Solving the Inviscid Burgers' Equation Numerically

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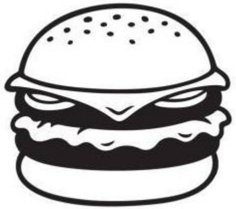
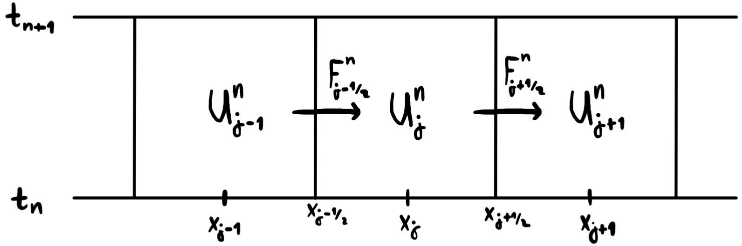
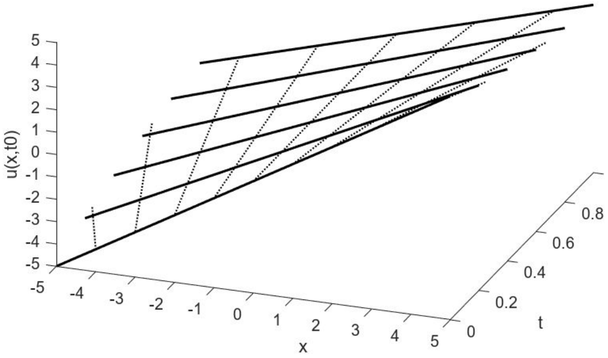


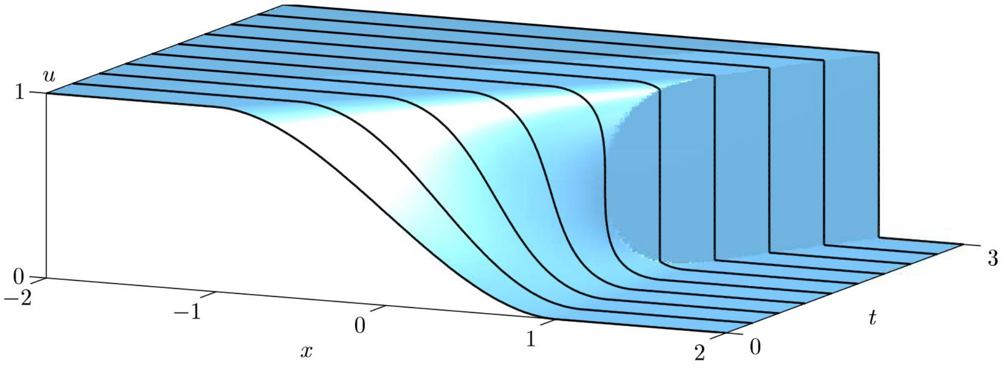




The Inviscid Burgers' Equation

- Modelling wave propagation
- Nonlinear Advection
- Shock formation (discontinuities)
- Applications include the modelling of fluids, sound waves, and traffic flow.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$



Trefethen. Shock Formation. 2011. The (Unfinished) PDE Coffee Table Book.

The (Viscous) Burgers' Equation

- $\epsilon > 0$ is the diffusion coefficient
- The diffusion term causes the wave to dissipate as it moves
- The diffusion term disallows the formation of shocks
- Sharp but non-vertical gradients
- Inviscid Burgers' arises when we let $\epsilon \to 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}$$

Conservation Laws

$$u_t + f(u)_x = 0 \qquad f(u) = \frac{1}{2}u^2$$

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)) \tag{5}$$

Conservation Laws

If we integrate (5) from time t_n to t_{n+1}, we get

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx - \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt \right]. \tag{10}$$

Dividing by Δx and applying (9) yields

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{\Delta x} \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt \right]. \tag{11}$$

Conservation Laws

$$\mathcal{F}(U_j^n, U_{j+1}^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt$$
$$\mathcal{F}(U_{j-1}^n, U_j^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt.$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_j^n)\right].$$

First-Order Godunov Scheme (REA)

- **Reconstruct** a piecewise polynomial p(x,tn) from the approximate cell averages

$$U_j^n \approx \overline{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)dx.$$

