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Endogenizing Epistemic Actions

Abstract.
Through a series of examples, we illustrate some important drawbacks that the *action model logic* framework suffers from in its ability to represent the dynamics of information updates. We argue that these problems stem from the fact that the *action model*, a central construct designed to encode agents' uncertainty about actions, is itself effectively common knowledge amongst the agents. In response to these difficulties, we motivate and propose an alternative semantics that avoids them by (roughly speaking) endogenizing the action model. We discuss the relationship between this new framework and action model logic, and provide a sound and complete axiomatization of several new logics that naturally arise.[1]

Keywords: epistemic logic, dynamic epistemic logic, higher-order uncertainty, action model logic

1. Introduction

Action Model Logic (AML) is a framework for reasoning about how knowledge and belief change on the basis of incoming information [2, 3, 9].[2] Information is conveyed in the form of "epistemic actions", with a canonical example being *public announcements* [8].[3] Unlike *Public Announcement Logic* (PAL), however, AML does not presume that all epistemic actions are public in the sense of becoming common knowledge, nor that such actions can always be distinguished from one another by the agents. Uncertainty about epistemic actions is explicitly encoded in a structure called an *action*

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[1]This paper subsumes and expands on a version published in the proceedings for TARK [15].

[2]Terminology varies; some authors instead use *Dynamic Epistemic Logic* to refer to this framework [12, §2.2.2]. We follow van Ditmarsch et al. [9] in using it instead as an umbrella term for a collection of thematically related logics of information change, including action model logic.

[3]In particular, despite its name, action model logic is not concerned with the full spectrum of possible actions in the world, but more narrowly with what might be thought of as information-containing signals, so-called "epistemic" actions. Moreover, issues of performability, situational aspects, and the moral status of actions, while certainly interesting topics of study, are not within the purview of AML and similarly outside the scope of this paper.

model; this allows for the representation of scenarios in which agents can be uncertain about which action has taken place.

By design, this formalism is well-equipped to capture uncertainty about actions themselves; however, we argue in this paper that the AML framework is ill-suited to the representation of *higher-order* uncertainty about actions. Roughly speaking, this is because the action model that captures uncertainty about actions is itself effectively common knowledge amongst the agents, making it awkward to encode, for example, one agent's uncertainty about another agent's uncertainty about actions.

We expose this difficulty through a series of motivating examples. We demonstrate that although AML can capture higher-order uncertainty about actions, this can only be done by expanding the action model in such a way as to essentially "pre-encode" the desired uncertainty; this makes choosing an appropriate action model for any given application problematically post hoc. Furthermore, we show that in such cases small variations in the background epistemic conditions require corresponding alterations to the action model in order to ensure that the "pre-encoded" uncertainty maintains the right form. These observations seriously undermine the practical applicability of AML as a tool for reasoning about information updates.

In response to these challenges, we formulate the semantics by "endogenizing" the action model; that is, we allow each agent's uncertainty about actions to be state-dependent, and therefore itself subject to uncertainty. Revisiting our examples, we show that these revised semantics completely circumvent the earlier difficulties; our semantics capture *formally* the informal process underlying the aforementioned post hoc expansion of the action model.

These revisions also prompt a change to the update mechanism which underlies the AML framework. The altered update which we present captures, intuitively, the idea that when an epistemic action is performed, agents learn about their (in)ability to distinguish that action from others.

The idea of representing higher-order uncertainty about epistemic actions by encoding extra information about the agents (and their perceptions of such actions) into the state space is a natural one; it also occurs in [14], in which Bolander et al. study a more general class of announcements that may not be entirely public in that (loosely speaking) some agents may not be "listening". In essence, their semantics work by encoding into each possible world whether or not each agent is "paying attention" at that world. As the authors show, such "attention-based" announcements can also be described using action models. The present work is more general in that we begin with the full action model framework rather than PAL, and we encode into each world the full spectrum of each agent's uncertainty regarding epistemic actions, not just whether or not they are attentive.

The rest of the paper is organized as follows. In Section 2, we motivate and define the basic AML framework and present a series of examples intended to illustrate its limitations. In Section 3, we present our new semantics and show how it deals with the limitations discussed in the previous section. We also consider a generalized model update mechanism that applies in cases where agents may be introspectively uncertain about their own ability to distinguish actions. In Section 4, we provide a sound and complete axiomatization by way of *reduction axioms*, extend the language to include common knowledge, and discuss completeness, with the technical details appearing in an appendix.

2. Action Model Logic

2.1. Introduction to Action Model Logic

We begin by reviewing the foundational definitions and motivations of AML, largely following van Ditmarsch et al. [9, Chapter 6]. This logic is an extension of standard epistemic logic, so we begin there.

In standard epistemic logic, we fix a countable set of primitive propositions PROP and a finite set of agents G. Let $\mathcal{L}_K$ denote the language recursively defined as follows:

$$\varphi := p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, K_j\varphi,$$

where $p \in$ PROP and $j \in G$. We read $K_j\varphi$ as "agent j knows that φ". Thus, $\mathcal{L}_K$ is a language for reasoning about the knowledge of the agents in G. The other Boolean connectives can be defined in the usual way; we write $\hat{K}_j$ to abbreviate $\neg K_j \neg$, and read $\hat{K}_j\varphi$ as "agent j considers it possible that φ".

An **epistemic model** is a structure of the form $M = \langle W, \{\sim_j : j \in G\}, V\rangle$, where:

- W is a (nonempty) set of *states*,
- each $\sim_j$ is an equivalence relation on W,
- V : PROP $\to 2^W$ is a *valuation function*.

Intuitively, V specifies for each primitive proposition those states where it is true, while the relations $\sim_j$ capture indistinguishability from the perspective of agent j. These intuitions are formalized in the following semantic clauses for the evaluation of formulas on pointed models:[4]

$$\begin{array}{lll} (M,w) \vDash p & \text{iff} & w \in V(p) \\ (M,w) \vDash \neg\varphi & \text{iff} & (M,w) \nvDash \varphi \\ (M,w) \vDash \varphi \wedge \psi & \text{iff} & (M,w) \vDash \varphi \text{ and } (M,w) \vDash \psi \\ (M,w) \vDash K_j\varphi & \text{iff} & (\forall w' \in [w]^j)((M,w) \vDash \varphi), \end{array}$$

where $[w]^j$ denotes the equivalence class of w under $\sim_j$. Thus, the Boolean connectives are interpreted as usual, and $K_j\varphi$ is true at w precisely when φ is true at all states that agent j cannot distinguish from w. Insisting that the indistinguishability relations be equivalence relations results in a logic of knowledge that is *factive* and *fully introspective*.[5] For a more thorough development of differing logics of knowledge we direct the reader to [1].

The framework of epistemic logic is appropriate for reasoning about scenarios involving uncertain agents. Consider the following simple example.

EXAMPLE 1. *Colleagues Anne and Bob are discussing whether a particular company policy passed in this morning's board meeting: both are ignorant of whether or not the policy passed. Taking* p *to be the proposition "the policy passed", this scenario may be represented with the simple epistemic model* M_0 *presented in Figure 1.*[6]

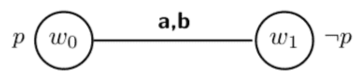

Figure 1. M_0 – Anne and Bob's initial uncertainty

In this diagram and others like it, the circles represents the states of the model (in this case, w_0 and w_1), the formulas listed beside a state are true

[4]A pointed model is a model M together with some state from the model, w. We sometimes write $w \vDash \varphi$, rather than $(M,w) \vDash \varphi$, when M is clear from context.

[5]That is, the logic witnesses the following validities: $\vDash K_j\varphi \rightarrow \varphi$, $\vDash K_j\varphi \rightarrow K_jK_j\varphi$, and $\neg K_j\varphi \rightarrow K_j\neg K_j\varphi$.

[6]Unless stated otherwise, all relations are equivalence relations, so reflexive loops and edges implied by transitivity are assumed to be present, even when suppressed in the diagrams.

at that state (in this case, p and $\neg p$ are true at w_0 and w_1, respectively), and edges between states are labelled with those agents that cannot distinguish them (in this case, the edge labelled with a and b indicates that neither Anne nor Bob can distinguish between these two possible states). Indeed, we have $M_0 \vDash (\neg K_a p \wedge \neg K_a \neg p) \wedge (\neg K_b p \wedge \neg K_b \neg p)$, so this model properly captures the ignorance of Anne and Bob regarding p.

On top of the basic epistemic framework, Action Model Logic adds a layer of structure that aims to capture the epistemic dynamics of information update. An **action model** is a structure of the form

$$A = \langle \Sigma, \{\approx_j : j \in G\}, Pre \rangle$$

where:

- Σ is a (nonempty) set of *epistemic actions*,
- $\approx_j$ is an equivalence relation on Σ,
- $Pre : \Sigma \to \mathcal{L}_K$ is a *precondition function.*

Intuitively, the relation $\approx_j$ captures indistinguishability of actions from the perspective of agent j, while the function Pre captures the background conditions $Pre(\sigma)$ that must hold for a given action σ to be successfully performed. In short, an action model specifies a set of epistemic actions that can be executed together with their preconditions and the extent to which they can be individuated by the agents.

To formalize these intuitions, we must define the process by which an epistemic model M is updated based on the performance of an epistemic action from A. This is captured in the **updated model**

$$M^A = \langle W_\Sigma, \{\sim'_j : j \in G\}, V' \rangle$$

defined as follows:

- $W_\Sigma = \{(w, \sigma) : w \vDash Pre(\sigma)\}$,
- $(w_0, \sigma_0) \sim'_j (w_1, \sigma_1)$ iff $w_0 \sim_j w_1$ and $\sigma_0 \approx_j \sigma_1$,
- $(w, \sigma) \in V'(p)$ iff $w \in V(p)$.

Thus, the states of the updated model consist of those state-action pairs (w, σ) such that the precondition of the action σ is satisfied by the state w in M; intuitively, (w, σ) represents the state of affairs w after σ has been performed. The definition of V' ensures that such "updated states" satisfy the same primitive propositions as they did before (corresponding to the intuition that epistemic actions can change the *information* agents have access to, but cannot change basic facts about the world). Finally, the definition of $\sim_j'$ specifies that updated state-action pairs are indistinguishable for agent j precisely when the constituent states and actions were indistinguishable for j in M and A, respectively. It is easy to see that M^A is an epistemic model.

The language for Action Model Logic, $\mathcal{L}_{K[A,\sigma]}$, extends the basic epistemic language with an update operator:

$$\varphi := p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, K_j\varphi \,|\, [A,\sigma]\varphi$$

$[A,\sigma]\varphi$ is read "if action σ (in the context A) can be performed, then after it is performed, φ is true".[7] This language therefore lets us reason about agents' knowledge and how it can change as a result of epistemic actions. Formulas of $\mathcal{L}_{K[A,\sigma]}$ can be interpreted in epistemic models as before, with the additional semantic clause for $[A,\sigma]\varphi$ given by:

$$(M,w) \vDash [A,\sigma]\varphi \quad \text{iff} \quad (M,w) \vDash Pre(\sigma) \;\text{ implies }\; (M^A,(w,\sigma)) \vDash \varphi$$

To make these definitions clear, we expand on our previous example.

EXAMPLE 2. *Anne and Bob are discussing whether a particular company policy passed (p). Their friend, Carl, comes along with news of p; however, he announces that for security purposes, he can only tell Bob whether or not p, and cannot tell Anne. He then takes one of two actions: he either tells Bob that p is true (σ_p), or he tells Bob that p is false ($\sigma_{\neg p}$). Anne watches, knowing that Carl is telling Bob whether or not p is true, but too far away to hear. Since we know Carl to be an honest fellow, the precondition of σ_p is p, and the precondition of $\sigma_{\neg p}$ is $\neg p$.*

This is all captured in the action model A_0 depicted in Figure 2. In diagrams of action models, the circles represent actions (in this case, σ_p and $\sigma_{\neg p}$), the formulas listed beside the actions are the corresponding preconditions (in this case, p and $\neg p$ for σ_p and $\sigma_{\neg p}$, respectively), and the edges represent action indistinguishability (in this case, Bob can distinguish the two actions, but Anne cannot).

If we perform the update procedure on M_0 and A_0, then we arrive at the updated model $M_0^{A_0}$ depicted in Figure 3. Thus we see that when we update the epistemic model with the action model, the result is a model where Bob

[7]The "if-then" construction here is interpreted as a standard material conditional, i.e., $[A,\sigma]\varphi$ is vacuously true when σ cannot be performed. When the action can be performed, then $[A,\sigma]\varphi$ is true just when actually performing the action results in φ being true. This is entirely analogous to the case for public announcements, where a proper gloss of $[\varphi]\psi$ might be "if the announcement φ can be truthfully made, then after φ is announced, ψ holds".

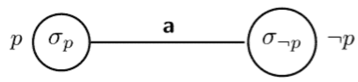

Figure 2. A_0 – Carl's communication to Bob

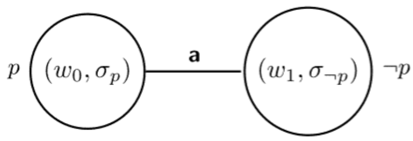

Figure 3. $M_0^{A_0}$ – After Carl's announcement

knows whether or not p, while Anne does not: $M_0^{A_0} \models K_b p \vee K_b \neg p$, *and* $M_0^{A_0} \models \neg(K_a p \vee K_a \neg p)$, *as desired.* ■

Note that Carl's announcement is not *public* since Anne cannot tell whether he announced p or $\neg p$ while Bob can.[8]. This demonstrates that AML can capture epistemic dynamics that PAL cannot.[9] Nonetheless, as we have claimed, AML suffers from limitations of its own; we turn now to a discussion of these limitations.

