Solving Imperfect-Recall Games: New Representations and Algorithms.

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20 December 2018

Outline of the Topics

Scaletta

- Argomento 1
  - bla bla
  - bla
- Argomento 2

Example of Normal-Form Game: Rock-Paper-Scissors

- $\mathcal{P} = \{1, 2\}$
- $A = \{A_1, A_2\}$ with $A_1 = \{R, P, S\}$, $A_2 = \{R, P, S\}$
- the utility functions are specified by the following matrix

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Extensive-Form Games

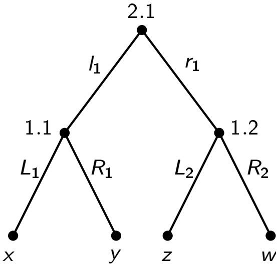
Definition

The extensive-form representation of a perfect-information game is a tuple $\langle \mathcal{P}, A, H, Z, P, \pi_c, u \rangle$, where:

- $\mathcal{P} = \{1, 2, ..., n\}$ is the finite set of players
- $A = \{A_1, A_2, ..., A_n\}$, where A_i is a finite set of actions of player i
- H is a finite set of histories (i.e., sequences of actions)
- $Z \subseteq H$ is the set of terminal histories
- $\mathrm{P} : H \rightarrow \mathcal{P}$ is the function returning the player acting at a given decision node
- π_c is the fixed strategy of a chance player
- $u = \{u_1, u_2, ...u_n\}$ is the set of utility functions in which $u_i : Z \rightarrow \mathbb{R}$

Example of Extensive-Form Perfect-Information Game

- $\mathcal{P} = \{1, 2\}$
- $A = \{A_1, A_2\}$, with $A_1 = \{L_1, R_1, L_2, R_2\}$ and $A_2 = \{l_1, r_1\}$



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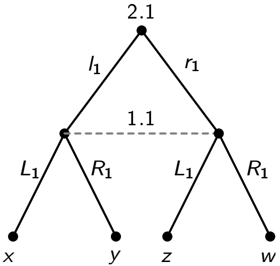
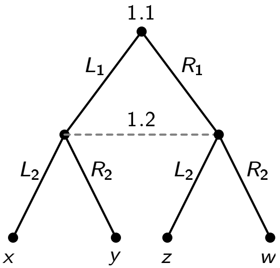
- $\mathcal{I}_1 = \{1.1\}$, $\mathcal{I}_2 = \{2.1\}$



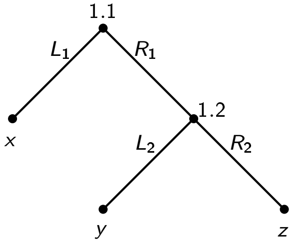
An information set I of player i is a subset of H_i such that, $\forall h, h' \in I$ the property $A(h) = A(h')$ holds.

Example of Extensive-Form Imperfect-Recall Game

- $\mathcal{P} = \{1\}$
- $A = \{A_1\}$ with $A_1 = \{L_1, R_1, L_2, R_2\}$
- $\mathcal{I} = \{1.1, 1.2\}$

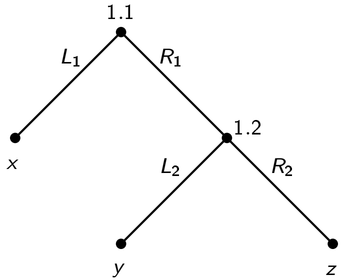


Normal-Form Strategies



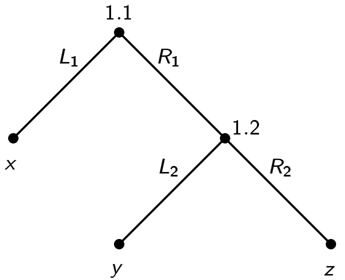
Normal-form plans	Outcome
$L_1 *$	x
$R_1 L_2$	y
$R_1 R_2$	z

Behavioral Strategies (Agent-Form Strategies)



$$\pi_{1.1} = \begin{cases} \frac{1}{2} & L_1 \\ \frac{1}{2} & R_1 \end{cases} \qquad \pi_{1.2} = \begin{cases} \frac{1}{2} & L_2 \\ \frac{1}{2} & R_2 \end{cases}$$

Sequence-Form Strategies



$$r_1 = \begin{cases} 1.0 & q_\emptyset \\ 0.2 & L_1 \\ 0.8 & R_1 \\ 0.4 & R_1 L_2 \\ 0.4 & R_1 R_2 \end{cases}$$

Choice of the Strategy Representation

Khun Theorem

Given an extensive-form game and player $i \in \mathcal{P}$, every normal-form strategy has an equivalent behavioral strategy if and only if i has perfect recall.