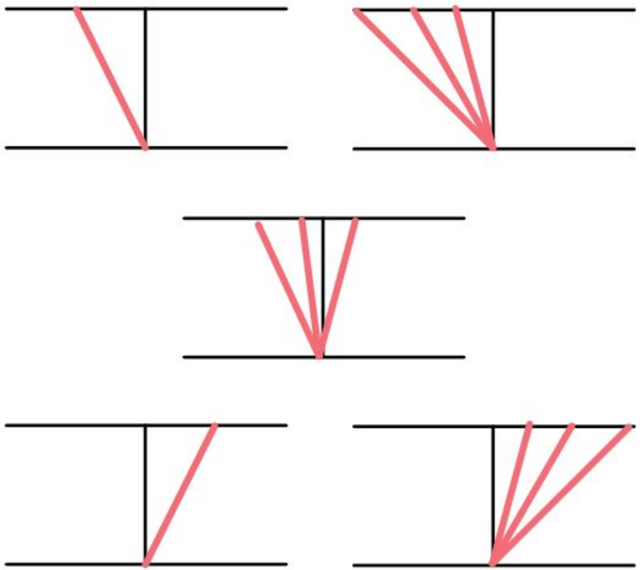
- **Evolve** the PDE from time tn to tn+1 using p(x,tn) as initial profile
- **Average** the new p(x,tn+1) to obtain new cell averages

$$U_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x,t_{n+1})dx.$$

First-Order Godunov Scheme (REA)

- We evolve by solving Riemann problems

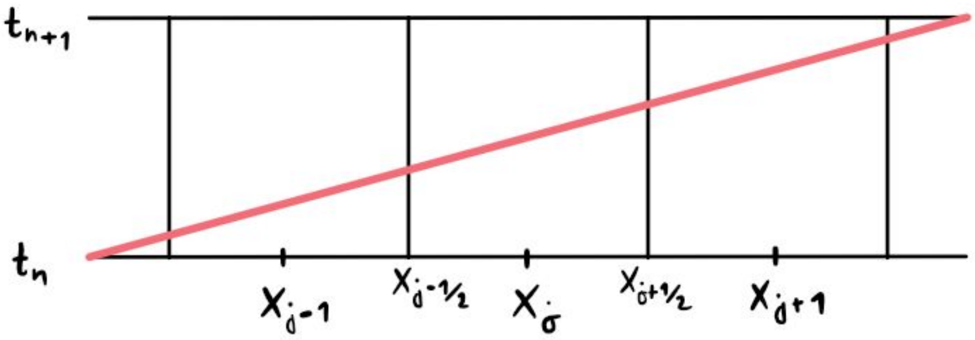
$$\mathcal{F}(u_L, u_R) = \begin{cases} \min_{u \in [u_L, u_R]} f(u) & u_L \leq u_R \\ \max_{u \in [u_R, u_L]} f(u) & u_L > u_R \end{cases}$$



Convergence

- CFL bounded by maximum wave-speed => variable time-step
- Bounds how far a characteristic can travel

$$\max |U_j^n| \frac{\Delta t_n}{\Delta x} \leq 1$$



Convergence

Theorem 12.3. *Suppose $Q^{(\Delta t)}$ is generated by a numerical method in conservation form with a Lipschitz continuous numerical flux, consistent with some scalar conservation law. If the method is TV-stable, i.e., if $\mathrm{TV}(Q^n)$ is uniformly bounded for all n, Δt with $\Delta t < \Delta t_0, n\,\Delta t \le T$, then the method is convergent, i.e., $\mathrm{dist}(Q^{(\Delta t)}, \mathcal{W}) \to 0$ as $\Delta t \to 0$.*

Riemann Problem

The Riemann problem is the IVP for Inviscid Burgers' with initial condition

$$u(x,0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x \geq 0 \end{cases}$$

<u>True Solutions:</u>

Shockwave Case: $u_L > u_R$

$$u(x,t) = \begin{cases} u_L & \text{if } x < st \\ u_R & \text{if } x \geq st \end{cases}$$

