The lead-lag transformation

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DataSig

A rough path between
mathematics and data science

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Outline

1. Definitions
2. Levy area
   - Defining the Levy area between paths
   - Levy area as a measure of auto-correlation
   - Levy area and signature
3. Recovering statistical properties of $\hat{\gamma}$ using lead-lag transform and signatures
4. Signatures as statistical properties

Definitions

Let Λ denote the lead-lag transform. We will see three variations of it:

1. **Discrete-discrete lead-lag**
   Λ_{dd} : {Sequences in $\mathbb{R}^d$} $\to$ {Sequences in $\mathbb{R}^2 d$}
2. **Discrete-continuous lead-lag**
   Λ_{dc} : {Sequences in $\mathbb{R}^d$} $\to$ {Continuous paths in $\mathbb{R}^2 d$}
3. **Continuous-continuous lead-lag**
   Λ_{cc} : {Sequences in $\mathbb{R}^d$} $\to$ {Continuous paths in $\mathbb{R}^2 d$}

Definition (p-lead lag transformation (Discrete-discrete version))

Let $\hat{\gamma} := \{x_{t_i}\}_{i\in\{1,\ldots,n\}}$ be a sequence of n points in $\mathbb{R}^d$. Define by the **discrete-discrete p-lead-lag transform** $\Lambda_{dd}(\hat{\gamma}; p)$ of $\hat{\gamma}$ the following sequence of points,

$$\Lambda_d(\hat{\gamma}, p) = \left\{(x_{t_{s+p-1}}, x_{t_s}), (x_{t_{s+p}}, x_{t_s})\right\}_{s\in\{1,\ldots,n-p\}} \tag{1}$$

for $p \in \{1, \ldots, n-1\}$.

Remark (Length and dimension)

$\Lambda_{dd}(\hat{\gamma}; p)$ transform a sequence of n points in $\mathbb{R}^d$ into one of $2n - 2p$ points in $\mathbb{R}^{2d}$.

Examples (Discrete-discrete version)

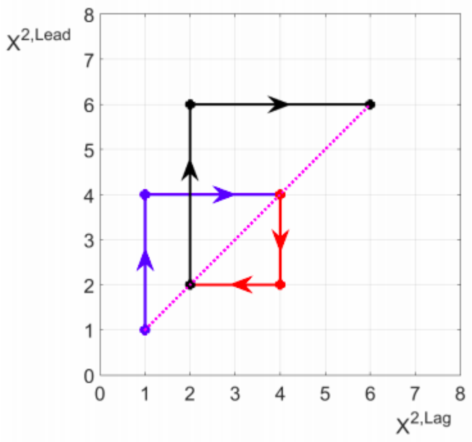
1. Take $\hat{\gamma} = (1, 4, 2, 6)$. Then,

$$\Lambda_{dd}(\hat{\gamma}, 1) = ((1,1),(4,1),(4,4),(2,4),(2,2),(6,2),(6,6)) \tag{2}$$

$$\Lambda_{dd}(\hat{\gamma}, 2) = ((4,1),(2,1),(2,4),(6,4),(6,2)) \tag{3}$$

$$\Lambda_{dd}(\hat{\gamma}, 3) = ((2,1),(6,1),(6,4)) \tag{4}$$

Examples (Discrete-discrete version)



Definition (continuous)

Definition (p-lead lag transformation (Discrete-continuous version))

Let $\hat{\gamma} := \{x_{t_i}\}_{i \in \{1,\dots,n\}}$ be a sequence of n points in $\mathbb{R}^d$. Define by the **discrete-continuous** p**-lead-lag transform** $\Lambda_{dc}(\hat{\gamma}; p)$ of $\hat{\gamma}$ the following continuous path in $\mathbb{R}^{2d}$,

$$\Lambda_{dc}(\hat{\gamma}; p)(t) := \tag{5}$$

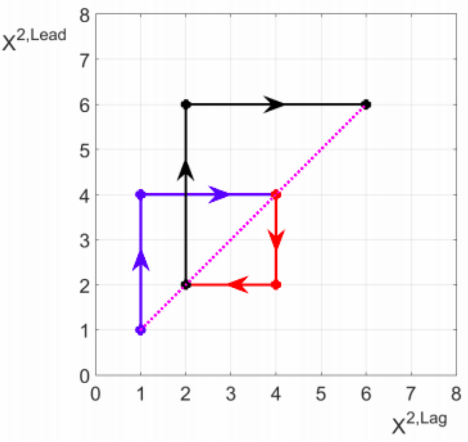
$$\begin{cases} (x_{t_{i+p-1}}, x_{t_i}), & t \in [i, i + \frac{1}{3}) \\ (x_{t_{i+p-1}} + 3(t - (i + \frac{1}{3}))(x_{t_{i+p}} - x_{t_{i+p-1}}), x_{t_i}), & t \in [i + \frac{1}{3}, i + \frac{2}{3}) \\ (x_{t_{i+p}}, x_{t_i} + 3(t - (i + \frac{2}{3}))(x_{t_{i+1}} - x_{t_i})), & t \in [i + \frac{2}{3}, i + 1) \end{cases} \tag{6}$$

for $p \in \{1, \dots, n-1\}$ and $t \in [1; n-p]$.

