

Chapter 5

5.1 Areas and Distance

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x]$$

$$A = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} [f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x]$$

$$A = \lim_{n \rightarrow \infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x]$$

- If function is increasing, use left endpoints for underestimation, right endpoints for overestimation. If function is decreasing, use right endpoints for underestimation, left endpoints for overestimation.
- Take the height of the i th rectangle to be the value of f at any number x_i^* in the i th subinterval $[x_{i-1}, x_i]$. The numbers $x_1^*, x_2^*, \dots, x_n^*$ are called **sample points**.

- $\Delta x = \frac{b - a}{n}$

- **Formula for the sum of the squares:** $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

- Use **sigma notation** to write sums with many terms more compactly. For example:

$$\sum_{i=1}^n f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

- The formula for the sum of the squares can be rewritten:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

- The area formulas can be rewritten:

Right Endpoints: $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$

Left Endpoints: $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1})\Delta x$

Midpoints $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$

5.2 The Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \dots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

- The sum $\sum_{i=1}^n f(x_i^*)\Delta x$ is called a **Riemann sum**
- A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x -axis and below the graph of f , and A_2 is the area of the region below the X -axis and above the graph of f .

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \dots + f(\bar{x}_n)]$$

where $\Delta x = \frac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Properties of the Integral

$$\int_a^b c \, dx = c(b - a), \quad \text{where } c \text{ is any constant}$$

$$\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

$$\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx, \quad \text{where } c \text{ is any constant}$$

$$\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

$$\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$$

Comparison Properties of the Integral

If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx \geq 0$$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$$

If we reverse a and b , then Δx changes from $\frac{(b - a)}{n}$ to $\frac{(a - b)}{n}$.

Therefore:

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$

If $a = b$, then $\Delta x = 0$ and so:

$$\int_a^a f(x) \, dx = 0$$

Formulas for sums of powers

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Rules for sigma notation

$$\sum_{i=1}^n c = nc$$

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f , that is, a function such that $F' = f$

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f , that is, $F' = f$

5.4 Indefinite Integrals and the Net Change Theorem

- The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus $\int f(x)dx = F(x)$ means $F'(x) = f(x)$
- Distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a *number*, whereas an indefinite integral $\int f(x)dx$ is a *function* (or family of functions).
- If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = \int f(x)dx \Big|_a^b$

Table of Indefinite Integrals

$$\int cf(x)dx = c \int f(x)dx$$

$$\int [f(x) + g(x)]dx =$$

$$\int k dx = kx + C$$

$$\int f(x) + \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

5.5 The Substitution Rule

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even [$f(-x) = f(x)$], then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

(b) If f is odd [$f(-x) = -f(x)$], then $\int_{-a}^a f(x) dx = 0$