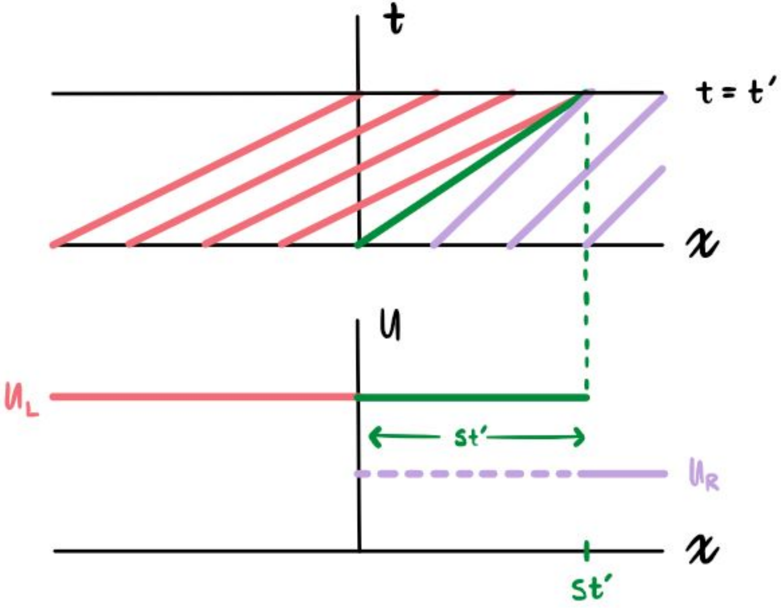
$$s = \frac{u_L + u_R}{2}$$

Rarefaction case: $u_L < u_R$

$$u(x,t) = \begin{cases} u_L & \text{if } x \leq u_L t \\ \frac{x}{t} & \text{if } u_L t < x < u_R t \\ u_R & \text{if } x \geq u_R t \end{cases}$$

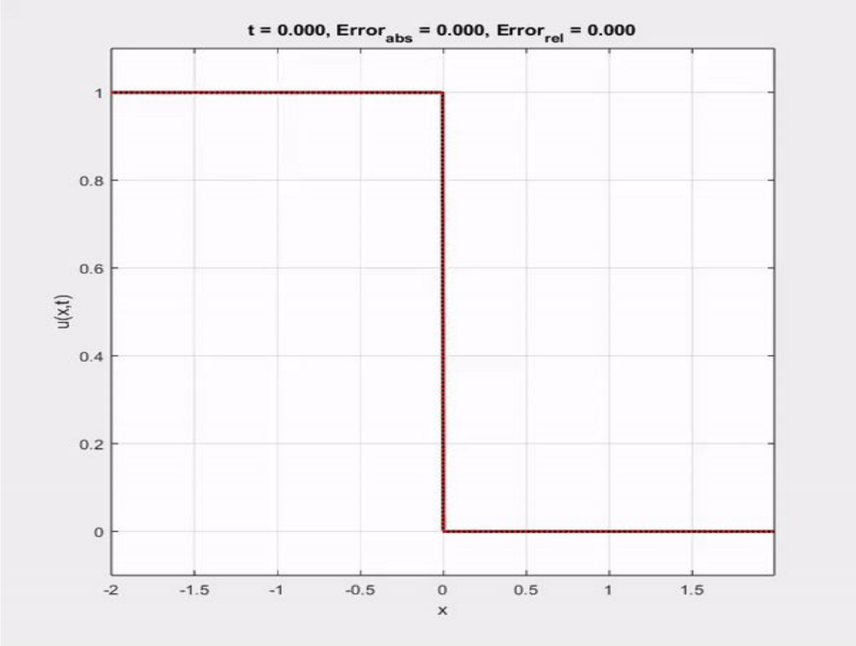
Riemann Shockwave Problem ($u_L > u_R$)

- u_L is moving faster than u_R
- u_L and u_R characteristic curves intersect
- Solution becomes discontinuous at the point in space where the curves intersect
- The discontinuity point moves through space as time progresses