2.2. Limitations of Action Model Logic

The limitations we have in mind can be summarized quite simply: the AML framework treats the action model as common knowledge. Indeed, the updated model M^A imports much of the structure of the action model: in it, all agents come to know what actions the *other* agents can distinguish. The result is that AML has difficulty capturing scenarios involving *higher-order uncertainty*—e.g., uncertainty about what other agents know. Although this limitation can be overcome, in a sense, by expanding the action model, we

[8]This particular sort of announcement, where not all parties are privy to the information exchanged, but all parties are *aware* of the information exchange, is sometimes referred to as "semi-private" in the literature [3, 13].

[9]In fact, AML subsumes PAL: to capture a public announcement of φ, consider the action model A_φ consisting of a single node σ with the precondition φ. Then, given an epistemic model M and a state w therein, one can show that the resulting models $(M^{A_\varphi}, (w, \sigma))$ and $(M|_\varphi, w)$ are bisimilar: updating with A_φ effectively deletes the states in M where φ is false.

will see that in general this is not an appealing solution. Moreover, the examples we consider demonstrate that the AML framework is not as modular as it might appear to be: adjustments to the epistemic model will often require corresponding adjustments to the action model to preserve the intended semantic interpretations. To illustrate these points, we return to the scenario presented in Example 2.

EXAMPLE 3. *Suppose again that Carl comes along and announces that he has news of whether p; however, instead of speaking so that only Bob can hear him, Carl speaks plainly for all to hear, but he delivers the message in French. As it happens, Bob speaks French and Anne does not.*

As before, the apparent actions that Carl might take are telling Bob that p and telling Bob that $\neg p$, which presumably have preconditions p and $\neg p$, respectively. Bob can distinguish these actions, as he speaks French, while Anne cannot. This reasoning produces the same action model A_0 depicted in Figure 2, which therefore produces the same updated model $M_0^{A_0}$ shown in Figure 3.

As expected, then, just as in Example 2, Bob ends up knowing whether or not p, and Anne does not. An unexpected result, however, is that Anne knows this former fact, and Bob knows the latter. That is, we have:

$$M_0^{A_0} \models K_a(K_b p \vee K_b \neg p)$$

and

$$M_0^{A_0} \models K_b(\neg(K_a p \vee K_a \neg p)).$$

The former says that Anne knows that Bob knows whether p, while the latter says that Bob knows that Anne is uncertain about p. But there was no assumption that Bob knows that Anne cannot understand Carl's message, nor that Anne knows that Bob can. That is, we did not explicitly stipulate whether or not either knew about the other's (in)ability to speak French. To capture this, the model must be refined. ∎

Loosely speaking, Bob's ability to speak French and Anne's inability to speak French are represented in the structure of the action model A_0 in Figure 2, and this is why they effectively become common knowledge in the updated model. But, of course, we may want to capture a scenario where one or both are *uncertain* about whether or not the other speaks French; indeed, uncertainties of this sort play an important role in everyday reasoning. For the sake of simplicity, let us begin by aiming only to remove the consequence in the updated model that Anne knows that Bob knows whether p.

EXAMPLE 4. *A relevant proposition here is that Bob speaks French; call this q. If we wish to account for Anne's uncertainty about q, we ought to expand the initial epistemic model M_0 to include the possible values this proposition might have. The result is the model M_1 depicted in Figure 4.*

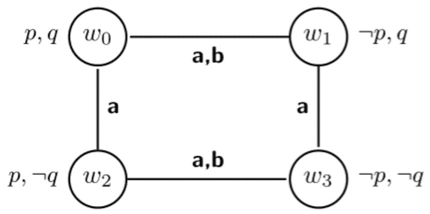

Figure 4. M_1 – An expanded model of Anne and Bob's initial uncertainty

Anne does not know whether or not p is true, and also does not know whether or not q is true; thus, there are a-edges between all 4 nodes. We assume that Bob does know whether or not he speaks French, but, as before, does not know whether or not p; thus there are horizontal b-edges, but no vertical b-edges.

It is easy to check that updating M_1 with A_0 produces an epistemic model $M_1^{A_0}$ in which Bob knows (at every state) the true value of p, and thus Anne knows that Bob knows this. So it seems we must modify the action model as well; for example, we can add a third action, σ, corresponding intuitively to the "unsuccessful" announcement in which Carl speaks his piece but no one (including Bob) understands him. The precondition for σ should therefore be $\neg q$: that Bob does not speak French. Furthermore, the preconditions for the actions σ_p and $\sigma_{\neg p}$ ought to be strengthened to include q, since these actions now represent Carl telling Bob p and $\neg p$, respectively, with the stipulation that Bob understands what was said. Bob will be able to distinguish any of these three actions, since he knows whether or not he speaks French, and, given that he speaks French, he knows which announcement Carl is making. Anne, on the hand, will not be able to distinguish any of the three actions—it all sounds the same to her. All this is captured by the action model A_1 given in Figure 5.

The updated model $M_1^{A_1}$ is shown in Figure 6.

As expected, we now have that Bob knows whether p only in those states where σ_p or $\sigma_{\neg p}$ was performed—that is, those states where q ("Bob speaks French") was true—and Anne does not know whether p at all. Furthermore,

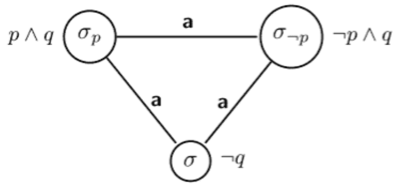

Figure 5. A_1 – An expanded action model

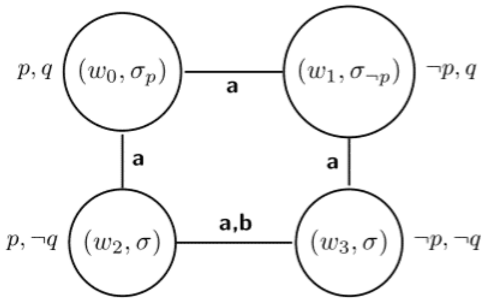

Figure 6. $M_1^{A_1}$ – After Carl's announcement

we have:

$$(w_0, \sigma_p) \vDash \hat{K}_a(K_b p \vee K_b \neg p) \wedge \hat{K}_a(\neg K_b p \wedge \neg K_b \neg p)$$

This reads: Anne considers it possible that Bob knows whether or not p, and also considers it possible that he does not. Thus, by expanding the initial epistemic model to include Anne's uncertainty about Bob's ability to speak French, as well as adding a third node to the action model corresponding (roughly speaking) to an "unsuccessful" announcement from Carl, we are able to capture the second-order uncertainty that we set out to capture. ■

One might reasonably feel some discomfort regarding the introduction of σ into the action model. On at least one intuition for what constitutes an "action", Carl's announcement (in French) of p ought to count as the *same* action regardless of who hears it or what languages they might understand. In other words, one might object to distinguishing σ_p from σ on the grounds that it builds into the ontology of actions properties that really have nothing to do with actions, but rather with agents.[10]

[10]A similar objection to AML is articulated in [11] Section 3.2.1; here, the authors

This philosophical objection could perhaps be swept aside if the underlying formalism actually did the job we wanted it to: it is hard to make the case that vague ontological concerns ought to receive priority over mathematical efficacy. What we now aim to demonstrate, however, is that this technique of expanding the action model is *not* an effective tool for capturing higher-order uncertainty.

EXAMPLE 5. *We alter the Example 4 in only one respect: we assume now that Anne does speak French. This requires no change to the initial epistemic model* M_1 *(since whether or not Anne speaks French is not represented explicitly in this model), but it does, intuitively, require us to re-work the action model* A_1*. In particular, the Anne-edge connecting* σ_p *and* $\sigma_{\neg p}$ *no longer seems appropriate, since now Anne can understand what Carl announces.*

Now we are faced with a somewhat awkward question—should there be an Anne-edge between σ_p *and* σ*? Intuitively, there should be, since* σ *is supposed to encode the fact that Bob does not understand Carl's announcement, and Anne is not supposed to be able to tell whether he does or not. Similar reasoning leads us to leave the Anne-edge between* $\sigma_{\neg p}$ *and* σ *in place, so the resulting relation fails to be transitive.*

Perhaps we could relax the requirements placed on the relations $\approx_i$ *in action models to accommodate this type of problem, but in fact there is a deeper issue here that suggests an alternative resolution: it is easy to see that any reflexive relation* $\approx'_a$ *for Anne on the set* $\{\sigma_p, \sigma_{\neg p}, \sigma\}$ *produces an action model* A'_1 *such that the updated model* $M_1^{A'_1}$ *satisfies* $(w_2, \sigma) \sim_a (w_3, \sigma)$*. But this misrepresents the situation: in state* w_2*, Carl's announcement must have been that the policy passed,* p*; as such, after his announcement, Anne should no longer be uncertain about* p*.*

The problem here lies with σ*: it was introduced originally to represent the possibility of an "unsuccessful" announcement by Carl. But in the present context, Carl's announcement is always at least partially successful, in that it always informs Anne of the truth value of* p*. The natural fix to this problem is another adjustment to the action model: we "split" the action* σ *into two actions,* $\sigma_{\neg p \neg q}$ *and* $\sigma_{p \neg q}$ *(and relabel the other actions for clarity). The new action model* A_2 *is shown in Figure 7.*

These new actions $\sigma_{p \neg q}$ *and* $\sigma_{\neg p \neg q}$ *might be thought of as corresponding to*

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discuss the property of AML which guarantees that any *Epistemic Temporal Logic* (ETL) model generated by AML (in the sense of the correspondence established in [10]) will satisfy the *No Miracles* principle. Indeed, the alternative semantics we propose below widens the class of ETL models which can be generated to include models which do *not* satisfy the No Miracles principle.

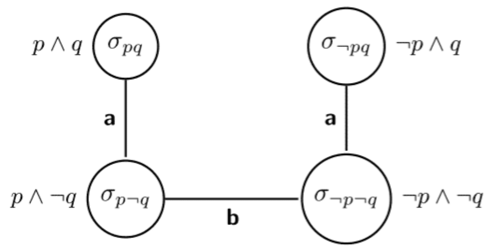

Figure 7. A_2 – The adjusted action model for when Anne speaks French.

situations where Carl announces p and Bob does not understand, and where Carl announces $\neg p$ and Bob does not understand, respectively. Anne can distinguish announcements based on their content, but not based on whether Bob understands them. Bob can distinguish announcements based on whether he understands them and, provided he understands them, based on their content as well. The updated model $M_1^{A_2}$ is given in Figure 8.

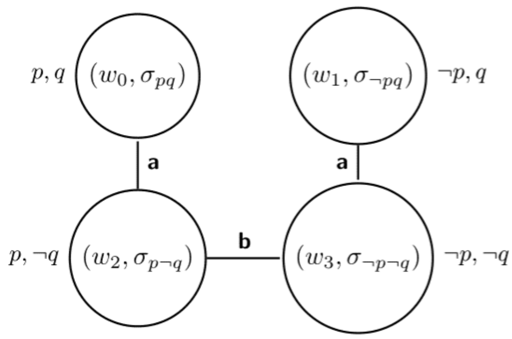

Figure 8. $M_1^{A_2}$ – The updated model when Anne speaks French.

As expected, Anne has learned whether or not p in the updated model (since she heard Carl), but she continues to be ignorant as to whether or not Bob speaks French (and, in turn, whether or not Bob has learned p). ■

Note the duplication of effort in the construction of A_2. The initial epistemic model M_1 already encodes the possibilities regarding Bob's ability to speak French and Anne's uncertainty about this. Yet our action model recapitulates this structure with actions that incorporate not just what Carl says, but also whether Bob understands it or not. Moreover, once our background assumptions are fixed (such as whether Anne understands French or not), edges (and nodes!) in A_2 are determined, essentially, by examining M_1 and reading off what the uncertainties ought to be. Thus, while AML gives the impression of a clean, modular division between epistemic states and actions, in practice the two seem to be quite tangled, with unavoidable redundancies in their representations. To drive this point home, we sketch one further example.

EXAMPLE 6. *Consider an expanded epistemic model M_2 in which we take not just Bob's but also Anne's knowledge of French as endogenous: that is, suppose we also wish to represent Bob as being uncertain of whether or not Anne speaks French. A simple model of such a scenario might consist in eight states representing the possible combinations of truth values for primitive propostions p, q, and r, where p and q are interpreted as before and r stands for the proposition "Anne speaks French".*

The action model A_2 of Example 5 is again inadequate. To see why, consider whether there ought to be an Anne-edge connecting σ_{pq} and $\sigma_{\neg pq}$. Intuitively, whether Anne can distinguish Carl announcing p from Carl announcing $\neg p$ depends on the state (i.e., it depends on whether or not Anne can speak French); it is not a fixed and unchanging truth that can be hard-coded into the model. As such, in order to capture this with a fixed action model we require, yet again, a proliferation of actions: e.g., actions of the form $\sigma_{pq\neg r}$, corresponding to something like Carl announcing p and Bob but not Anne understanding it. ∎

These examples make it clear that the AML formalism is not well-suited to the practical task of building models to represent scenarios in which second-order knowledge is relevant: in addition to specifying the initial epistemic model, one must construct alongside it an elaborate space of actions fine-tuned to the specifics of the epistemic setting. These actions, rather than corresponding in a natural way with concrete events in the world (like Carl making an announcement), are individuated by details about agents' perceptions of them that seem less like part of the actions themselves and more like part of the background epistemic situation.[11] We turn now to our proposed solution to this problem, which simplifies the action space considerably and imports the representation of higher-order uncertainty into the epistemic model, which is designed to handle it.

