Chapter 4

4.1 Maximum and Minimum Values

Let c be a number in the domain D of a function f. Then $f(c)$ is the
- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

- An absolute maximum or minimum is also called a **global** maximum or minimum.
- The maximum and minimum values of f are called **extreme values** of f.

The number $f(c)$ is a
- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

- If we say that something is true **near** c, we mean that it is true on some **open interval** containing c.

The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

- An extreme value can be taken on more than once.
- A function need not posess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

- Even when $f'(c) = 0$ there need not be a maximum or minimum at c. Furthermore, there may be an extreme value even when $f'(c)$ does not exist. Such numbers are called **critical numbers**.

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

- If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the *critical numbers* of f in (a, b).
2. Find the values of f at the *endpoints* of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

4.2 The Mean Value Theorem

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

- Recall that the slope of the secant line AB of points $A(a, f(a))$ and $B(b, f(b))$ is $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$ and $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$.

Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

4.3 How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is *increasing* on that interval.

(b) If $f'(x) < 0$ on an interval, then f is *decreasing* on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive or negative on both sides of c), then f has no local maximum or minimum at c.

Definition of Concavity If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Concavity Test

(a) If $f"(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f"(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition of the Inflection Point A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose $f"$ is continuous near c.

(a) If $f'(c) = 0$ and $f"(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f"(c) < 0$, then f has a local maximum at c.