1 Random Measures

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ be a measurable space.

Definition 1.1. . A **random measure** $\nu : \Omega \times \mathcal{E} \to \mathbb{R}$ is a function such that

(i) $\nu(\,\cdot\,, F)$ is $\mathbb{P}$-measurable for any fixed $F \in \mathcal{E}$.

(ii) $\nu(\omega, \,\cdot\,)$ is a (signed-)measure on $(E, \mathcal{E})$, for any fixed $\omega \in \Omega$.
Source: Jacod, Shiryaev 1.3 Definition (page 65)

Remark 1.1. In either a later paper or draft of this paper, I plan on defining a vector-valued and eventually operator-valued random measures. But for now, to keep everything simple, I will keep the random measure exposition in this section real valued (i.e., so-called "signed measures").

Proposition 1.2. Let $M : \mathcal{E} \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ be a countably additive vector-valued measure. $\nu_M : \Omega \times \mathcal{E} \to \mathbb{R}$ defined by $\nu_M(\omega, E) := M(E)(\omega)$ is a random measure (up to $\mathbb{P}$-a.s. equivalence). Conversely, for every random measure ν, $M_\nu : \mathcal{E} \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ defined by $M_\nu(E) := \nu(\,\cdot\,, E)$ is a $\mathcal{L}^0(\mathbb{P}; \mathbb{R})$-valued countably additive measure.
Source: modification of Applebaum 2.3.1 (page 103)

IMPORTANT NOTE: The proof below is mostly rigorous, though the step where the sets $P(\omega)$ and $N(\omega)$ are chosen has been claimed as conjecture, though the **Hahn Decomposition Theorem** gives rise to a way to prove the existence of such sets in the more general context of random measures.

Outline of Proof. The converse is trivial since conditions *(i)* and *(ii)* of **Definition 1.1** guarentee that M_ν is a well-defined mapping on $\mathcal{E} \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ such that given disjoint $\{E_n\} \subset \mathcal{E}$ that

$$\sum_{n=1}^{\infty} M_\nu(E_n) = M_\nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \text{ pointwise on } \Omega\text{, (and hence in } \mathbb{P} \text{ as well)}$$

which hence establishes that M_ν is a vector-valued measure. It then remains to prove that M being a vector-valued measure $\implies \nu_M$ is a random measure.

Note that by construction ν_M satisfies condition *(i)* of **Definition 1.1**. To show condition *(ii)* of

Definition 1.1, note that $\nu_M(\,\cdot\,, F)$ is defined up to $\mathcal{L}^0(\mathbb{P};\mathbb{R})$-equivalence class, so given a pairwise disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$, it shall suffice to show that

$$\sum_{n=1}^{\infty} \nu_M(\,\cdot\,, E_n) = \sum_{n=1}^{\infty} M(E_n) = M\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \nu_M\left(\,\cdot\,, \bigcup_{n\in\mathbb{N}} E_n\right), \quad (1.1)$$

for almost every $\omega \in \Omega$, since the value $\nu_M\left(\,\cdot\,, \bigcup_{n\in\mathbb{N}} E_n\right)$ is $\mathbb{P}$-a.s. equivalent to a $\mathbb{P}$-measurable function such that (1.1) holds for all $\omega \in \Omega$.

Next, note that given a pairwise disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$, we find

$$\sum_{n=1}^{\infty} M(E_n) = M\left(\bigcup_{n\in\mathbb{N}} E_n\right) \quad (1.2)$$

absolutely in $\mathbb{P}$. We claim and try to verify that in fact (1.2) holds absolutely $\mathbb{P}$-a.s. Define $\nu_M^+, \nu_M^- : \Omega \times \mathcal{E} \to \mathbb{R}_+$ by

$$\nu_M^+(\omega, \,\cdot\,) := \nu_M(\omega, (\,\cdot\,) \cap P(\omega)),\ \nu_M^-(\omega, \,\cdot\,) := -\nu_M(\omega, (\,\cdot\,) \cap N(\omega)), \quad (1.3)$$

where $P(\omega)$ and $N(\omega)$ are the a.e. $\subset$ -largest elements in $\mathcal{E}$ such that $P(\omega)$ and $N(\omega)$ are disjoint $P(\omega) \cup N(\omega) = E$ and

$\nu_M(\omega, F) \geq 0$ for every $F \in \mathcal{E}$ such that $E \subset P(\omega)$ and
$\nu_M(\omega, F) \leq 0$ for every $F \in \mathcal{E}$ such that $E \subset N(\omega)$,

which exist via a similar argument to the **Hahn Decomposition Theorem** (that we will prove in a later draft); refer to *Folland* § *3.3 (page 86)*.

We then define $M^-, M^+ : \mathcal{E} \to \mathcal{L}^0(\mathbb{P}; \mathbb{R}_+)$ by

$M^+(F) := M(F \cap P(\,\cdot\,)),\ M^-(F) := -M(F \cap N(\,\cdot\,))$, for every $E \in \mathcal{E}$,

Note that $M = M^+ - M^-$ by definition, and that for every sequence

$\left\{\sum_{n=1}^{N_k} M^+(E_n)\right\}_{k\in\mathbb{N}}$, $\left\{\sum_{n=1}^{N_k} M^-(E_n)\right\}_{k\in\mathbb{N}}$ of $\left\{\sum_{n=1}^{N} M^+(E_n)\right\}_{N\in\mathbb{N}}$, $\left\{\sum_{n=1}^{N} M^-(E_n)\right\}_{N\in\mathbb{N}}$, there exists a further subsequence $\left\{\sum_{n=1}^{N_{k_j}} M^+(E_n)\right\}_{j\in\mathbb{N}}$, $\left\{\sum_{n=1}^{N_{k_j}} M^-(E_n)\right\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty}\sum_{n=1}^{N_{k_j}} M^+(E_n) = M^+\left(\bigcup_{n\in\mathbb{N}} E_n\right),\ \lim_{j\to+\infty}\sum_{n=1}^{N_{k_j}} M^-(E_n) = M^-\left(\bigcup_{n\in\mathbb{N}} E_n\right)\ \mathbb{P}\text{-a.s.,}$$

which imply in particular that $\sum_{n=1}^{\infty} M^+(E_n)$, $\sum_{n=1}^{\infty} M^-(E_n)$ converge to $M^+\left(\bigcup_{n\in\mathbb{N}} E_n\right)$ and $M^-\left(\bigcup_{n\in\mathbb{N}} E_n\right)$, respectively $\mathbb{P}$-a.s., since the sequences $\left\{\sum_{n=1}^{N} M^+(E_n)\right\}_{N\in\mathbb{N}}$, $\left\{\sum_{n=1}^{N} M^-(E_n)\right\}_{N\in\mathbb{N}}$ are pointwise monotonic.

From this, it follows that

$$\sum_{n=1}^{\infty} M(E_n) = \sum_{n=1}^{\infty} M^+(E_n) - \sum_{n=1}^{\infty} M^-(E_n) = M^+\left(\bigcup_{n\in\mathbb{N}} E_n\right) - M^-\left(\bigcup_{n\in\mathbb{N}} E_n\right) = M\left(\bigcup_{n\in\mathbb{N}} E_n\right),$$

abolutely $\mathbb{P}$-a.s. and we conclude that (1.1) holds for almost every $\omega\in\Omega$. □

Remark 1.2.

(i) What a vector valued measure M on $(E,\mathcal{E})$ exactly is needs additional clarification, especially in this context. Usually, a vector-valued measure is defined in a Banach space $\mathcal{X}$ and is countably additive if for a pairwise disjoint sequence $\{E_n\}_{n\in\mathbb{N}}\subset\mathcal{E}$, we have

$$M\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} M(E_n) \text{ in } \mathcal{X},$$

i.e., we have $\sum_{n=1}^{N} M(E_n) \to M\left(\bigcup_{n=1}^{\infty} E_n\right)$ in norm $||\cdot||_X$ as $N \to +\infty$. We could also do this for a complete vector-valued metric space (X, d), i.e., if M is X-valued, then M is a **countably additive** measure if for a pairwise disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$, we have

$$\sum_{n=1}^{N} M(E_n) \to M\left(\bigcup_{n=1}^{\infty} E_n\right) \text{ in } d \text{ as } N \to +\infty.$$

(ii) The set of random measures and the set of $\mathfrak{L}^0(\mathbb{P};\mathbb{R})$-valued measures are in fact isomorphic in the category Set, which is what we showed with **Proposition 1.2**. Denote this isomorphism from random measures to $\mathfrak{L}^0(\mathbb{P};\mathbb{R})$-valued measures $M_{(-)}$ and denote the isomorphism from $\mathfrak{L}^0(\mathbb{P};\mathbb{R})$-valued measures to the set of random measures $\nu_{(-)}$.

(iii) (Source: Folland § 2.4, page 61-63) Note that $\mathfrak{L}^0(\mathbb{P};\mathbb{R})$ is a complete metric space (according to **Thoerem 2.30** and **Exercise 32** of *Folland § 2.4*), with a metric defined as follows:

$$d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(X, Y) := \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]. \quad (1.4)$$

(iv) The next definitions and propositions are ones that I came up with that generalize notions we study, in lieu of literature that deals with such matters (that I hope to find in the future). hence there are no sources to cite. The hope is to make these defintiions as self-contained as possible.

(v) Given an $\mathbb{R}$-valued random measure ν, we shall define ν^+ and ν^- to be the positive and negative parts, respectively, of the random measure ν to be as defined in the outline of the proof of **Proposition 1.2**, i.e. we have

$$\nu^+(\omega, \cdot) := \nu(\omega, (\cdot) \cap P(\omega)),\ \nu^-(\omega, \cdot) := -\nu(\omega, (\cdot) \cap N(\omega)),$$

where $P(\omega)$ and $N(\omega)$ are the a.e. $\subset$-largest elements in $\mathcal{E}$ such that $P(\omega)$ and $N(\omega)$ are disjoint $P(\omega) \cup N(\omega) = E$ and

$\nu(\omega, F) \geq 0$ for every $F \in \mathcal{E}$ such that $E \subset P(\omega)$ and
$\nu(\omega, F) \leq 0$ for every $F \in \mathcal{E}$ such that $E \subset N(\omega)$,

Note that ν^+ and ν^- are defined analogously to the conventions in *Folland* § *3.1* for deterministic measures. We then note that $\nu = \nu^+ - \nu^-$ by convention and we define $|\nu| := \nu^+ + \nu^-$. As stated before, we shall verify that these conventions are well-defined in a future draft.

Definition 1.3.

(i) A random measure ν is said to be **finite** if $\nu(\omega, \cdot)$ is a finite measure (i.e., $|\nu|(\omega, E) < +\infty$) almost surely for $\omega \in \Omega$.

(ii) A random measure ν is said to be **σ-finite** if there exists a sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and $|\nu|(\omega, E_n) < +\infty$, for each $n \in \mathbb{N}$, almost surely for $\omega \in \Omega$.

(iii) We say that a set $F \in \mathcal{E}$ occurs **ν-almost everywhere** (abbreviated "ν-a.e.") if almost surely for $\omega \in \Omega$, we have $|\nu|(\omega, E \setminus F) = 0$.

