Chapter 1

1.1 Functions and Models

- A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.
- **The Vertical Line Test** A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.
- **Piecewise defined functions** are functions that are defined by different formulas in different parts of their domains

If f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is an **even function**.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**.

A function f is called **increasing** on an interval I if

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I

It is called **decreasing** on I if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I

1.2 Mathematical Models

Linear Models

If we say that y is a **linear function** of x, we mean that the graph of the function is a line. This means we can use the slope-intercept form of the equation to write a formula for the function as

$$y = f(x) = mx + b$$

A characteristic feature of linear functions is that they grow at a constant rate.

Polynomials

- A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1}x^{n+1} + ... + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, ...a_n$ are constants called the **coefficients** of the polynomial.

- The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$.
- If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n.
- A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a **linear function**.
- A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the paraboly $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$.
- A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d \quad a \neq 0$ and is called a **cubic function**.

Power Functions

- A function of the form

$$f(x) = x^a$$

where a is a constant, is called a **power function**.

- ***a* = *n*, where *n* is a positive integer**.
  - The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. As n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$.
- ***a* = 1/*n*, where *n* is a positive integer**
  - The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For even values of n, the domain is $[0, \infty)$ and its graph is the upper half of the parabola $x = y^2$. For odd values of n, the domain is $\mathbb{R}$.
- ***a* = −1**
  - The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is a hyperbola with the coordinate axes as its asymptotes.

Rational Functions

- A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain consists of all values of x such that $Q(x) \neq 0$.

Algebraic Functions

- A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.
- Any rational function is automatically an algebraic function.

Trigonometric Functions

- For both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus for all values of x, we have

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1$$

or in terms of absolute values, $\quad |\sin x| \leq 1 \qquad |\cos x| \leq 1$

- The zeros of the sine function occur at the integer multiples of π, that is $\sin x = 0$ when $x = n\pi$ and n is an integer.
- An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

- The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

It is undefined whenever $\cos x = 0$. Its range is $(-\infty, \infty)$. It has period π:

$$\tan(x + \pi) = \tan x$$

- The remaining trig functions cosecant, secant, and cotangent are the reciprocals of the sine, cosine, and tangent functions.

Exponential Functions

- **Exponential functions** are the functions of the form

$$f(x) = a^x$$

where the base a is a positive constant.

- The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$

Logarithmic Functions

- The **logarithmic functions**

$$f(x) = \log_a x$$

where the base a is a positive constant, are the inverse functions of the exponential functions.

- The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

1.3 Transformations of Functions

Vertical and Horizontal Shifts

Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting

Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

1.5 Exponential Functions

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant.

Law of Exponents

If a and b are positive numbers and x and y are any real numbers, then

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

- The choice of base a is influenced by the way the graph of $y = a^x$ crosses the y-axis
- The **natural exponential function**, or e, is a base such that the slope of the tangent line to $y = a^x$ at $(0, 1)$ is exactly 1.

1.6 Inverse functions and Logarithms

A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

- **Caution:** Do not mistake the -1 in f^{-1} for an exponent. $f^{-1}(x)$ does *not* mean $\dfrac{1}{f(x)}$
- The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y.

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

By substituting for y and x, we get the following **cancellation equations**:

$$f^{-1}(f(x)) = x \text{ for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \text{ for every } x \text{ in } B$$

How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y.
The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Logarithmic Functions

- If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test.
- It therefore has an inverse function f^{-1} which is called the **logarithmic function with base** a and is denoted by $\log_a$
- If we use the formulation of an inverse function

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

then we have

$$\log_a x = y \Leftrightarrow a^y = x$$

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

- The cancellation equations, when applied to the functions $f(x) = a^x$ and $f^{-1}(x) = log_a x$, become

$$\log_a(a^x) = x \text{ for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \text{ for every } x > 0$$

- The logarithmic function log_a has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$
- Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r\ \log_a x$ where r is any real number

Natural Logarithms

- The most convenient choice of a base is the number e, called the **natural logarithm**. It has special notation:

$$\log_e x = \ln\ x$$

- If we put $a = e$ and replace $\log_e$ with ln, then the defining properties of the natural logarithm function become

$$\ln x = y \Leftrightarrow e^y = x$$

$$\ln(e^x) = x \quad x \in \mathbb{R}$$

$$e^{\ln\ x} = x \quad x > 0$$

If we set $x = 1$, we get

$$\ln\ e = 1$$

Change of Base Formula For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln\ x}{\ln\ a}$$

Inverse Trigonometric Functions

- **Inverse sine function (arcsin)**
  - Since the definition of an inverse function says that

    $$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

    we have

    $$\sin^{-1} x = y \Leftrightarrow \sin y = x \ \text{ and } \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

  - The cancellation equations then become

    $$\sin^{-1}(\sin x) = x \ \text{ for } \ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

    $$\sin(\sin^{-1} x) = x \ \text{ for } \ -1 \leq x \leq 1$$

  - The inverse sine function has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

- **Inverse cosine function (arccos)**
  - The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

    $$\cos^{-1} x = y \Leftrightarrow \cos y = x \ \text{ and } \ 0 \leq y \leq \pi$$

  - The cancellation equations are

    $$\cos^{-1}(\cos x) = x \ \text{ for } \ 0 \leq x \leq \pi$$

    $$\cos(\cos^{-1} x) = x \ \text{ for } \ -1 \leq x \leq 1$$

  - The inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$

- **Inverse tangent function (arctan)**
  - The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$
  - Thus the inverse tangent function is defined as the inverse of the function $f(x) = \tan x, -\pi/2 < \pi/2$ and is denoted by $\tan^{-1}$ or arctan

    $$\tan^{-1} x = y \Leftrightarrow \tan y = x \ \text{ and } \ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

- **Other inverse trig functions**

  $$y = \csc^{-1} x (|x| \geq 1) \Leftrightarrow \csc y = x \ \text{ and } \ y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

  $$y = \sec^{-1} x (|x| \geq 1) \Leftrightarrow \sec y = x \ \text{ and } \ y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

  $$y = \cot^{-1} x (x \in \mathbb{R}) \Leftrightarrow \cot y = x \ \text{ and } \ y \in (0, \pi)$$