- PERFECT-RECALL GAMES
  - **Khun Theorem** holds
  - **Mixed Strategies**: the size of their space is exponential
  - **Behavioral Strategies**: the size of their space is linear
- IMPERFECT-RECALL GAMES
  - **Khun Theorem** is not valid
  - **Mixed Strategies**: more expressive
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Realization equivalence: Two strategies of player $i \in \mathcal{P}$ are realization equivalent if, for every strategy of the opponents, they induce the same probability distribution over the outcomes of the game

Definition (Immediate Inflation)

Let $\mathcal{I}_i$ and $\mathcal{I}'_i$ be two possible information partitions of player $i \in \mathcal{P}$. We say that $\mathcal{I}'_i$ is an immediate inflation of $\mathcal{I}_i$ iff there exist $I \in \mathcal{I}_i$ and $I_1, I_2 \in \mathcal{I}'_i$ such that:

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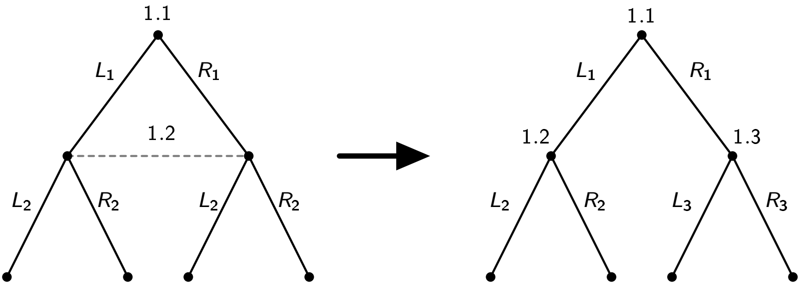

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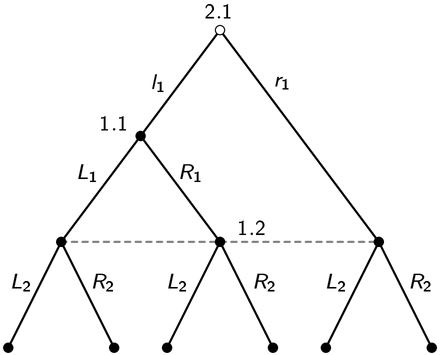

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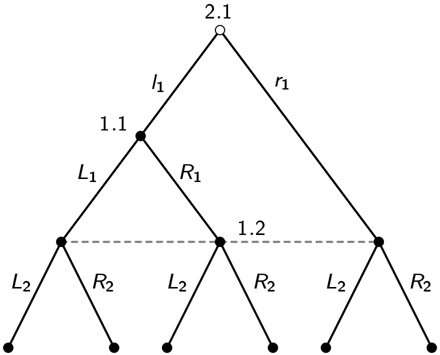

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Can we turn an imperfect-recall game into an equivalent perfect-recall one?

- Team: set of players sharing the same objectives
- Personality: given player $i \in \mathcal{P}$ with information partition $\mathcal{I}_i$, a personality $\tilde{\mathcal{I}}_i^k \subseteq \mathcal{I}_i$ is a subset of information sets such that an hypothetical player j with $\mathcal{I}_j = \tilde{\mathcal{I}}_i^k$ would have perfect recall

- Property:
  Given Γ where $i \in \mathcal{P}$ is imperfect recall, there is a bijection between the set of NE of Γ and the set of TMECor of $\tilde{\Gamma}$

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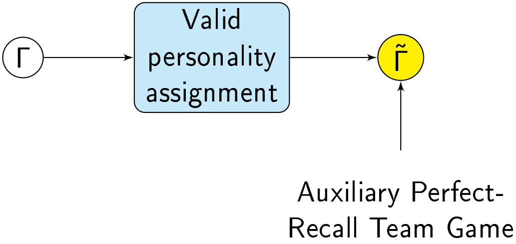