Definition 1.4. We further define $\mathcal{L}^0(\nu; \mathbb{R})$, or the **space of random ν-measurable functions**, as the set of functions $Q : \Omega \times E \to \mathbb{R}$ such that the following two conditions are met:

(i) For every $e \in E$, the function $Q(\cdot, e) : \Omega \to \mathbb{R}$ is a random variable, i.e. in $\mathcal{L}^0(\mathbb{P}; \mathbb{R})$.
(ii) The function $Q' : E \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ defined by $Q'(e) := Q(\cdot, e) : \Omega \to \mathbb{R}$ is a measurable function from $(E, \mathcal{E})$ to $\left(\mathcal{L}^0(\mathbb{P}; \mathbb{R}), \mathcal{B}\left(\mathcal{L}^0(\mathbb{P}; \mathbb{R})\right)\right)$, where $\mathcal{B}\left(\mathcal{L}^0(\mathbb{P}; \mathbb{R})\right)$ is the Borel σ-algebra on the metric space.

We moreover state that $Q, R \in \mathcal{L}^0(\nu; \mathbb{R})$ are equal **ν-almost everywhere** (or "$Q = R$ ν-a.e." or "$Q \overset{\nu\text{-a.e.}}{=} R$" for short) $Q' \overset{\mathbb{P}\text{-a.s.}}{=} R'$ occurs ν-a.e., i.e., if for almost surely for $\omega \in \Omega$, we have

$|\nu|(\omega, d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q', R') \neq 0) = 0$.

Remark 1.3.

(i) IMPORTANT NOTATION NOTE: From this point forward, given a random function Q, we define $Q' : E \to \mathcal{L}^0(\mathbb{P}; \mathbb{R})$ by $Q'(e) := Q(\cdot, e) : \Omega \to \mathbb{R}$ as we did in **Definition 1.4** ***(ii)***. Moreover, we later (particularly in **Proposition 1.9** ***(ii)*** and *Remark 1.6*) define $Q^* : \Omega \to \mathbb{R}^E$

by $Q^*(\omega) = Q(\omega, \cdot): E \to \mathbb{R}$.

(ii) Random functions $Q: \Omega \times E \to \mathbb{R}$ are related to stochastic processes $\{X_t\}_{t \in T}$, i.e., a family of random variables indexed by T, in the sense that every random function IS a stochastic process, since Q can be concieved as the family $\{Q(\cdot, e)\}_{e \in E}$ of random variables. In fact, a function $X: \Omega \times T \to \mathbb{R}$ satisfying only *condition (i)* of **Definition 1.4** is precisely the definition of a stochastic process, keeping in mind that the concept of currying makes the sets $\mathbb{R}^{\Omega \times T}$ and $\left(\mathbb{R}^{\Omega}\right)^T$ (which we take full advantage of by switching between Q and Q' to talk about the same "random function") isomorphic in the category Set. *Condition (ii)* is a necessary condition to make Q into a "measurable function" with respect to the "measure" ν on the measurable set $(E, \mathcal{E})$, which is what we need to further do our measure-theoretic analysis.

(iii) With stochastic processes, we additionally often consider them to be $\mathbb{R}^T$-valued random variables, i.e., measurable functions $X: \Omega \to \mathbb{R}^T$ from (Ω, Σ) to $\left(\mathbb{R}^T, \mathcal{B}\left(\mathbb{R}^T\right)\right)$. This makes intuitive sense, because currying (as discussed before), as well as the fact that $\Omega \times T \cong T \times \Omega$ gives us the isomorphism $\mathbb{R}^{\Omega \times T} \cong \left(\mathbb{R}^T\right)^{\Omega}$ in the category Set. However, it's not immediately obvious that X is measurable from (Ω, Σ) to $\left(\mathbb{R}^T, \mathcal{B}\left(\mathbb{R}^T\right)\right)$ iff $\{X(\cdot)(t)\}_{t \in T}$ is a family of random variables, i.e., $\mathcal{L}^0(\mathbb{P}; \mathbb{R})^T \cong \mathcal{L}^0\left(\mathbb{P}; \mathbb{R}^T\right)$ as categories in Set, and this doesn't hold in the general case where $X: \Omega \to S^T$ is an arbitrary set S-valued random variable (though with additional regularity topological conditions on S it does). This will be discussed in more detail in a future draft.

(iv) There should definitely be concrete examples of random functions, specifically. While I do in this draft mention examples (refer to **Examples 1.12**) of random measures and what integration over random functions look like, I don't provide concrete examples (or nonexamples) of specific random functions. This will change in a future draft.

(v) Note that for any fixed $\omega \in \Omega$, we find by **Definition 1.4** that $Q(\omega, \cdot): E \to \mathbb{R}$ an $(E, \mathcal{E})$-measurable function, since it turns out the mapping $(-)(\omega): \mathcal{L}^0(\mathbb{P}; \mathbb{R}) \to \mathbb{R}$ defined by $(X)(\omega) = X(\omega)$ is a measurable mapping (as verify in an outline of a proof) between $\mathcal{B}\left(\mathcal{L}^0(\mathbb{P}; \mathbb{R})\right)$ and $\mathcal{B}(\mathbb{R})$, making $Q(\omega, \cdot)$ measurable, as the composition $Q(\omega, \cdot) = (-)(\omega) \circ Q'$ of measurable functions (noting by *condition (ii)* that Q' is $(E, \mathcal{E})$-measurable).

One can verify that $(-)(\omega)$ is measurable as follows: Note that for each $n \geq 1$, $(-)_n(\omega) : \mathfrak{L}^0(\mathbb{P}; \mathbb{R}) \to \mathbb{R}$ defined by

$$(X)_n(\omega) := \begin{cases} X(\omega) & \text{if } |X(\omega)| \leq n d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(X, 0), \\ \text{sgn}(X(\omega)) \cdot n d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(X, 0) & \text{else,} \end{cases}$$

where $\text{sgn}(x) = \dfrac{x}{|x|}$, forms a sequence of functions that are continuous on $\mathfrak{L}^0(\mathbb{P}; \mathbb{R})$, hence measurable on $\mathcal{B}\left(\mathfrak{L}^0(\mathbb{P}; \mathbb{R})\right)$, that appoximate $(-)(\omega)$ pointwise on $\mathfrak{L}^0(\mathbb{P}; \mathbb{R})$, and hence $(-)(\omega)$ is measurable.

(vi) We find that ν-a.e. equivalence of random functions is an equivalence relation, i.e., we find the reflexive, symmetric, and transitive properties hold for $\overset{\nu\text{-a.e.}}{=}$ as a relation, and this can be verified in similar fashion to how almost everywhere equivalence of measurable functions is established for a deterministic measure.

(vii) There are certain behaviors of $Q, R \in \mathfrak{L}^0(\nu; \mathbb{R})$ that imply ν-a.e. equivalence, for example, if $\mathbb{P}$-a.s. for fixed $\omega \in \Omega$, we have $Q(\omega, e) = R(\omega, e)$ $\nu(\omega, \cdot)$-a.e., then $Q = R$ ν-a.e. I believe the converse holds as well, but this will be verified in a future draft.

(viii) One can easily verify $\mathfrak{L}^0(\nu; \mathbb{R})$ vector space (up to ν-a.e. equivalence) under pointwise addition and scalar multiplcation, i.e., addition and scalar multiplication defined by

$$(Q + R)(\omega, e) := Q(\omega, e) + R(\omega, e),$$
$$(cQ)(\omega, e) := cQ(\omega, e),$$

for $Q, R \in \mathfrak{L}^0(\nu; \mathbb{R})$ and $c \in \mathbb{R}$. I additionally hope to show in the next draft that $\mathfrak{L}^0(\nu; \mathbb{R})$ is a topological vector space, and more generally metric space if ν is finite, in the space of convergence in ν.

Definition 1.5.

(i) For a sequence $\{Q_n\}_{n \in \mathbb{N}} \subset \mathfrak{L}^0(\nu; \mathbb{R})$, we say that $\{Q_n\}_{n \in \mathbb{N}}$ **converges in ν**, and write $Q_n \xrightarrow{\nu} Q$, if for every $\epsilon > 0$, we have

$$|\nu|(\cdot, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to +\infty. \quad (1.5)$$

(ii) For a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0(\nu;\mathbb{R})$, we say that $\{Q_n\}_{n\in\mathbb{N}}$ **converges ν-a.e.**, and write $Q_n \xrightarrow{\nu\text{-a.e.}} Q$, if the set $\left\{e : Q_n{}'(e) \xrightarrow{\mathbb{P}} Q'(e)\right\}$ occurs ν-a.e., i.e., almost surely for $\omega \in \Omega$, we have $|\nu|\left(\omega, Q_n{}' \not\xrightarrow{\mathbb{P}} Q'\right) = 0$.

(iii) We shall call a random ν-measurable function $S : \Omega \times E \to \mathbb{R}$ a **ν-simple random function** if it is of the form.

$$S(\omega, e) := \sum_{k=1}^{N} a_k \mathbb{1}_{\Lambda_k}(\omega) \cdot \mathbb{1}_{F_k}(e) = \sum_{k=1}^{N} a_k \mathbb{1}_{\Lambda_k \times F_k}(\omega, e),\ a_k \in \mathbb{R},\ \Lambda_k \in \Sigma,\ F_k \in \mathcal{E}$$

We shall call the set of ν-simple random functions $\mathcal{S}(\nu;\mathbb{R})$.

Remark 1.4. Note how the definition of simple random functions only includes random variables of the form $a_k \mathbb{1}_{\Lambda_k}(\omega)$, and not $X_k(\omega)$, for each $k \geq 1$, which seems more like the natural collection of functions to use when the deterministic version is $\sum_{k=1}^{N} a_k \mathbb{1}_{F_k},\ a_k \in \mathbb{R},\ F_k \in \mathcal{E}$. It turns out that the collection $\mathcal{S}(\nu;\mathbb{R})$ of functions is dense in $\mathfrak{L}^0(\nu;\mathbb{R})$ in ν, and even ν-a.e., i.e., for every $Q \in \mathfrak{L}^0(\nu;\mathbb{R})$ there exists a sequence $\{S_n\}_{n\in\mathbb{N}}$ such that $S_n \xrightarrow{\nu\text{-a.s.}} Q$ as $n \to +\infty$, and hence by **Proposition 1.9 *(iii)*** we also have $S_n \xrightarrow{\nu} Q$ as $n \to +\infty$. It moreover turns out that we can choose $\{S_n\}_{n\in\mathbb{N}}$ such that (1.8) holds for Q, i.e., we have $S_n{}'(e) \xrightarrow{\mathbb{P}\text{-a.s.}} Q'(e)$ ν-a.e. We'll prove all these claims in a future draft.

For the next proposition proposition, given a sequence of sets $\{F_j\}_{j\in\mathbb{N}}$, we shall use the notation

$$\limsup_j F_j = \bigcap_{j=1}^{\infty} \bigcup_{k\geq j} F_k,$$
$$\liminf_j F_j = \bigcup_{j=1}^{\infty} \bigcap_{k\geq j} F_k.$$

Proposition 1.6. *(Borel-Cantelli Lemma for Random Measures)* Suppose ν is a finite random measure, $\{F_j\}_{j\in\mathbb{N}} \subset \mathcal{E}$, and $\{Q_j\}_{j\in\mathbb{N}}$ is a sequence of random ν-measurable functions.