[11] We thank an anonymous reviewer for pointing out that a robust theory of action identity is lacking here. Nonetheless, we believe our contribution at least sharpens the notion of action identity underlying dynamic epistemic logics by highlighting the need to distinguish an epistemic action from its (epistemic) circumstances and effects.

3. Action-Epistemic Logic

3.1. Adjusted Semantics

We propose a new semantics for modeling information update that subsumes AML and is able to capture higher-order uncertainty without the proliferation of actions illustrated in the previous section. In essence, we endogenize the action model, making the distinguishability of the actions state-dependent. It is therefore natural in our revised framework to drop the notion of a separate action model altogether.

In addition to a countable collection of primitive propositions PROP and a finite set of agents G, fix a set Σ of *epistemic actions* together with a *precondition* function $Pre : \Sigma \to \mathcal{L}_K$ specifying the precondition for each action as before.[12] A **action-epistemic model** (over Σ and Pre) is a tuple

$$M = \langle W, \{\sim_j : j \in G\}, \{f_j : j \in G\}, V \rangle$$

where $\langle W, \{\sim_j : j \in G\}, V \rangle$ is an epistemic model, and for each agent j, $f_j : W \to 2^{\Sigma \times \Sigma}$ is a function from states to relations on actions. Intuitively, $(\sigma, \sigma') \in f_j(w)$ means that at state w, if action σ is performed, then agent j cannot rule out σ' being the action performed. The f_j functions constitute the crucial novel component of our framework which allows action indistinguishability to vary from state to state.

We will use action-epistemic models to interpret the language $\mathcal{L}^{\Sigma}_{K[\sigma]}$ recursively defined by

$$\varphi := p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, K_j\varphi \,|\, [\sigma]\varphi,$$

where $\sigma \in \Sigma$ and $[\sigma]\varphi$ is read, as before, "if action σ can be performed, then after the action is performed, φ is true".[13]

Of course, in order to interpret the update modalities, we must define the notion of an updated model. Let

[12] What exactly is the intended interpretation of an epistemic action $\sigma \in \Sigma$ in this new setting? In contrast to the action model logic framework, where we saw by example that "epistemic actions" can be quite abstract objects, here we can simply and concretely interpret σ as a signal carrying the information $Pre(\sigma)$. Of course, the details of how the agents might interpret this signal depend on what they know and what signals they can distinguish from one another, as encoded in the action-epistemic models, presented below.

[13] We no longer need to "tag" the update modality with an action model since our framework does not employ action models; this turns out to have important implications in our proof of completeness.

$$M^+ = \langle W^+, \{\sim_j^+ : j \in G\}, \{f_j^+ : j \in G\}, V^+ \rangle,^{14}$$

where:

$$\begin{array}{rcl} W^+ & = & \{(w,\sigma) \,:\, w \vDash Pre(\sigma)\} \\ (w,\sigma) \sim_j^+ (w',\sigma') & \Leftrightarrow & w \sim_j w' \text{ and } (\sigma,\sigma') \in f_j(w) \\ f_j^+((w,\sigma)) & = & f_j(w) \\ (w,\sigma) \in V^+(p) & \Leftrightarrow & w \in V(p). \end{array}$$

Note that the clause for $\sim_j^+$ very closely resembles the clause for $\sim_j'$ above, in action model logic; here, we require that $(\sigma,\sigma') \in f_j(w)$, which serves as the analogue to the requirement $\sigma \approx_j \sigma'$. Thus, although we have relativized the indistinguishability of actions to states, we have not really altered the mechanism which produces the update model (though we return to this point in Section 3.4). It is easy to see that M^+ is itself a action-epistemic model over Σ and Pre provided the relations $\sim_j^+$ are equivalence relations. In general, this need not be the case, but we can formulate natural conditions which suffice to ensure that each $\sim_j^+$ is an equivalence relation.

An **action-epistemic frame** is a tuple $F = \langle W, \{\sim_j : j \in G\}, \{f_j : j \in G\} \rangle$, i.e., it is an action-epistemic model without a valuation function. We say that action-epistemic model M is based on frame F just when $M = \langle F, V \rangle$ for some V. Consider the following frame properties:

(Act-Equiv) Each $f_j(w)$ is an equivalence relation on Σ.

(Act-Intro) If $w \sim_j w'$, then $f_j(w) = f_j(w')$.

PROPOSITION 7. *If M is based on a frame which satisfies (Act-Equiv) and (Act-Intro), then M^+ is an action-epistemic model.*

PROOF. As noted, it suffices to show that $\sim_j^+$ is an equivalence relation. Reflexivity follows immediately from reflexivity of $\sim_j$ and $f_j(w)$. For symmetry, suppose that $(w,\sigma) \sim_j^+ (w',\sigma')$. Then $w \sim_j w'$ so also $w' \sim_j w$; moreover, by (Act-Intro) we have $(\sigma,\sigma') \in f_j(w) = f_j(w')$, so (Act-Equiv) implies $(\sigma',\sigma) \in f_j(w')$, whence $(w',\sigma') \sim_j^+ (w,\sigma)$. For transitivity, suppose that $(w,\sigma) \sim_j^+ (w',\sigma')$ and $(w',\sigma') \sim_j^+ (w'',\sigma'')$. Clearly $w \sim_j w''$.

[14] This notation does not specify *which* action σ the update is being performed with respect to, since (as in AML) our update procedure effectively performs all available updates simultaneously.

Moreover, we have $(\sigma, \sigma') \in f_j(w)$ and $(\sigma', \sigma'') \in f_j(w')$; by (Act-Intro) $f_j(w) = f_j(w')$, and by (Act-Equiv) this relation is transitive, so we deduce that $(\sigma, \sigma'') \in f_j(w)$, which shows that $(w, \sigma) \sim_j^+ (w'', \sigma'')$, as desired.[15] ∎

We consider (Act-Equiv) a natural assumption, given a straightforward reading of 'indistinguishability'.[16] (Act-Intro) states that an agent is introspective with respect to their ability to distinguish actions; that is, if they are actually able/unable to distinguish actions σ and σ', then at all states which they consider possible, they are able/unable to distinguish actions σ and σ'. One does not need to look far for examples where this does not hold; we revisit this principle in Section 3.4, where we consider how to update action-epistemic models which do not satisfy it.

We presently only consider action-epistemic models M which satisfy both (Act-Equiv) and (Act-Intro); thus M^+ will again be an action-epistemic model, and we may define

$$(M, w) \models [\sigma]\varphi \quad \text{iff} \quad (M, w) \models Pre(\sigma) \text{ implies } (M^+, (w, \sigma)) \models \varphi.$$

This completes our specification of the new semantics for epistemic actions. It is not hard to see that update by an action model is essentially a special case of update in this framework: given an action model A, one simply defines each f_j to be the constant function such that $f_j(w) = \approx_j$. This directly realizes the intuition that the action model is common knowledge in AML.

3.2. Revisiting the examples

We now revisit the problematic examples of Section 2.2 and show that the new framework we have developed actually addresses the deficiencies we demonstrated. Let $\Sigma = \{\sigma_p, \sigma_{\neg p}\}$ and set $Pre(\sigma_p) = p$ and $Pre(\sigma_{\neg p}) = \neg p$,

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[15]In fact, (Act-Intro) is stronger than it needs to be to establish Proposition 7: since $\sim_j$ is assumed to be symmetric; weakening (Act-Intro) to:

(Act-Intro$^-$) If $w \sim_j w'$, then $f_j(w) \subseteq f_j(w')$

has no effect. Of course, this argument would not work for weaker epistemic logics where agents are not assumed to be negatively introspective and $\sim_j$ is not assumed to be symmetric.

[16]This same reading motivates the assumption that $\sim_j$ is an equivalence relation. Of course, there are counterexamples to this principle—for instance, scenarios where the appropriate indistinguishability relation ought to violate transitivity—but for present purposes we ignore such cases.

corresponding to the two intuitive actions of Carl announcing p or announcing $\neg p$, respectively. Recall that in Example 4, Carl delivers his message in French, we assume that Anne does not know French, and we let q represent the proposition that Bob knows French. We therefore define the action-epistemic model $\tilde{M}_1$ that we will use to reason about this scenario by extending the epistemic model M_1 depicted in Figure 4. In particular, we set

- $f_a(w_0) = f_a(w_1) = f_a(w_2) = f_a(w_3) = \Sigma \times \Sigma$,
- $f_b(w_0) = f_b(w_1) = id_\Sigma$, and
- $f_b(w_2) = f_b(w_3) = \Sigma \times \Sigma$,

where id_X denotes the identity relation on X. Thus, this action-epistemic model encodes the fact that Anne can never distinguish the two actions, whereas Bob can distinguish them just in case he speaks French.

It is easy to see that the epistemic part of $\tilde{M}_1^+$ looks exactly like the model $M_1^{A_1}$ depicted in Figure 6, except with the nodes (w_2, σ) and (w_3, σ) relabeled (w_2, σ_p) and $(w_3, \sigma_{\neg p})$, respectively. In other words, our update produces the "right" epistemic results, and it does so using a simple and natural set of actions and without requiring any fine-tuning of the model beyond the basic association between states where Bob speaks French and states where he can distinguish Carl's two possible announcements.

Next consider the scenario of Example 5, which is just like the previous one except it is assumed that Anne *does* speak French. To capture this, we need only change one line of the previous specifications for $\tilde{M}_1$:

- $f_a(w_0) = f_a(w_1) = f_a(w_2) = f_a(w_3) = id_\Sigma$.

This corresponds directly to the assumption that Anne knows French (i.e., this is valid in the model), so she can always distinguish the two actions in question. Call this action-epistemic model $\tilde{M}_1'$. Now as before, it is straightforward to check that the epistemic part of $\tilde{M}_1'^+$ looks exactly like the model $M_1^{A_2}$ given in Figure 8, provided we replace every instance of σ_{pq} and $\sigma_{p\neg q}$ with σ_p, and every instance of $\sigma_{\neg pq}$ and $\sigma_{\neg p\neg q}$ with $\sigma_{\neg p}$. So again, without the confusion of defining new, abstract actions, our framework reproduces the intended epistemic consequences of Carl's announcement.

Finally, it is not hard to figure out how to define a action-epistemic model $\tilde{M}_2$ extending the epistemic model M_2 of Example 6:

- $f_a(w) = \begin{cases} id_\Sigma & \text{if } w \vDash r \\ \Sigma \times \Sigma & \text{if } w \vDash \neg r, \end{cases}$
- $f_b(w) = \begin{cases} id_\Sigma & \text{if } w \vDash q \\ \Sigma \times \Sigma & \text{if } w \vDash \neg q. \end{cases}$

3.3. Comparing updates

Any action-epistemic update from a model which satisfies (Act-Intro) and (Act-Equiv) can be simulated by an action model:

THEOREM 8. *Given any action-epistemic model M over Σ which satisfies (Act-Intro) and (Act-Equiv), there is an action model A^M such that for any $(w, \sigma) \in M^+$, there is a bisimilar world in the updated model M^{A^M}.*[17]

PROOF. Let M be an action-epistemic model over Σ which satisfies (Act-Intro) and (Act-Equiv); we define A^M and construct a bisimulation between M^+ and M^{A^M}. Define $A^M = \langle \Sigma^M, \{\approx_j^M : j \in G\}, Pre^M \rangle$ as follows:[18]

$$\begin{array}{rcl} \Sigma^M & = & \{\sigma^w : (w, \sigma) \in M^+\} \\ \sigma^w \approx_j^M \sigma'^{w'} & \Leftrightarrow & (\sigma, \sigma') \in f_j(w) \\ Pre^M(\sigma^w) & = & th(w) \end{array}$$

We now show that the relation B which relates each $(w, \sigma) \in M^+$ to the state $(w, \sigma^w) \in M^{A^M}$ is a bisimulation. It is clear, from the definition of A^M, that $(w, \sigma^w) \in M^{A^M}$; it is also clear that the propositional requirement for bisimulations is satisfied by this relation. ($\Rightarrow$) suppose that $(w, \sigma) \sim_j^+ (w', \sigma')$. Then $(\sigma, \sigma') \in f_j(w)$, and so $\sigma^w \approx_j^M \sigma'^{w'}$ in A^M. Since it must also be true that $w \sim_j w'$ in M, we have that $(w, \sigma^w) \sim'_j (w', \sigma'^{w'})$ in M^{A^M}. ($\Leftarrow$) suppose that $(w, \sigma^w) \sim'_j (w', \sigma'^{w'})$ in M^{A^M}. Then we must have that $w \sim_j w'$ in M and $\sigma^w \approx_j^M \sigma'^{w'}$ in A^M. By the definition of $\approx_j^M$, this implies that $(\sigma, \sigma') \in f_j(w)$ and so it follows that $(w, \sigma) \sim_j^+ (w', \sigma')$ in M^+. ■

Note that, in general, Σ^M will be significantly larger than Σ. The existence of an appropriate A^M should not come as a surprise; action model logic captures a very general notion of update. We state below the conditions required of M_1, M_2 so that there is an action model A such that M_1^A is isomorphic to M_2. These conditions are relatively weak, and show that action model logic is capable of simulating a wide class of transformations on epistemic models.