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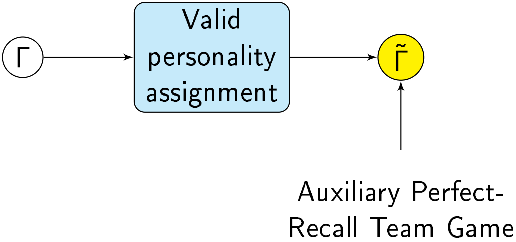

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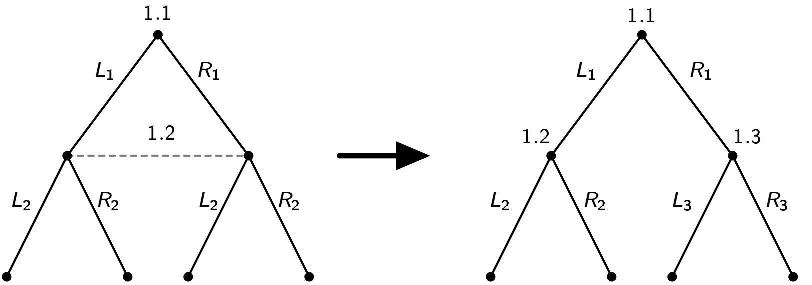

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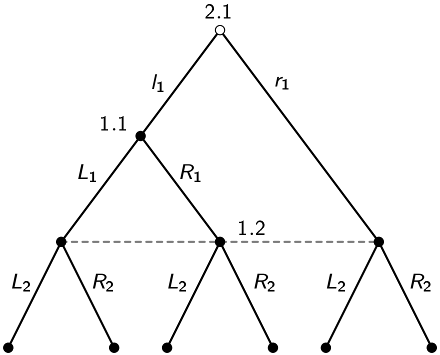

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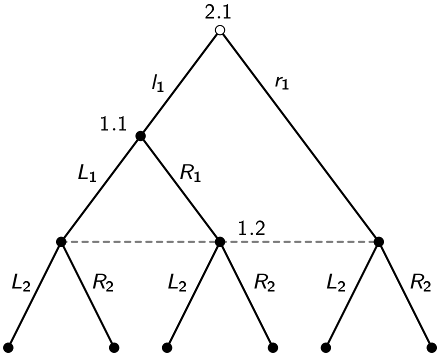

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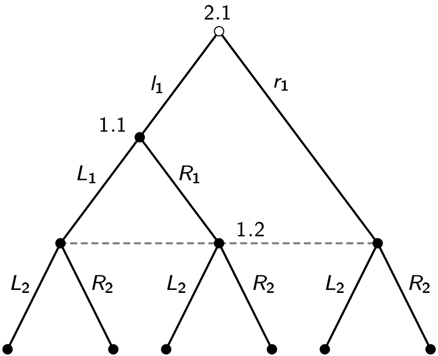
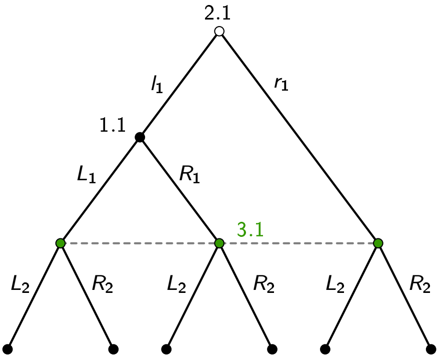


Auxiliary Team Game

$\tilde{\mathcal{I}}_1^1 = \{1.1\}$ $\tilde{\mathcal{I}}_1^2 = \{3.1\}$ $\mathcal{I}_2 = \{2.1\}$



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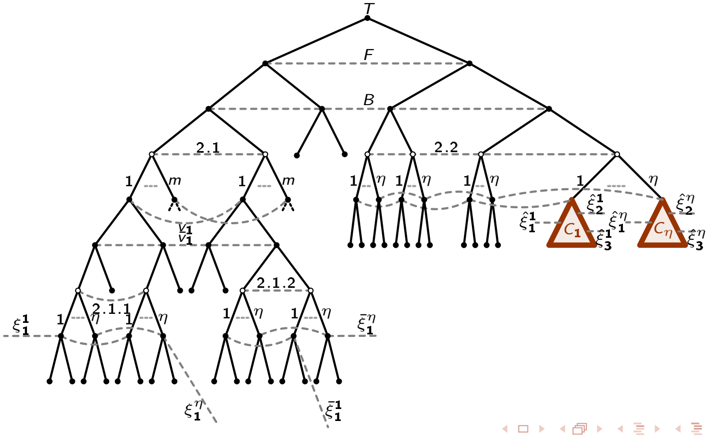