(i) Suppose

$$\sum_{j=1}^{\infty} |\nu|(\,\cdot\,, F_j) = \sum_{j=1}^{\infty} M_{|\nu|}(F_j) < +\infty, \ \mathbb{P}\text{-a.s.} \quad (1.6)$$

Then $\liminf_j E \setminus F_j$ occurs ν-a.e., i.e. we find

$$|\nu|\Big(\,\cdot\,,\ \limsup_j F_j\Big),\ \text{or } |\nu|\big(\,\cdot\,, F_j \text{ occurs i.o.}\big) = 0.\ \ \mathbb{P}\text{-a.s.}$$

(ii) We find that if, for every $\epsilon > 0$, we have

$$\sum_{j=1}^{\infty} |\nu|(\,\cdot\,, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_j{}', Q') \geq \epsilon) = \sum_{j=1}^{\infty} M_{|\nu|}(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_j{}', Q') \geq \epsilon) < +\infty, \ \mathbb{P}\text{-a.s.}$$

then $Q_j{}' \longrightarrow Q'$ pointwise ν-a.e., i.e., we have

$$|\nu|\Big(\omega, Q_j{}' \xrightarrow{\mathbb{P}}\!\!\!\!\!\!/\ \ Q'\Big) = 0,\ \mathbb{P}\text{-a.s. for } \omega \in \Omega$$

(iii) Given a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that $\epsilon_j > 0$ and $\epsilon_j \longrightarrow 0$, we find that if

$$\sum_{j=1}^{\infty} |\nu|(\,\cdot\,, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_j{}', Q') \geq \epsilon_j) = \sum_{j=1}^{\infty} M_{|\nu|}(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_j{}', Q') \geq \epsilon_j) < +\infty, \ \mathbb{P}\text{-a.s.}$$

then $Q_j \xrightarrow{\nu\text{-a.e.}} Q$.

(iv) We find that if, for every $\epsilon > 0$, we have

$$\sum_{j=1}^{\infty} |\nu|(\omega, |Q_j(\omega,\,\cdot\,) - Q(\omega,\,\cdot\,)| \geq \epsilon) = \sum_{j=1}^{\infty} M_{|\nu|}(|Q_j - Q| \geq \epsilon) < +\infty, \ \mathbb{P}\text{-a.s.} \quad (1.7)$$

then

$Q_n(\omega, \cdot) \to Q(\omega, \cdot)$ as $n \to +\infty$ $\nu(\omega, \cdot)$-a.e. $\quad (1.8)$

(v) Given a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that $\epsilon_j > 0$ and $\epsilon_j \to 0$, we find that if

$$\sum_{j=1}^{\infty} |\nu|(\omega, |Q_j(\omega, \cdot) - Q(\omega, \cdot)| \geq \epsilon_j) = \sum_{j=1}^{\infty} M_{|\nu|}(|Q_j - Q| \geq \epsilon_j) < +\infty, \ \mathbb{P}\text{-a.s.},$$

then (1.8) holds.

Outline of proof. To prove *(i)*, we apply the original **Borel Cantelli Lemma** (refer to *Billingsley, page 59*) on every $\omega \in \Omega$ such that (1.6) holds to show that $|\nu|\left(\omega, Q_j{}' \not\xrightarrow{\mathbb{P}} Q'\right) = 0$, for almost every $\omega \in \Omega$. *(ii)* and *(iv)* are then consequences of *(i)* and *(iii)* and *(v)* is are immediate consequences of *(ii)* and *(iv)*, respectively ◻

Remark 1.5. Note that the hypothesis of (1.6) (as well as the analogous equations can be replaced with the analogous hypothesis of

$$\sum_{j=1}^{\infty} |\nu|(\cdot, F_j) = \sum_{j=1}^{\infty} M_{|\nu|}(F_j) < +\infty \text{ in } \mathbb{P},$$

since $M_{|\nu|}(F_j) \geq 0$, for all $j \geq 1$, so convergence in $\mathbb{P}$ of the random series implies convergence $\mathbb{P}$-a.s. I don't state this hypothesis in the original theorem, because it's more annoying of a hypothesis to use directly in the proof.

Definition 1.7. Given a sequence of random $\{Q_n\}_{n\in\mathbb{N}}$ of random ν-measurable functions, we state that Q_n converges to Q uniformly, and we write $Q_n \xrightarrow{u} Q$ if $Q_n{}' \xrightarrow{u} Q'$ in the uniform topology (with respect to the metric space $d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}$ on the space $\mathcal{L}^0(\mathbb{P};\mathbb{R})$ of random variables), i.e., for every $\epsilon > 0$, we have

$d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}'(e), Q'(e)) < \epsilon$, for all $e \in E$, e.v. for $n \in \mathbb{N}$,

or equivalently we have

$\rho(Q_n{}'(e), Q'(e)) < \epsilon$ e.v. for $n \in \mathbb{N}$,

where ρ is the uniform metric defined on $\mathcal{L}^0(\mathbb{P};\mathbb{R})^E$ by

$$\rho(R', T') := \sup_{e \in E} d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(R'(e), T'(e)), \text{ for all } R', T' \in \mathcal{L}^0(\mathbb{P};\mathbb{R})^E$$

Proposition 1.8. *(Egorov's Thoerem for Random Measures)* Suppose ν is a finite random measure and $\{Q_n\}_{n\in\mathbb{N}}$ is a sequence of random ν-measurable functions such that $Q_n \xrightarrow{\nu\text{-a.e.}} Q$. Then the following holds:

(i) For all $\epsilon > 0$, there exists $F \in \mathcal{E}$ such that almost surely for $\omega \in \Omega$, we have $Q_n \xrightarrow{u} Q$ on F and $|\nu|(\omega, E \setminus F) < \epsilon$.

(ii) For all $\epsilon_1, \epsilon_2 > 0$, there exists $F \in \mathcal{E}$ such that

$$\lim_{n\to\infty} \mathbb{P}[|Q_n(\cdot, e) - Q(\cdot, e)| \geq \epsilon_2] = 0 \text{ uniformly for } e \in F, \quad (1.9)$$

and $|\nu|(\omega, E \setminus F) < \epsilon_1$.

Outline of Proof.

(i) Note that

$$\left\{e : Q_n(e) \overset{\mathbb{P}}{\not\to} Q'(e)\right\} = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} \left\{e : d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n'(e), Q'(e)) \geq k^{-1}\right\}.$$

Then for all $q \in \mathbb{Q}^+$, almost surely for $\omega \in \Omega$ we have

$$\lim_n \sup |\nu|\left(\omega, d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n', Q') \geq k^{-1}\right) = |\nu|\left(\omega, \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} \left\{e : d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n'(e), Q'(e)) \geq k^{-1}\right\}\right) = 0.$$

Let $\epsilon > 0$. For each $k \geq 1$, we can recursively choose $N_k > N_j$ for $1 \leq j < k$ sufficiently large so that almost surely for $\omega \in \Omega$ we have

$$|\nu|\left(\omega, \bigcup_{n\geq N_k} \left\{e : d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n'(e), Q'(e)) \geq k^{-1}\right\}\right)$$
$$= \sup_{n\geq N_k} |\nu|\left(\omega, d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n', Q') \geq k^{-1}\right) < \frac{\epsilon}{2^k}. \quad (1.10)$$

Set

$$F_{n,k} := \left\{e : d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}'(e), Q'(e)) < k^{-1}\right\},\ F := \bigcap_{k=1}^{\infty} \bigcap_{n \geq N_k} F_{n,k}$$

For all $e \in F$, we have for every $\epsilon' > 0$ some $k^{-1} \leq \epsilon'$ and hence for every $n \geq N_k$, we have

$d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}'(e), Q'(e)) < k^{-1} \leq \epsilon'$,

which shows that $Q_n \xrightarrow{u} Q$ on F, and by (1.10), we find almost surely for $\omega \in \Omega$ that

$$|\nu|(\omega, E \setminus F) = |\nu|\left(\bigcup_{k=1}^{\infty} \bigcup_{n \geq N_k} [E \setminus F_{k,n}]\right) \leq \sum_{k=1}^{\infty} \left[\sup_{n \geq N_k} |\nu|\left(\omega, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq k^{-1}\right)\right]$$
$$< \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

(ii) Follows from the fact that we've shown by *part (i)* that given $\epsilon_1 > 0$, we have some $F \in \mathcal{E}$ such that $\nu(\omega, E \setminus F) < \epsilon_1$, almost surely for $\omega \in \Omega$, and $Q_n{}' \xrightarrow{u} Q'$ uniformly on F with respect to the metric space $d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}$. In other words, we have $\lim_{n \to \infty} d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}'(e), Q'(e)) \to 0$ uniformly, for all $e \in F$. We can then use that fact to show (using a similar argument to the fact that convergence in $d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})} \implies$ convergence in $\mathbb{P}$, which we shall elaborate in the next draft) that for any $\epsilon_2 > 0$, we have (1.9) occur. $\square$

Propostion 1.9. Let $\{Q_n\}_{n \in \mathbb{N}} \subset \mathfrak{L}^0(\nu; \mathbb{R})$ and $Q \in \mathfrak{L}^0(\nu; \mathbb{R})$.

(i) If almost surely for $\omega \in \Omega$, we have (1.8), then $Q_n \xrightarrow{\nu\text{-a.e.}} Q$.

(ii) The converse of *(i)* is false (even in the case where ν is finite), but if ν is finite $Q_n \xrightarrow{\nu\text{-a.e.}} Q$, and if $\left\{Q_n^*\right\}_{n \in \mathbb{N}}, Q^* : \Omega \to \mathbb{R}^E$ is measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathcal{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$, where $\rho_{E,\mathbb{R}}$ is the uniform metric on $\mathbb{R}^E$ and R^* is defined by $R^*(\omega) = R(\omega, \cdot)$, for any $R \in \mathfrak{L}^0(\nu; \mathbb{R})$, then there exists a subsequence $\{Q_{n_k}\}_{k \in \mathbb{N}}$ such that (1.8) holds.

(iii) If $Q_n \xrightarrow{\nu\text{-a.e.}} Q$, then $Q_n \xrightarrow{\nu} Q$.

Proof.

(i) Suppose almost surely for $\omega \in \Omega$, (1.8) holds. Then for almost every $\omega \in \Omega$, we have

$$|\nu|\Big(\omega, Q_n{}'(e) \overset{\mathbb{P}}{\not\longrightarrow} Q'(e)\Big) \le |\nu|\Big(\omega, Q_n{}'(e) \overset{\mathbb{P}\text{-a.s.}}{\not\longrightarrow} Q'(e)\Big) = 0.$$

(ii) Choose a collection $\{A_n\}_{n\in\mathbb{N}} \subset \Omega$ of independent sets such that $\mathbb{P}[A_n] = \dfrac{1}{n}$. Set $(E, \mathcal{E}) := (\{*\}, \{\varnothing, \{*\}\})$, and define $\nu : \Omega \times \mathcal{E} \to \mathbb{R}$ by

$\nu(\omega, \varnothing) := 0,\ \nu(\omega, \{*\}) := 1$, for all $\omega \in \Omega$.