[17] We write M^{A^M} to mean the epistemic component of M updated according to the procedure from action model logic on action model A^M.

[18] We write $th(w)$ to mean the theory of w in M in the basic epistemic language.

We call an epistemic model M *non-redundant* just in case any two states in the model disagree on some formula of the basic epistemic language. With this terminology in hand, we make this point more precisely:[19]

THEOREM 9. *If $M_1 = \langle W_1, \sim_1, V_1\rangle$ and $M_2 = \langle W_2, \sim_2, V_2\rangle$ are finite and non-redundant, then the following conditions are equivalent: (1) there is an action model A such that M_1^A is isomorphic to M_2 (2) there is a function $g : W_2 \to W_1$ such that (i) $w \in V_2(p)$ iff $g(w) \in V_1(p)$ and (ii) if $w \sim_2 w'$, then $g(w) \sim_1 g(w')$.*

PROOF. (1) $\Rightarrow$ (2). Suppose that (1) is true of M_1, M_2; let Φ be an isomorphism from M_1^A to M_2. Define $g : W_2 \to W_1$ as follows:[20]

$$g(w) := pr_1(\Phi^{-1}(w))$$

We show that g satisfies (i) and (ii).

$$w \in V_2(p) \text{ iff } \Phi^{-1}(w) \in V_{M^A}(p) \text{ iff } pr_1(\Phi^{-1}(w)) \in V_1(p)$$

$$w \sim_2 w' \text{ iff } \Phi^{-1}(w) \sim_1^A \Phi^{-1}(w') \ \Rightarrow\ pr_1(\Phi^{-1}(w)) \sim_1 pr_1(\Phi^{-1}(w))$$

(2) $\Rightarrow$ (1). Let g witness that (2) is true of M_1, M_2. We define an action model $A = \langle \Sigma, \{\approx_j : j \in G\}, Pre\rangle$ as follows:

$$\Sigma = W_2$$
$$w \approx w' \text{ iff } w \sim_2 w'$$
$$Pre(w) = \bigwedge_{w' \in W_1, w' \neq g(w)} \varphi_{M_1, g(w), w'}$$

where $\varphi_{M,x,x'}$ is a formula which (M, x) satisfies but (M, x') does not (this is guaranteed to exist when M is non-redundant). Note that in the resulting update model, M_1^A, states are of the form $(g(w), w)$ for $w \in W_2$. Now the desired isomorphism Φ is simply the second projection pr_2:

$$pr_2((g(w), w)) = w$$

We verify that pr_2 is an isomorphism. It is easy to see that it is a bijection; suppose that $(g(w), w) \in V_1^A(p)$. This is equivalent to $g(w) \in V_1(p)$ which

[19] In the theorem, we use epistemic models with only one agent, but this result may by easily generalized to a multi-agent setting.

[20] Where pr_1 is the first projection: $pr_1((x, y)) = x$.

in turn is equivalent to $w \in V_2(p)$. Suppose that $w \sim_2 w'$. By the properties of g, $g(w) \sim_1 g(w')$, and by the definition of A, $w \approx w'$. Thus, $(g(w), w) \sim_1^A (g(w'), w')$. The other direction is similar. ■

Note that when proving (2) $\Rightarrow$ (1) above, we chose an appropriate A by essentially "copying" M_2. This is reminiscent of the reasoning invoked in Example 5 above: we examined what the resulting epistemic relations should be, and encoded them in the action model. In any scenario which satisfies the conditions of Theorem 9, this strategy (replicating the desired epistemic model in an action model) will work.

3.4. Generalized Semantics

In all of the examples we have considered so far, (Act-Equiv) and (Act-Intro) have been satisfied, and so the update model produced from our representation of the scenario has been again an action-epistemic model. As noted above, however, counterexamples to (Act-Intro) are easy to come by, such as the following scenario:

EXAMPLE 10. *Anne and Bob are communicating about the viability of a company which is going public tomorrow. For security, they are encrypting their messages to one another. Bob is waiting for an email from Anne—the email will tell him that he should purchase stock (p) or that he should not purchase ($\neg p$). Unfortunately, Bob is not great at remembering or recording passwords – he has an idea for what the password to his private key might be, but he is not sure, and will not know until he tries it. Thus, when Anne's message arrives, Bob is unsure that he will be able to decrypt the message; if he recalls the password correctly (q), then he will decrypt it, and otherwise ($\neg q$) he will not.*

We represent the scenario with the action-epistemic model M:

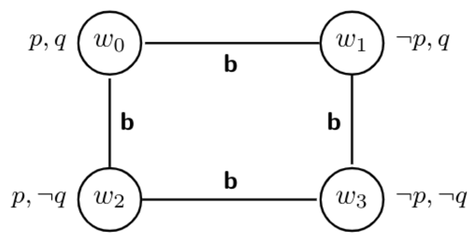

Figure 9. M, the scenario before Bob receives the message

where $\Sigma = \{\sigma_p, \sigma_{\neg p}\}$, $Pre(\sigma_p) = p$, $Pre(\sigma_{\neg p}) = \neg p$, *and:*

$$f_b(w_0) = f_b(w_1) = id_\Sigma$$
$$f_b(w_2) = f_b(w_3) = \Sigma \times \Sigma$$

Note that this model violates (Act-Equiv) because $w_0 \sim_b w_2$, but:

$$(\sigma_p, \sigma_{\neg p}) \notin f_b(w_0) \text{ and } (\sigma_p, \sigma_{\neg p}) \in f_b(w_2)$$

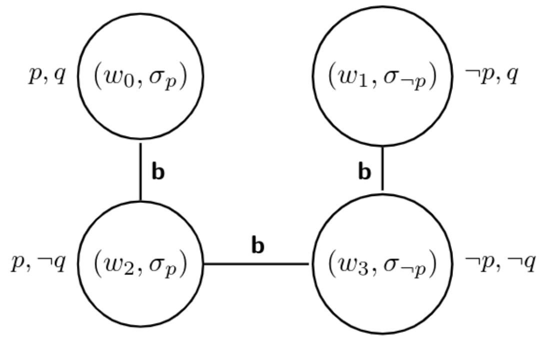

Figure 10. The epistemic part of M^+, the scenario after Bob receives the message

Now, consider the epistemic component of the ensuing update model M^+, seen in Figure 10. As we would expect, in the subset of states where q is false (that is, where Bob does not recall the correct password), Bob cannot distinguish the state where p is true from the state where p is false. On the other hand, Bob can distinguish w_0 and w_1, because q is true there. However, the update model also contains an edge between w_0 and w_2—this is to say that Bob cannot distinguish the state where he forgot his password, and received (but did not understand) the message that p, from the state where he did not forget his password, and received (and did understand) the message that p.

This seems wrong, however—surely Bob knows, after receiving the message that p, whether he remembered his password or not. This should be apparent to him as soon as he decrypts (or fails to decrypt) the message. Perhaps more obviously: if Bob rules out the possibility that $\sigma_{\neg p}$ was performed (this possibility will be ruled out at, for example, (w_0, σ_p)), then Bob should not consider possible any state where he thinks $\sigma_{\neg p}$ might have been performed (this is a live possibility for Bob at, for example, (w_2, σ_p)). We generalize this intuition as follows: after the update, two states (w, σ) and (w', σ') should be indistinguishable to Bob only if the epistemic actions Bob *might have mistaken for σ at w are the same actions Bob might have mistaken for σ' at w'. We codify this in a generalized semantics for the updated relation:*[21]

$$(w,\sigma) \sim_j^{\#} (w',\sigma') \quad \Leftrightarrow \quad w \sim_j w' \text{ and } f_j(w)[\sigma] = f_j(w')[\sigma']$$

We first note that this is indeed a generalization of the update semantics for AEL (and so also AML): $\sim_j^{\#}$ specializes to $\sim_j^{+}$ whenever both (Act-Equiv) and (Act-Intro) are satisfied.

PROPOSITION 11. *If M satisfies (Act-Equiv) and (Act-Intro), then $\sim_j^{\#} = \sim_j^{+}$.*

PROOF. Suppose that M satisfies (Act-Equiv) and (Act-Intro). We show that under the assumption that $w \sim_j w'$, the condition (#) $f_j(w)[\sigma] = f_j(w')[\sigma']$ is equivalent to the condition (+) $(\sigma, \sigma') \in f_j(w)$. The forwards direction is straightforward; since $f_j(w')$ is reflexive, $\sigma' \in f_j(w')[\sigma'] = f_j(w)[\sigma]$ and so we conclude (+). For the backwards direction, we assume (+) and show (#). Note that the reflexivity of $f_j(w)$ and $f_j(w')$, taken together with (+), implies that $\sigma' \in f_j(w)[\sigma]$ and $\sigma \in f_j(w')[\sigma']$. ($\subseteq$) suppose that $\sigma'' \in f_j(w)[\sigma]$. This implies that σ, σ', and σ'' are all in the same cell of the partition $f_j(w)$. By (Act-Intro), $f_j(w) = f_j(w')$, and so $\sigma'' \in f_j(w')[\sigma']$. The ($\supseteq$) direction is similar. ∎

For this reason, we drop the notation $\sim_j^{\#}$ and simply redefine $\sim_j^{+}$ (from the definition of the updated model):

$$(w,\sigma) \sim_j^{+} (w',\sigma') \quad \Leftrightarrow \quad w \sim_j w' \text{ and } f_j(w)[\sigma] = f_j(w')[\sigma']$$

The generalized semantics also obviate the need for assuming (Act-Intro):

PROPOSITION 12. *If M satisfies (Act-Equiv), then M^+ is an action-epistemic model.*

Thus AEL is capable of representing scenarios where action-introspection fails—that is, where an agent is not aware of their abilities to distinguish epistemic actions. We now revisit Example 10; the update model for M in Figure 9 is now M^+ in Figure 11.

The edge between (w_0, σ_p) and (w_2, σ_p) has been removed in virtue of the fact that $\sigma_{\neg p} \in f_b(w_0)[\sigma_p]$ but $\sigma_{\neg p} \notin f_b(w_2)[\sigma_p]$. This represents the fact that when Bob receives message σ_p, he either learns that he can distinguish σ_p, or he learns that he cannot.

[21] We write $R[x]$ to indicate the image of x in relation R: $R[x] = \{y | (x,y) \in R\}$.

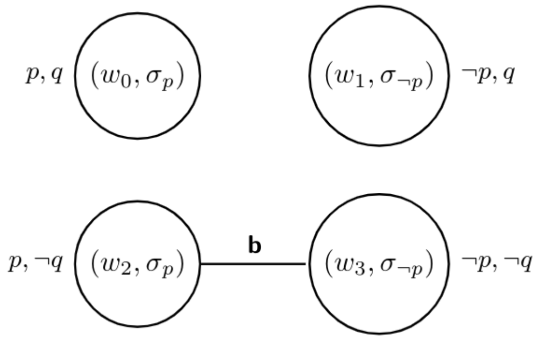

Figure 11. M^+, the scenario after Bob receives the message

We might express the content of this generalization as follows: when an agent j successfully perceives some action σ, they learn not only the content of σ, but also the fact that σ was perceivable to them. Indeed, we may express this fact by expanding our language; we introduce the constants $\xi_{j,\sigma,\sigma'}$ for every $\sigma, \sigma' \in \Sigma$ and $j \in G$, and give the semantics for these constants as follows:

$$(M, w) \vDash \xi_{j,\sigma,\sigma'} \text{ iff } (\sigma, \sigma') \in f_j(w)$$

With this in hand, we may express the above fact as the following proposition:

PROPOSITION 13. *For any $\sigma, \sigma' \in \Sigma$ and $j \in G$, the following formula is valid: $[\sigma](K_j \xi_{j,\sigma,\sigma'} \vee K_j \neg \xi_{j,\sigma,\sigma'})$.*

PROOF. Suppose that $(M^+, (w, \sigma)) \vDash \xi_{\sigma,\sigma'',j}$, and consider any (w', σ') such that $(w, \sigma) \sim_j^+ (w', \sigma')$. By the definition of $\sim_j^+$, we know that $f_j(w)[\sigma] = f_j(w')[\sigma']$. Thus since $(\sigma, \sigma'') \in f_j(w)$, we have that $\sigma'' \in f_j(w')[\sigma']$. By reflexivity, $\sigma \in f_j(w)[\sigma] = f_j(w')[\sigma']$, and since $f_j(w')$ is an equivalence relation, we have that $(\sigma, \sigma'') \in f_j(w')$. The case for $\neg \xi_{\sigma,\sigma'',j}$ is similar. ■

Interestingly, even under the generalized semantics, any action-epistemic update may be simulated by an action model:

THEOREM 14. *Given any action-epistemic model M over Σ which satisfies (Act-Equiv), there is an action model A^M such that for any $(w, \sigma) \in M^+$, there is a bisimilar world in the updated model M^{A^M}.*[22]

[22]We write M^{A^M} to mean the epistemic component of M updated according to the procedure from action model logic on action model A^M.

