

# Chapter 1

## 1.1 Functions and Models

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- A **function**  $f$  is a rule that assigns to each element  $x$  in a set  $D$  exactly one element, called  $f(x)$ , in a set  $E$ .
- **The Vertical Line Test** A curve in the  $xy$ -plane is the graph of a function of  $x$  if and only if no vertical line intersects the curve more than once.
- **Piecewise defined functions** are functions that are defined by different formulas in different parts of their domains

If  $f$  satisfies  $f(-x) = f(x)$  for every number  $x$  in its domain, then  $f$  is an **even function**.

If  $f$  satisfies  $f(-x) = -f(x)$  for every number  $x$  in its domain, then  $f$  is called an **odd function**.

A function  $f$  is called **increasing** on an interval  $I$  if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on  $I$  if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

## 1.2 Mathematical Models

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### Linear Models

- If we say that  $y$  is a **linear function** of  $x$ , we mean that the graph of the function is a line.
- This means we can use the slope-intercept form of the equation to write a formula for the function as

$$y = f(x) = mx + b$$

- A characteristic feature of linear functions is that they grow at a constant rate.

## Polynomials

- A function  $P$  is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n+1} + \dots + a_2 x^2 + a_1 x + a_0$$

where  $n$  is a nonnegative integer and the numbers  $a_0, a_1, \dots, a_n$  are constants called the **coefficients** of the polynomial.

- The domain of any polynomial is  $\mathbb{R} = (-\infty, \infty)$ .
- If the leading coefficient  $a_n \neq 0$ , then the **degree** of the polynomial is  $n$ .
- A polynomial of degree 1 is of the form  $P(x) = mx + b$  and so it is a **linear function**.
- A polynomial of degree 2 is of the form  $P(x) = ax^2 + bx + c$  and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola  $y = ax^2$ . The parabola opens upward if  $a > 0$  and downward if  $a < 0$ .
- A polynomial of degree 3 is of the form  $P(x) = ax^3 + bx^2 + cx + d$   $a \neq 0$  and is called a **cubic function**.

## Power Functions

- A function of the form

$$f(x) = x^a$$

where  $a$  is a constant, is called a **power function**.

- **$a = n$ , where  $n$  is a positive integer.**
  - The general shape of the graph of  $f(x) = x^n$  depends on whether  $n$  is even or odd. If  $n$  is even, then  $f(x) = x^n$  is an even function and its graph is similar to the parabola  $y = x^2$ . If  $n$  is odd, then  $f(x) = x^n$  is an odd function and its graph is similar to that of  $y = x^3$ . As  $n$  increases, the graph of  $y = x^n$  becomes flatter near 0 and steeper when  $|x| \geq 1$ .
- **$a = 1/n$ , where  $n$  is a positive integer**
  - The function  $f(x) = x^{1/n} = \sqrt[n]{x}$  is a **root function**. For even values of  $n$ , the domain is  $[0, \infty)$  and its graph is the upper half of the parabola  $x = y^2$ . For odd values of  $n$ , the domain is  $\mathbb{R}$ .
- **$a = -1$** 
  - The graph of the **reciprocal function**  $f(x) = x^{-1} = 1/x$  is a hyperbola with the coordinate axes as its asymptotes.

## Rational Functions

- A **rational function**  $f$  is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P$  and  $Q$  are polynomials.

- The domain consists of all values of  $x$  such that  $Q(x) \neq 0$ .

## Algebraic Functions

- A function  $f$  is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.
- Any rational function is automatically an algebraic function

## Trigonometric Functions

- For both the sine and cosine functions the domain is  $(-\infty, \infty)$  and the range is the closed interval  $[-1, 1]$ . Thus for all values of  $x$ , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

or in terms of absolute values,  $|\sin x| \leq 1$   $|\cos x| \leq 1$

- The zeros of the sine function occur at the integer multiples of  $\pi$ , that is  $\sin x = 0$  when  $x = n\pi$  and  $n$  is an integer.
- An important property of the sine and cosine functions is that they are periodic functions and have period  $2\pi$ . This means that, for all values of  $x$ ,

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

- The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

It is undefined whenever  $\cos x = 0$ . Its range is  $(-\infty, \infty)$ . It has period  $\pi$ :

$$\tan(x + \pi) = \tan x$$

- The remaining trig functions cosecant, secant, and cotangent are the reciprocals of the sine, cosine, and tangent functions.