Define $Q_n : \Omega \times E \to \mathbb{R}$ by $Q_n(\omega, e) = \mathbb{1}_{A_n}(\omega)$, and note that $Q_n(\,\cdot\,, *) = \mathbb{1}_{A_n}$, which is a random variable and moreover $Q_n{}'$ is the mapping $* \mapsto \mathbb{1}_{A_n}$, which is continuous on the trivial topology $\mathcal{T} := \{\varnothing, \{*\}\}$, and $\mathcal{B}(\mathcal{T}) = \{\varnothing, \{*\}\}$, hence a measurable mapping from $(E, \mathcal{E})$ to $(\mathcal{L}^0(\mathbb{P}; \mathbb{R}), \mathcal{B}\big(\mathcal{L}^0(\mathbb{P}; \mathbb{R})\big)$. We find that $Q_n(\,\cdot\,, *) = \mathbb{1}_{A_n} \xrightarrow{\mathbb{P}} 0$ in measure, and it follows that $Q_n \xrightarrow{\nu\text{-a.e.}} 0$.

However, we find that for $\epsilon := 1$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbb{1}_{A_n} \ge \epsilon] = \sum_{n=1}^{\infty} \mathbb{P}[\mathbb{1}_{A_n} = 1] = \sum_{n=1}^{\infty} \mathbb{P}[A_n] = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

hence by the **Second Borel Cantelli Lemma** (refer to *Billingsley, page 60*) we find that

$$\mathbb{P}\Big[Q_n(\,\cdot\,, *) \not\to 0\Big] \ge \mathbb{P}\Big[A_n \text{ occurs i.o.}\Big] = 1 \ne 0,$$

and it follows that (1.8) does not hold.

To prove that if ν is finite and $Q_n \xrightarrow{\nu\text{-a.e.}} Q$ then there exists a subsequence such that (1.8) holds. we shall first, using **Egorov's Thoerem for Random Measures** (i.e. **Proposition 1.8 *(ii)***), choose $F_k \in \mathcal{E}$, for every $k \ge 1$, such that

$$\lim_{n \to +\infty} \mathbb{P}\Big[|Q_n(\cdot, e) - Q(\cdot, e)| \geq k^{-1} \Big] = 0$$

uniformly for every $e \in F_k$ and $|\nu|(\omega, E \setminus F_k) < 2^{-k}$ almost surely for $\omega \in \Omega$. Recursively choose n_k sufficiently large such that $n_k > n_j$ for all $1 \leq j < k$, and

$$\mathbb{P}\Big[|Q_{n_k}(\cdot, e) - Q(\cdot, e)| \geq k^{-1} \Big] < 2^{-k}, \text{ for all } e \in F_k$$

It follows by a variant of the **Borel Cantelli Lemma** (that I shall fully state and prove in the next draft)--which utilizes the hypothesis that $\left\{Q_n^*\right\}_{n \in \mathbb{N}}, Q^* : \Omega \to \mathbb{R}^E$ is measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathfrak{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$--that

$$|Q_{n_k}(\omega, e) - Q(\omega, e)| < k^{-1}, \text{ for all } e \in F_k, \text{ e.v. for } k \geq 1 \;\; \mathbb{P}\text{-a.s.} \quad (1.11)$$

It follows that eventually for $k \geq 1$, we have

$$\left\{|Q_{n_k}(\omega, \cdot) - Q(\omega, \cdot)| \geq k^{-1}\right\} \subset E \setminus F_k \;\; \mathbb{P}\text{-a.s.}$$

and hence

$$|\nu|\left(\cdot, |Q_{n_k}{}' - Q'| \geq k^{-1}\right) = O\left(2^{-k}\right), \;\; \mathbb{P}\text{-a.s.}$$

We've thereby shown that

$$\sum_{k=1}^{\infty} |\nu|\left(\cdot, |Q_{n_k} - Q| \geq k^{-1}\right) = \sum_{k=1}^{\infty} M_{|\nu|}\left(|Q_{n_k} - Q| \geq k^{-1}\right) < +\infty, \;\; \mathbb{P}\text{-a.s.},$$

and our conclusion immediately follows from the **Borel-Cantelli Lemma for Random Measures** (i.e., **Proposition 1.6 *(v)***).

(iii) Suppose $Q_n \xrightarrow{\nu\text{-a.e.}} Q$. We want to show that for all $\epsilon > 0$ we have $|\nu|(\cdot, d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. First note that

$$|\nu|\left(\cdot, {Q_n}' \xrightarrow{\mathbb{P}}\!\!\!\!\!\!/\;\, Q'\right) = |\nu|\left(\cdot, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}({Q_n}', Q') \not\to 0\right)$$

$$= \sup_{\epsilon \in \mathbb{Q}^+} \limsup_{n \to +\infty} [|\nu|(\cdot, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}({Q_n}', Q') \geq \epsilon)].$$

Then by hypothesis, we find almost surely for $\omega \in \Omega$, we have

$$\sup_{\epsilon \in \mathbb{Q}^+} \limsup_{n \to +\infty} [|\nu|(\omega, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}({Q_n}', Q') \geq \epsilon)] = |\nu|\left(\omega, {Q_n}' \xrightarrow{\mathbb{P}}\!\!\!\!\!\!/\;\, Q'\right) = 0.$$

It follows that for all $\epsilon > 0$ we have

$$\lim_{n \to +\infty} [|\nu|(\cdot, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}({Q_n}', Q') \geq \epsilon)] = 0, \text{ for all } \epsilon > 0,$$

and our conclusion is reached. □

Remark 1.6.

(i) As stated in the of **Proposition 1.9** ***(ii)*** (1.11) requires a variant of the Borel-Cantelli Lemma that states that if there is a sequence of random functions $\{Q_k\}_{k \in \mathbb{N}}$ such that Q_k^*, for each $k \in \mathbb{N}$, is measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathcal{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$, and a sequences $\{M_k\}_{k \in \mathbb{N}}, \{\delta_k\}_{k \in \mathbb{N}} \subset (0, +\infty)$ such that

(a) $\mathbb{P}[|Q_k(\cdot, e)| \geq \delta_k] \leq M_k$ for all $e \in E$,

(b) $\sum_{k=1}^{\infty} M_k < +\infty$,

.

then $\mathbb{P}$-a.s. for $\omega \in \Omega$, we have

$|Q_k(\omega, e)| < \delta_k$, for all $e \in E$, e.v., for $k \geq 1$,

or equivalently we have

$$\mathbb{P}\left[||Q_k^*||_{\rho_{E,\mathbb{R}}} < \delta_k \text{ e.v., for } k \geq 1\right] = 1.$$

To outline the proof of this (because it's a fundamental step to prove **Proposition 1.9** ***(ii)***), note that for every $k \geq 1$, we have

$||Q_k^*||_{\rho_{E,\mathbb{R}}} \geq \delta_k \iff |Q_k(\cdot, e)| \geq \delta_k$ for some $e \in E$.

For every $k \geq 1$ and $\omega \in \Omega$ can then choose a sequence $\{e_{k,m}(\omega)\}_{m \in \mathbb{N}}$ such that

$|Q_k(\omega, e_{k,m}(\omega))| \nearrow ||Q_k^*||_{\rho_{E,\mathbb{R}}}$ as $m \to +\infty$

and observe that for every $k \geq 1$, we have

$$\mathbb{P}\Big[\,||Q_k^*||_{\rho_{E,\mathbb{R}}} \geq \delta_k\Big] = \lim_{m \to +\infty} \mathbb{P}[|Q_k(\omega, e_{k,m}(\omega))| \geq \delta_k] = \sup_{e \in E} \mathbb{P}[|Q_k(\omega, e)| \geq \delta_k] \leq M_k.$$

Then we have $\sum_{k=1}^{\infty} \mathbb{P}\Big[\,||Q_k^*||_{\rho_{E,\mathbb{R}}} \geq \delta_k\Big] < +\infty$ and we conclude by the **Borel-Cantelli Lemma** that $\mathbb{P}$-a.s. we have

$||Q_k^*||_{\rho_{E,\mathbb{R}}} < \delta_k$ e.v., for $k \geq 1$.

(ii) The hypothesis of **Proposition 1.9** ***(ii)*** that $\left\{Q_n^*\right\}_{n \in \mathbb{N}}$, Q^* is measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathcal{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$ may seem like a strong hypothesis, except that in practice, not only does it probably hold random that functions that are $\mathbb{P}$-a.s. continuous processes. It very-likely holds that ALL random functions are measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathcal{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$ as a result of $S(\nu; \mathbb{R})$ (which when looked at as S^*, given $S \in S(\nu; \mathbb{R})$, are measurable from (Ω, Σ) to $\left(\mathbb{R}^E, \mathcal{B}\left(\left(\mathbb{R}^E, \rho_{E,\mathbb{R}}\right)\right)\right)$) being $\mathbb{P}$-a.s. ν-a.e.-dense in $\mathcal{L}^0(\nu; \mathbb{R})$ (refer to *remark 1.7* to understand convergence $\mathbb{P}$-a.s. ν-a.e), i.e., for any $Q \in \mathcal{L}^0(\nu; \mathbb{R})$, there exists a sequence $\{S_n\} \subset S(\nu; \mathbb{R})$ such that (1.8) holds, which equivalently occurs if $\mathbb{P}$-a.s. for $\omega \in \Omega$, we have $S_n^*(\omega) \to Q^*(\omega)\ \nu(\omega, \cdot)$-a.e. We shall prove this more in depth in a future draft.

Proposition 1.10. Let $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^0(\nu; \mathbb{R})$ and $Q \in \mathcal{L}^0(\nu; \mathbb{R})$. *(i)*-*(iii)* are equivalent and *(v)* $\implies$ *(iv)* $\implies$ *(i)* , with the converse implications holding if ν is finite.

(i) $Q_n \xrightarrow{\nu} Q$

(ii) For all $\epsilon_1 > 0$, we have $M_{|\nu|}(d_{\mathcal{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon_1) \xrightarrow{\mathbb{P}} 0$ as $n \to +\infty$, i.e., for all $\epsilon_2 > 0$, we have

$$\lim_{n\to+\infty} \mathbb{P}[|M_{|\nu|}(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon_1)| \geq \epsilon_2] = 0, \quad (1.12)$$

(iii) For all $\epsilon > 0$, we have

$$\lim_{n\to+\infty} d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(M_{|\nu|}(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon), 0) = 0. \quad (1.13)$$

(iv) For every subsequences $\{Q_{n_k}\}_{k\in\mathbb{N}}$, there exists a further subsequence $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that$Q_{n_{k_j}} \xrightarrow{\nu\text{-a.e.}} Q$.

(v) For every subsequences $\{Q_{n_k}\}_{k\in\mathbb{N}}$, there exists a further subsequence $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that almost surely for $\omega \in \Omega$, (1.8) holds.

Proof.