Given an action-epistemic model M over Σ, we define A^M as before, with one modification:

$$\sigma^w \approx_j^M \sigma'^{w'} \;\Leftrightarrow\; (\sigma, \sigma') \in f_j(w) \text{ and } f_j(w)[\sigma] = f_j(w')[\sigma']$$

4. Completeness

It is well known that both public announcement logic and action model logic reduce to epistemic logic; for any formula of $\mathcal{L}_{K[\varphi]}$ or $\mathcal{L}_{K[A,\sigma]}$, there is a formula of $\mathcal{L}_K$ which is equivalent in all epistemic models. In this sense, PAL and AML are no more expressive than EL: they merely offer (very convenient!) shorthand for formulas corresponding to epistemic dynamics.

Unlike PAL and EL, the language of action-epistemic logic does not offer a reduction to the basic epistemic language—it is easy to produce pairs of action-epistemic models whose epistemic parts are bisimilar but which satisfy different formulas of $\mathcal{L}^{\Sigma}_{K[\sigma]}$ at bisimilar states. This rules out the tactic—used to establish the completeness of PAL and AML—of reducing the language to that of EL, and appealing to EL's completeness. Instead, we adapt this tactic by reducing to a stronger language, and showing completeness for that language.

Let $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ and $\mathcal{L}_{K\xi}$ denote the languages $\mathcal{L}^{\Sigma}_{K[\sigma]}$ and $\mathcal{L}_K$, respectively, augmented with the additional primitive formulas $\xi_{j,\sigma,\sigma'}$ whose semantics we gave at the end of the previous section.

Intuitively, $\xi_{j,\sigma,\sigma'}$ says that if action σ is performed, agent j cannot rule out that σ' was the action performed. Thus, $\mathcal{L}_{K\xi}$ can talk about both knowledge of the agents and action indistinguishability.

Somewhat surprisingly, $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ is reducible to $\mathcal{L}_{K\xi}$: every formula of $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ is equivalent to a formula in $\mathcal{L}_{K\xi}$, and this equivalence can be captured by reduction schemes that allow us to provide a sound and complete axiomatization of $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ with respect to the class of all action-epistemic models that satisfy (Act-Equiv). It can be shown, however, that $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ is strictly more expressive than $\mathcal{L}^{\Sigma}_{K[\sigma]}$, so this result leaves something to be desired. This is the subject of ongoing research. We turn to the axiomatization of $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ now.

4.1. Soundness and completeness of $\mathcal{L}_{K\xi}$

Fix a finite set of actions Σ together with a precondition function $Pre : \Sigma \to \mathcal{L}_K$. We begin by axiomatizing $\mathcal{L}_{K\xi}$ and then turn to $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$. Consider the following axioms:

1. All instantiations of propositional tautologies
2. $K_j(\varphi \to \psi) \to (K_j\varphi \to K_j\psi)$
3. $K_j\varphi \to \varphi$
4. $K_j\varphi \to K_jK_j\varphi$
5. $\neg K_j\varphi \to K_j\neg K_j\varphi$
6. ξ axioms:
   (a) $\xi_{j,\sigma,\sigma}$
   (b) $\xi_{j,\sigma,\sigma'} \to \xi_{j,\sigma',\sigma}$
   (c) $\xi_{j,\sigma,\sigma'} \to (\xi_{j,\sigma',\sigma''} \to \xi_{j,\sigma,\sigma''})$
7. From φ and $\varphi \to \psi$, infer ψ.
8. From φ, infer $K_j\varphi$.

Soundness is easy, while completeness can be established by a fairly standard canonical model construction.

Let $M^c = \langle W^c, \{\sim_j^c : j \in G\}, \{f_j^c : j \in G\}, V^c\rangle$ be defined as follows:

- $M^c = \{\Gamma \subseteq \mathcal{L}_{K\xi} : \Gamma \text{ is maximally consistent}\}$
- $\Gamma \sim_j^c \Delta$ iff $\{K_j\varphi : K_j\varphi \in \Gamma\} = \{K_j\varphi : K_j\varphi \in \Delta\}$
- $(\sigma, \sigma') \in f_j^c(\Gamma)$ iff $\xi_{j,\sigma,\sigma'} \in \Gamma$
- $\Gamma \in V^c(p)$ iff $p \in \Gamma$.

We note that M^c is, as desired, an action-epistemic model over (Σ, Pre) which satisfies (Act-Equiv). That $\sim_j^c$ is an equivalence relation follows from standard proofs of the completeness of $\mathsf{S5}$. Axioms 6a-c guarantee that M^c satisfies condition (Act-Equiv).

The equivalence

$$(M^c, \Gamma) \vDash \varphi \text{ iff } \varphi \in \Gamma$$

(i.e., the Truth Lemma) is proved in the standard way by structural induction.

4.2. Soundness and completeness of $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$

Let pre_σ abbreviate the epistemic formula $Pre(\sigma)$, and consider the following additional axioms and inference rule:

9. Action axioms:
   (a) $[\sigma](\varphi \to \psi) \to ([\sigma]\varphi \to [\sigma]\psi)$
   (b) $[\sigma]p \leftrightarrow (pre_\sigma \to p)$
   (c) $[\sigma]\neg\varphi \leftrightarrow (pre_\sigma \to \neg[\sigma]\varphi)$
   (d) $[\sigma](\varphi \wedge \psi) \leftrightarrow ([\sigma]\varphi \wedge [\sigma]\psi)$
   (e) $[\sigma]K_j\varphi \leftrightarrow \Big(pre_\sigma \to \bigwedge_{\sigma' \in \Sigma} \big(\xi_{j,\sigma,\sigma'} \to [j, \sigma, [\sigma']\varphi]\big)\Big)$
10. From φ, infer $[\sigma]\varphi$.

Axiom 9e uses the following abbreviations:

$$\xi_{j,\sigma,\Sigma'} := \bigwedge_{\sigma' \in \Sigma'} \xi_{j,\sigma,\sigma'} \wedge \bigwedge_{\sigma'' \notin \Sigma'} \neg\xi_{j,\sigma,\sigma''}$$

$$[j, \sigma, \varphi] := \bigwedge_{\Sigma' \subseteq \Sigma} \big(\xi_{j,\sigma,\Sigma'} \to K_j(\xi_{j,\sigma,\Sigma'} \to \varphi)\big)$$

$\xi_{j,\sigma,\Sigma'}$ is true at w when Σ' is exactly the set of actions which j cannot distinguish from σ at w. We will write $\Sigma_{j,w,\sigma}$ to indicate the unique subset of Σ such that $w \vDash \xi_{j,\sigma,\Sigma_{j,w,\sigma}}$.[23] $[j, \sigma, \varphi]$ is true at w if, for any state w' which (1) is j-reachable from w and (2) satisfies $\xi_{j,\sigma,\Sigma_{j,w,\sigma}}$, w' satisfies φ.

The soundness of most of the above axioms is immediate; we show the soundness of (9e).

($\Rightarrow$) Suppose that $(M, w) \vDash [\sigma]K_j\varphi$ and assume that $(M, w) \vDash pre_\sigma$ (otherwise the equivalence is trivial). Let $\sigma' \in \Sigma$ be such that $w \vDash \xi_{j,\sigma,\sigma'}$. Consider any state w' such that $w \sim_j w'$ and $w' \vDash \xi_{j,\sigma,\Sigma_{j,w,\sigma}}$. We wish to show that $w' \vDash [\sigma']\varphi$. By supposition, $(M^+, (w, \sigma)) \vDash K_j\varphi$. Since $w' \vDash \xi_{j,\sigma,\Sigma_{w,\sigma}}$, and $\sigma' \in \Sigma_{j,w,\sigma}$, we have that $f_j(w)[\sigma] = f_j(w')[\sigma']$. Thus, $(w, \sigma) \sim^+_j (w', \sigma')$; it follows that $(w', \sigma') \vDash \varphi$, which means that $w' \vDash [\sigma']\varphi$.

($\Leftarrow$) Suppose that $w \vDash \Big(pre_\sigma \to \bigwedge_{\sigma' \in \Sigma} \big(\xi_{j,\sigma,\sigma'} \to [j, \sigma, [\sigma']\varphi]\big)\Big)$, and that $w \vDash pre_\sigma$. We wish to show that $(w, \sigma) \vDash K_j\varphi$. Consider any $(w', \sigma') \sim^+_j (w, \sigma)$; we will show that $(w', \sigma') \vDash \varphi$. Since $(\sigma, \sigma') \in f_j(w)$, we know that

[23] Note that this is trivially identical to the set $f_j(w)[\sigma]$.

$w \models \xi_{j,\sigma,\sigma'}$, so by supposition, $w \models [j, \sigma, [\sigma']\varphi]$. By the semantics of $\sim_j^+$, we know that $f_j(w)[\sigma] = f_j(w')[\sigma']$; this implies that $w' \models \xi j, \sigma, \Sigma_{j,w,\sigma}$, and so $w' \models [\sigma']\varphi$. This then implies that $(w', \sigma') \models \varphi$, as desired.

For completeness, consider the following translation:

$$\begin{array}{rcl}
t(p) & = & p \\
t(\xi_{j,\sigma,\sigma'}) & = & \xi_{j,\sigma,\sigma'} \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(K_a\varphi) & = & K_a t(\varphi) \\
t([\sigma]p) & = & pre_\sigma \rightarrow p \\
t([\sigma]\neg\varphi) & = & pre_\sigma \rightarrow \neg t([\sigma]\varphi) \\
t([\sigma](\varphi \wedge \psi)) & = & t([\sigma]\varphi) \wedge t([\sigma]\psi) \\
t([\sigma]K_j\varphi) & = & pre_\sigma \rightarrow \bigwedge_{\sigma' \in \Sigma} \big(\xi_{j,\sigma,\sigma'} \rightarrow [j, \sigma, t([\sigma']\varphi)]\big) \\
t([\sigma][\sigma']\varphi) & = & t([\sigma]t([\sigma']\varphi))
\end{array}$$

Proposition 15. *For all formulas $\varphi \in \mathcal{L}^{\Sigma}_{K[\sigma]\xi}$, $t(\varphi)$ is provably equivalent to φ and $t(\varphi) \in \mathcal{L}_{K\xi}$.*

Proof. We proceed by induction on the *action nesting depth* of φ, defined in the obvious way:

$$\begin{array}{rcl}
d(p) & = & 0 \\
d(\xi_{j,\sigma,\sigma'}) & = & 0 \\
d(\neg\varphi) & = & d(\varphi) \\
d(\varphi \wedge \psi) & = & \max(d(\varphi), d(\psi)) \\
d(K_j\varphi) & = & d(\varphi) \\
d([\sigma]\varphi) & = & d(\varphi) + 1.
\end{array}$$

The case $d(\varphi) = 0$ is immediate. So suppose the result holds for all formulas with nesting depth less than n, and let $\varphi \in \mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ be such that $d(\varphi) \leq n$.

We now proceed via a subinduction on the *weight* of φ, defined as follows:

$$\begin{array}{rcl} w(p) & = & 1 \\ w(\xi_{j,\sigma,\sigma'}) & = & 1 \\ w(\neg\varphi) & = & w(\varphi)+1 \\ w(\varphi\wedge\psi) & = & max(w(\varphi),w(\psi))+1 \\ w(K_j\varphi) & = & w(\varphi)+1 \\ w([\sigma]\varphi) & = & w(\varphi)+1 \end{array}$$

The base case where $w(\varphi)=1$ is again immediate. So suppose inductively the result holds for formulas of weight less than $w(\varphi)$. The proof now breaks into cases depending on the structure of φ, since this determines which recursive clause of the definition of t is relevant. The inductive steps corresponding to the Boolean connectives and the K_a modalities are straightforward, so we move to the case where $\varphi=[\sigma]\psi$; this in turn naturally breaks into several subcases depending on the structure of ψ:

- If $\psi=p$, then $t(\varphi)=t([\sigma]p)=pre_\sigma\rightarrow p$ and we are done by axiom (9b).
- If $\psi=\neg\chi$, then $t(\varphi)=t([\sigma]\neg\chi)=pre_\sigma\rightarrow\neg t([\sigma]\chi)$. Clearly $w([\sigma]\chi)<w(\varphi)$, so by the inductive hypothesis we know that $t([\sigma]\chi)\in\mathcal{L}_{K\xi}$ and is provably equivalent to $[\sigma]\chi$. It follows immediately that $t(\varphi)\in\mathcal{L}_{K\xi}$ and, by axiom (9c), that $t(\varphi)$ is provably equivalent to φ.
- If $\psi=\chi_1\wedge\chi_2$, then $t(\varphi)=t([\sigma]\chi_1)\wedge t([\sigma]\chi_2)$. Clearly $w([\sigma]\chi_1)<w(\varphi)$ and $w([\sigma]\chi_2)<w(\varphi)$, so by the inductive hypothesis we know that $t([\sigma]\chi_1)\in\mathcal{L}_{K\xi}$ and $t([\sigma]\chi_2)\in\mathcal{L}_{K\xi}$, and they are provably equivalent to $[\sigma]\chi_1$ and $[\sigma]\chi_2$, respectively. It follows immediately that $t(\varphi)\in\mathcal{L}_{K\xi}$ and, by axiom (9d), that $t(\varphi)$ is provably equivalent to φ.
- If $\psi=K_j\chi$, then:

  $$t(\varphi)=t([\sigma]K_j\chi)=pre_\sigma\rightarrow\bigwedge_{\sigma'\in\Sigma}\big(\xi_{j,\sigma,\sigma'}\rightarrow[j,\sigma,t([\sigma']\chi)]\big)$$

  For each $\sigma'\in\Sigma$, $w([\sigma']\chi)<w(\varphi)$, so by the inductive hypothesis we know that $t([\sigma']\chi)\in\mathcal{L}_{K\xi}$ and is provably equivalent to $[\sigma']\chi$. It follows immediately that $t(\varphi)\in\mathcal{L}_{K\xi}$ and, by axiom (9e), that $t(\varphi)$ is provably equivalent to φ, as desired.
- Finally, if $\psi=[\sigma']\chi$, then $t(\varphi)=t([\sigma][\sigma']\chi)=t([\sigma]t([\sigma']\chi))$. Let $\tilde{\psi}=t([\sigma']\chi)$; clearly $w([\sigma']\chi)<w(\varphi)$, so by the inductive hypothesis we know that $\tilde{\psi}=t([\sigma']\chi)\in\mathcal{L}_{K\xi}$ and $\tilde{\psi}$ is provably equivalent to $[\sigma']\chi$.