### Exponential Functions

- **Exponential functions** are the functions of the form

$$f(x) = a^x$$

where the base  $a$  is a positive constant.

- The domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$

### Logarithmic Functions

- The **logarithmic functions**

$$f(x) = \log_a x$$

where the base  $a$  is a positive constant, are the inverse functions of the exponential functions.

- The domain is  $(0, \infty)$ , the range is  $(-\infty, \infty)$ , and the function increases slowly when  $x > 1$ .

## 1.3 Transformations of Functions

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### Vertical and Horizontal Shifts

Suppose  $c > 0$ . To obtain the graph of

$y = f(x) + c$ , shift the graph of  $y = f(x)$  a distance  $c$  units upward

$y = f(x) - c$ , shift the graph of  $y = f(x)$  a distance  $c$  units downward

$y = f(x - c)$ , shift the graph of  $y = f(x)$  a distance  $c$  units to the right

$y = f(x + c)$ , shift the graph of  $y = f(x)$  a distance  $c$  units to the left

## Vertical and Horizontal Stretching and Reflecting

Suppose  $c > 1$ . To obtain the graph of

$y = cf(x)$ , stretch the graph of  $y = f(x)$  vertically by a factor of  $c$

$y = (1/c)f(x)$ , shrink the graph of  $y = f(x)$  vertically by a factor of  $c$

$y = f(cx)$ , shrink the graph of  $y = f(x)$  horizontally by a factor of  $c$

$y = f(x/c)$ , stretch the graph of  $y = f(x)$  horizontally by a factor of  $c$

$y = -f(x)$ , reflect the graph of  $y = f(x)$  about the  $x$ -axis

$y = f(-x)$ , reflect the graph of  $y = f(x)$  about the  $y$ -axis

Given two functions  $f$  and  $g$ , the **composite function**  $f \circ g$  (also called the **composition** of  $f$  and  $g$ ) is defined by

$$(f \circ g)(x) = f(g(x))$$

## 1.5 Exponential Functions

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An **exponential function** is a function of the form

$$f(x) = a^x$$

where  $a$  is a positive constant.

### Law of Exponents

If  $a$  and  $b$  are positive numbers and  $x$  and  $y$  are any real numbers, then

$$a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y} \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x$$

- The choice of base  $a$  is influenced by the way the graph of  $y = a^x$  crosses the  $y$ -axis
- The **natural exponential function**, or  $e$ , is a base such that the slope of the tangent line to  $y = a^x$  at  $(0, 1)$  is exactly 1.

## 1.6 Inverse functions and Logarithms

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A function  $f$  is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

### Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Let  $f$  be a one-to-one function with domain  $A$  and range  $B$ . Then its **inverse function**  $f^{-1}$  has domain  $B$  and range  $A$  and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any  $y$  in  $B$ .

- **Caution:** Do not mistake the  $-1$  in  $f^{-1}$  for an exponent.  $f^{-1}(x)$  does *not* mean  $\frac{1}{f(x)}$
- The reciprocal  $1/f(x)$  could, however, be written as  $[f(x)]^{-1}$

The letter  $x$  is traditionally used as the independent variable, so when we concentrate on  $f^{-1}$  rather than on  $f$ , we usually reverse the roles of  $x$  and  $y$ .

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

By substituting for  $y$  and  $x$ , we get the following **cancellation equations**:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

## How to Find the Inverse Function of a One-to-One Function $f$

**Step 1** Write  $y = f(x)$ .

**Step 2** Solve this equation for  $x$  in terms of  $y$  (if possible).

**Step 3** To express  $f^{-1}$  as a function of  $x$ , interchange  $x$  and  $y$ .

The resulting equation is  $y = f^{-1}(x)$ .

