Solution to the Inviscid Burgers' Equation by the Lax-Friedrichs Scheme*

Andre Gormann
agormann@sfu.ca

Ethan MacDonald
jem21@sfu.ca

1 Introduction

We have chosen to use the Lax-Friedrichs scheme to approximate solutions of the Inviscid Burgers equation. The scheme is defined by the update rule

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

The scheme combines upadtes in the spacial domain with time-stepping. Our reasoning for choosing the Lax-Friedrichs scheme is X Y Z. We will be using periodic boundary conditions because UMMMMMMMMMM WAVES.

2 Body

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_j^n)\right)$$

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

$$\mathcal{F}(U_j^n, U_{j+1}^n) := \frac{\Delta x}{2\Delta t}(U_j^n - U_{j+1}^n) + \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$$

$$U_j^n \sim \bar{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx$$

$$\mathcal{F}(U_j^n, U_{j+1}^n) \sim \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt$$

$$\mathcal{F}(\bar{u}, \bar{u}) = f(\bar{u})$$

$$u_t + f(u)_x = 0$$

*Placeholder title!

$$F^n_{j-1/2} = \mathcal{F}(U^n_{j-1}, U^n_j) \qquad F^n_{j+1/2} = \mathcal{F}(U^n_j, U^n_{j+1})$$

$$\frac{U^{n+1}_j - U^n_j}{\Delta t} + \frac{F^n_{j+1/2} - F^n_{j-1/2}}{\Delta x} = 0$$

3 Matrix Form of the Lax-Friedrichs Scheme

The Lax-Friedrichs scheme

$$U^{n+1}_j = \frac{1}{2}\left(U^n_{j-1} + U^n_{j+1}\right) - \frac{\Delta t}{2\Delta x}\left(f(U^n_{j+1}) - f(U^n_{j-1})\right)$$

can be converted into a matrix form

$$\vec{U}^{n+1} = A\vec{U}^n - B\vec{f}(\vec{U}^n)$$

where

$$A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B = \frac{\Delta t}{2\Delta x}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}$$

If we instead use the conservation form of the Lax-Friedrichs scheme

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U^n_j, U^n_{j+1}) - \mathcal{F}(U^n_{j-1}, U^n_j)\right)$$

$$\mathcal{F}(U^n_j, U^n_{j+1}) := \frac{\Delta x}{2\Delta t}(U^n_j - U^n_{j+1}) + \frac{1}{2}\left(f(U^n_j) + f(U^n_{j+1})\right)$$

we get the following matrix form

$$\vec{U}^{n+1} = \vec{U}^n - C\vec{\mathcal{F}}(\vec{U}^n)$$

$$\vec{\mathcal{F}}(\vec{U}^n) = D\vec{U}^n + E\vec{f}(\vec{U}^n)$$

where

$$
C = \frac{\Delta x}{\Delta t} \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & -1 \end{bmatrix},
$$

$$
D = \frac{\Delta x}{2\Delta t} \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}, \quad E = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}
$$

In either case, we can construct the matrices in MATLAB by using the *diag()* command on vectors containing the values of the non-zero diagonals, then filling in the values in the bottom left and top right corners as necessary, and lastly multiplying by the respective coefficient.

We will be using the matrices defined in this section to implement our periodic boundary conditions. The entries in the bottom left and top right corners of the matrices handle the cases where a value from beyond the periodic boundaries is needed.

4 Conclusion

References

Iserles, A. (2009). *A first course in the numerical analysis of differential equations*. Cambridge University Press.

LeVeque, R. J. (1992). *Numerical methods for conservation laws*. Birkhauser.

LeVeque, R. J. (2002). *Finite volume methods for hyperbolic problems*. Cambridge University Press.

Trefethen, L. N. (2000). *Spectral methods in matlab*. Society for Industrial; Applied Mathematics. https://doi.org/10.1137/1.9780898719598