Numerical Analysis of Burgers' Equation*

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1 Introduction[1]

1.1 The Inviscid Burgers' Equation

We have elected to study the Burgers' equation, or more correctly, the inviscid Burgers' equation.[2] Given $u \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a domain, the general form is written as

$$\partial_t u(\boldsymbol{x}, t) + u(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} u(\boldsymbol{x}, t) = 0 \tag{1}$$

where $\nabla_{\boldsymbol{x}}$ denotes the gradient with respect to the spatial variable $\boldsymbol{x} \in \mathbb{R}^n$. For pragmatic reasons though, we will be focusing on the $n = 1$ case. Then (1) simplifies to

$$\partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0. \tag{2}$$

There are two key observations to make. The first is that (2) is really a statement about the directional derivative, that is

$$\nabla u(x, t) \cdot (u(x, t), 1) = 0^{[3]} \tag{3}$$

so the derivative of u in the direction of $(u, 1)$ is 0 - in other words, u is constant in this direction. This is a consequence of (2) being first-order. While on the surface it may seem problematic that (2) is quasilinear (and so the direction $(u, 1)$ is varied), this does not complicate the finding of an analytic solution.

1.2 The Method of Characteristics

Given data on some curve $\Gamma \subset \overline{\Omega}$, we are looking specific parametric curves $(x(t), t)$ which connect points $(x, t) \in \Omega$ to Γ. We want these curves to be precisely those which are parallel to the vector $(u, 1)$, that is

$$\frac{dx}{dt} = \frac{u(x(t), t)}{1} = u(x(t), t)$$

*Placeholder title!

[1] This entire section has been modified from the content in Chapter 2 of Choksi, 2022. Specifically, sections 2.2-2.4.

[2] We may decide later on to study the viscous Burgers' equation.

[3] Technically we should be normalizing so that this is a unit vector.

Now supposing that u solves (2), let $z(t)$ denote the value of u along a characteristic, i.e.

$$z(t) = u(x(t), t)$$

Then by the chain rule

$$\frac{dz}{dt} = \partial_x u(x(t), t)\frac{dx}{dt}u(x(t), t) + \partial_t u(x(t), t)$$

but $x'(t) = u(x, t)$, so

$$\frac{dz}{dt} = \partial_t u(x(t), t) + u(x, t)\partial_x u(x(t), t)$$

which is precisely 0 by (2). Hence, we have the following coupled system of ODEs

$$\begin{cases} x'(t) = z(t) = u(x(t), t) \\ z'(t) = 0 \end{cases} \tag{4}$$

Integrating the second term, we get that

$$z(t) = z_0$$

for some $z_0 \in \mathbb{R}$. But $z(t) = u(x(t), t)$, so then $u(x(t), t) = z_0$. This corroborates our findings with (3). Now by integrating the first term, we get

$$x(t) = z_0 t + x_0 \tag{5}$$

where $x_0 \in \mathbb{R}$. Evaluating at $t = 0$, we have that $x(0) = x_0$. Now assuming we are prescribed some initial condition $u(x, 0) = g(x)$, we have that (5) becomes

$$x(t) = g(x_0)t + x_0 \tag{6}$$

which are exactly those characteristic curves we initially sought.

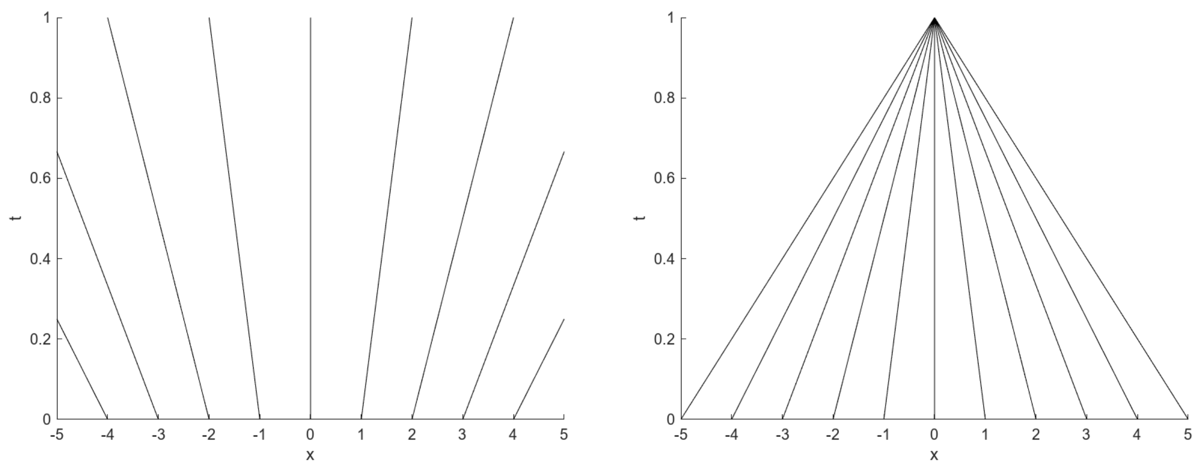
2 Numerical Analysis

3 Conclusion

References

Choksi, R. (2022). *Partial differential equations: A first course*. American Mathematical Society.

Kutz, J. N. (2013). *Data-driven modeling & scientific computation: Methods for complex systems & big data*. Oxford University Press.



(a) A subfigure

(b) A subfigure

Figure 1: A figure with two subfigures