

## Support Vector Machines

In their simplest form, SVMs are **linear classifiers** that do **risk minimization**.

$$R(\Theta) \leq J(\Theta) = R_{emp}(\Theta) + \sqrt{\frac{h \times (\log(\frac{2N}{h}) + 1) - \log(\frac{\eta}{4})}{N}} \quad (1)$$

$h$ : capacity of a classifier

$N$ : number of training points

- Vapnik and Chernovenkis proved that with probability  $1 - \eta$  previous equation holds true.
- Vapnik and Chernovenkis proved that with probability  $1 - \eta$  previous equation holds true.
- SVM algorithm minimizes both  $h$  and empirical risk at the same time by increasing separation margin (less flexibility)
- Vapnik and Chernovenkis proved that with probability  $1 - \eta$  previous equation holds true.
- SVM algorithm minimizes both  $h$  and empirical risk at the same time by increasing separation margin (less flexibility)
- Let's derive the equations

## Derivation

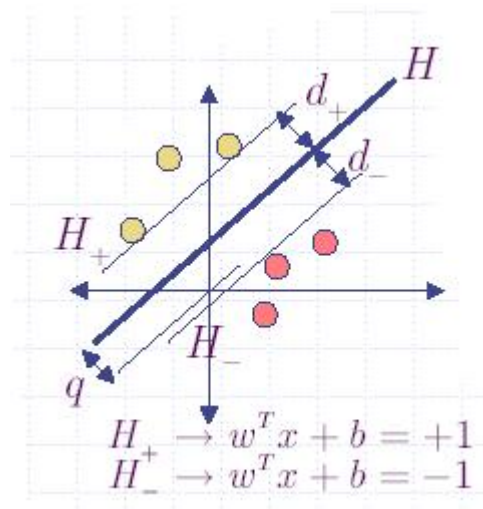


Figure 1:

Decision plane:  $w^T x + b = 0$

Let's define  $q = \min_x \|x - 0\|$

- We will later use the formula for  $q$  on  $H^+$  and  $H^-$ .
- For  $H$ :  $q = \min_x \|x - 0\|$  subject to  $w^T x + b = 0$
- Lagrange:  $\min_x \frac{1}{2} \|x - 0\|^2 + \lambda(w^T x + b)$
- Take gradient  $(\frac{\partial}{\partial x})$  set to 0
- After some algebra:  $q = \frac{|b|}{\|w\|}$
- Define:
  - $H^+ = w^T x + b = +1$
  - $H^- = w^T x + b = -1$
- This is without loss of generality; We can still adjust  $b$  &  $w$
- Calculate  $q^+$  and  $q^-$ 
  - $q^+ = \frac{|b-1|}{\|w\|}$
  - $q^- = \frac{|-b-1|}{\|w\|}$
- The margin then is
  - $m = q^+ + q^- = \frac{|b-1-b-1|}{\|w\|} = \frac{|-2|}{\|w\|} = \frac{2}{\|w\|}$

For maximal margin, increase  $m$  (maximize  $\frac{2}{\|w\|}$ ) or minimize  $\|w\|$ !

Constraints

We want points classified so that + and - points are in the correct side of the hyperplanes;

$$\begin{aligned} w^T x + b &\geq +1, \forall y_i = +1 \\ w^T x + b &\leq -1, \forall y_i = -1 \end{aligned}$$

Combine the two

$$y_i(w^T x + b) - 1 \geq 0 \tag{2}$$

Putting it all together

$$\min \frac{1}{2} \|w\|^2 \text{ subject to } y_i(w^T x_i + b) - 1 \geq 0 \tag{3}$$

This is a quadratic program!

qp

- Python language has cvxopt package
- Matlab Optimization Toolbox has qp() function.

- Or Steve Gunn's SVM Toolbox has another qp written in C
- SVMLight has its own qp
- qp functions usually expect a problem in  $\frac{1}{2}x^T Px + q^T x$  format
- We can massage previous equation to fit the equation above

Dual

- For SVM purposes, working with the dual is easier.
- Form the Lagrange (again), take derivative, set equal to zero
- This gives us the KKT point

$$L_p = \frac{1}{2}||w||^2 - \sum_i \alpha_i (y_i (w^T x_i + b) - 1) \quad (4) \quad (\text{eq:primal})$$

$$\frac{\partial}{\partial w} L_p = w - \sum_i \alpha_i y_i x_i = 0$$

$$w = \sum_i \alpha_i y_i x_i \quad (5) \quad (\text{eq:wdual})$$

$$\frac{\partial}{\partial b} L_p = - \sum_i \alpha_i y_i = 0 \quad (6) \quad (\text{eq:cdual})$$

Plugging equation ~~5~~ <sup>(eq:wdual)</sup> and ~~6~~ <sup>(eq:cdual)</sup> into primal equation ~~4~~ <sup>(eq:primal)</sup>:

$$\text{Maximize } L_D = \sum_i \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i^T x_j \quad (7) \quad (\text{eq:svm})$$

constraints

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i \geq 0$$

qp

- qp() again! But now the variable(s) we are solving for are  $\alpha_i$ 's, not  $x$ 's.
- Massage ~~7~~ <sup>(eq:svm)</sup> into  $\frac{1}{2}x^T Px + q^T x$  form
- This can be achieved by setting  $P_{i,j}$  to be  $-y_i y_j x_i^T x_j$
- Call qp
- The solution is a list of  $\alpha$ 's

### Calculating $b$

- Due to KKT condition, for each nonzero  $\alpha_i$ , the corresponding constraint in the primal problem is tight (an equality)
- Then for each non-zero  $\alpha_i$ , calculate  $b$  using  $w^T x_i + b = y_i$ .
- Each  $b$  from non-zero  $\alpha_i$  will be approximately equal to other  $b$ 's. It is numerically safer to average all  $b$ 's for final  $b$ .