(i) $\iff$ *(ii)* Immediate from the fact that $M_\nu(\{|Q_n - Q| \geq \epsilon_1\}) = \nu(\cdot, \{|Q_n - Q| \geq \epsilon\})$.

(ii) $\iff$ *(iii)* Immediate from *Remark 1.2 (ii)*.

(v) $\implies$ *(iv)* Immediate from **Proposition 1.9 *(i)***.

(iv) $\implies$ *(ii)* Suppose *(iv)* holds. Let $\epsilon_1, \epsilon_2 > 0$, and set

$$a_n := \mathbb{P}[|M_{|\nu|}(\{d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_n{}', Q') \geq \epsilon_1\})| \geq \epsilon_2]$$

Given a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$, by hypothesis we can choose a further subsequence $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{Q_{n_k}\}_{k\in\mathbb{N}}$ such that $Q_{n_{k_j}} \xrightarrow{\nu\text{-a.e.}} Q$, which by **Proposition 1.9 *(iii)*** implies $Q_{n_{k_j}} \xrightarrow{\nu} Q$ as $j \to +\infty$, hence $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ satisfies *property (ii)*, which shows that $\lim_{j\to+\infty} a_{n_{k_j}} = 0$.

In the preceding paragraph, we have shown that given a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$, there exists a further subsequence $\{a_{n_{k_j}}\}$ such that $\lim_{j\to+\infty} a_{n_{k_j}} = 0$. Since $\{a_n\}_{n\in\mathbb{N}}$ is bounded (i.e., $0 \leq a_n \leq 1$), we find by sequential compactness that condition (1.12) is met (i.e. $\lim_{n\to+\infty} a_n = 0$), and our conclusion is reached.

(i) $\implies$ *(iv) if ν is finite.* Suppose *(i)* holds.. Let $\{Q_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence. By the fact that we showed *(i)* $\iff$ *(ii)* (and in particular, *(i)* $\implies$ *(ii)*), we can choose a subsequence $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that

$$\mathbb{P}\Big[\,|M_{|\nu|}\Big(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big)|\geq 2^{-j}\Big] < 2^{-j}.$$

Since

$$\sum_{j=1}^{\infty}\mathbb{P}\Big[\,|M_{|\nu|}\Big(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big)|\geq 2^{-j}\Big] < +\infty,$$

we find by the **Borel-Cantelli Lemma** that

$$\mathbb{P}\Big[M_{\nu}\Big(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big) \geq 2^{-j} \text{ i.o.}\Big] = 0,$$

hence we have

$$M_{|\nu|}\Big(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big)| < 2^{-j} \text{ e.v. } \mathbb{P}\text{-a.s.},$$

and it follows that we have

$$\sum_{j=1}^{\infty}|\nu|\Big(\,\cdot\,, d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big) = \sum_{j=1}^{\infty}M_{|\nu|}\Big(d_{\mathfrak{L}^0(\mathbb{P};\mathbb{R})}(Q_{n_{k_j}}{}', Q') \geq 2^{-j}\Big) < +\infty,\ \mathbb{P}\text{-a.s.}$$

Since ν is finite, we conclude by the **Borel-Cantelli Lemma for Random Measures** (i.e. **Proposition 1.5 *(iii)***) we find that almost surely for $\omega \in \Omega$, we have $Q_{n_{k_j}} \xrightarrow{\nu\text{-a.e.}} Q$.

(iv) $\implies$ *(v) if ν is finite.* Immediate from **Proposition 1.9 *(ii)***. ◻

Remark 1.7. The above proposition motivates another concept (and even more notation to go with it!). For a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathfrak{L}^0(\nu;\mathbb{R})$, we state that $\{Q_n\}_{n\in\mathbb{N}}$ **converges to some $\mathbf{Q \in L^0(\nu;\mathbb{R})}$ $\mathbb{P}$-a.s, ν-a.e.** if (1.8) holds, i.e., instead of $Q_n{}'(e) \xrightarrow{\mathbb{P}} Q'(e)$ ν-a.e. (which is the definition of $Q_n \xrightarrow{\nu\text{-a.e.}} Q$) we have $Q_n{}'(e) \xrightarrow{\mathbb{P}\text{-a.s.}} Q'(e)$ ν-a.e.

Definition 1.11. Let ν be a σ-finite random measure. The random integral of $Q \in \mathcal{L}^0(\nu;\mathbb{R})$ (if it exists) with respect to a random measure ν is defined as follows:

(i) For simple random functions of the form $S(\omega,s) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_k}(\omega)\cdot \mathbb{1}_{F_k}(s)$, for $A_1, \ldots, A_N \in \Sigma,\quad x_1, \ldots, x_k \in \mathbb{R}$, and disjoint $F_1, \ldots, F_k \in \mathcal{E}$, we have

$$\int S(\cdot,s)d\nu(\cdot,s) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_k}(\cdot)\nu(\cdot,F_k).$$

(ii) In general, for Q such that

$$\int |Q(\omega,e)|d|\nu|(\omega,e) < +\infty, \quad (1.14)$$

a.s. for $\omega \in \Omega$, we define $\int Q(\cdot,s)d\nu(\cdot,s)$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\mathbb{P};\mathbb{R})$ (if it exists) of random integrals of a sequence of simple functions S_n such that $|S_n| \leq |Q|$ a.s. and $S_n \xrightarrow{\nu\text{-a.s.}} Q$, i.e. we have

$$\int S_n d\nu \xrightarrow{\mathbb{P}} Y. \quad (1.15)$$

Any Q such that (1.14) holds and $Y \in \mathcal{L}^0(\mathbb{P};\mathbb{R})$ exists such that (1.15) is satisfied for some sequence $\{S_n\}_{n\in\mathbb{N}}$ of simple functions and we call ν-integrable, and the set of such functions we call $\mathcal{L}^1(\nu;\mathbb{R})$.

Remark 1.8.

(i) Although **Definition 1.11** is a valid definition--since **The Dominated Convergence Theorem for Random Measures (DCTRM)** (i.e., **Theorem 1.13 *(ii)***) shows that the limit exists and can be passed on between the limit of the integral also proves that the definition is valid--the definition is not the most ideal one. For one, it's not immediately clear from the definition itself that a limit $Y \in \mathcal{L}^0(\mathbb{P};\mathbb{R})$ such that (1.15) holds for some sequence of simple functions S_n such that $|S_n| \leq |Q|$ a.s. and $S_n \xrightarrow{\nu\text{-a.s.}} Q$ does exist in $\mathbb{P}$, even though the aforementioned DCTRM shows such a limit indeed does exist. For another, it's not immediately clear that the integral is even well-defined in the sense that the limit Y is uniquely-determined (up to $\mathbb{P}$-a.s. equivalence) from the choice of S_n with the property $|S_n| \leq |Q|$ a.s. converging to Q ν-a.s. In other words, without additional work (in particular,

the very tedious work of proving of the DCTRM) verifying otherwise, it may be possible multiple values of the integral exist!

A potentially better definition, which I'll define in a future draft (especially if it's in fact better), utilizes simple functions (as first done in *part (i)*) followed the value established by "convergence in $\mathfrak{L}^1(\nu; \mathbb{R})$", i.e., convergence in (what turns out to be) the complete norm

$$||Q||_{\mathfrak{L}^1(\nu;\mathbb{R})} := \int |Q(\omega, e)| d|\nu|(\omega, e).$$

Note that work is still needed to be done to show that the definition is well-defined (in particular, to show that $\mathfrak{L}^1(\nu; \mathbb{R})$ is in fact a Banach space and the integral is a bounded linear map from $\mathfrak{L}^1(\nu; \mathbb{R})$ to $\mathfrak{L}^0(\mathbb{P}; \mathbb{R})$), but more intuitive work that uses already well-established concepts in Functional Analysis of dense subspaces of Banach Spaces that translate fairly elegently to this setting, and don't require any additional technical work like that of the DCTRM. This shall again be done in a future draft.

(ii) $\mathfrak{L}^1(\nu; \mathbb{R})$ is a vector subspace of $\mathfrak{L}^0(\nu; \mathbb{R})$. This is verified by the fact that $0 \in \mathfrak{L}^1(\nu; \mathbb{R})$ for $Q, R \in \mathfrak{L}^1(\nu; \mathbb{R})$, almost surely for $\omega \in \Omega$, we have

$$\begin{aligned}\int |Q(\omega, \cdot) + R(\omega, \cdot)| d|\nu|(\omega, \cdot) &\le \int |Q(\omega, \cdot)| + |R(\omega, \cdot)| d|\nu|(\omega, \cdot) \\ &= \int |Q(\omega, \cdot)| d|\nu|(\omega, \cdot) + \int |R(\omega, \cdot)| d|\nu|(\omega, \cdot) \\ &< +\infty.\end{aligned}$$

(iii) Random integrals of random functions with respect to random measures are a more general case of a stochastic Integral, as we shall soon see in the next section. For now, let's provide some examples:

Examples 1.12.

(i) First, we can define $\nu(\omega, E) := m(E)$, where $\mathcal{E} := \mathcal{B}(\mathbb{R}_+)$ and m is the Lebesgue measure. The integral of a random function Q over this random measure is simply the Borel extension of the Riemann integral over a stochastic process, i.e., we have

$$\int_{(a,b)} Q dm := \int_a^b Q dt.$$

(ii) (Source: Applebaum 2.3.1, page 104 and Folland § 1.4, page 30-31) Next, we can define $\nu(\omega, E) := \mu_{X(\omega)}(E)$, where $(X_t)_{t\in\mathbb{R}_+}$ is an a.s. cadlag process and for a cadlag function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, we define μ_γ to be the Borel extension of the (signed-)premeasure

$\mu_\gamma(a, b) := \gamma(b) - \gamma(a)$. (refer to *Folland page 30-31*)

The random integral over this random measure is in turn the Borel extension of the Ito integral over X, i.e., we have

$$\int_{(a,b)} Q d\mu_X := \int_a^b Q dX.$$

Note that this example is generalized from the example given from Applebaum (which assumes that X is a Levy process, but all we need for this example to work is that it's a.s. Cadlag). Note that this example does usually also involve additional filtration structure where we additionally require $Q(\cdot, t) : \Omega \to \mathbb{R}$, for every $t \in \mathbb{R}_+$ to be $\mathcal{F}_t$-measurable, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$is a filtration to which X is adapted. This kind of extra condition imposed on processes that are stochastically integrated in this way is something that will be addressed in a future draft.

(iii) (Source: Applebaum 2.3.1, page 104-105 and 2.3.2, page 106) Next, we have a **Poisson Random measure** with respect to $(E, \mathcal{A}, \mu)$, for some measure space with σ-finite measure μ. The Poisson random measure with inensity measure μ is a random measure $(\omega, A) \mapsto N_A(\omega)$ such that

(a) $\forall A \in \mathcal{A}$, N_A is a Poisson random variable with rate $\mu(A)$.

(b) If sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$ don't intersect then the corresponding random variables from *(a)* are mutually independent.

