Solving the Inviscid Burgers' Equation Numerically

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1 Introduction

The 1-D inviscid Burgers' equation is a first-order hyperbolic partial differential equation (PDE) of the form

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0 \tag{1}$$

where $u \in C^1(\Omega)$, and $x, t \in \Omega \subset \mathbb{R} \times \mathbb{R}^+$. Note that a more compact form of (1) is $u_t + uu_x = 0$. The latter formulation is what is commonly seen in literature.

This equation has a brother named the *viscous* Burgers' equation (or simply referred to as Burgers' equation), which takes the form

$$u_t + uu_x = \epsilon u_{xx} \tag{2}$$

where $\epsilon > 0$ is the diffusion coefficient. We mention this because the inviscid Burgers' equation can be interpreted as resulting from letting $\epsilon \to 0$ in (2). This is important because it informs us what the 'correct' behavior of (1) should be.

The quasilinear equation (1) is not the only formulation of the inviscid Burgers' equation, and in a certain sense it is actually the 'wrong' one to study. This is because under a few reasonable assumptions, a completely smooth initial profile modeled by (1) will devolve into a discontinuous one in finite time. This is unsettling because then (1) fails to hold; the partial derivative of a discontinuous function does not exist!

Instead, we rewrite (1) as

$$u_t + f(u)_x = 0 \tag{3}$$

where

$$f(u) = \frac{1}{2}u^2 \tag{4}$$

is known as the flux function[1]. This is known as the conservation form of the inviscid Burgers' equation. If we integrate (3) over $[a, b]$, where $[a, b] \subset \Omega$, then we get

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)) \tag{5}$$

where we have exchanged differentiation and integration. The form (5) is known as the integral form of (3), and it is where the 'conservative' notion comes from[2].

Importantly, this integral form has no problems admitting profiles u with spatial discontinuities (we assume it is not also discontinuous in time). It is this formulation (along with (3)) that we will be studying and developing our numerical schemes for, not the quasilinear form.

2 Theory

We will begin by introducing some notation that will be used throughout this section. Supposing that $\Omega = [a, b] \times \mathbb{R}^+$, we discretize the interval $[a, b]$ into a vector of N points x_j by defining a fixed mesh-width[3] $\Delta x = (b - a)/N$ so that

$$x_j = a + j\Delta x. \tag{6}$$

Note that we are assuming periodic boundary conditions so that $x_0 = x_N$, and so we have exactly N points. For reasons that will soon become clear, we are also interested in the half-steps $x_{j\pm1/2}$ defined by

$$x_{j\pm1/2} = x_j \pm \frac{\Delta x}{2}.$$

Time will simply be denoted by t_n, with no explicit formula. This is because for stability purposes (see Section 2.4), we require a variable time-step.

We denote the *pointwise values* of the true solution u which solves (3) exactly at the mesh point (x_j, t_n) by

$$u_j^n = u(x_j, t_n).$$

The *cell averages* about the mesh point (x_j, t_n) are then defined by

$$\overline{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx. \tag{7}$$

[1] Because the inviscid Burgers' equation can be put into this form, we are able to leverage a whole host of theory concerning advection conservation laws.

[2] A more explicit derivation of (5) can be found in Section 2.1 of LeVeque, 2002.

[3] It is possible to use a variable mesh-width, but we have elected not to.

2.1 Finite Volume Methods for Conservation Laws

For finite difference methods, at each time t_n, we are computing a vector $U^n \in \mathbb{R}^N$ where the j-th component U_j^n approximates the pointwise true solution u_j^n. In light of (5) though, it is perhaps more natural to instead view U_j^n as approximating the *cell average* $\overline{u}_j^n$. This gives rise to what is known as a *finite volume method*.

If we integrate (5) from time t_n to t_{n+1}, we get

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})dx = &\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)dx \\ &- \left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt\right].\end{aligned}$$

Dividing by Δx and applying (7) yields

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt\right]. \tag{8}$$

The goal of a successful finite volume method then is to accurately model the flux through the boundaries of each cell. Explicitly, we want to find some numerical flux function $\mathcal{F}$ so that

$$\mathcal{F}(U_j^n, U_{j+1}^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt \qquad \mathcal{F}(U_{j-1}^n, U_j^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt.$$

To this end, we say that a numerical method is in *conservation form* if

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_j^n)\right]. \tag{9}$$

Note that while this derivation comes about through the introduction of control volumes, we can also understand it through the lens of finite differences. Specifically, from (3) we get the relation

