Chapter 5

5.1 Areas and Distance

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x]$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1})\Delta x]$$

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x]$$

- If function is increasing, use left endpoints for underestimation, right endpoints for overestimation. If function is decreasing, use right endpoints for underestimation, left endpoints for overestimation.
- Take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. The numbers $x_1^*, x_2^*, ..., x_n^*$ are called **sample points**.
- $\Delta x = \dfrac{b-a}{n}$
- **Formula for the sum of the squares**: $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- Use **sigma notation** to write sums with many terms more compactly. For example:

  $$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x$$

- The formula for the sum of the squares can be rewritten:

  $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- The area formulas can be rewritten:

  Right Endpoints: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

  Left Endpoints: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$

  Midpoints $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

5.2 The Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, ..., x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum**
- A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the X-axis and above the graph of f.

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$

Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + ... + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \quad = \quad$ midpoint of $[x_{i-1}, x_i]$

Properties of the Integral

$$\int_a^b c\,dx = c(b-a), \quad \text{where } c \text{ is any constant}$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx, \quad \text{where } c \text{ is any constant}$$

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Comparison Properties of the Integral

If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0$$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

If we reverse a and b, then Δx changes from $\dfrac{(b-a)}{n}$ to $\dfrac{(a-b)}{n}$.
Therefore:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

If $\quad a = b, \quad$ then $\quad \Delta x = 0 \quad$ and so:

$$\int_a^a f(x)dx = 0$$

Formulas for sums of powers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Rules for sigma notation

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

The Fundamental Theorem of Calculus Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$

5.4 Indefinite Integrals and the Net Change Theorem

- The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus $\int f(x)dx = F(x)$ means $F'(x) = f(x)$
- Distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a *number*, whereas an indefinite integral $\int f(x)dx$ is a *function* (or family of functions).
- If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$

Table of Indefinite Integrals

$$\int cf(x)dx = c\int f(x)dx \qquad \int [f(x) + g(x)]dx =$$

$$\int k\ dx = kx + C \qquad \int f(x) + \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\ \ (n \neq -1) \qquad \int \frac{1}{x}dx = ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{ln\ a} + C$$

$$\int sinx\ dx = -cosx + C \qquad \int cosx\ dx = sinx + C$$

$$\int sec^2x\ dx = tanx + C \qquad \int csc^2x\ dx = -cotx + C$$

$$\int secx\ tanx\ dx = secx + C \qquad \int cscx\ cotx\ dx = -cscx + C$$

$$\int \frac{1}{x^2+1}dx = tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + C$$

$$\int sinhx\ dx = coshx + C \qquad \int coshx\ dx = sinhx + C$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

5.5 The Substitution Rule

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)dx = \int f(u)du$$

The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$