So we have $t(\varphi) = t([\sigma]t([\sigma']\chi)) = t([\sigma]\tilde{\psi})$. Now the weight of $\tilde{\psi}$ may be very large, so we can't apply our inner inductive hypothesis again here. However, since $\tilde{\psi} \in \mathcal{L}_{K\xi}$, it is easy to see that $d([\sigma]\tilde{\psi}) = 1 < d(\varphi)$, so we can appeal to our *outer* inductive hypothesis to conclude that $t(\varphi) = t([\sigma]\tilde{\psi}) \in \mathcal{L}_{K\xi}$ and is provably equivalent to $[\sigma]\tilde{\psi}$. Moreover, since $\tilde{\psi}$ is provably equivalent to $[\sigma']\chi$, using axiom (9a) it is easy to show that $[\sigma]\tilde{\psi}$ is provably equivalent to $[\sigma][\sigma']\chi$, whence $t(\varphi)$ is provably equivalent to φ, as desired. ∎

The argument for completeness now runs as follows: consider some valid formula $\varphi \in \mathcal{L}^{\Sigma}_{K[\sigma]\xi}$. By 15, $t(\varphi)$ is provably equivalent to φ and so must also be valid (by soundness). Since $t(\varphi) \in \mathcal{L}_{K\xi}$, the completeness of $\mathcal{L}_{K\xi}$ ensures there is some proof of $t(\varphi)$ using axioms 1-9; we may combine this proof with the proof of the equivalence of φ and $t(\varphi)$ to get a proof of φ using the axioms for $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$, which concludes our argument.

4.3. Completeness with Act-Intro

If we restrict our attention to the class of models which satisfy (Act-Intro), then we may modify our axiom system in the following respect to get a sound and complete axiomatization.
We add the following axiom to 6:[24]

$$6(d) \quad \xi_{j,\sigma,\sigma'} \to K_j \xi_{j,\sigma,\sigma'}$$

The soundness of 6(d) in the presence of (Act-Intro) is easy to see: suppose that $\xi_{j,\sigma,\sigma'}$ is true at M, w, and that $w \sim_j w'$. This implies that $(\sigma, \sigma') \in f_j(w)$, and by Act-Intro, that $f_j(w) = f_j(w')$; thus, $M, w' \vDash \xi_{j,\sigma,\sigma'}$, as desired.

We also confirm that the resulting canonical model satisfies (Act-Intro): suppose that $\Gamma \sim^c_j \Gamma'$ and that $(\sigma, \sigma') \in f^c_j(\Gamma)$. By definition, this implies that $\xi_{j,\sigma,\sigma'} \in \Gamma$ and by axiom 6(d), $K_j \xi_{j,\sigma,\sigma'} \in \Gamma$. Then by the definition of $\sim^c_j$, $K_j \xi_{j,\sigma,\sigma'} \in \Gamma'$, and by axiom 3, $\xi_{j,\sigma,\sigma'} \in \Gamma'$. Thus, $(\sigma, \sigma') \in f^c_j(\Gamma')$, as desired; the other direction is analogous.

In the presence of 6(d), 9(e) becomes equivalent to a much simpler form:

$$9(e*) \quad [\sigma]K_j\varphi \leftrightarrow \Big(pre_\sigma \to \bigwedge_{\sigma' \in \Sigma} \big(\xi_{j,\sigma,\sigma'} \to K_j[\sigma']\varphi\big)\Big)$$

[24]We might think of this as 'positive introspection' of the ξ formulas. Negative introspection, given by the schema $\neg\xi_{j,\sigma,\sigma'} \to K_j \neg\xi_{j,\sigma,\sigma'}$, is derivable from this axiom in the presence of the S5 axioms for K_j.

4.4. Adding Common Knowledge

In this section, we discuss the consequences of introducing *common knowledge* to our language and semantics. We first discuss how the introduction of common knowledge impacts related languages and their axiomatizations, and then show that in our case, a completeness proof is achievable.

Common knowledge is an important concept in the study of knowledge and strategic reasoning. Standard game-theoretic analyses assume common knowledge of rationality between players—determining a player's best strategy is made tractable by the assumption that all other players are rational, and that this fact is common knowledge. Some research has systematically weakened this assumption to determine its effect on strategy analysis. Outside of these fields of research, common knowledge is ubiquitous in everyday reasoning. The efficacy of conversations, for instance, is contingent on some method of communication which is common knowledge to all participating parties.

As such, common knowledge is a valuable concept to represent in an epistemic logic. We turn to a formal characterization of the concept now. Intuitively, 'it is common knowledge to everyone in group B that φ' will be true just when all the conjuncts of 'everyone in B knows φ, and everyone in B knows that everyone in B knows φ, and ... ' are simultaneously true. We can make this precise by introducing shorthand for 'everybody in group B knows that φ' and its iterations:

$$E_B\varphi = \bigwedge_{j \in B} K_j\varphi$$

$$E_B^n\varphi = \underbrace{E_B E_B \dots E_B}_{n \text{ times}} \varphi$$

Using this, we can provide semantics for the common knowledge operator, C_B:

$$(M, w) \models C_B\varphi \text{ iff } (\forall n \in \mathbb{N})\ (M, w) \models E_B^n\varphi$$

It is helpful to think of the semantics of this operator in terms of 'edge traversal': $C_B\varphi$ is true at a state w when every state reachable via only B-edges is a state which satisfies φ.

Several prominent epistemic languages have been extended with a common knowledge operator—for instance, $\mathcal{L}_K$, $\mathcal{L}_{K[\varphi]}$, and $\mathcal{L}_{K[A,\sigma]}$ are extended to $\mathcal{L}_{KC}$, $\mathcal{L}_{K[\varphi]C}$, and $\mathcal{L}_{K[A,\sigma]C}$, respectively, by adding the clause $C_B\varphi$, where

$B \subseteq G$. We define the languages $\mathcal{L}_{KC\xi}$ and $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$ by extending $\mathcal{L}_{K\xi}$ and $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$, respectively, in the same way.

The logics associated with $\mathcal{L}_{KC}$, $\mathcal{L}_{K[\varphi]C}$, and $\mathcal{L}_{K[A,\sigma]C}$ have been given sound and complete axiomatizations in [9, Chapter 7]. The inclusion of a common knowledge operator, however, necessitates a dramatically different tactic when establishing completeness for logics which include dynamic operators. To provide a sound and complete axiom system for PAL *without* common knowledge, a straightforward reduction suffices—one shows that every formula in $\mathcal{L}_{K[\varphi]}$ is equivalent to a formula in $\mathcal{L}_K$ via a set of reduction axioms, and invokes completeness of the axiom system $S5$.[25] The resulting axiom system for PAL consists in the axioms of $S5$ plus these reduction axioms. One might think that a similar approach can be taken for $\mathcal{L}_{K[\varphi]C}$: we show that every formula in $\mathcal{L}_{K[\varphi]C}$ is equivalent to a formula in $\mathcal{L}_{KC}$, and appeal to a sound and complete axiomatization of the $\mathcal{L}_{KC}$. This proves impossible, however: interestingly, the introduction of C_B to both languages does not preserve the reducibility of $\mathcal{L}_{K[\varphi]}$ to $\mathcal{L}_K$. In particular, $[\psi]C_B\varphi$ cannot be transformed into an equivalent formula of $\mathcal{L}_{KC}$. Intuitively, the reason is that this formula will be true at w just when all states which are reachable by $\psi - B$ paths—paths consisting of only B-edges and ψ-states—satisfy φ after the update. However, there is no mechanism in $\mathcal{L}_{KC}$ for quantifying over this set. Note that as a consequence of this irreducibility, $\mathcal{L}_{K[A,\sigma]C}$ must also be irreducible to $\mathcal{L}_{KC}$, since public announcement logic can be expressed within action model logic.

Since completeness by reduction is not an option for PAL with common knowledge (PALC), the standard method—the construction of a canonical model—is used to show completeness. This method is used successfully for both PALC and ALC (AML with common knowledge), but it requires substantial adaptation to accommodate common knowledge. This is because, roughly, the infinitary nature of $C_B\varphi$ makes both logics non-compact, and so the standard canonical model does not suffice. This problem is overcome by catering the canonical model to the formula in question; for details, see [9, Chapter 7].

Turning to action-epistemic logic (AELC) with common knowledge, we see that completeness by reduction is also out of reach, due to the simple fact that ALC can be captured in AELC, and $\mathcal{L}_{K[A,\sigma]C}$ cannot be reduced to $\mathcal{L}_{KC}$.[26] Instead, we might try to construct canonical models in the same

[25] A method of proving completeness of PAL without reduction axioms is presented in [16].

[26] This establishes only that $\mathcal{L}^{\Sigma}_{K[\sigma]C}$ cannot be reduced to $\mathcal{L}_{KC}$; however, it follows

way done for PALC and ALC.

Unfortunately, this, too, proves problematic. As mentioned above, in this construction, the canonical model we construct is particular to the unprovable formula for which a counterexample is desired. The problem arises in the proof of the *Truth Lemma*, which asserts, roughly, that our canonical model for φ has the property necessary to serve as a counterexample to φ.[27] The difficulty arises with the treatment of formulas of the form $[\alpha][\beta]\varphi$—these cases require that we look to the *update* canonical model, which lies outside of the scope of the inductive hypothesis. Several details are relevant to this problem that we do not mention here, but as of now, we see no way forward in this proof.

Another option, however, is afforded by an alternative proof in [4]. In this short article, Kooi and van Benthem discuss the difficulties with proving completeness when common knowledge is introduced to dynamic logics. They introduce a new operator, called *relativized common knowledge*, expressed by the formula $C_B(\psi, \varphi)$. $C_B(\psi, \varphi)$ is true at a state w just when every ψ-B-path (all paths consisting of only B-edges, and encountering only ψ-states) from w ends in a state which satisfies φ.[28] Although the operator resists a completely natural interpretation, it offers a solution to the reducibility problem discussed above: public announcement logic with relativized common knowledge (PALRC) is reducible to epistemic logic with relativized common knowledge (S5RC). This is because the relativized common knowledge operator captures precisely the conditions for when $[\psi]C_B\varphi$ holds, which was the problematic case in the attempt at reduction above. Axiomatizing S5RC is a straightforward adaptation of the axiomatization for S5C; with this, completeness has been established for PALRC. Somewhat counterintuitively, it is by *strengthening* PALC that an appropriate reduction proof is found, and completeness established. This is similar to the approach taken to establish completeness for AEL.

While relativized common knowledge works well in the case for PAL, it will not work for AML or AEL. This is because $[\alpha]C_B\varphi$ evades reduction, even when relativized common knowledge is in play. The conditions for the satisfaction of $[\alpha]C_B\varphi$ involve paths in both the epistemic model *and* the action model—or in the case of action-epistemic logic, paths in W and

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immediately that $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$ cannot be reduced to $\mathcal{L}_{K[A,\sigma]C}$.

[27] In more detail: the states of the canonical model are maximal consistent sets of formulas from the language in question. The Truth Lemma shows that if a formula ψ is a member of some state Γ, then in the canonical model, $\Gamma \models \psi$.

[28] Note that the standard common knowledge operator may be recovered by simply setting $\psi = \top$.

properties of $\{f_j | j \in G\}$—and the language without an update operator is not powerful enough to capture these conditions, even with relativized common knowledge.