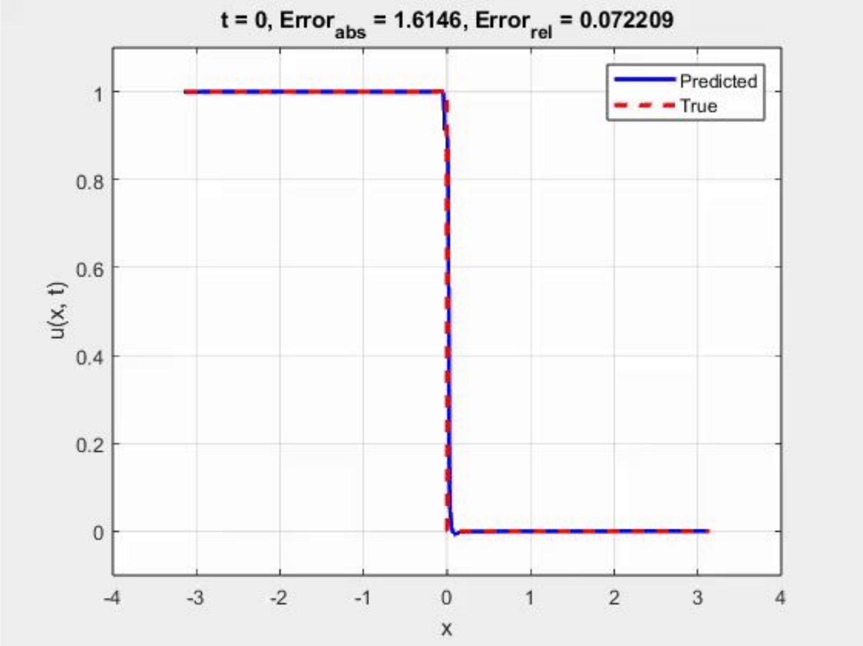


Riemann Shockwave Problem ($u_L > u_R$)

Numerical Solution

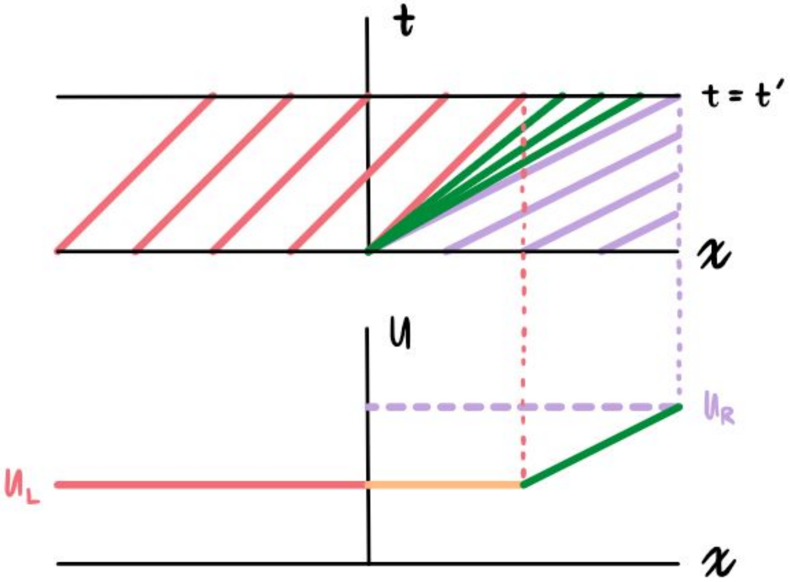


PINN Solution



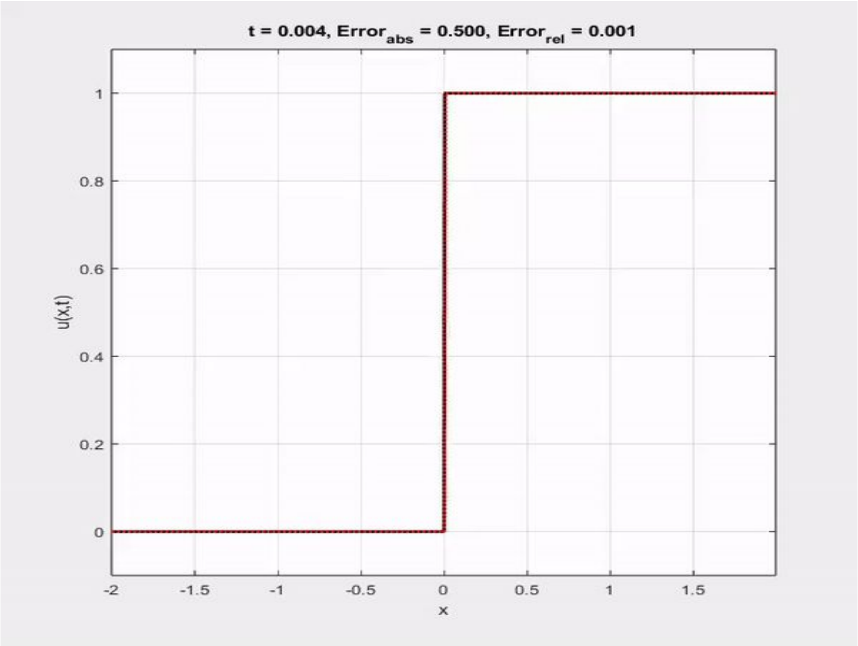
Riemann Rarefaction Wave Problem ($u_L < u_R$)

