Numerical Analysis of Burgers' Equation*

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1 Introduction[1]

1.1 The Inviscid Burgers' Equation

We have elected to study the Burgers' equation, or more correctly, the inviscid Burgers' equation.[2] Given $u \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a domain, the general form is written as

$$\partial_t u(\boldsymbol{x}, t) + u(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} u(\boldsymbol{x}, t) = 0 \tag{1}$$

where $\nabla_{\boldsymbol{x}}$ denotes the gradient with respect to the spatial variable $\boldsymbol{x} \in \mathbb{R}^n$. For pragmatic reasons though, we will be focusing on the $n = 1$ case. Then (1) simplifies to

$$\partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0. \tag{2}$$

There are two key observations to make. The first is that (2) is really a statement about the directional derivative, that is

$$\nabla u(x, t) \cdot (u(x, t), 1) = 0^{[3]} \tag{3}$$

so the derivative of u in the direction of $(u, 1)$ is 0 - in other words, u is constant in this direction. This is a consequence of (2) being first-order. While on the surface it may seem problematic that (2) is quasilinear (and so the direction $(u, 1)$ is varied), this does not complicate the finding of an analytic solution.

1.2 The Method of Characteristics

Given data on some curve $\Gamma \subset \overline{\Omega}$, we are looking specific parametric curves $(x(t), t)$ which connect points $(x, t) \in \Omega$ to Γ. We want these curves to be precisely those which are parallel to the vector $(u, 1)$, that is

$$\frac{dx}{dt} = \frac{u(x(t), t)}{1} = u(x(t), t)$$

Now supposing that u solves (2), let $z(t)$ denote the value of u along a characteristic, i.e.

$$z(t) = u(x(t), t)$$

*Placeholder title!

[1] This entire section has been modified from the content in Chapter 2 of Choksi, 2022. Specifically, sections 2.2-2.4.

[2] We may decide later on to study the viscous Burgers' equation.

[3] Technically we should be normalizing so that this is a unit vector.

Then by the chain rule

$$\frac{dz}{dt} = \partial_x u(x(t),t)\frac{dx}{dt}u(x(t),t) + \partial_t u(x(t),t)$$

but $x'(t) = u(x,t)$, so

$$\frac{dz}{dt} = \partial_t u(x(t),t) + u(x,t)\partial_x u(x(t),t)$$

which is precisely 0 by (2). Hence, we have the following coupled system of ODEs

$$\begin{cases} x'(t) = z(t) = u(x(t),t) \\ z'(t) = 0 \end{cases} \tag{4}$$

Integrating the second term, we get that

$$z(t) = z_0$$

for some $z_0 \in \mathbb{R}$. But $z(t) = u(x(t),t)$, so then $u(x(t),t) = z_0$. This corroborates our findings with (3). Now by integrating the first term, we get

$$x(t) = z_0 t + x_0 \tag{5}$$

where $x_0 \in \mathbb{R}$. Evaluating at $t = 0$, we have that $x(0) = x_0$. Now assuming we are prescribed some initial condition $u(x,0) = g(x)$, we have that (5) becomes

$$x(t) = g(x_0)t + x_0 \tag{6}$$

which are exactly those characteristic curves we initially sought.

