Numerical Analysis of Burgers' Equation*

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1 Introduction

1.1 The Inviscid Burgers' Equation

We have elected to study the Burgers' equation, or more correctly, the inviscid Burgers' equation.[1] Given $u \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a domain, the general form is written as

$$\partial_t u(\boldsymbol{x}, t) + u(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} u(\boldsymbol{x}, t) = 0 \tag{1}$$

where $\nabla_{\boldsymbol{x}}$ denotes the gradient with respect to the spatial variable $\boldsymbol{x} \in \mathbb{R}^n$. For pragmatic reasons though, we will be focusing on the $n = 1$ case. Then (1) simplifies to

$$\partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0. \tag{2}$$

There are two key observations to make. The first is that (2) is really a statement about the directional derivative, that is

$$\nabla u(x, t) \cdot (u(x, t), 1) = 0^{[2]} \tag{3}$$

so the derivative of u in the direction of $(u, 1)$ is 0 - in other words, u is constant in this direction. This is a consequence of (2) being first-order. While on the surface it may seem problematic that (2) is quasilinear (and so the direction $(u, 1)$ is varied), this does not complicate the finding of an analytic solution.

1.2 The Method of Characteristics

Given data on some curve $\Gamma \subset \overline{\Omega}$, we are looking for parametric curves $(x(t), t)$ which connect points $(x, t) \in \Omega$ to Γ. The upshot is that on these *characteristic curves*, the PDE (2) degenerates into an ODE. So

$$\frac{dx}{dt} = u(x(t), t) \tag{4}$$

*Placeholder title!

[1] We may decide later on to study the viscous Burgers' equation.

[2] Technically we should be normalizing so that this is a unit vector.

Now supposing that u solves (2), let $z(t)$ denote the value of u along a characteristic, i.e.

$$z(t) = u(x(t), t)$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \partial_x u(x(t), t)\frac{dx}{dt}u(x(t), t) + \partial_t u(x(t), t) \\ &= \partial_t u(x(t), t) + u(x(t), t)\partial_x u(x(t), t)\end{aligned}$$

References

Choksi, R. (2022). *Partial differential equations: A first course.* American Mathematical Society.

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