Solution to the Inviscid Burgers' Equation by the Lax-Friedrichs Scheme[*]

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1 Introduction

2 Body

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_{j+1}^n)\right)$$

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

$$\mathcal{F}(U_j^n, U_{j+1}^n) := \frac{\Delta t}{2\Delta x}(U_j^n - U_{j+1}^n) + \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$$

$$U_j^n \sim \bar{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx$$

$$\mathcal{F}(U_j^n, U_{j+1}^n) \sim \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt$$

$$\mathcal{F}(\bar{u}, \bar{u}) = f(\bar{u})$$

$$u_t + f(u)_x = 0$$

$$F_{j-1/2}^n = \mathcal{F}(U_{j-1}^n, U_j^n) \qquad F_{j+1/2}^n = \mathcal{F}(U_j^n, U_{j+1}^n)$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = 0$$

[*]Placeholder title!

3 Matrix Form

The Lax-Friedrichs scheme

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

can be converted into a matrix form

$$\vec{U}^{n+1} = A\vec{U}^n - B\vec{f}(\vec{U}^n)$$

where

$$A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B = \frac{\Delta t}{2\Delta x}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}$$

If we instead use the conservation form of the Lax-Friedrichs scheme

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U_j^n, U_{j+1}^n) - \mathcal{F}(U_{j-1}^n, U_j^n)\right)$$

$$\mathcal{F}(U_j^n, U_{j+1}^n) := \frac{\Delta x}{2\Delta t}(U_j^n - U_{j+1}^n) + \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$$

we get the following matrix form

$$\vec{U}^{n+1} = \vec{U}^n - C\vec{\mathcal{F}}(\vec{U}^n)$$

$$\vec{\mathcal{F}}(\vec{U}^n) = D\vec{U}^n + E\vec{f}(\vec{U}^n)$$

where

$$C = \frac{\Delta x}{\Delta t}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & -1 \end{bmatrix},$$

$$D = \frac{\Delta x}{2\Delta t}\begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}, \quad E = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}$$

In either case, we can construct the matrices in MATLAB by using the *diag()* command on vectors containing the values of the non-zero diagonals, then filling in the values in the bottom left and top right corners as necessary.

4 Conclusion