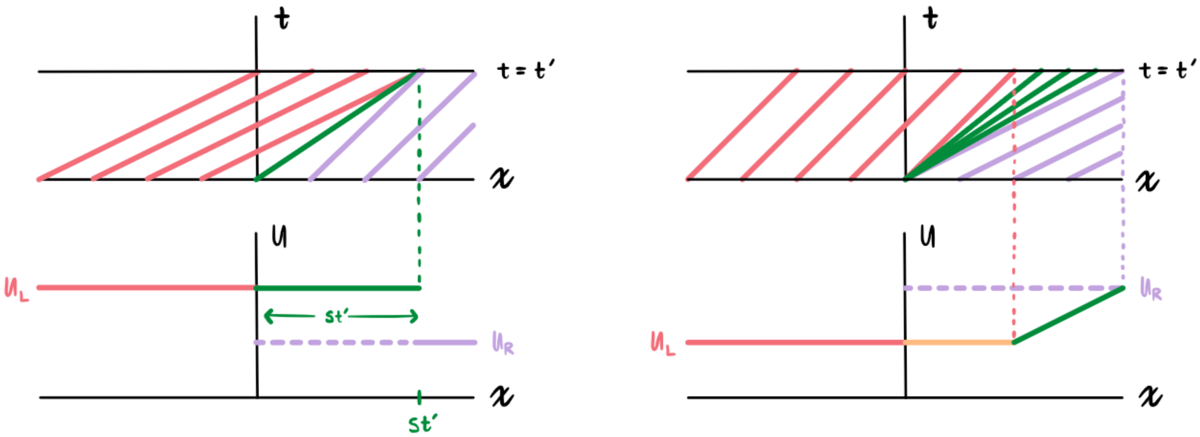
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = 0$$

where $F_{j\pm 1/2}^n \sim \mathcal{F}$ as before.

2.2 The REA Algorithm

The reconstruct-evolve-average algorithm (or REA for short)

1.
2.
3.



(a) caption 1

(b) caption 2

Figure 1: main caption

2.3 The Riemann Problem and Discontinuous Solutions

$$F(u_L, u_R) = \begin{cases} f(u_L) & s \geq 0 \\ f(u_R) & s < 0 \end{cases} \tag{10}$$

$$s = \frac{f(u_R) - f(u_L)}{u_R - u_L} \tag{11}$$

$$F(u_L, u_R) = \begin{cases} \min_{u_l \leq u \leq u_R} f(u) & u_l \leq u_R \\ \max_{u_l \leq u \leq u_R} f(u) & u_l > u_R \end{cases} \tag{12}$$

2.4 Convergence

2.5 High-Resolution Methods

shit i dont fully understand how to implement

3 Experiments

where our code crashed and burns

3.1 'Easy' Problems

basic riemann problems

3.2 'Hard' Problems

smooth curvy boies

4 Conclusion

this shit was way harder than i thought it would be

A Appendix

A.1 Code

All project code can be found on our GitHub page: https://github.com/agormann/MACM416-project. As a courtesy, we have also included our code in this document, below.

[insert code]

A.2 Analytical Solution via the Method of Characteristics

We wish to solve the 1-D inviscid Burgers' equation analytically.

Given data on some curve $\Gamma \subset \overline{\Omega}$, we are looking specific parametric curves $(x(t), t)$ which connect points $(x, t) \in \Omega$ to Γ. We want these curves to be precisely those which are parallel to the vector $(u, 1)$, that is

$$\frac{dx}{dt} = \frac{u(x(t), t)}{1} = u(x(t), t)$$

Now supposing that u solves (2), let $z(t)$ denote the value of u along a characteristic, i.e.

$$z(t) = u(x(t), t)$$

Then by the chain rule

$$\frac{dz}{dt} = \partial_x u(x(t), t)\frac{dx}{dt}u(x(t), t) + \partial_t u(x(t), t)$$

but $x'(t) = u(x, t)$, so

$$\frac{dz}{dt} = \partial_t u(x(t), t) + u(x, t)\partial_x u(x(t), t)$$

which is precisely 0 by (2). Hence, we have the following coupled system of ODEs

$$\begin{cases} x'(t) = z(t) = u(x(t), t) \\ z'(t) = 0 \end{cases} \tag{13}$$

Integrating the second term, we get that

$$z(t) = z_0$$

for some $z_0 \in \mathbb{R}$. But $z(t) = u(x(t), t)$, so then $u(x(t), t) = z_0$. This corroborates our findings with (3). Now by integrating the first term, we get

$$x(t) = z_0 t + x_0 \tag{14}$$

where $x_0 \in \mathbb{R}$. Evaluating at $t = 0$, we have that $x(0) = x_0$. Now assuming we are prescribed some initial condition $u(x, 0) = g(x)$, we have that (5) becomes

$$x(t) = g(x_0)t + x_0 \tag{15}$$

which are exactly those characteristic curves we initially sought.

References

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LeVeque, R. J. (1992). *Numerical methods for conservation laws*. Birkhauser.

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