SVD Factorization for Tall-and-fat Matrices on Map/Reduce Architectures

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Abstract

We demonstrate an implementation for an approximate rank-k SVD factorization, combining well-known randomized projection techniques with previously implemented map/reduce solutions in order to compute steps of the random projection based SVD procedure, such QR and SVD. We structure the problem in a way that it reduces to Cholesky and SVD factorizations on $k \times k$ matrices computed on a single machine, greatly easing the computability of the problem.

1 Introduction

[1] presents many excellent techniques for utilizing map/reduce architectures to compute QR and SVD for the so-called tall-and-skinny matrices. The idea is based on the fact that QR factorization can be turned into an $A^T A$ computation problem computer in parallel, en masse using map/reduce, and through this to a Cholesky decomposition performed on a single machine. Since

$$A^T A = (QR)^T (QR) = R^T Q^T QR = R^T R$$

and because Cholesky factorization of an $n \times n$ symmetric positive definite matrix is

$$A = LL^T$$

where L is an $n \times n$ lower triangular matrix, and R is upper triangular, we can conclude if we factorize A into L and L^T, this implies $LL^T = RR^T$, we have a method of calculating R of QR using Cholesky factorization on $A^T A$. The key observation here is $A^T A$ computation results an $n \times n$ matrix and

if A is "skinny" then n is relatively small (in the thousands), and Cholesky decomposition can be executed on a small $n \times n$ matrix on a single computer. Q is computed simply as $Q = AR^{-1}$. This again is relatively cheap because R is $n \times n$, the inverse is computed locallly, matrix multiplication with A can be performed through map/reduce.

SVD is an additional step. SVD decomposition is

$$A = U\Sigma V^T$$

If we expand it with $A = QR$

$$QR = U\Sigma V^T$$

$$R = Q^T U\Sigma V^T$$

Let's call $\tilde{U} = Q^T U$

$$R = \tilde{U}\Sigma V^T$$

This means if we run a local SVD on R (we just calculated above with Cholesky) which is an $n \times n$ matrix, we will have calculated $\tilde{U}$, the real Σ, and real V^T.

Now we have a map/reduce way of calculating QR and SVD on $m \times n$ matrices where n is small.

1.1 Approximate rank-k SVD

Switching gears, we look at another method for calculating SVD. The motivation is computing SVD if n is large, creating a "fat" matrix which might have columns in the billions would require reducing the dimensionality of the problem. According to [2], one way to achieve is through random projection. First we draw an $n \times k$ Gaussian random matrix Ω. Then we calculate

$$Y = A\Omega$$

We perform QR decomposition on Y

$$Y = QR$$

Then form $k \times n$ matrix

$$B = Q^T A$$

Then we can calculate SVD on this small matrix

$$B = \hat{U}\Sigma V^T$$

Then form the matrix

$$U = Q\hat{U}$$

The main idea is based on

$$A = QQ^T A$$

if replace Q which comes from random projection Y,

$$A \approx \tilde{Q}\tilde{Q}^T A$$

Q and R of the projection are close to that of A. In the multiplication above R is called B where $B = \tilde{Q}^T A$, and,

$$A \approx \tilde{Q}B$$

then, as in [1], we can take SVD of B and apply the same transition rules to obtain an approximate U of A.

This approximation works because of the fact that projecting points to a random subspace preserves distances between points, or in detail, projecting the n-point subset onto a random subspace of $O(\log n/\epsilon^2)$ dimensions only changes the interpoint distances by $(1 \pm \epsilon)$ with positive probability [3]. It is also said that Y is a good representation of the span of A.

1.2 Combining Both Methods

What if n is also very large? In this case local Cholesky or SVD computations would take a long time as well. Our idea was using approximate k-rank SVD where $k << n$, before map/reduce based QR and SVD methods presented in section 1, to reduce dimension before using map/reduce methods presented in Section 1, this way, we are again able to work with small matrices locally, $k \times k$ this time on which Cholesky, SVD can be performed. Below we outline each map/reduce job.

Algorithm 1: Random Projection Job

> **input** : A
> **output**: Y
> **function** *MAP(key, value)*
> > Tokenize *value* and pick out id value pairs
> > result ← zeros(1,k)
> > **for** *each jth token $\in$ value* **do**
> > > Initialize seed with j
> > > r ← generate k random numbers
> > > result ← result + $r \cdot token[j]$
> >
> > **end**
> > **emit** *key, result*
>
> **function** *REDUCE(key, value)*
> > noop
> > **return**

Each value of A will arrive to the algorithm as a key and value pair. Key is line number or other identifier per row of A. Value is a collection of id value pairs where id is column id this time, and value is the value for that column. Sparsity is handled through this format, if an id for a column does not appear in a row of A, it is assumed to be zero. The resulting Y matrix has dimensions $m \times k$.

Algorithm 2: $A^T A$ Cholesky Job

> **input** : Y
> **output**: R
> **function** *MAP(key k, val a)*
> > Tokenize *value* and pick out id value pairs
> > result ← zeros(1,k)
> > **for** *i, row in enumerate($a^T a$)* **do**
> > > **emit** *i, row*
> >
> > **end**
>
> **function** *REDUCE(key, value)*
> > **emit** (k,sum(v_j^k)
>
> **function** *FINAL LOCAL REDUCE (key, value)*
> > **emit** (k,sum(v_j^k)

References

[1] Gleich, Benson, Demmel, *Direct QR factorizations for tall-and-skinny matrices in MapReduce architectures*

[2] N. Halko, *Randomized methods for computing low-rank approximations of matrices*

[3] S. Dangupta, A. Gupta *An Elementary Proof of a Theorem of Johnson and Lindenstrauss*