- u_L is moving slower than u_R
- u_L and u_R characteristic curves never intersect, so a shock never forms
- The characteristic curves become increasingly rarified as time progresses
- The solution is sloped between u_L and u_R
- This slope becomes longer and steeper as time progresses

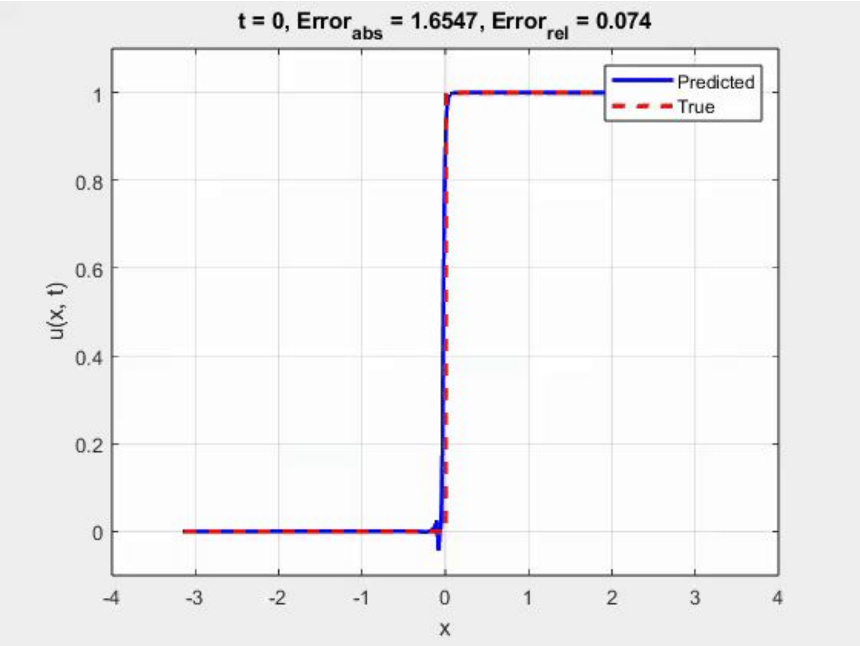


Riemann Rarefaction Wave Problem ($u_L < u_R$)

