Solving the Inviscid Burgers' Equation Numerically

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Contents

1 Introduction

The 1-D inviscid Burgers' equation is a first-order hyperbolic partial differential equation (PDE) of the form

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0 \tag{1}$$

where $u \in C^1(\Omega)$, and $x, t \in \Omega \subset \mathbb{R} \times \mathbb{R}^+$. Note that a more compact form of (1) is $u_t + uu_x = 0$. The latter formulation is what is commonly seen in literature.

This equation has a brother named the *viscous* Burgers' equation (or simply referred to as Burgers' equation), which takes the form

$$u_t + uu_x = \epsilon u_{xx} \tag{2}$$

where $\epsilon > 0$ is the diffusion coefficient. We mention this because the inviscid Burgers' equation can be interpreted as resulting from letting $\epsilon \to 0$ in (2). This is important because it informs us what the 'correct' behavior of (1) should be.

The quasilinear equation (1) is not the only formulation of the inviscid Burgers' equation, and in a certain sense it is actually the 'wrong' one to study. This is because under a few reasonable assumptions, a completely smooth initial profile modeled by (1) will devolve into a discontinuous

one in finite time. This is unsettling because then (1) fails to hold; the partial derivative of a discontinuous function does not exist!

Instead, we rewrite (1) as

$$u_t + f(u)_x = 0 \tag{3}$$

where

$$f(u) = \frac{1}{2}u^2 \tag{4}$$

is known as the flux function. This is known as the conservation form of the inviscid Burgers' equation. If we integrate (3) over $[a, b]$, where $[a, b] \subset \Omega$, then we get

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)) \tag{5}$$

where we have exchanged differentiation and integration. The form (5) is known as the integral form of (3), and it is where the 'conservative' notion comes from.

Importantly, this integral form has no problems admitting profiles u with spatial discontinuities (we assume it is not also discontinuous in time). It is this formulation (not (1)) that we will be studying and developing our numerical schemes for.

2 Theory

yahoo theory

2.1 Finite Volume Methods for Conservation Laws

i heart fvm

2.2 The REA Algorithm

more like diarrhea amirite

2.3 The Riemann Problem

i got 99 problems and riemann is one

2.4 Convergence

2.5 High-Resolution Methods

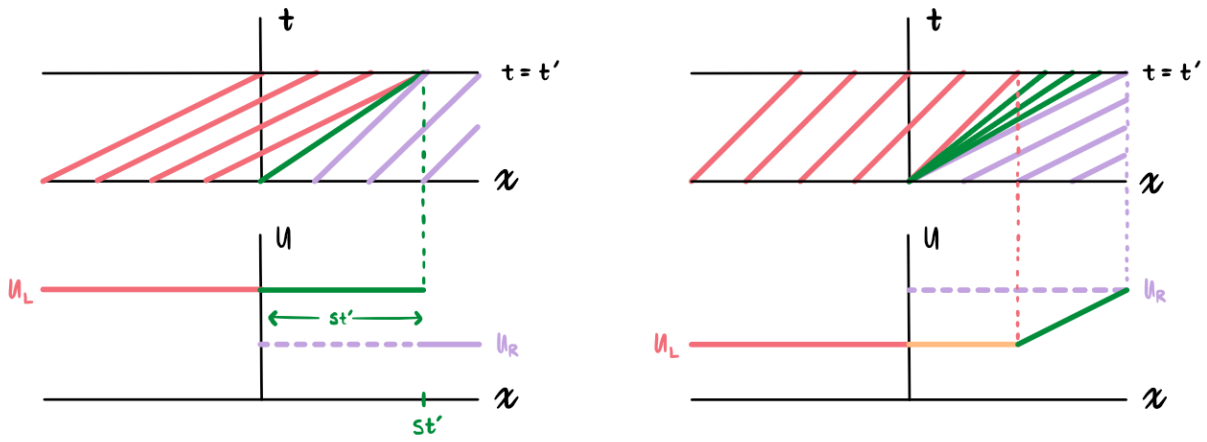
shit i dont fully understand how to implement

3 Experiments

where our code crashed and burns

3.1 'Easy' Problems

basic riemann problems



(a) caption 1

(b) caption 2

Figure 1: main caption

3.2 'Hard' Problems

smooth curvy boies

4 Conclusion

this shit was way harder than i thought it would be

A Appendix

A.1 Code

All project code can be found on our GitHub page: https://github.com/agormann/MACM416-project. As a courtesy, we have also included our code in this document, below.

[insert code]

A.2 Analytical Solution via the Method of Characteristics

We wish to solve the 1-D inviscid Burgers' equation analytically.

Given data on some curve $\Gamma \subset \overline{\Omega}$, we are looking specific parametric curves $(x(t), t)$ which connect points $(x, t) \in \Omega$ to Γ. We want these curves to be precisely those which are parallel to the vector $(u, 1)$, that is

$$\frac{dx}{dt} = \frac{u(x(t), t)}{1} = u(x(t), t)$$

Now supposing that u solves (2), let $z(t)$ denote the value of u along a characteristic, i.e.

$$z(t) = u(x(t), t)$$

Then by the chain rule

$$\frac{dz}{dt} = \partial_x u(x(t), t)\frac{dx}{dt}u(x(t), t) + \partial_t u(x(t), t)$$

but $x'(t) = u(x, t)$, so

$$\frac{dz}{dt} = \partial_t u(x(t), t) + u(x, t)\partial_x u(x(t), t)$$

which is precisely 0 by (2). Hence, we have the following coupled system of ODEs

$$\begin{cases} x'(t) = z(t) = u(x(t), t) \\ z'(t) = 0 \end{cases} \tag{6}$$

Integrating the second term, we get that

$$z(t) = z_0$$

for some $z_0 \in \mathbb{R}$. But $z(t) = u(x(t), t)$, so then $u(x(t), t) = z_0$. This corroborates our findings with (3). Now by integrating the first term, we get

$$x(t) = z_0 t + x_0 \tag{7}$$

where $x_0 \in \mathbb{R}$. Evaluating at $t = 0$, we have that $x(0) = x_0$. Now assuming we are prescribed some initial condition $u(x, 0) = g(x)$, we have that (5) becomes

$$x(t) = g(x_0)t + x_0 \tag{8}$$

which are exactly those characteristic curves we initially sought.

