$\begin{aligned}
&\left\{\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \dots \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right\} \\
&= \left[\mathbb{1}_{\pi_{k+1}^{-1}[q_{i-1},q_i)}\left(\vec{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] \circ \phi_i\right)\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \\
&= \left(\left(\vec{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] \circ \phi_i\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1}, q_i)\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)
\end{aligned}$$

$$= \left(\left[\left[\left(\overrightarrow{\overrightarrow{X}}_n \circ \phi_i^{-1} \circ \phi_i, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right] \circ \phi_i\right)\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1}, q_i)\right)$$

$$\cup\, \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)$$

$$= \left(\left[\left[\left(\overrightarrow{\overrightarrow{X}}_n \circ \phi_i^{-1}, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right]\right) \circ \phi_i\right]^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right) \cap \pi_{k+1}^{-1}[q_{i-1}, q_i)\right)$$

$$\cup\, \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)$$

$$= \phi_i^{-1}\left(\left(\overrightarrow{\overrightarrow{X}}_n \circ \phi_i^{-1}, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right]\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)$$

$$= \phi_i^{-1}\left(\bigcap_{j=0}^{k}\left[\left(\overline{X}_{n,j} \circ (\pi_0, \ \ldots, \pi_{k+1}) \circ \phi_i^{-1}\right)^{-1}(\mathrm{C}_j)\right] \cap \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]^{-1}(\mathrm{C}_{k+1})\right)$$

$$\cup\, \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)$$

$$= \phi_i^{-1}\left(\bigcap_{j=0}^{k}\left[(\overline{X}_{n,j} \circ (\pi_0, \ \ldots, \pi_{k+1}))^{-1}(\mathrm{C}_j)\right] \cap \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]^{-1}(\mathrm{C}_{k+1})\right)$$

$$= \phi_i^{-1}\left(\left(\overrightarrow{\overrightarrow{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right]\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right). \tag{4.8}$$

It follows by (4.4) and (4.8) that
%POSSIBLY CITE FOLLAND SOURCE
%POSSIBLY DELETE CLAIM AND REDIRECT IT TO PROPOSITION 1.4
%INSTEAD OF CLAIM SHOW THAT IMAGE DERIVATION WORKS

$$\widetilde{\mathbb{P}}\left[\mathbb{1}_{\widetilde{X}_{k+1}=x_i}\left(\widetilde{X}_{n,0}, \ \ldots \widetilde{X}_{n,k+1}\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right]$$

$$= \widetilde{\mathbb{P}}\left[\phi_i^{-1}\left(\left(\overrightarrow{\overrightarrow{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right]\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)\right]$$

$$= \widetilde{\mathbb{P}}\left[\phi_i^{-1}\left(\left(\overrightarrow{\overrightarrow{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right]\right)^{-1}\left(\prod_{j=0}^{k+1} \mathrm{C}_j\right)\right)\right] + \widetilde{\mathbb{P}}\left[\mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j}\left(\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right)\right]$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \widetilde{\mathbb{P}}\left[\left(\vec{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot \widetilde{\mathbb{P}}\left[\pi_{k+1}^{-1}[q_{i-1}, q_i)^c\right]$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \widetilde{\mathbb{P}}\left[\left(\vec{\widetilde{X}}_n, \widehat{X}_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot (1 - m_{[0,1)}[q_{i-1}, q_i))$$

$$= \mathbb{P}[X_{k+1} = x_i] \cdot \mathbb{P}_{X_{k+1}^{-1}(\{x_i\})}\left[\left(\vec{X}_n, X_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot (1 - \mathbb{P}[X_{k+1} = x_i])$$

$$= \mathbb{P}\left[\left\{\left(\vec{X}_n, X_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right\} \cap \{X_{k+1} = x_i\}\right] + \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \cdot \mathbb{P}[X_{k+1} \neq x_i]$$

$$= \mathbb{P}\left[\left(\left\{\left(\vec{X}_n, X_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right\} \cap \{X_{k+1} = x_i\}\right) \cup \mathbb{1}_{\vec{0} \in \prod_{j=0}^{k+1} \mathrm{C}_j} \{X_{k+1} \neq x_i\}\right]$$

$$= \mathbb{P}\left[\mathbb{1}_{X_{k+1} = x_i}\left(\vec{X}_n, X_{n,k+1}\left[X_{k+1}^{-1}(\{x_i\})\right]\right) \in \prod_{j=0}^{k+1} \mathrm{C}_j\right],$$

and (4.6) immediately follows.

Finally, to prove (ii) (in the general case)
%USE SIMILAR STRATEGY TO THEOREM 4.1

%INCORPORATE A SIMILAR STRATEGY AS LEMMA 4.2 AND THEOREM 4.1

%Theorem 2.3-Lemma 2.5

5 Generalizing Skorohod's Thoerem for Arbitrary Random Variables

Corollary 5.1. Suppose $\left\{(X_{n,j})_{j=0}^{\infty}\right\}_{n\in\mathbb{N}} \subset \ell^{\infty}\left(\mathfrak{L}^{\infty}(\mathbb{P}; \mathcal{X})\right)$, for every $n \in \mathbb{N}$ and $(X_j)_{j=0}^{\infty} \in \ell^{\infty}\left(\mathfrak{L}^{\infty}(\mathbb{P}; \mathcal{X})\right)$ such that $X_{n,j}, X_j$ for each $n \geq 1, j \geq 0$ has a countable image of isolated points. Then *Skorohod's Representation Thoerem* holds such that

$$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{\mathbb{N}\cup\{0\}}, \mathcal{B}\left([0,1)^{\mathbb{N}\cup\{0\}}\right), \bigotimes_{k=0}^{\infty} m_{[0,1)}\right).$$

Outline of proof.

%REPEATEDLY USE LEMMA 4.4

Corollary 5.2. For any $\{X_n\}_{n=1}^{\infty} \subset \mathcal{L}^0(\mathbb{P}; \mathcal{X})$, *Skorohod's Representation Thoerem* holds such that

$$\left(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mathbb{P}}\right) := \left([0,1)^{\mathbb{N}\cup\{0\}}, \mathcal{B}\left([0,1)^{\mathbb{N}\cup\{0\}}\right), \bigotimes_{k=0}^{\infty} m_{[0,1)}\right).$$

Outline of proof.

%USE COROLLARY 5.1 AND THOEREM 3.1

%Corollary 2.6-Corollary 2.7