- The graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  about the line  $y = x$ .

## Logarithmic Functions

- If  $a > 0$  and  $a \neq 1$ , the exponential function  $f(x) = a^x$  is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test.
- It therefore has an inverse function  $f^{-1}$  which is called the **logarithmic function with base  $a$**  and is denoted by  $\log_a$
- If we use the formulation of an inverse function

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

then we have

$$\log_a x = y \Leftrightarrow a^y = x$$

Thus, if  $x > 0$ , then  $\log_a x$  is the exponent to which the base  $a$  must be raised to give  $x$ .

- The cancellation equations, when applied to the functions  $f(x) = a^x$  and  $f^{-1}(x) = \log_a x$ , become

$$\log_a(a^x) = x \text{ for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \text{ for every } x > 0$$

- The logarithmic function  $\log_a$  has domain  $(0, \infty)$  and range  $\mathbb{R}$ . Its graph is the reflection of the graph of  $y = a^x$  about the line  $y = x$
- Since  $\log_a 1 = 0$ , the graphs of all logarithmic functions pass through the point  $(1, 0)$ .

**Laws of Logarithms** If  $x$  and  $y$  are positive numbers, then

1.  $\log_a(xy) = \log_a x + \log_a y$

2.  $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3.  $\log_a(x^r) = r \log_a x$  where  $r$  is any real number

### Natural Logarithms

- The most convenient choice of a base is the number  $e$ , called the **natural logarithm**. It has special notation:

$$\log_e x = \ln x$$

- If we put  $a = e$  and replace  $\log_e$  with  $\ln$ , then the defining properties of the natural logarithm function become

$$\ln x = y \Leftrightarrow e^y = x$$

$$\ln(e^x) = x \quad x \in \mathbb{R}$$

$$e^{\ln x} = x \quad x > 0$$

If we set  $x = 1$ , we get

$$\ln e = 1$$

**Change of Base Formula** For any positive number  $a$  ( $a \neq 1$ ), we have

$$\log_a x = \frac{\ln x}{\ln a}$$



## Inverse Trigonometric Functions

- **Inverse sine function (arcsin)**

- Since the definition of an inverse function says that

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

we have

$$\sin^{-1}x = y \Leftrightarrow \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

- The cancellation equations then become

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \leq x \leq 1$$

- The inverse sine function has domain  $[-1, 1]$  and range  $[-\pi/2, \pi/2]$

- **Inverse cosine function (arccos)**

- The restricted cosine function  $f(x) = \cos x$ ,  $0 \leq x \leq \pi$ , is one-to-one and so it has an inverse function denoted by  $\cos^{-1}$  or  $\arccos$ .

$$\cos^{-1}x = y \Leftrightarrow \cos y = x \quad \text{and} \quad 0 \leq y \leq \pi$$

- The cancellation equations are

$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \leq x \leq \pi$$

$$\cos(\cos^{-1}x) = x \quad \text{for} \quad -1 \leq x \leq 1$$

- The inverse cosine function has domain  $[-1, 1]$  and range  $[0, \pi]$

- **Inverse tangent function (arctan)**

- The tangent function can be made one-to-one by restricting it to the interval  $(-\pi/2, \pi/2)$
- Thus the inverse tangent function is defined as the inverse of the function  $f(x) = \tan x$ ,  $-\pi/2 < \pi/2$  and is denoted by  $\tan^{-1}$  or  $\arctan$

$$\tan^{-1}x = y \Leftrightarrow \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

- **Other inverse trig functions**

$$y = \csc^{-1}x (|x| \geq 1) \Leftrightarrow \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x (|x| \geq 1) \Leftrightarrow \sec y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \cot^{-1}x (x \in \mathbb{R}) \Leftrightarrow \cot y = x \quad \text{and} \quad y \in (0, \pi)$$