### Classifier Done

For each new point  $x$ , we can use  $\text{sign}(x^T w + b)$  as our classifier. The result,  $-1$  or  $+1$  will tell us which class this new point belongs to.

### Sample Output

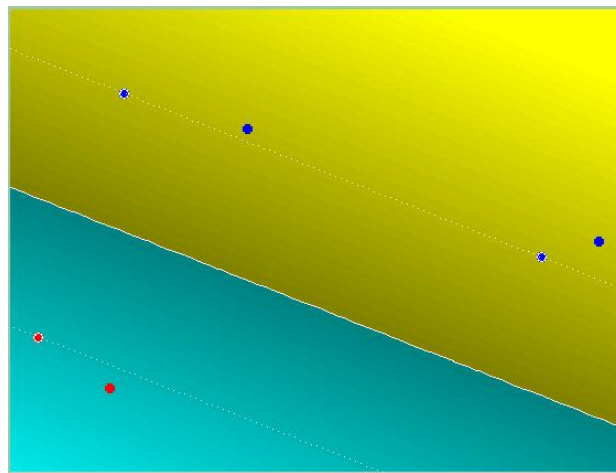


Figure 2:

### Kernels

- We talked about linear boundaries so far
- SVMs can also form non-linear boundaries
- Simple: Just preprocess input data with a basis function into higher dimensions
- Rest of the algorithm is unchanged

### Nonlinear Kernel

### Slack

- Sometimes the problem might be inseperable

- A few points might throw off the classifier
- We can introduce “slack” into a classifier
- For example, allow data to fall on the wrong side with  $w^T + b \geq -0.03$  for  $y_i = +1$
- But we don’t want too many of such points, hence penalize the “quantity” of such slack points

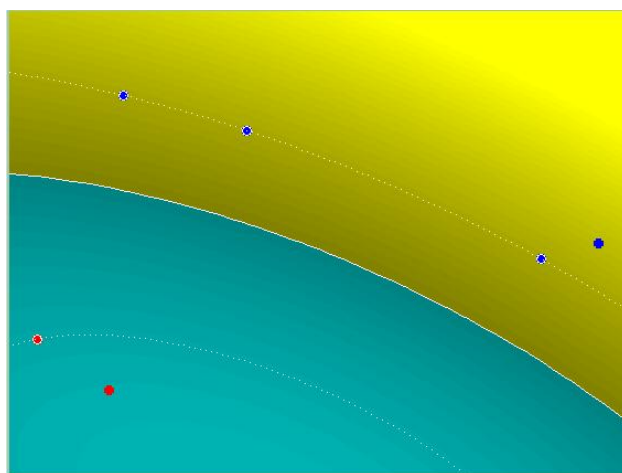


Figure 3:

```

import numpy as np
from numpy import linalg
import cvxopt
import cvxopt.solvers

def svm(X, y):
    n_samples, n_features = X.shape

    # Gram matrix
    K = np.zeros((n_samples, n_samples))
    for i in range(n_samples):
        for j in range(n_samples):
            K[i,j] = np.dot(X[i], X[j])

    P = cvxopt.matrix(np.outer(y,y) * K)
    q = cvxopt.matrix(np.ones(n_samples) * -1)
    A = cvxopt.matrix(y, (1,n_samples))
    b = cvxopt.matrix(0.0)

    G = cvxopt.matrix(np.diag(np.ones(n_samples) * -1))
    h = cvxopt.matrix(np.zeros(n_samples))

    # solve QP problem
    solution = cvxopt.solvers.qp(P, q, G, h, A, b)

    print solution

    # Lagrange multipliers
    a = np.ravel(solution['x'])

    print "a", a

    # Support vectors have non zero lagrange multipliers
    ssv = a > 1e-5
    ind = np.arange(len(a))[ssv]
    a = a[ssv]
    sv = X[ssv]
    sv_y = y[ssv]
    print "%d support vectors out of %d points" % (len(a), n_samples)
    print "sv", sv
    print "sv_y", sv_y

    # Intercept
    b = 0
    for n in range(len(a)):
        b += sv_y[n]
        b -= np.sum(a * sv_y * K[ind[n],ssv])
    b /= len(a)

    # Weight vector
    w = np.zeros(n_features)
    for n in range(len(a)):
        w += a[n] * sv_y[n] * sv[n]

    print "a", a
    return w, b, sv_y, sv, a

if __name__ == "__main__":

    def test():
        X = np.array([[3.,3.],[4.,4.],[7.,7.],[8.,8.]])
        y = np.array([1.,1.,-1.,-1.])
        w, b, sv_y, sv, a = svm(X, y)
        print "w", w
        print "b", b
        print 'test points'
        print np.dot([2.,2.], w) + b # > 1
        print np.dot([9.,9.], w) + b # < -1

    test()

```

Note: We are maximizing the dual  $L_d$ , but we are still calling the minimizer `qp()` function. Therefore the  $q$ 's, which represent the summation of all  $\alpha$ 's are negated as seen above in `np.ones(n_samples) * -1`. The quadratic part already has the negated statement  $-\frac{1}{2}$  in the beginning, so the rest of does not have to change.

#### References

<http://www.mblondel.org/journal/2010/09/19/support-vector-machines-in-python>

Jebara, T., Machine Learning Lecture, Columbia University