Numerical Solution



PINN Solution

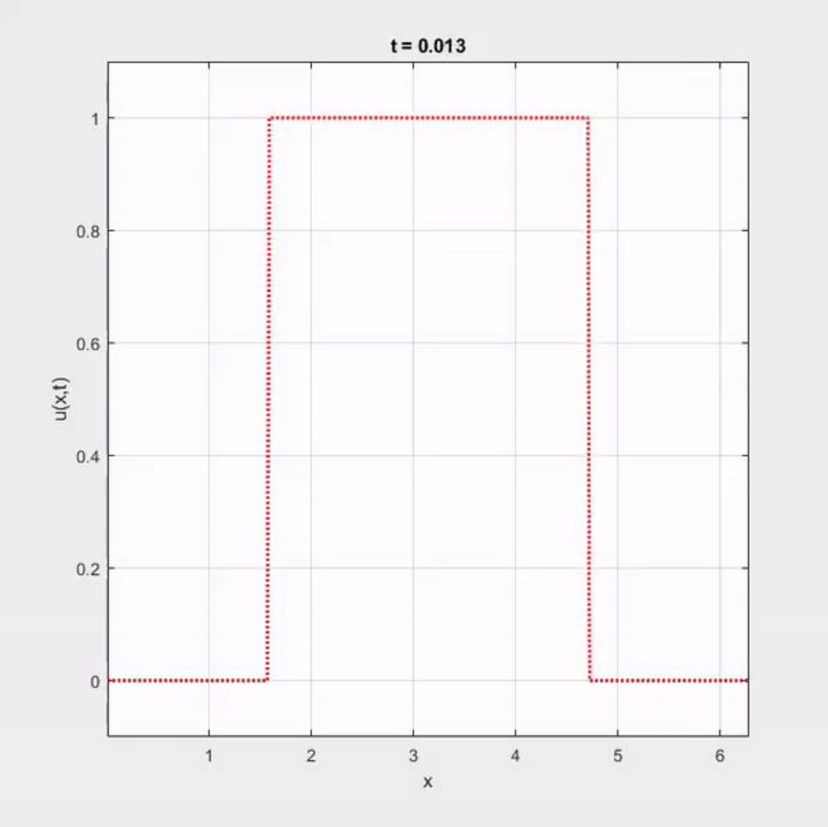


Square Wave Problem

IVP:
$$\begin{cases} u_t + uu_x = 0, \quad x \in [0, 2\pi],\ t > 0 \\ u(x,0) = u_0(x) \\ u(0,t) = u(2\pi, t), \quad t > 0 \end{cases}$$

Where

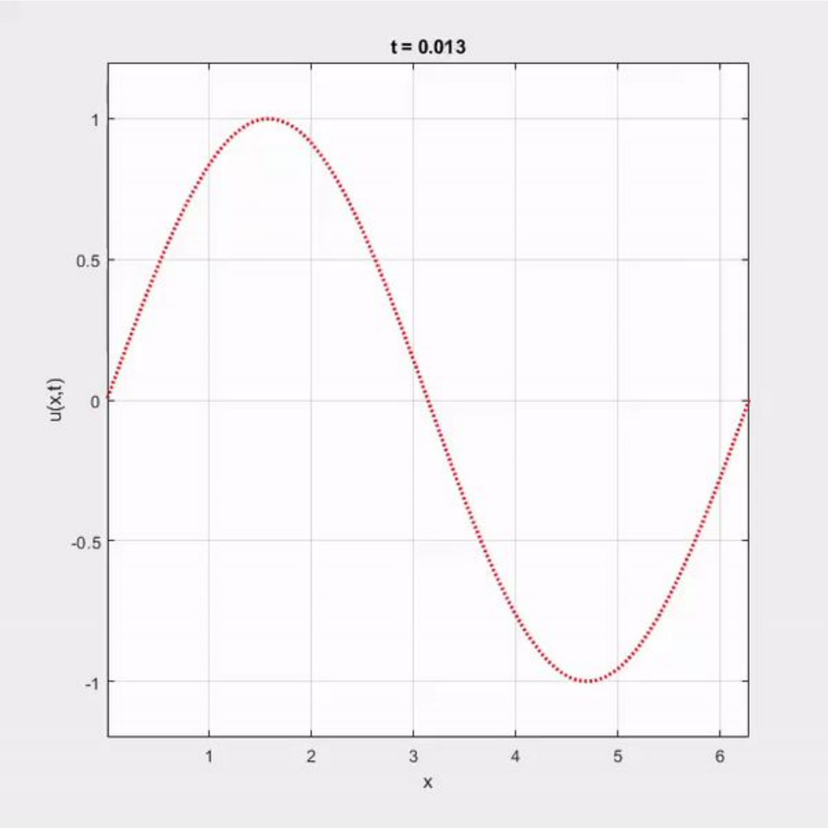
$$u_0(x) = \begin{cases} 1 & x \in [\pi/2, 3\pi/2] \\ 0 & \text{otherwise} \end{cases}$$