The poisson random measure gives rise to integration with respect to the Poisson random measure on some $Q \in \mathcal{L}^0(N; \mathbb{R})$. In practice, since Poisson distributions are discrete, and hence integrals over a Poisson random measure end up being series, as so

$$\int_A Q dN = \sum_{x\in A} Q(\cdot, x) N(\cdot, \{x\}) \quad (1.16)$$

As will be mentioned in future reports, there is a very intimate relationship between Poisson random measures (over Borel σ-algebras in particular). and Levy processes, and so integration over poisson processes will often occur.

Theorem 1.13. Suppose $Q \in \mathcal{L}^1(\nu; \mathbb{R})$

(i) If there exists a sequence $Q_n \in \mathcal{L}^1(\nu; \mathbb{R})$ such that

$$\int |Q_n - Q| d|\nu| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to +\infty,$$

then $\int Q_n d\nu \xrightarrow{\mathbb{P}} \int Q d\nu$ as $n \to +\infty$.

(ii) (Dominated Convergence Theorem for Random Measures) If ν is σ-finite, and $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\nu; \mathbb{R})$ is a sequence of random functions such that $|Q_n| \leq |Q|$ and $Q_n \xrightarrow{\nu\text{-a.s.}} Q$, then

$$\int Q_n d\nu \xrightarrow{\mathbb{P}} \int Q d\nu.$$

Proof.

(i) Note that for fixed $\omega \in \Omega$, we have

$$\left| \int Q_n(\omega, s) d\nu(\omega, s) - \int Q(\omega, s) d\nu(\omega, s) \right| \leq \int |Q_n(\omega, s) - Q(\omega, s)| d|\nu|(\omega, s),$$

hence given $\epsilon > 0$, we have

$$\mathbb{P}\left[\left| \int Q_n d\nu - \int Q d\nu \right| \geq \epsilon \right] \leq \mathbb{P}\left[\int |Q_n - Q| d|\nu| \geq \epsilon \right],$$

and our conclusion immediately follows.

(ii) Suppose ν is σ-finite and $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\nu; \mathbb{R})$ is a sequence of random functions such that $|Q_n| \leq |Q|$ and $Q_n \xrightarrow{\nu\text{-a.s.}} Q$. Using *part (i)*, it shall suffice to show that given a subsequence $\{Q_{n_k}\}$, we find that there exists a further subsequence $\{Q_{n_{k_j}}\}$ such that

$$\int |Q_{n_{k_j}} - Q| d|\nu| \xrightarrow{\mathbb{P}\text{-a.s.}} 0 \text{ as } j \to +\infty. \qquad (1.17)$$

First we prove this in the special case that ν is finite. Given a subsequence $\{Q_{n_k}\}$, we use **Proposition 1.10** (and the special case hypothesis that ν is finite) to choose a further subsequence $\{Q_{n_{k_j}}\}$ such that $Q_{n_{k_j}}'(e) \xrightarrow{\mathbb{P}\text{-a.s.}} Q'(e)$ ν-a.e. as $j \to +\infty$. Since it follows by hypothesis that $|Q - Q_{n_{k_j}}| \leq 2|Q|$ $\mathbb{P}$-a.s. we find by the (deterministic) **Dominated Convergence Theorem** that almost surely for fixed $\omega \in \Omega$ that

$$\int |Q_{n_{k_j}}(\omega, s) - Q(\omega, s)| d|\nu|(\omega, s) \to 0 \text{ as } j \to +\infty,$$

and we have shown that $\int |Q_n - Q| d\nu \xrightarrow{\mathbb{P}} 0$ in the special case that ν is finite.

Now, given a subsequence $\{Q_{n_k}\}$, we prove the existence of a further subsequence such that (1.10) holds in the more general case that ν is σ-finite. Choose a $\subset$-increasing sequence $\{E_m\}_{m\in\mathbb{N}} \subset \mathcal{E}$ such that $E = \bigcup_{m\in\mathbb{N}} E_m$ and almost surely for $\omega \in \Omega$ we have $|\nu|(\omega, E_m) < +\infty$ for each $n \geq 1$. For each $m \geq 1$, let ν_m be the random measure defined by $\nu_m(\,\cdot\,, F) := \nu(\,\cdot\,, F \cap E_m)$ and note that each ν_m is a finite measure. Since $\mathbb{1}_{E_m}|Q_n(\omega, s) - Q(\omega, s)| \nearrow |Q_n(\omega, s) - Q(\omega, s)|$ pointwise as $m \to +\infty$, for each $n \geq 1$, we find, almost surely for fixed $\omega \in \Omega$, by the **Monotone Convergence Theorem** (applied to $\nu(\omega, \,\cdot\,)$ for fixed $\omega \in \Omega$), we have

$$\begin{aligned}\lim_{m \to +\infty} \int |Q_n(\omega, s) - Q(\omega, s)| d|\nu_m|(\omega, s) &= \lim_{m \to +\infty} \int \mathbb{1}_{E_m}(s)|Q_n(\omega, s) - Q(\omega, s)| d|\nu|(\omega, s) \\ &= \int |Q_n(\omega, s) - Q(\omega, s)| d|\nu|(\omega, s)\end{aligned}$$

and we've shown that

$$\int |Q_n - Q| d|\nu_m| \nearrow \int |Q_n - Q| d|\nu| \ \mathbb{P}\text{-a.s. as } m \to +\infty \text{ for each } n \geq 1 \qquad (1.18)$$

Next, Note that for each $m \geq 1$, we have $\int |Q_n - Q| d|\nu_m| \xrightarrow{\mathbb{P}} 0$ as $n \to +\infty$. For each $j \geq 1$, recursively choose $n_{k_j} > n_{k_{j-1}}$ such that

$$\mathbb{P}\left[\int |Q_{n_{k_j}} - Q|d|\nu_j| \geq j^{-1}\right] < 2^{-j}.$$

For each $m \geq 1$ and $\epsilon > 0$, we find by (1.18) that

$$\mathbb{P}\left[\int |Q_{n_{k_j}} - Q|d|\nu_m| \geq \epsilon\right] = O\left(\mathbb{P}\left[\int |Q_{n_{k_j}} - Q|d|\nu_j| \geq j^{-1}\right]\right) = O\left(2^{-j}\right) \text{ as } j \to +\infty,$$

and it immediately follows by the (deterministic) **Borel-Cantelli Lemma** that

$$\int |Q_{n_{k_j}} - Q|d|\nu_m| \xrightarrow{\mathbb{P}\text{-a.s.}} 0 \text{ as } j \to +\infty, \text{ for each } m \geq 1.$$

Using (1.18) and diagonalization of $\mathbb{P}$-a.s. convergence (see *Remark 1.9 (ii)*) it follows that

$$\int |Q_{n_{k_j}} - Q|d|\nu| \xrightarrow{\mathbb{P}\text{-a.s.}} 0 \text{ as } j \to +\infty,$$

and we conclude that $\{Q_{n_{k_j}}\}_{j\in\mathbb{N}}$ as chosen satisfies (1.17), which completes the proof. □

Remark 1.9.

(i) Note that the **Dominated Convergence Theorem for Random Measures** (i.e., **Theorem 1.13 *(ii)***) hypothesizes that the random measure ν is σ-finite, as defined in **Definition 1.3. *(ii)***. It's possible for this theorem to be generalized using some sort of "locally finite" condition (i.e., $\mathbb{P}$-a.s. for fixed $\omega \in \Omega$, for every $e \in E$, existence of some $F \ni e$ such that $\nu(\omega, F) < +\infty$) and then proving the theorem using nets in place of sequences.

(ii) Diagonalization of $\mathbb{P}$-a.s. convergence tells us that if we have a double-indexed sequence $\{X_{m,j}\}_{j,m\in\mathbb{N}} \subset \mathfrak{L}^0(\mathbb{P};\mathbb{R})$ such that $X_{m,j} \xrightarrow{\mathbb{P}\text{-a.s.}} X_j$ as $m \to +\infty$, for each $j \geq 1$ for some $\{X_j\}_{j\in\mathbb{N}} \subset \mathfrak{L}^0(\mathbb{P};\mathbb{R})$, then we have $X_j \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $j \to +\infty$, for some $X \in \mathfrak{L}^0(\mathbb{P};\mathbb{R})$, if and only if $X_{j,j} \xrightarrow{\mathbb{P}\text{-a.s.}} X$ as $j \to +\infty$. This holds for essentially every mode of convergence, and usually proved using some kind of $\epsilon/2$ trick. We shall discuss this mode of convergence in a future paper.

Sources:

Stochastic Differential Equations and Diffusion Processes 2nd ed. Chapter I § *8*

Ikeda, Watanabe

Lévy Processes and Stochastic Calculus § *2.3.1-* § *2.3.2*
Applebaum

Real Analysis, Modern Techniques § *1.4* § *2.4,* § *3.1*
Folland

Probability and Measure, 3rd edition, § *4*
Billingsley

2 Stochastic Integration

For the two following defintions, let $\mathcal{X}$ be a Banach space, and let P be an $\mathcal{X}$-valued stochastic process.

Definition 2.1. The (Riemann) Integral of a (Banach-valued) stochastic process P over time on the interval (a, b) is defined as follows:

(i) For simple processes of the form $P := S(\omega, t) = \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(\omega)\mathbb{1}_{(t_k, t_{k-1})}(t)] \cdot x_k$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $x_1, \ldots, x_k \in \mathcal{X}$, and $t_0 := a < t_1 < \cdots < t_N := b$ $(1 \leq k \leq N)$, we have

$$\int_a^b S dt := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(t_k - t_{k-1})] \cdot x_k.$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_{\mathcal{X}} dt < +\infty, \quad (2.1)$$

a.s. for $\omega \in \Omega$, we define $\int_a^b P dt$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; \mathcal{X})$ (if it exists) of stochastic integrals of sequences of simple processes $\{S_n\}_{n \in \mathbb{N}}$ such that $||S_n||_{\mathcal{X}} \leq ||P||_{\mathcal{X}}$ $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dt \xrightarrow{\mathbb{P}} Y. \quad (2.2)$$

Any process such that (2.2) exists we call **(Riemann) integrable over** (a, b).
Note: I haven't really found a great source that corresponds well to this specific definition, but I have a source for a **Definition 2.6**, which more or less generalizes both this definition and a later definition; I'll look for a good source in a future draft.

For the next definition, let X be real-valued cadlag process.

Definition 2.2. A **stochastic (Stiltjes) Integral (or an ito integral) of a (Banach-valued) stochastic process** P **with respect to** X **on the interval** (a, b) is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on P), $x_1, \ldots, x_N \in X$, and $t_0 := a < t_1 < \cdots < t_N := b$ ($1 \leq k \leq N$), we have

$$\int_a^b SdX := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(X(t_k) - X(t_{k-1}))] \cdot x_k.$$

(ii) For any process P such that

$$\int_a^b ||P(\omega, t)||_X d|\mu_{X(\omega)}|(\omega, t) < +\infty, \quad (2.3)$$

a.s. for $\omega \in \Omega$, where $\mu_{X(\omega)}$ is given in **Examples 1.3 (ii)**, we define $\int_a^b PdX$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; X)$ (if it exists) of stochastic integrals of sequences of simple processes $\{S_n\}_{n \in \mathbb{N}}$ such that $||S_n||_X \leq ||P||_X$ for all $n \in \mathbb{N}$ and $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dX \xrightarrow{\mathbb{P}} Y. \quad (2.4)$$

Any function such that (2.4) exists, we call **Ito integrable** with respect to X over (a, b).
Note: I haven't really found a great source that corresponds well to this specific definition, but I have a source for a **Definition 2.6**, which more or less generalizes both this definition and a later definition; I'll look for a good source in a future draft.