In [6], van Eijck demonstrates another means of showing completeness for ALC. This method is via *Propositional Dynamic Logic* (PDL), a logic designed for the description of programs which is prominent in computer science. In the language of PDL, $\mathcal{L}_{PDL}$, modalities correspond to programs; formulas are evaluated in Kripke structures, and the execution of a program is indicated by a relation over the possible states of affairs. The set of programs which may be invoked by formulas is closed under certain operations; these closure conditions ensure that if each agent's relation is included in the set as a program, then there exists a program in the set which may be interpreted as the relation for common knowledge for some subset of agents. Van Eijck shows that $\mathcal{L}_{K[A,\sigma]C}$ may be expressed using $\mathcal{L}_{PDL}$ extended with *program transformations*, which are functions on the space of programs. A sound and complete axiomatization is already known for PDL with program transformations; van Eijck combines this axiomatization with his reducibility result to provide an complete axiomatization for ALC.[29]

We take a similar tactic in the appendix. We extend PDL to *action-epistemic PDL* by introducing an update operator to the logic and interpreting formulas on action-epistemic models which satisfy (Act-Equiv). The language of PDL supplemented with update operators will be notated by $\mathcal{L}^{\Sigma}_{PDL[\sigma]}$; the language which additionally includes ξ formulas will be $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$. This language is rich enough to capture $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$. We then show, using an adaptation of van Eijck's proof, that the dynamic modalities included in $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ do not increase expressive power:

THEOREM 16. *There is a truth-preserving translation from $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ to $\mathcal{L}_{PDL\xi}$.*

Lastly, we append the reduction axioms used in this proof onto a complete axiomatization for PDL (with ξ operators) for a complete axiomatization of $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$.

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[29] An abbreviated form of this proof is also presented in [7]. This proof was inspired by one similar in spirit presented in [4] – this proof uses *automata* instead of program transformations.

5. Conclusion

We conclude with a summary of the results presented here, and some remarks about future work.

In Section 2, we outlined some difficulties action model logic faces with respect to the representation of higher-order uncertainty (e.g., "Anne does not know whether Bob knows that p"): the examples we presented demonstrate that in order to capture scenarios featuring higher-order uncertainty using the AML framework, we need to inflate the action model with artificial actions in a way which essentially "pre-encodes" the desired update model. These actions do not correspond to concrete epistemic actions like "Carl announcing p", but instead are individuated by facts which are already encoded in the epistemic model. Thus, the AML framework loses its apparent modularity in the context of representing higher-order uncertainty.

In Section 3, we proposed a solution to this problem in the form of alternative semantics. In particular, we relativized the indistinguishability of actions to the worlds in which they are performed via the introduction of indistinguishability functions f_j for each agent j. *Action-Epistemic Logic* (AEL), as we call it, succinctly represents the examples which were problematic for AML. While AEL updates can be simulated in the AML framework (as can any standard epistemic update, cf. Theorem (9)), AEL recovers the modularity discussed above, and more to the point, it does not require us to know in advance what the intended update model looks like in order to define the update procedure.

The introduction of the functions f_j allowed us to articulate the principle of *action introspection*—that agents have correct beliefs about their own ability to distinguish potential actions—which is assumed implicitly in the AML framework. In Section 3.4, we generalized our semantics in order to account for scenarios where action introspection may fail. This generalization required an adaptation to the update mechanism to reflect the fact the agents may learn something about their own epistemic abilities after the performance of an epistemic action; this is captured in Proposition 13. Endogenizing the agents' epistemic abilities into the model allows us to study the interplay between agents' knowledge and their ability to receive information, a topic we continue to explore in ongoing work.

In Section 4, we proved completeness for AEL via a standard reduction method: we extended the basic epistemic language to $\mathcal{L}_{K\xi}$, provided an axiomatization, and showed that the dynamic language $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ can be reduced to $\mathcal{L}_{K\xi}$. We also discussed why this is *not* possible when a common knowledge operator is introduced to the language; in the appendix, we provide an alternative completeness proof using Propositional Dynamic Logic. Whether completeness for $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$ can be proven without appeal to a stronger language is a question we are currently investigating.

A natural extension of the AEL framework permits the representation of sequences of epistemic actions. The relationship between such sequences of AEL models and more traditional frameworks for reasoning about temporal and epistemic statements is the subject of ongoing work. Finally, as noted in the introduction, we have focused here only on *epistemic* actions; however, there is an extensive literature on the theory of actions more generally. In particular, there are alternative frameworks—STIT logic ("sees-to-it-that"), for instance, or situational semantics—whose purview intersects with that of AML and AEL; in future work, we plan to explore the relationship between the topics treated in this paper and these more general frameworks [5].[30]

A. Completeness with Common Knowledge

In this appendix, we show completeness for $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$. We do this in the following steps:

1. We define $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ and show that $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$ may be captured within it.
2. We show that $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ may be reduced to $\mathcal{L}_{PDL\xi}$.
3. We provide a sound and complete axiomatization for $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ via $\mathcal{L}_{PDL\xi}$.

Defining $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$

We begin by defining the language of PDL, and then extend the language with an update operator and the ξ formulas. As before, this language is relative to a fixed finite set of agents G, and a fixed finite set of actions Σ with preconditions Pre. We enumerate $\Sigma : \Sigma = \{\sigma_0, \ldots, \sigma_{n-1}\}$. i and k will be used to denote the indices of actions, and will range over $[0, n)$; as before, j will range over the set of agents, G. $\mathcal{L}_{PDL}$ is generated by the following:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\pi]\varphi$$

where $\pi \in \Pi$; Π is the set of programs generated inductively with the following clauses:

$$j \in G \mid \pi; \pi' \mid \pi \cup \pi' \mid \pi^* \mid ?\varphi$$

[30] We thank an anonymous reviewer for emphasizing this broader context.

where φ is a formula in $\mathcal{L}_{PDL}$. Here $\pi;\pi'$ denotes sequential composition, $\pi \cup \pi'$ denotes non-deterministic choice, π^* denotes arbitrary iteration, and $?\varphi$ denotes a test. For any $B \subseteq G$, we abbreviate $\bigcup_{j \in B} j$ with B. $\mathcal{L}_{PDL\xi}$ includes all previous formula clauses as well as the clause $\xi_{j,\sigma,\sigma'}$ for $j \in G$, $\sigma, \sigma' \in \Sigma$. $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ includes this additional clause and the clause $[\sigma]\varphi$ for $\sigma \in \Sigma$ and $\varphi \in \mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$. Lastly, a program transformation is a function $r : \Pi \to \Pi$.

We interpret the formulas of $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ in an action-epistemic model, $M = \langle W, \{\sim_j : j \in G\}, \{f_j : j \in G\}, V\rangle$, over Σ and Pre. Semantics for boolean operators, the update operator, and $\xi_{j,\sigma,\sigma'}$ are as before, and semantics for the remaining formulas are as follows:[31]

- $(M, w) \vDash [\pi]\varphi$ iff $(\forall v \in R^M_\pi(w))((M, v) \vDash \varphi)$

where R^M_π is defined inductively, using the relations $\sim_j$ in M, as follows:[32]

- $R^M_j = \sim_j$
- $R^M_{\pi;\pi'} = \{(w, w'')|(\exists w')((w, w') \in \pi \ \wedge \ (w', w'') \in \pi')\}$
- $R^M_{\pi\cup\pi'} = \{(w, w')|(w, w') \in \pi \ \vee \ (w, w') \in \pi'\}$
- $R^M_{\pi^*} = \{(w, w')|(\exists m \geq 0)(\exists w_0, \ldots, w_m)(w = w_0 \wedge w' = w_m \wedge (\forall i \in [1, m])(w_i, w_{i+1}) \in \pi)\}$
- $R^M_{?\varphi} = \{(w, w)|M, w \vDash \varphi\}$

Thus, $R^{M^+}_\pi$ will also be defined inductively using the relations $\sim^+_j$.

This language is sufficient to capture the language $\mathcal{L}^{\Sigma}_{K[\sigma]C\xi}$ via the following interpretation (this interpretation is used in [6] and [7]): we interpret all formulas as themselves except in the following cases:

$$K_j\varphi := [j]\varphi$$

$$C_B\varphi := [B^*]\varphi$$

[31] The overlap in notation between a program operator $[\pi]$ and an epistemic action $[\sigma]$ is unfortunate; however, we will make it clear which we mean, with every use.

[32] In this (admittedly somewhat idiosyncratic) use of PDL, each agent corresponds to an "atomic program", in the sense that the relation that corresponds to the program is interpreted as the agent's epistemic accessibility relation. Mutual knowledge amongst groups of agents can then be encoded using the nondeterministic union ($\cup$) program constructor, and iterated knowledge (of the sort relevant for defining common knowledge) can be encoded using the sequencing (;) program constructor.

Reducing to $\mathcal{L}_{PDL\xi}$

Now we show that $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ may be reduced to $\mathcal{L}_{PDL\xi}$. We do this by providing a truth-preserving translation t from the former to the latter, defined as follows on the formulas and programs of $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$:

$$
\begin{array}{rcl}
t(p) & = & p \\
t(\xi_{j,\sigma,\sigma'}) & = & \xi_{j,\sigma,\sigma'} \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t([\pi]\varphi) & = & [t(\pi)]t(\varphi) \\
t([\sigma_i]p) & = & pre_{\sigma_i} \to p \\
t([\sigma_i]\neg\varphi) & = & pre_{\sigma_i} \to \neg t([\sigma_i]\varphi) \\
t([\sigma_i](\varphi \wedge \psi)) & = & t([\sigma]\varphi) \wedge t([\sigma_i]\psi) \\
t([\sigma_i][\pi]\varphi) & = & \displaystyle\bigwedge_{j=0}^{n-1} T_{\sigma_i,\sigma_k}(\pi)[\sigma_k]t(\varphi) \\
t([\sigma_i][\sigma_k]\varphi) & = & t([\sigma_i]t([\sigma_k]\varphi))
\end{array}
$$

$$
\begin{array}{rcl}
t(j) & = & j \\
t(\pi;\pi') & = & t(\pi);t(\pi') \\
t(\pi \cup \pi') & = & t(\pi) \cup t(\pi') \\
t(\pi^*) & = & t(\pi)^* \\
t(?\varphi) & = & ?t(\varphi)
\end{array}
$$

Almost all formula cases are direct analogues of the translation of $\mathcal{L}^{\Sigma}_{K[\sigma]\xi}$ to $\mathcal{L}_{K\xi}$. The new case, for $[\sigma_i][\pi]\varphi$, uses a class of *program transformations*, which we motivate now. Intuitively, this formula is true at (M, w) when (assuming that $w \vDash Pre_{\sigma_i}$) for any pair (w', σ_k) in M^+ such that $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_{\pi}$, we have that $(M^+, (w', \sigma_k)) \vDash \varphi$. We need to characterize this condition while remaining within $\mathcal{L}_{PDL\xi}$; to do this, we need a way of quantifying over worlds which are possibly the result of executing π at (w, σ_i) in the update model. We take these possibilities by cases, according to which action σ_k was performed at the result (w', σ_k). Thus, we create a transformation T_{σ_i,σ_k} on programs parameterized by the starting action σ_i and ending action σ_k. We want every member (w, w') of $R^M_{T_{\sigma_i,\sigma_k}(\pi)}$ to correspond to a tuple $((w, \sigma_i), (w', \sigma_k)) \in W^+ \times W^+$ which occurs in the interpretation of π in the update model. That is, we want a program $T_{\sigma_i,\sigma_k}(\pi)$ such that $(w, w') \in R^M_{T_{\sigma_i,\sigma_k}(\pi)}$ if and only if $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_\pi$.

We define T_{σ_i,σ_k} on programs inductively.

$$
\begin{aligned}
T_{\sigma_i,\sigma_k}(j) &= \bigcup_{\sigma_k \in \Sigma' \subseteq \Sigma} ?(pre_{\sigma_i} \wedge \xi_{j,\sigma_i,\Sigma'}); j; ?(\xi_{j,\sigma_k,\Sigma'}) \\
T_{\sigma_i,\sigma_k}(\pi \cup \pi') &= T_{\sigma_i,\sigma_k}(\pi) \cup T_{\sigma_i,\sigma_k}(\pi') \\
T_{\sigma_i,\sigma_k}(\pi; \pi') &= \bigcup_{\sigma_m \in \Sigma} (T_{\sigma_i,\sigma_m}(\pi); T_{\sigma_m,\sigma_k}(\pi')) \\
T_{\sigma_i,\sigma_k}(\pi^*) &= K_{\sigma_i,\sigma_k,n}(\pi) \\
T_{\sigma_i,\sigma_k}(?\varphi) &= \begin{cases} ?(\varphi \wedge [\sigma_i]\varphi) & \text{if } i = k \\ ?\bot & \text{otherwise} \end{cases}
\end{aligned}
$$

In the last clause, we use another transformation, $K_{\sigma_i,\sigma_k,m}$, which is defined recursively over m. The intuition behind this transformation is that $K_{\sigma_i,\sigma_k,n}(\pi)$ should contain $(w, w') \in R^M_\pi$ if and only if $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_{\pi^*}$.[33] We interpret $K_{\sigma_i,\sigma_k,m}(\pi)$ as a program which contains (w, w') if there is a sequence of members in $R^{M^+}_{\pi^*}$ which lead from (w, σ_i) to (w', σ_k) without crossing any tuples containing actions which have index m or greater.

$$
K_{\sigma_i,\sigma_k,0}(\pi) = \begin{cases} ?\top \cup T_{\sigma_i,\sigma_k}(\pi) \text{ if } i = k \\ T_{\sigma_i,\sigma_k}(\pi) \text{ if } i \neq k \end{cases}
$$

$K_{\sigma_i,\sigma_k,m+1}(\pi) =$

$$
\begin{cases}
K_{\sigma_m,\sigma_m,m}(\pi)^* \text{ if } i = k = m \\
K_{\sigma_m,\sigma_m,m}(\pi)^*; K_{\sigma_m,\sigma_k,m}(\pi) \text{ if } i = m \neq k \\
K_{\sigma_i,\sigma_m,m}(\pi); K_{\sigma_m,\sigma_m,m}(\pi)^* \text{ if } i \neq k = m \\
K_{\sigma_i,\sigma_k,m}(\pi) \cup (K_{\sigma_i,\sigma_m,m}(\pi); K_{\sigma_m,\sigma_m,m}(\pi)^*; K_{\sigma_m,\sigma_k,k}(\pi)) \text{ if } i \neq k \neq m
\end{cases}
$$

We show that the clauses given define an operator with the desired behavior, summarized in the following lemma:

---

[33] Recall that n is the size of Σ.

