Solution to the Inviscid Burgers' Equation by the Lax-Friedrichs Scheme*

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1 Introduction

2 Body

The problem we will be considering is

$$\begin{cases} u_t + uu_x = 0 \qquad x \in [-L, L] \qquad t \geq 0 \\ u(x,0) = u_0(x) \\ u(-L,t) = u(L,t) \end{cases} \tag{1}$$

We begin by discretizing the interval $[-L, L]$ by defining a mesh width $\Delta x = 2L/N$ so that

$$x_j = -L + j\Delta x \qquad x_{j+1/2} = x_j + \frac{\Delta x}{2}$$

We will also denote the time step by Δt_n so that

$$t_n = n\Delta t_n$$

The exact formula for Δt_n will remain undefined for now as it must be a variable time-step.

We denote the *pointwise values* of the true solution at the mesh point (x_j, t_n) by

$$u_j^n = u(x_j, t_n)$$

and the *cell average* of the true solution

$$\bar{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx - \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dx - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dx\right]$$

*Placeholder title!

We say that a method is in *conservation form* if

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_{j+1}^n)\right]$$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dx - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dx\right]$$

So the numerical flux function $\mathcal{F}$ plays the role of an average flux through $x_{j\pm1/2}$ over the time interval $[t_n, t_{n+1}]$

$$\mathcal{F}(U_j^n, U_{j+1}^n) \sim \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt \qquad \mathcal{F}(U_{j-1}^n, U_j^n) \sim \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt$$

2.1 Lax-Friedrichs

We have

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

We can write this in conservation form by taking

$$\mathcal{F}(U_j^n, U_{j+1}^n) := \frac{\Delta t}{2\Delta x}(U_j^n - U_{j+1}^n) + \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$$

3 Conclusion