Remark (Trivial recovery)

The discrete-continuous lead lag path $\Lambda_{dc}(\hat{\gamma}; p)(t)$ evaluated at the times $\{k + \frac{s}{3}\}_{k \in \{1,\dots,n\}, s \in \{1,2\}}$ exactly recovers the discrete-discrete lead lag $\Lambda_{dd}(\hat{\gamma})$.

Examples (Discrete-continuous version)



Definition (continuous-continuous)

Definition (p-lead lag transformation (continuous version))

Let $\gamma : [a, b] \to \mathbb{R}^d$ be a path in $\mathbb{R}^d$. The **continuous-continuous** $p-$**lead-lag path** Λ_{cc} of γ is simply

$$\Lambda_{cc}(\gamma; p)(t) := (\gamma(t + p), \gamma(t)) \tag{7}$$

for $p \in [a; b]$ and $t \in [a, b - p]$.

Remark (Λ_{cc} is the limit of Λ_{dc} for large n)

Let $\hat{\gamma} := \{x_{t_i}\}_{i \in \{1,\dots,n\}}$ be a set of n points and consider its linear interpolation γ_L^n. If there exists a path $\gamma : [a,b] \to \mathbb{R}^d$ such that $\hat{\gamma}$ is the evaluation of γ on the set of times $\{t_i\}_{i \in \{1,\dots,n\}}$, then

$$\lim_{n\to\infty} \Lambda_{dc}(\hat{\gamma}_n; \lfloor \frac{p_n}{n} \rfloor)(\frac{t}{n}) = \Lambda_{cc}\big(\gamma; (b-a)c + a\big)\big((b-a)t + a\big), \ \forall t \in [0;1] \tag{8}$$

where p_n is such that $\frac{p_n}{n} = c$ for a constant $c \in [0,1]$, $n \in \mathbb{N}$.

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Levy area and convexity

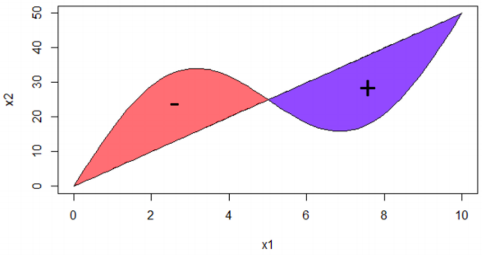
- One can think of the **correlation of γ'' with itself in terms of the Levy area A of its lead-lag path** with respect to the straight line linking endpoints.

Levy area

Definition

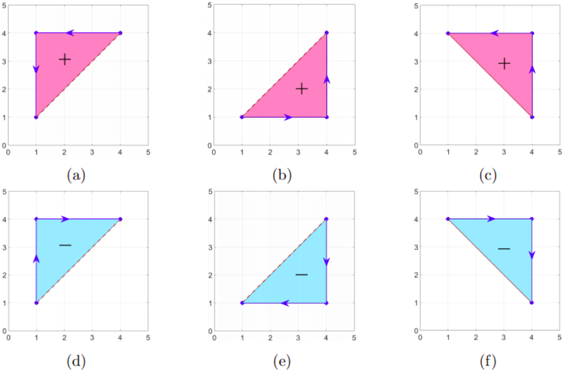
Denote by $A_{[0,t]}$ the Levy area drawn by (X_1, X_2) between $[0, t]$ wrt. straight dashed line linking start and end points,

$$A_{[0,t]} := \frac{1}{2}\left(\int_0^t X_1(s)dX_2(s) - \int_0^t X_2(s)dX_1(s)\right) \tag{9}$$



Signs of the Levy area in terms of the windig number

Using **Green's theorem** (to express A as a double integral with respect to the area) **and then Kelvin-Stokes theorem** (to express the integral of the signed area w.r.t. to windig number of the curve around the areas) **gives the following sign rules,**

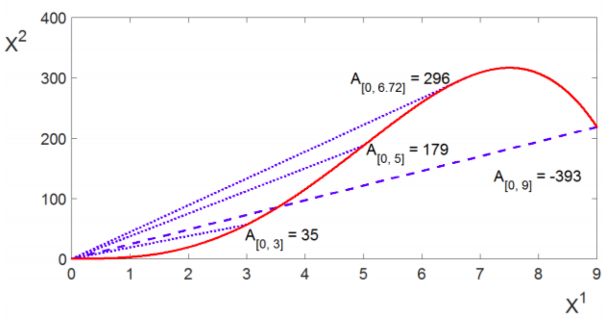


(a) (b) (c)
(d) (e) (f)

Levy area and auto-correlation

Let $\gamma : [a, b] \to \mathbb{R}$ be a path in $\mathbb{R}$.