LEMMA 17. *Suppose that* $(w, w') \in R^M_{T_{\sigma_i,\sigma_k}(\pi)}$ *iff* $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_\pi$. *Then* $(w, w') \in R^M_{K_{i,k,m}(\pi)}$ *just when there is a (possibly empty) sequence of* π-*steps*[34] *in* M^+ *from* (w, σ_i) *to* (w', σ_k) *which does not have any intermediate states with an action* σ_l *where* $l \geq m$.[35]

$m = 0$:: Suppose that $i = k$. Then $K_{\sigma_i,\sigma_k,0}(\pi) = ?\top \cup T_{\sigma_i,\sigma_k}(\pi)$. Consider $(w, w) \in R^M_{?\top}$ – this corresponds to the empty sequence of π steps from (w, σ_i) to itself. Suppose $(w, w') \in R^M_{T_{\sigma_i,\sigma_k}(\pi)}$; this is equivalent, by our premise, to $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_\pi$, which is a single π-step and so trivially satisfies the requirement on intermediate states.

Suppose that $i \neq k$. Then $K_{\sigma_i,\sigma_k,0}(\pi) = T_{\sigma_i,\sigma_k}(\pi)$, and the second of the two cases considered above applies.

For the induction step, we assume that the result holds up to m; we show it for $m + 1$. We consider the same cases used to define $K_{\sigma_i,\sigma_k,m}$ above.

Suppose that $i = k = m$. Any π-sequence from (w, σ_k) to (w', σ_k) in M^+ which does not pass through any intermediate states of the form (u, σ_l) for $l \geq k + 1$ may be viewed as a chain of π-sequences with the following characteristics:

- Each π-sequence begins with a state of the form (u, σ_k), and ends with a state of the form (v, σ_k).
- The first sequence begins at (w, σ_k), and the last sequence ends at (w', σ_k).
- No sequence encounters an intermediate state of the form (u, σ_l) with $l \geq k$.

By the induction hypothesis, each π-sequence of the described sort from (u, σ_k) to (v, σ_k) exists iff $(u, v) \in R^M_{K_{\sigma_k,\sigma_k,k}(\pi)}$. Thus, the full chain π-sequences from (w, σ_k) to (w', σ_k) in M^+ exists iff $(w, w') \in R^M_{(K_{\sigma_k,\sigma_k,k}(\pi))^*} = R^M_{K_{\sigma_i,\sigma_k,k+1}(\pi)}$.

Suppose that $i = m \neq k$. Any π from (w, σ_i) to $(w', \sigma_k) \in M^+$ which does not pass through any intermediate states of the form (u, σ_l) for $l \geq m + 1$ may be viewed as a chain of π-sequences with the following characteristics:

[34] In the following, we will use the term 'π-sequence' to denote a chain of π steps. More formally, a π-sequence in an action-epistemic model M' is a sequence of states in M', $(s_0, \ldots, s_b)$, such that for $i \in [0, b)$, $(s_i, s_{i+1}) \in R^{M'}_\pi$.

[35] The case of zero π steps corresponds to the trivial π^* path.

- Every π-sequence in the chain, *except for the last*, begins with a state of the form (u, σ_m), and ends with a state of the form (v, σ_m). The last π-sequence in the chain begins with a state of the form (u, σ_m), and ends with a state of the form (v, σ_k).
- The first sequence of the chain begins at (w, σ_m), and the last sequence ends at (w', σ_k).
- No sequence encounters an intermediate state of the form (u, σ_l) with $l \geq m$.

By the induction hypothesis, a π-sequence of the first sort (i.e., not including the last sequence of the chain) from (u, σ_m) to (v, σ_m) exists iff $(u, v) \in R^M_{(K_{\sigma_m,\sigma_m,m})(\pi)^*}$, and a π-sequence of the second sort (i.e., the last sequence in the chain) from (u, σ_m) to (v, σ_k) exists iff $(u, v) \in R^M_{K_{\sigma_m,\sigma_k,m}(\pi)}$. Thus, the full chain from (w, σ_i) to (w', σ_k) exists if and only if $(w, w') \in R^{M+}_{(K_{\sigma_m,\sigma_m,m}(\pi))^*;K_{\sigma_m,\sigma_k,m}(\pi)} = R^{M+}_{K_{\sigma_i,\sigma_k,m+1}(\pi)}$.
The other cases work similarly. ■

An easy consequence of this lemma is the following:

LEMMA 18. *Suppose that* $(w, w') \in R^M_{T_{\sigma_i,\sigma_k}(\pi)}$ *iff* $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_\pi$. *Then* $(w, w') \in R^M_{K_{\sigma_i,\sigma_k,n}(\pi)}$ *iff* $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_{\pi^*}$.

We use this lemma to show that our translation is truth-preserving in the case for $[\sigma_i][\pi]\varphi$, which is stated in the following theorem:

THEOREM 19. $M, w \vDash [\sigma_i][\pi]\varphi$ *if and only if* $M, w \vDash \bigwedge_{\sigma_k \in \Sigma} [T_{\sigma_i,\sigma_k}(\pi)][\sigma_k]\varphi$.

We show this by induction over the complexity of π.

$\pi = j \in G$:: suppose the left hand side, and take some $\sigma_k \in \Sigma$. We show that $M, w \vDash [\bigcup_{\sigma_k \in \Sigma' \subseteq \Sigma} ?(pre_{\sigma_i} \wedge \xi_{j,\sigma_i,\Sigma'}); j; \xi_{j,\sigma_k,\Sigma'}][\sigma_k]\varphi$. Suppose that $M, w \vDash pre_{\sigma_i} \wedge \xi_{j,\sigma_i,\Sigma'}$, and consider any w' such that $w \sim_j w'$ and $M, w \vDash \xi_{j,\sigma_k,\Sigma'}$; suppose that $w' \vDash pre_{\sigma_k}$. We know that $(w, \sigma_i) \in M^+$ and $(w, \sigma_i) \sim^+_j (w', \sigma_k)$, and so by hypothesis, $(w', \sigma_k) \vDash \varphi$. Thus $w' \vDash [\sigma_k]\varphi$ as desired.

Suppose the right hand side, and that $w \vDash pre_{\sigma_i}$. Consider some (w', σ_k) such that $(w, \sigma_i) \sim^+_j (w', \sigma_k)$. This implies that for some $\Sigma' \subseteq \Sigma$ with $\sigma_k \in \Sigma'$, we have that $w \vDash \xi_{j,\sigma_i,\Sigma'}$ and $w' \vDash \xi_{j,\sigma_k,\Sigma'}$. Thus, $(w, w') \in R^M_{T_{\sigma_i,\sigma_k}(j)}$, and so by our hypothesis, $w' \vDash [\sigma_k]\varphi$, and $(w', \sigma_k) \vDash \varphi$.

$\pi = \ ?\psi$:: suppose the left hand side. Since $T_{i,k}(?\psi)$ is non-trivial only when $i = k$, we only need to show that $w \vDash [?(pre_{\sigma_i} \wedge [\sigma_i]\psi)][\sigma_i]\varphi$. Suppose

$w \vDash pre_{\sigma_i} \wedge [\sigma_i]\psi$; then $(w, \sigma_i) \vDash \psi$ and so by the hypothesis, $(w, \sigma_i) \vDash \varphi$, as desired.

Suppose the right hand side, and that $w \vDash pre_{\sigma_i}$, and that $(w, \sigma_i) \vDash \psi$. By hypothesis, $w \vDash [?(pre_{\sigma_i} \wedge [\sigma_i]\psi)][\sigma_i]\varphi$. w satisfies the test formula, and so $(w, \sigma_i) \vDash \varphi$ and we are done.

$\pi = \pi_1; \pi_2$::

We show by a sequence of equivalent formulas, starting with the right hand side:

$$\bigwedge_{\sigma_k \in \Sigma} [\bigcup_{\sigma_m \in \Sigma} (T_{\sigma_i,\sigma_m}(\pi_1); T_{\sigma_m,\sigma_k}(\pi_2))][\sigma_k]\varphi$$

$$\bigwedge_{\sigma_k \in \Sigma} \bigwedge_{\sigma_m \in \Sigma} [(T_{\sigma_i,\sigma_m}(\pi_1); T_{\sigma_m,\sigma_k}(\pi_2))][\sigma_k]\varphi$$

$$\bigwedge_{\sigma_m \in \Sigma} [T_{\sigma_i,\sigma_m}(\pi_1)] \bigwedge_{\sigma_k \in \Sigma} [T_{\sigma_m,\sigma_k}(\pi_2)][\sigma_k]\varphi$$

$$\bigwedge_{\sigma_m \in \Sigma} [T_{\sigma_i,\sigma_m}(\pi_1)][\sigma_m][\pi_2]\varphi$$

$$[\sigma_m][\pi_1][\pi_2]\varphi$$

$$[\sigma_m][\pi_1; \pi_2]\varphi$$

Where lines three and four are justified by the induction hypothesis. The case where $\pi = \pi_1 \cup \pi_2$ is similar.

$\pi = \pi^*$. The induction hypothesis establishes the premise of Lemma 18, by which we conclude that $(w, w') \in R^M_{K_{i,k,n}}$ if and only if $((w, \sigma_i), (w', \sigma_k)) \in R^{M^+}_{\pi^*}$. From this the desired conclusion follows straightforwardly. ■

Now we give the reduction scheme from $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ to $\mathcal{L}_{PDL\xi}$.

$$t(p) = p$$

$$t(\xi_{j,\sigma,\sigma'}) = \xi_{j,\sigma,\sigma'}$$

$$t(\neg\varphi) = \neg t(\varphi)$$

$$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$$

$$t([\pi]\varphi) = [r(\pi)]t(\varphi)$$

$$t([\pi]\varphi) = [r(\pi)]t(\varphi)$$
$$t([\sigma]p) = pre_\sigma \to p$$
$$t([\sigma]\neg\varphi) = pre_\sigma \to \neg[\sigma]t(\varphi)$$
$$t([\sigma](\varphi \wedge \psi)) = t([\sigma]\varphi) \wedge t([\sigma]\psi)$$
$$t([\sigma_i][\pi]\varphi) = \bigwedge_{\sigma_k \in \Sigma} T_{i,k}(r(\pi))t([\sigma_k]\varphi)$$
$$t([\sigma][\sigma']\varphi = t([\sigma]t([\sigma']\varphi))$$

$$r(a) = a$$
$$r(?\varphi) = \ ?t(\varphi)$$
$$r(\pi; \pi') = r(\pi); r(\pi')$$
$$r(\pi \cup \pi') = r(\pi) \cup r(\pi')$$
$$r(\pi^*) = r(\pi)^*$$

The correctness of this translation may be proved as before, by defining an appropriate weight and depth on formulas. We then induct over the depth and weight of the formula; all cases are identical to those before except for $[\pi]\varphi$ and $[\sigma_i][\pi][\varphi]$. For these cases, we may induct over the structure of π. We leave this proof to the reader. With this, we have shown Theorem 16.

Completeness for $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$

Our axiom system is all of the axioms and rules of PDL, axioms 6a-c, 7 from above (where 7 has been suitably adapted to the present language: $\xi_{k,\sigma,\sigma'} \to [\pi]\xi_{k,\sigma,\sigma'}$), plus the following reduction axioms:

$$[\sigma]p \leftrightarrow pre_\sigma \to p$$

$$[\sigma]\neg\varphi \leftrightarrow pre_\sigma \to \neg[\sigma]\varphi$$

$$[\sigma](\varphi \wedge \psi) \leftrightarrow [\sigma]\varphi \wedge [\sigma]\psi$$

$$[\sigma][\pi]\varphi \leftrightarrow \bigwedge_{\sigma_k \in \Sigma} T_{\sigma_i,\sigma_k}(\pi)[\sigma_k]\varphi$$

The soundness of the last axiom was establish by Theorem 19. With this, we have established completeness for $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$ with respect to the above collection of axioms: take any non-theorem φ of $\mathcal{L}^{\Sigma}_{PDL[\sigma]\xi}$. We know that φ is provably equivalent to $t(\varphi)$, which is in the language $\mathcal{L}_{PDL\xi}$. Since the axiomatization for $\mathcal{L}_{PDL\xi}$ is complete (we include axioms 6a-c and 7 in its adjusted form), there is some action-epistemic model (M, w) such that $M, w \vDash \neg t(\varphi)$, and we are done.

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