A.3 Conservation Laws

We will begin by introducing notation and then move on to discussion about the theory. First we discretize the interval $[-L, L]$ into a vector of N points x_j by defining a mesh width $\Delta x = 2L/N$ so that

$$x_j = -L + j\Delta x.$$

Note that we are assuming periodic boundary conditions so that $x_0 = x_N$, and so we have exactly N points. For reasons that will soon become clear, we are also interested in the half-steps $x_{j\pm 1/2}$ defined by

$$x_{j\pm 1/2} = x_j \pm \frac{\Delta x}{2}.$$

We will also denote the time step by Δt_n so that

$$t_n = n\Delta t_n.$$

The exact formula for Δt_n will remain undefined for now as it must be a variable time-step.

We denote the *pointwise values* of the true solution at the mesh point (x_j, t_n) by

$$u_j^n = u(x_j, t_n).$$

For finite difference methods, at each time step t_n we are computing a vector $U^n \in \mathbb{R}^N$ where the j-th component U_j^n approximates the true solution u_j^n. Specifically for conservation laws though, it is perhaps more natural to instead view U_j^n as approximating the *cell average* $\bar{u}_j^n$ (see Figure 1) about the mesh point (x_j, t_n) where

$$\bar{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx. \tag{9}$$

The motivation for this interpretation comes from the integral form of the conservation law[1] (2),

$$\frac{d}{dt}\int_{x_j-1/2}^{x_j+1/2} u(x,t)dx - \left[f(u(x_{j-1/2}, t)) - f(u(x_{j+1/2}, t))\right] = 0,$$

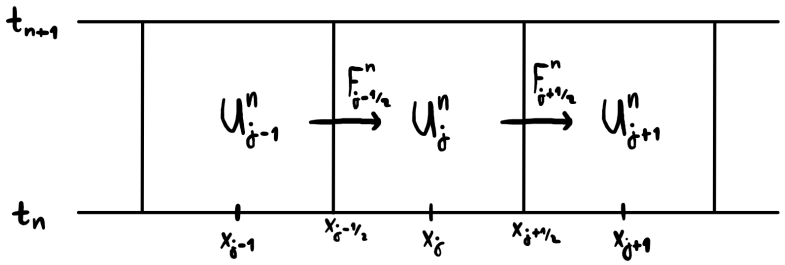
for which a derivation can be found in LeVeque, 1992, pp. 14-16. Integrating from t_n to t_{n+1} yields

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})dx = &\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx \\ &- \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dx - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dx\right].\end{aligned}$$

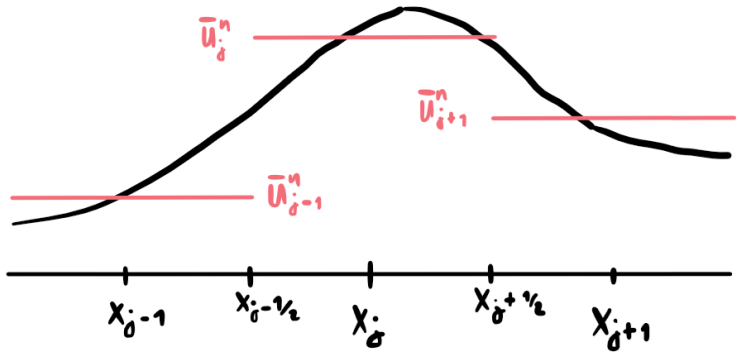
Now dividing by Δx and using the definition of (3) results in

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dx - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dx\right]. \tag{10}$$

[1] Interestingly, we are somewhat doing things backwards. The conservation law is derived from the integral form, and the PDE that satisfies it is a consequence of this integral form.



(a) Illustration of the cell averages. Inspired from Fig 4.1, p.65 of LeVeque, 2002.



(b) Illustration of the step-wise nature the cell-average approximation to $u(x_j, t_n)$ has. Inspired from Fig 17.9, p.418 of Iserles, 2009.

Figure 2

The goal of a successful numerical scheme then is to accurately model the flux through the boundaries of each cell. Explicitly, we want to find some numerical flux function $\mathcal{F}$ so that

$$\mathcal{F}(U^n_j, U^n_{j+1}) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt \qquad \mathcal{F}(U^n_{j-1}, U^n_j) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt.$$

To this end, we say that a numerical method is in *conservation form* if

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U^n_j, U^n_{j+1}) - \mathcal{F}(U^n_{j-1}, U^n_j)\right].$$

Note that while this derivation comes about through the introduction of control volumes, and hence falls under the umbrella of *finite volume methods*, we can also understand it through the lens of finite differences. Specifically, from (2) we get the relation

$$\frac{U^{n+1}_j - U^n_j}{\Delta t} + \frac{F^n_{j+1/2} - F^n_{j-1/2}}{\Delta x} = 0$$

where $F^n_{j\pm 1/2} \sim \mathcal{F}$ as before. Hence these methods can be understood to be first-order accurate in time and 'somewhat' second-order accurate in space. We say 'somewhat' because the reality is complicated.

References

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LeVeque, R. J. (1992). *Numerical methods for conservation laws*. Birkhauser.

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