Chapter 2

2.1 The Tangent and Velocity Problems

The Tangent Problem

We can find the equation of the tangent line t as soon as we know its slope m. We need two points to compute the slope, but with one point P we can compute an approximation to m by choosing a nearby point $Q(a, f(a))$ and computing the slope m_{PQ} of the secant line PQ

The slope of the tangent line is the *limit* of the slopes of the secant lines, expressed as

$$\lim_{P \to Q} m_{PQ} = m$$

Then, we use the point-slope form of the equation of a line to write the equation of the tangent line through (x, y) as

$$y - y_1 = m(x - x_1)$$

This can also be represented as

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The Velocity Problem

There is a close connection between the tangent problem and the problem of finding velocities. If we consider the points $P(a, f(a))$ and $Q(a + h, f(a + h))$, the slope of the secant line PQ is the same as the average velocity over the time interval $[a, a + h]$. Therefore the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

The **average velocity** is equal to the slope of the secant line

$$\bar{v} = \frac{f(a + h) - f(a)}{h} \qquad \text{or} \qquad \bar{v} = \frac{\Delta s}{\Delta t}$$

The **instantaneous velocity** is equal to the slope of the tangent line

$$v = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad \text{or} \qquad v = \frac{ds}{dt}$$

2.2 The Limit of a Function

Definition of Limits

Suppose $f(x)$ is defined when x is near the number a (this means that f is defined on some open interval that contains a, except possibly at a itself). Then we write

$$\lim_{x\to a} f(x) = L$$

which is read as "$f(x)$ approaches L as x approaches a".

Left and Right Hand Limits

The left-hand limit of $f(x)$ as x approaches a (or, the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x < a$.

$$\lim_{x\to a^-} f(x) = L$$

Right hand limits are represented as

$$\lim_{x\to a^+} f(x) = L$$

By combining the definitions of one-sided and two-sided limits, we see that

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large(positive) or large(negative) by taking x sufficiently close, but not equal to a.

(Note that the symbol ∞ does not represent a number, and thus the Limit Laws cannot be applied to infinite limits)

Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

2.3 Calculating Limits Using the Limit Laws

Limit Laws

Suppose that c is a constant and these limits exist

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

Then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[c\, f(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)\, g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

    (if n is even, assume that $a > 0$)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

    (if n is even, assume that $a > 0$)

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = g(x)$ provided the limits exist

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

2.5 Continuity

A function f is **continuous** at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

This implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

We say that f has a **discontinuity** at a if f is not continuous at a

A function f is **continuous from the right** at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continous from the right* or *continuous from the left*).

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

a. Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

b. Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

The following types of functions are continuous at every number in their domains:

polynomials	rational functions	root functions
trigonometric functions	inverse trig functions	
exponential functions	logarithmic functions	

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

2.6 Limits at Infinity; Horizontal Asymptotes

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Similarly, let f be a function defined on some interval $(a, -\infty)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Examples

An example of a curve with two horizontal asymptotes is $y = tan^{-1}x$

$$\lim_{x \to -\infty} tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} tan^{-1}x = \frac{\pi}{2}$$

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote.

$$\lim_{x \to -\infty} e^x = 0$$

2.7 Derivatives and Rates of Change

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$ where $x \neq a$, and compute the slope of the secant line PQ. Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P.

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Velocity

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. We define the **velocity** (or **instantaneous velocity** $v(a)$ at time $t = a$ to be the limit of these average velocities

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P.

Derivatives

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

In other words, the tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

2.8 The Derivative as a Function

Instead of considering the derivative of function f at a fixed number a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

by replacing a by a variable x we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We can regard f' as a new function, called the **derivative of** f and defined that this equation, and we know that the value of f' at x, $f'(x)$ can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The domain of f' is the set $\{x \mid f'(x)\text{exists}\}$ and may be smaller than the domain of f.

Leibniz Notation

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D\,f(x) = D_x f(x)$$

The symbols D and d/dx are **differention operators** because they indicate the operation of **differention**. It is especially useful when used in conjunction with increment notation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x \to a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x \to a}$$

which is a synonym for $f'(a)$

A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

If f is differentiable at a, then f is continuous at a. The converse is false; that is, there are functions that are continuous but not differentiable (such as the Weierstrass function).

How can a Function Fail to be Differentiable?

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there (in trying to compute $f'(a)$, we find that the left and right limits are different).

A second possibility is that if f is not continuous at a, then f is not differentiable at a.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

This means that the tangent lines become steeper and steeper as $x \to a$.

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Note that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows:

If $s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the cuve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

The third derivative of the position function (the derivative of the acceleration function) is called the **jerk** and is the rate of change of acceleration.

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$