Sine Wave Problem

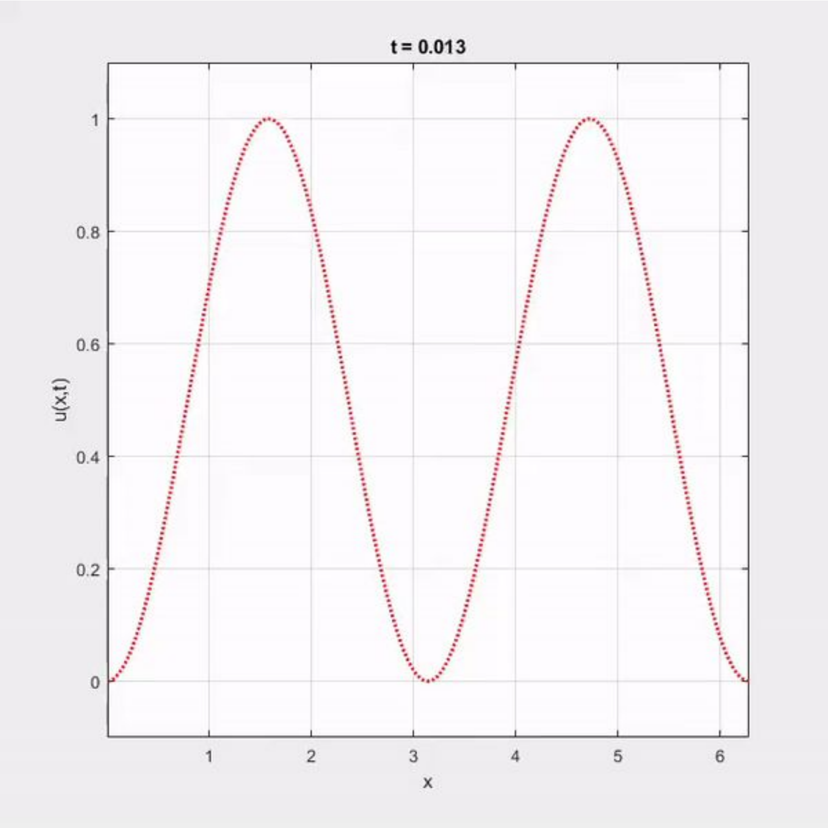
IVP:

$$\begin{cases} u_t + uu_x = 0, \quad x \in [0, 2\pi],\ t > 0 \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi, t), \quad t > 0 \end{cases}$$

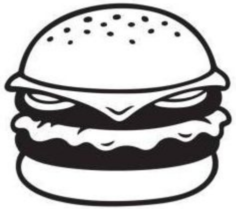


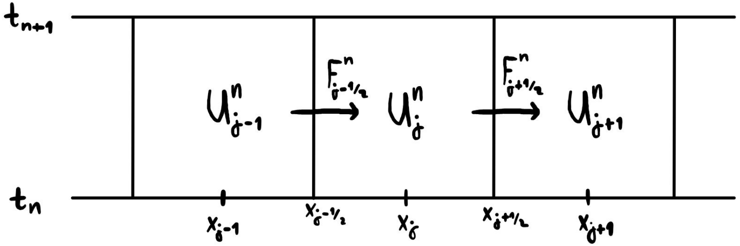
Sine-Squared Wave Problem

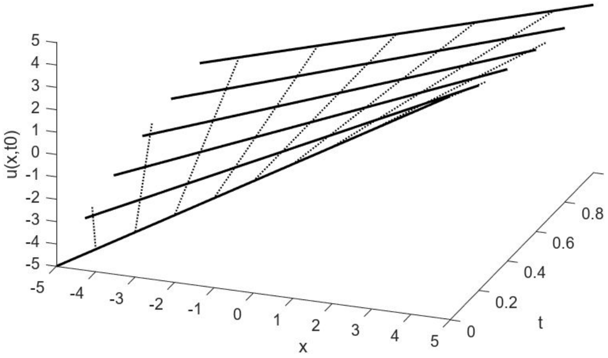
IVP:
$$\begin{cases} u_t + uu_x = 0, \quad x \in [0, 2\pi],\ t > 0 \\ u(x, 0) = \sin^2(x) \\ u(0, t) = u(2\pi, t), \quad t > 0 \end{cases}$$



Q & A




t_{n+1}
t_n
U^n_{j-1}
$F^n_{j-1/2}$
U^n_j
$F^n_{j+1/2}$
U^n_{j+1}
x_{j-1}
$x_{j-1/2}$
x_j
$x_{j+1/2}$
x_{j+1}


u(x,t0)
5 4 3 2 1 0 -1 -2 -3 -4 -5
-5 -4 -3 -2 -1 0 1 2 3 4 5
x
0 0.2 0.4 0.6 0.8 1
t