Consider the p-lead-lag transform $\Lambda_{cc}(\gamma; p) : [a; b - p] \to \mathbb{R}^{2d}$ of γ whose components are denoted as $\Lambda_{cc}(\gamma; p) = (X_1, X_2)$.



(Plot from[1])

[1] Ilya Chevyrev and Andrey Kormilitzin. "A primer on the signature method in machine learning". In: *arXiv preprint arXiv:1603.03788* (2016).

Levy area and auto-correlations between second derivatives:

$$\frac{d}{dt}A_{[0,t]} \stackrel{\geq}{\leq} 0 \quad \Longrightarrow \quad sign(\gamma''(t)\gamma''(t-p)) = \pm 1 \tag{10}$$

if both second-derivatives are non-null and well-defined.

Finally, Levy area and signature are related.
By definition 7, the Levy area A^{ij} between the components X_i and X_j of a path γ can be expressed in terms of the 2-order signature terms as

$$A^{ij}_{[0,t]} = \frac{1}{2}\left(S(\gamma)^{2;ij} - S(\gamma)^{2;ji}\right) \tag{11}$$

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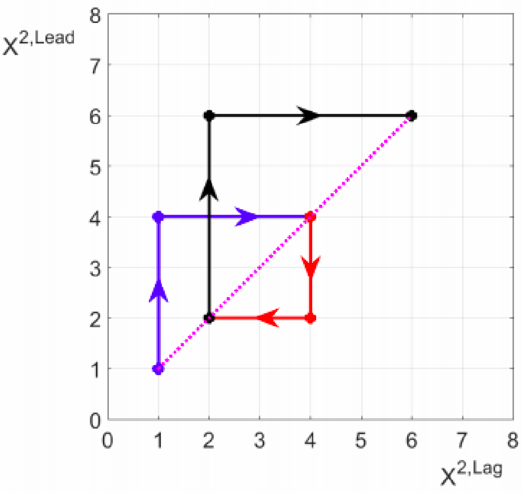
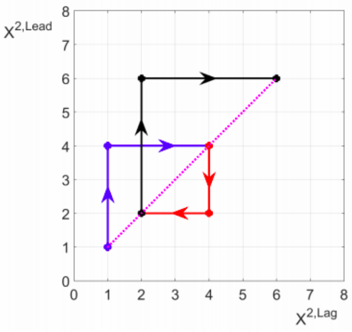
Recall the following lead-lag path





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The total Levy area $A_{[0,T]}$ of the rectangles is

$$|A_{[0,T]}| = \frac{1}{2}[(X_2^2 - X_1^2)(X_2^2 - X_1^2) + (X_3^2 - X_2^2)(X_3^2 - X_2^2) + 2) \tag{12}$$

$$+ (X_4^2 - X_3^2)(X_4^2 - X_3^2)] \tag{13}$$

$$= \frac{1}{2}[(4-1)^2 + (2-4)^2 + (6-2)^2] \tag{14}$$

Definition (Quadratic variation (discrete version))

Let $\hat{\gamma} := \{X_i\}_{i \in \{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$. Then, the quadratic variation $QV(\hat{\gamma})$ of $\hat{\gamma}$ is

$$QV(\hat{\gamma}) := \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 \tag{15}$$

For a lead-lag path γ, the Levy area and its quadratic variation are linked,

$$A_{[0,T]} = \frac{1}{2} QV(X) \tag{16}$$

Cumulative lead-lag transform

We now present two transformations that will be used together to recover statistical properties of $\hat{\gamma}$:

1. the $k-$cumulative sum transformation
2. the zero augmentation.

Lead lag and cumulative sum

Definition (k-cumulative sum transformation)

Let $\hat{\gamma} := \{X_i\}_{i \in \{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$. Define the k-cumulative sum transformation C_k as,

$$C_k(\hat{\gamma}) := \{X_1^k, X_1^k + X_2^k, \dots, \sum_{i=1}^{n} X_i^k\} \tag{17}$$

Remark

$C_k(\hat{\gamma})$ preserves the length of the sequence.

Zero augmentation

Definition (Zero augmentation)

Let $\hat{\gamma} := \{X_i\}_{i \in \{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$. Define the **zero augmentation** as $A : \{X_1, X_2, \dots\} \mapsto \{0, X_1, X_2, \dots\}$.

Remark

The zero augmentation makes the signature sensitive to translations of the time series (if path is translated, the first order term clearly changes).