Remark 2.1.

(i) Note that **Definition 2.2** actually generalizes **Definition 2.1** in the sense that Definition 2.1

is a special case of **Definition 2.2** in the case where X is the deterministic idenity function $t \mapsto t$ on $\mathbb{R}_+$.

(ii) Note that in both preceding definitions, we work specifically with simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}}(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on S), $x_1, \ \ldots \ , x_N \in \mathcal{X}$, and $t_0 := a < t_1 < \cdots < t_N := b$ $(1 \leq k \leq N)$, as opposed to simple processes in general, which are more generally of the form $\sum_{k=1}^{N} X_k(\omega) \cdot \mathbb{1}_{(t_{k-1}, t_k)}(t)$, where $X_k \in \mathcal{F}_{t_{k-1}}$. In a later draft, we shall define Riemann integrability of such processes in the more conventional sense, and show this, we shall talk about the classic notions of $\mathcal{L}^1$ convergence for stochastic processes, and how it generalizes $\mathcal{L}^1$ convergence for deterministic functions defined over an interval $[0, T]$ or $\mathbb{R}_+$.

(iii) Note that **Definition 2.1** and **Definition 2.2** have not been generalized by the previous section (unfortunately), since the previous section deals with real-valued integrable functions over a random measure and this section deals with Banach-valued Riemann integration over a stochastic process, which is a random measure with additional filtration structure. Hopefully in later reports, we shall talk about integration of random measures in a Banach valued setting, as well as stochastic Lebesgue integration.

(iv) Note that **Definition 2.1** and **Definition 2.2** as presented are incomplete in the sense that it only accounts for limits of one specific simple function. In the case of pure Riemann integration (without the introduction of random measures over the analogous measurable space), this does not exactly work as ideal. In a later draft, we shall remedy this problem by providing a more rigorous definition of Riemann integrability of a stochastic process.

Example 2.4. The most common example of such a stochastic Stiltjes Integral is the one with respect to the Wiener process W, i.e. $\int_0^T P_t dW_t$. There are lots of nice properties involving this integral, and variants of that integral, such as the Ito Isometry, which we shall now prove. *Note:* As before, I haven't really found a great source that corresponds well to this specific example. And as before, I'll look for a good source in a future draft.

Propositon 2.5. *(Ito Isometry; Hilbert Space Version)* Let $W : [0, T] \times \Omega \to \mathbb{R}$ denote the canonical real-valued Weiner process defined up to time $T > 0$, and let $X : [0, T] \times \Omega \to \mathcal{H}$ be an a.s. $L^2(0, T)$ stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then

$$\mathbb{E}\left(\left|\left|\int_0^T X_t dW_t\right|\right|_{\mathcal{H}}^2\right) = \mathbb{E}\left[\int_0^T ||X_t||_{\mathcal{H}}^2 dt\right]. \quad (2.6)$$

Source: Ito Isometry (from [Wikipedia link here])

Remark 2.2.

(i) The proposition in the cited wikipedia article tells us that

$$\mathbb{E}\left(\left(\int_0^T X_t dW_t\right)^2\right) = \mathbb{E}\left[\int_0^T (X_t)^2 dt\right],$$

for an adapted *real-valued* process X_t. Noting that

$||a||_{\mathcal{H}}^2 = a^2$

for $a \in \mathcal{H}$ in the special case where the Hilbert space $\mathcal{H}$ is one dimensional (i.e. we have $\mathcal{H} \cong \mathbb{R}$), we find that the proposition cited is (more or less) this proposition in the single dimensional case. The proof moreover is also very similar.

(ii) It's worth noting that the proof of this proposition relies on simple functions of the elegant form

$S := \sum_{k=1}^{N} x_k \mathbb{1}_{A_{t_{k-1}}} \cdot \mathbb{1}_{(t_{k-1}, t_k)}$, for $A_{t_{k-1}} \in \mathcal{F}_{t_{k-1}}$ (where $\mathcal{F}_t$ is the filtration on S), $x_1, \ldots, x_N \in X$, and $t_0 := a < t_1 < \cdots < t_N := b$ ($1 \leq k \leq N$).

The previous section tries to define random measures, which is in some ways (though not completely in other ways) a generalization of the Stiltjes Integral. So far I've only defined random measures in the real-value case. The hope is that this framework of stochastic processes can be talked about more generally in the context of (Bochner/Pettis) integration of a Hilbert-valued (and maybe even Banach-valued function) function over a random measure, which in turn allows us to talk about SDE's in a more general framework.

(iii) To expand on a specific instance where the idea of *part (ii)* gives us a pretty elegant (and potentially useful) result, it's worth noting that I'm pretty sure the Ito-Isometry also holds for the situation of processes adapted to Levy Processes that are integrated with respect to Levy

Processes. I shall verify this claim in a future draft.

Proof. Let

$$S(t) = \sum_{j=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_k, t_{k-1})} a_k \quad \left(t_0 := 0 < t_1 < \cdots < t_N := T,\ a_k \in \mathcal{H},\ A_{t_{k-1}} \in \mathcal{F}^W_{t_{k-1}} \right)$$

be a simple function. Since $\mathbb{1}_{A_{t_{k-1}}},\ W_{t_k} - W_{t_{k-1}}$ $(1 \leq j \leq N)$ are independent and $W_{t_j} - W_{t_{j-1}}, W_{t_k} - W_{t_{k-1}}$ $(j \neq k)$ are independent, we have

$$\begin{aligned}
\mathbb{E}\left(\left| \left| \int_0^T S(t) dW_t \right| \right|_{\mathcal{H}}^2 \right) &= \mathbb{E}\left(\left\langle \int_0^T S(t) dW_t, \int_0^T S(t) dW_t \right\rangle_{\mathcal{H}} \right) \\
&= \mathbb{E}\left(\left\langle \sum_{j=1}^{N} [\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] a_j, \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})] a_k \right\rangle_{\mathcal{H}} \right) \\
&= \mathbb{E}\left(\sum_{k=1}^{N} \sum_{j=1}^{N} ([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \langle a_j, a_k \rangle_{\mathcal{H}} \right) \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}\left(\left[\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})^2 \right] \right) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}([\mathbb{1}_{A_{t_{j-1}}} (W_{t_j} - W_{t_{j-1}})] \cdot [\mathbb{1}_{A_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})]) \\
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}\left[(W_{t_j} - W_{t_{j-1}})^2 \right] \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \mathbb{E}[W_{t_j} - W_{t_{j-1}}] \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \mathbb{E}[W_{t_k} - W_{t_{k-1}}] \\
&= \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] (t_j - t_{j-1}) \\
&\quad + \sum_{j \neq k} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{E}[\mathbb{1}_{A_{t_{j-1}}}] \cdot 0 \cdot \mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}] \cdot 0
\end{aligned}$$

$$= \mathbb{E}\left[\sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} (t_j - t_{j-1}) \right]$$

$$= \mathbb{E}\left[\int_0^T \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} dt \right]$$

$$= \mathbb{E}\left[\int_0^T \sum_{j=1}^{N} \langle a_j, a_j \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} + \sum_{j \neq k}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_k, t_{k-1})} dt \right]$$

$$= \mathbb{E}\left[\int_0^T \sum_{k=1}^{N} \sum_{j=1}^{N} \langle a_j, a_k \rangle_{\mathcal{H}} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})} dt \right]$$

$$= \mathbb{E}\left[\int_0^T \left\langle \sum_{j=1}^{N} a_j \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_j, t_{j-1})}, \sum_{k=1}^{N} a_k \mathbb{1}_{A_{t_{j-1}}} \mathbb{1}_{(t_k, t_{k-1})} \right\rangle_{\mathcal{H}} dt \right]$$

$$= \mathbb{E}\left[\int_0^T ||S(t)||_{\mathcal{H}}^2 dt \right].$$

Then for any integrable process X_t such that $\int_0^T X_t dW_t$ is $L^2(\mathbb{P}; \mathcal{H})$, we can choose a sequence of simple processes $S_n(t)$ such that almost surely we have

$$\lim_{n \to +\infty} S_n(t) = X_t, \ \lim_{n \to +\infty} \int_0^T S_n(t) dt = \int_0^T X_t dt, \ \lim_{n \to +\infty} \int_0^T S_n(t) dW_t = \int_0^T X_t dW_t, \quad (2.6)$$

and then satisfy (2.6) by passing the limit using the **Dominated Convergence Theorem**. □

Now let X be an $\mathcal{X}$-valued stochastic process, and P, be an operator-valued, i.e. $L(\mathcal{X})$-valued, stochastic process. We will now define stochastic integration under this setting:

Definition 2.6. A **stochastic (Stiltjes) Integral (or an ito integral) of a (Operator-valued) stochastic process P with respect to X on the interval (a, b)** is defined as follows:

(i) For simple processes of the form $S(\omega, t) := \sum_{k=1}^{N} [\mathbb{1}_{A_{t_{k-1}}}(\omega) \mathbb{1}_{(t_{k-1}, t_k)}(t)] \cdot \Phi_k$, for $A_{t_{k-1}} \in \mathcal{F}_{t_k}$ (where $\mathcal{F}_t$ is the filtration on P), $\Phi_1, \ldots, \Phi_N \in L(\mathcal{X})$, and $t_0 := a < t_1 < \cdots < t_N := b$ (

$1 \leq k \leq N$), we have

$$\int_a^b SdX := \sum_{k=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \Phi_k(X(t_k) - X(t_{k-1})).$$

(ii) For any process P such that

$$\int_a^b ||P(\omega,t)||_{L(X)} d||\mu_{X(\omega)}||(\omega,t) < +\infty, \quad (2.7)$$

a.s. for $\omega \in \Omega$, where $\mu_{X(\omega)}$ is given in **Examples 1.3 (ii)**, we define $\int_a^b PdX$ to be the $\mathbb{P}$-limit $Y \in \mathcal{L}^0(\Omega; L(X))$ (if it exists) of stochastic integrals of sequences of simple S_n processes such that $||S_n||_{L(X)} \leq ||P||_{L(X)}$ and $S_n \xrightarrow{\mathbb{P}} P$, i.e.

$$\int_a^b S_n dX \xrightarrow{\mathbb{P}} Y. \quad (2.8)$$

Any function such that (2.8) exists, we call **Ito integrable** with respect to X over (a,b).
Source: Lototsky, Rozovskey (page 128-131) and Spring 2022-M647 Lecture 3 (revised) Definition 3.2.3

Remark 2.3.

(i) Note that **Definition 2.4** is a generalization of the definition of *A643 Spring 2021 Lecture 4/M647 Spring 2022 Lecture 3* that it came from, as well as the *Da Prato, Zabczyk* $\S$ *4.2 (page 90-96)* source that it came from, in the sense that the integration is over any stochastic process X as opposed to simply a Wiener process W.