1.3 Interpretation

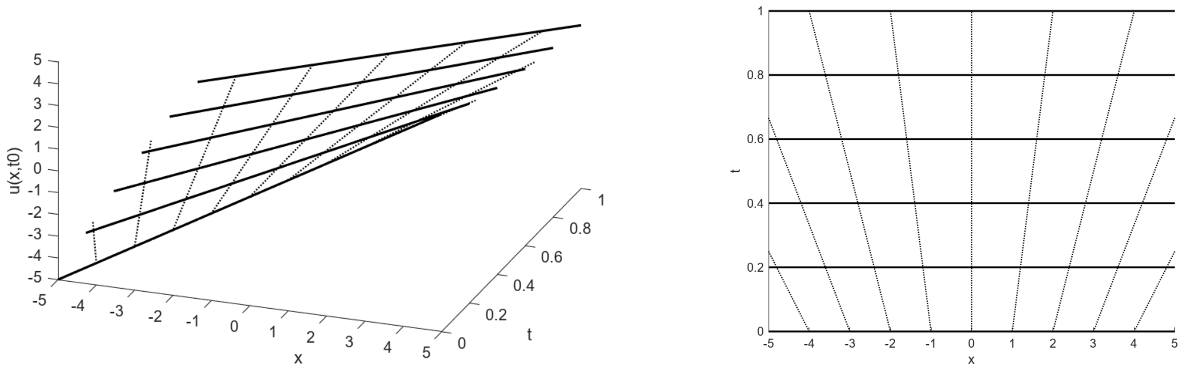
For those familiar with the linear transport equation (or advection equation), the physical interpretation of (2) is quite similar; it models the space-time propagation of an initial signal $u(x,0) = g(x)$. The major difference being that the wave-speed is not only not constant - it is proportional to the solution itself (see Figures 1 and 2), and hence can be different along the signal. A consequence of this non-linearity is that the solution to (6) is not always well-defined. Specifically, it can be shown that if $g(x)$ is decreasing on any $E \subset \Gamma$, the solution u will become multivalued at some time t (see Figure 2). This is known as a shock or shockwave. A useful proxy for this behavior is the characteristic curves (Figure 2b); they converge to the point of discontinuity.

2 Numerical Analysis

3 Conclusion

References

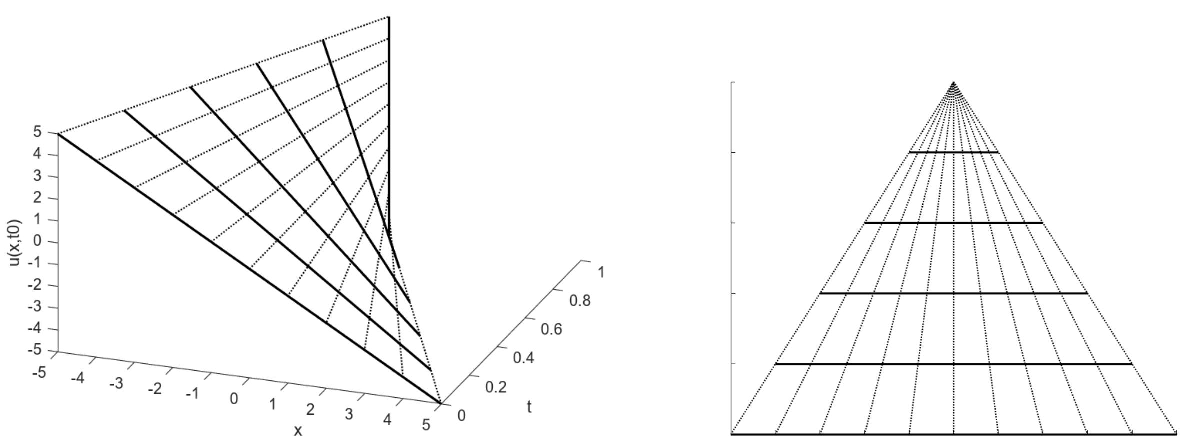
Choksi, R. (2022). *Partial differential equations: A first course*. American Mathematical Society.



(a) Plot of solution curves at fixed $t \in [0, 1]$ with increments of 0.2.

(b) Characteristic curves (see (6)) at fixed $x_0 \in [-5, 5]$ with increments of 1.

Figure 1: Solution to (2) with the non-decreasing initial condition $u(x, 0) = x$.



(a) Plot of solution curves at fixed $t \in [0, 1]$ with increments of 0.2.

(b) Characteristic curves at fixed $x_0 \in [-5, 5]$ with increments of 1.

Figure 2: Solution to (2) with the initial condition $u(x, 0) = -x$.

Iserles, A. (2009). *A first course in the numerical analysis of differential equations*. Cambridge University Press.

Kutz, J. N. (2013). *Data-driven modeling & scientific computation: Methods for complex systems & big data*. Oxford University Press.

Trefethen, L. N., & Embree, K. (2001). 51. inviscid burgers equation. In *The (unfinished) pde coffee table book*. Unpublished.