Cumulative lead-lag transform

We will show how we recover statistical properties of interest using the k-cumulative lead-lag transform Λ^k

Definition (Cumulative lead-lag transform)

Let $\hat{\gamma} := \{X_i\}_{i \in \{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$. Define the k-**cumulative lead lag transformation** as

$$\Lambda^k(\hat{\gamma}) := (C_k \circ A)(\hat{\gamma}) \tag{18}$$

$$= \{0, X_1^k, X_1^k + X_2^k, \dots, \sum_{i=1}^{n-1} X_i^k\} \tag{19}$$

Recovering statistical properties of $\hat{\gamma}$ with cumulative lead-lag

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Lemma (Cumulative lead lag and statistical properties)

Let $\hat{\gamma} := \{X_i\}_{i\in\{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$ and $\Lambda^k(\hat{\gamma}) := \{\tilde{X}_i^j\}_{i\in\{0,\dots,n\},j\in\{1,2\}}$ its 1-cumulative lead lag transformation. Then, ***the total increment $\Delta\Lambda^1(\hat{\gamma})$ and quadratic variation $QV(\Lambda^1(\hat{\gamma}))$ of $\Lambda^1(\hat{\gamma})$ characterise the mean and variance of $\hat{\gamma}$,***

$$\Delta\Lambda^1(\hat{\gamma})_j = \sum_{i=1}^{n} X_i, \quad j \in \{1,2\}, \tag{20}$$

$$QV(\Lambda^1(\hat{\gamma}))_j = \sum_{i=0}^{N-1} (\tilde{X}_{i+1}^j - \tilde{X}_i^j)^2 = \sum_{i=1}^{N} X_i^2, \quad j \in \{1,2\}. \tag{21}$$

Corollary (Cumulative lead-lag to recover mean and variance)

In particular,

$$Mean(X) = \frac{\Delta\Lambda^1(\hat{\gamma}))}{N} \tag{22}$$

$$Var(X) = \frac{1}{N}\left(QV(\Lambda^1(\hat{\gamma}) - \frac{1}{N}(\Delta\Lambda^1(\hat{\gamma})^2\right) \tag{23}$$

Outline

Signature as statistical properties

Let $\hat{\gamma} := \{X_i\}_{i\in\{1,\dots,n\}}$ be a set of n points in $\mathbb{R}^d$ and $\Lambda^1(\hat{\gamma}) := \{\tilde{X}_i^j\}_{i\in\{0,\dots,n\},j\in\{1,2\}}$ its 1-cumulative lead lag transformation. Denote the 2-truncated signature of the latter path as

$$S(\Lambda^1(\hat{\gamma})|_{L=2} = \left(1, S^{1;1}, S^{1;2}, S^{2;11}, S^{2;12}, S^{2;21}, S^{2;22}\right) \tag{24}$$

Lemma (Truncated signature of 1-cumulative lead lag)

The signature of the 1*-cumulative lead lag transformed path satisfies*

$$S^{1;1} = S^{1;2} = \sum_{i=1}^{N} X_i \tag{25}$$

$$S^{2;11} = S^{2;22} = \frac{1}{2}\left(\sum_{i=1}^{n} X_i^2\right) \tag{26}$$

$$S^{2;12} = \frac{1}{2}\left[\left(\sum_{i}^{n} X_i\right)^2 + \sum_{i}^{n} X_i^2\right] \tag{27}$$

$$S^{2;21} = \frac{1}{2}\left[\left(\sum_{i}^{n} X_i\right)^2 - \sum_{i}^{n} X_i^2\right] \tag{28}$$

Similar equations can be derived for the signature of the raw path $\hat{\gamma}$,

Lemma (Truncated signature of γ)

$$S^{1;1} = S^{1;2} = \sum_{i=1}^{N} X_{i+1} - X_i \tag{29}$$

$$S^{2;11} = S^{2;22} = \frac{1}{2}\left(\sum_{i=1}^{n}(X_{i+1} - X_i)\right)^2 \tag{30}$$

$$S^{2;12} = \frac{1}{2}\left[\left(\sum_{i}^{n}(X_{i+1} - X_i)\right)^2 + \sum_{i}^{n}(X_{i+1} - X_i)\right] \tag{31}$$

$$S^{2;21} = \frac{1}{2}\left[\left(\sum_{i}^{n}(X_{i+1} - X_i)\right)^2 + \sum_{i}^{n}(X_{i+1} - X_i)\right] \tag{32}$$

$$\tag{33}$$

Using the above equations, simple algebra yields the following relationships:

Corollary (Signature as statistical properties)

$$Mean(X) = \frac{1}{n} S^{1;1} \tag{34}$$

$$Var(X) = -\frac{n+1}{n^2} S^{2;1,2} + \frac{n-1}{n^2} S^{2;21} \tag{35}$$

Bibliography