(ii) Note furthermore that **Definition 2.4** is a generalization of **Definition 2.2** (as well as **Definition 2.1**), in the sense that the integration **Definition 2.2** can be imbedded into the instance of **Definition 2.4** where $X := \mathbb{R}$, using the canonical isometry $\mathbb{R} \to \mathcal{L}(\mathbb{R})$ defined by $a \mapsto T_a$, where $T_a : \mathbb{R} \to \mathbb{R}$ is defined by $T_a(x) = ax$, hence integrating over real-valued stochastic processes can be looked at as integrating over $\mathcal{L}(\mathbb{R})$-valued stochastic processes.

(iii) As a generalization of **Definition 2.2**, the ideas and caveats mentioned in *Remark 2.1 (ii)-(iv)* also (more or less) apply to **Definition 2.4**, although integrating vector-valued functions over random measures have not been established yet, and neither has integrating operator-valued functions over vector valued measures (not even in the deterministic setting!)

(iv) In a later draft, I shall provide some concrete examples of the kind of integration mentioned in **Definition 2.4**.

(v) For this next version of Ito's Isometry (i.e. **Proposition 2.7**), we need the $L(\mathcal{U}, \mathcal{H})$-valued process P on $(0, +\infty)$ to additionally be almost surely **Hilbert-Schmidt** operator-valued on $(0, T)$. In other words, for any $t \in (0, T)$, we assume that almost surely for $\omega \in \Omega$, we have

$$||P_t(\omega)||_{L_2(\mathcal{U},\mathcal{H})} := \sqrt{\sum_{i\in I} ||P_t(\omega)e_i||^2_{\mathcal{H}}} < +\infty,$$

where $\{e_i : i \in I\}$ is an orthonormal basis of $\mathcal{H}$. We shall moreover let $L_2(\mathcal{U}, \mathcal{H})$ denote the set of Hilbert-Schmidt opeators, call $||\cdot||_{L_2(\mathcal{U},\mathcal{H})}$ the **Hilbert-Schmidt norm**, and note that $L_2(\mathcal{U}, \mathcal{H})$ is a Hilbert Space on the inner-product

$$\langle A, B\rangle_{L_2(\mathcal{U},\mathcal{H})} := \sum_{i\in I} \langle Ae_i, Be_i\rangle.$$

We finally define an a.s. Hilbert-Schmidt operator-valued process to to be **X-admissible on $[0, T]$ if**

$$\int_0^T \langle Q_X P_{t t} P_t\rangle_{L_2(\mathcal{U},\mathcal{H})} dt < +\infty.$$

We shall talk about Hilbert-Schmidt Operators in more detail in a future report.
Source: Hilbert–Schmidt Operator (from Wikipedia link here), *Lototsky, Rozovskey (Page 130)*, and *Da Prato, Zabczyk (page 90-91)*

We shall now state and outline the proof of the operator-valued version of the Ito Isometry.

Proposition 2.7. *(Ito Isometry; Operator Version):* For an $\mathcal{H}$-valued wiener process W and an a.s. $L_2(\mathcal{U}, \mathcal{H})$-valued process X that is Wiener-adapted to W and W-admissible on $[0, T]$, we have

$$\begin{aligned}\mathbb{E}\left[\left|\left|\int_0^T X_t dW_t\right|\right|^2_{\mathcal{H}}\right] &= \mathbb{E}\int_0^T ||X_t Q^{1/2}||^2_{L_2(\mathcal{U},\mathcal{H})} dt,\\ &= \mathbb{E}\int_0^T ||\Psi(s)||^2_{L_2(\mathcal{U}_0,\mathcal{H})} dt, \qquad (2.9)\end{aligned}$$

where Q is the covariance operator of W and $\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$.
Source: Spring 2022-M647 Lecture 3 (revised) Proposition 3.2.4

Remark 2.4.

(i) Note that **Proposition 2.7** is a generalization of **Proposition 2.5**, observing that $L_2(\mathcal{H}, \mathbb{R}) \cong \mathcal{H}^* \cong \mathcal{H}$ and naturally, the proof goes through a similar derivation in the case of simple functions, as well as a similar convergence theorem/localization trick in the general case. The main difference is the fact that the dimensions The main difference is that the infinite dimensions cause there to be additional steps in the derivation, involving utilizing the covariance operator and interactions with the real-valued Brownian motion terms $\{\beta_n\}_{n\in\mathbb{N}}$, with both themselves and other terms.

(ii) On the subject of convergence thoerems/localization tricks, there is still work to be done on showing that the limit works in the way it does, either through usig a convergence theorem or localization. In this draft, I rather handwavily give reference to the Dominated Convergence Theorem, which definitely holds in the Bochner integration setting (though I have not yet verified this) and hence holds in the desired stochastic sense. But it might be more intuitively and flavorfully satisfying to verify the argument using the more general functional analytic trick of localization. Maybe I'll use this in a future draft.

Outline of proof. Let $\Psi(t)$ be a simple process of the form

$$\Psi(t) := \sum_{k=1}^{N} \mathbb{1}_{A_{t_{k-1}}} \mathbb{1}_{(t_{k-1}, t_k)} \Phi_k$$

Let Q be the covariance operator with eigenvalue coefficients $\{v_n\}_{n\in\mathbb{N}}$ of an orthonormal eigenbasis $(u_n)_{n\in\mathbb{N}}$, i.e., we have

$$W(t) := \sum_{n=1}^{\infty} \sqrt{v_n} \beta_n(t) u_n$$

for pairwise independent sequence of real-valued Brownian motions $\{\beta_n(t)\}_{n\in\mathbb{N}}$, and it follows that

$$\mathbb{E}\left(\left|\left|\int_0^T \Psi(t)dW_t\right|\right|_{\mathcal{H}}^2\right) = \mathbb{E}\left(\left|\left|\sum_{k=1}^N \mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sum_{n=1}^{\infty}\sqrt{\nu_n}\beta_n(t_k)u_n - \sum_{n=1}^{\infty}\sqrt{\nu_n}\beta_n(t_{k-1})u_n\right)\right|\right|_{\mathcal{H}}^2\right)$$

$$= \mathbb{E}\left(\left|\left|\sum_{k=1}^N\sum_{n=1}^{\infty}\mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sqrt{\nu_n}(\beta_n(t_k)-\beta_n(t_{k-1}))u_n\right)\right|\right|_{\mathcal{H}}^2\right)$$

$$= \mathbb{E}\left(\left\langle\sum_{k=1}^N\sum_{n=1}^{\infty}\mathbb{1}_{A_{t_{j-1}}}\Phi_j\left(\sqrt{\nu_m}(\beta_m(t_j)-\beta_m(t_{j-1}))u_m\right), \sum_{k=1}^N\sum_{n=1}^{\infty}\mathbb{1}_{A_{t_{k-1}}}\Phi_k\left(\sqrt{\nu_n}(\beta_n(t_k)-\beta_n(t_{k-1}))u_n\right)\right\rangle_{\mathcal{H}}\right)$$

$$= \mathbb{E}\left(\sum_{k=1}^N\sum_{j=1}^N\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right)$$

$$= \sum_{k=1}^N\sum_{j=1}^N\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right]$$

Since $\mathbb{1}_{A_{t_{k-1}}}$, $W_{t_k} - W_{t_{k-1}}$ $(1 \le k \le N)$ are independent (and hence $\mathbb{1}_{A_{t_{k-1}}}$ and $\beta_n(t_k) - \beta_n(t_{k-1})$ are independent for all $n \ge 1$), $W_{t_j} - W_{t_{j-1}}, W_{t_k} - W_{t_{k-1}}$ $(j \ne k)$ are independent (and hence $\beta_n(t_j) - \beta_n(t_{j-1})$ and $\beta_n(t_k) - \beta_n(t_{k-1})$ are independent for all $n \ge 1$), and β_n, β_m $(m \ne n)$ are independent we find in the situation that $j \ne k$ or $m \ne n$, we have

$$\mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right] = 0,$$

and in the situation where $j = k$ and $m = n$ we have

$$\mathbb{E}\left[\mathbb{1}_{A_{t_{j-1}}}\mathbb{1}_{A_{t_{k-1}}}\sqrt{\nu_m}\sqrt{\nu_n}((\beta_m(t_j)-\beta_m(t_{j-1}))(\beta_n(t_k)-\beta_n(t_{k-1}))\langle\Phi_j u_m, \Phi_k u_n\rangle_{\mathcal{H}}\right]$$
$$= \mathbb{E}\left[\mathbb{1}_{A_{t_{k-1}}}v_n(\beta_n(t_k)-\beta_n(t_{k-1}))^2||\Phi_k u_n||_{\mathcal{H}}^2\right]$$
$$= ||\sqrt{v_n}\Phi_k u_n||_{\mathcal{H}}^2\mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}]\mathbb{E}\left[(\beta_n(t_k)-\beta_n(t_{k-1}))^2\right]$$
$$= ||\sqrt{v_n}\Phi_k u_n||_{\mathcal{H}}^2\mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}](t_k - t_{k-1}).$$

Since it can be shown (which we shall do in a later report about compact operators) that

$$||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})} = \sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}$$

it follows that

$$\begin{aligned}
\mathbb{E}\left(\left|\left|\int_0^T \Psi(t)dW_t\right|\right|^2_{\mathcal{H}}\right) &= \sum_{k=1}^{N}\sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{E}[\mathbb{1}_{A_{t_{k-1}}}](t_k - t_{k-1}) \\
&= \mathbb{E}\left[\sum_{k=1}^{N}\sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{1}_{A_{t_{k-1}}}(t_k - t_{k-1})\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{N}\int_0^T\sum_{n=1}^{\infty} ||\sqrt{v_n}\Phi_k u_n||^2_{\mathcal{H}}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)dt\right] \\
&= \mathbb{E}\left[\int_0^T\sum_{k=1}^{N}\left[||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)dt\right]\right],
\end{aligned}$$

and through functional analysis techniques on compact operators (in particular taking advantage of the fact that $L_2(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert Space, for any two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$,) , we can further show that

$$\sum_{k=1}^{N}\left[||Q^{1/2}\Phi_k||^2_{L_2(\mathcal{U},\mathcal{H})}\mathbb{1}_{A_{t_{k-1}}}\mathbb{1}_{(t_{k-1},t_k)}(t)\right] = ||Q^{1/2}\Psi(t)||^2_{L_2(\mathcal{U},\mathcal{H})},$$

and condition (2.9) is met for simple functions.

Then for any integrable process X_t such that $\int_0^T X_t dW_t$ is $L^1(\mathbb{P}; L_2(\mathcal{U}, \mathcal{H}))$, we proceed in similar fashion to the general case of the outline of the proof of **Propositon 2.5,** where we choose a sequence of simple processes $\Psi_n(t)$ satisfying an analogous condition to (2.6), and then show that (2.9) is satisfied by passing the limit using the **Dominated Convergence Theorem**. □

Sources:

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Stochastic Partial Differential Equations § *3.3*
Lototsky, Rozovskey

Stochastic equations in infinite dimensions § *4.2*
Da Prato, Zabczyk

Ito Isometry (from Wikipedia link here)

Hilbert–Schmidt Operator (from Wikipedia link here)