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3-P over Γ_ϕ is satisfiable if and only if ϕ is satisfiable

Theorem

MIN-P is NP-hard.

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MIN-P is NP-hard even for the class of completely inflated games.

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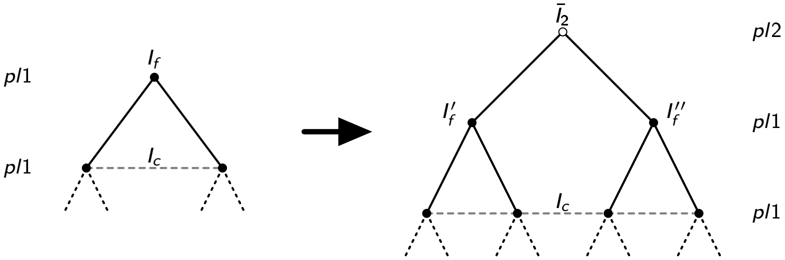
Making a Game-Tree Robust with respect to Inflation



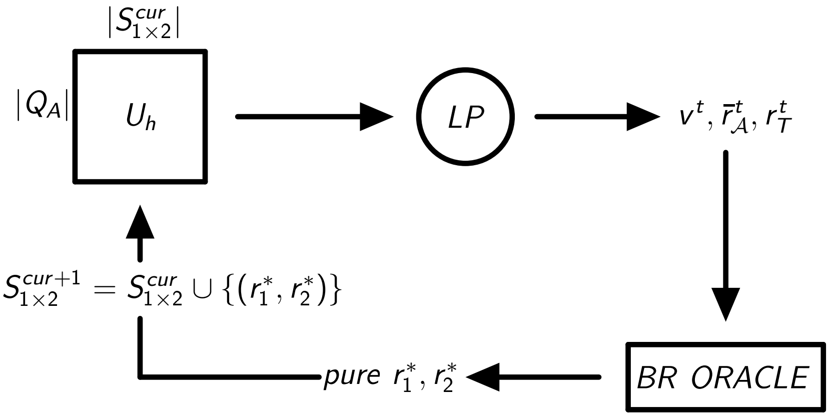
ILP for Minimizing the Number of Personalities

In this section we describe an ILP in order to find a valid minimum personality assignment

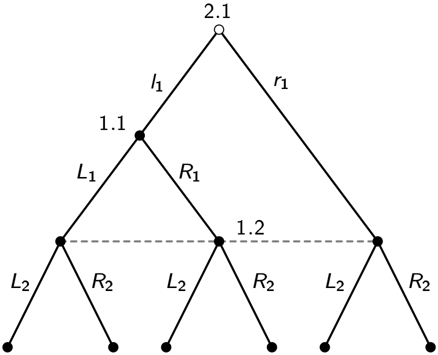
$$
\begin{aligned}
&\min_{w,P} \ \mathbb{1}^T \mathbf{w} \\
&\text{s.t.} \ \mathbb{1}^T P_{\cdot j} = 1 && \forall j \in \{1, \dots, |\mathcal{I}_1|\} \\
&\qquad w_i \geq \frac{1}{|\mathcal{I}_1|} P_{i\cdot} \mathbb{1} && \forall i \in \{1, \dots, |\mathcal{I}_1|\} \\
& && \forall i \in \{1, \dots, |\mathcal{I}_1|\}, \\
&\qquad P_{i,j} + P_{i,k} - 1 \leq M_{j,k} && \forall (j,k) \in \{1, \dots, |\mathcal{I}_1|\} \times \{1, \dots, |\mathcal{I}_1|\} \\
& && \text{s.t.} \ k > j \\
&\qquad \mathbf{w} \in \{0,1\}^{|\mathcal{I}_1|} \\
&\qquad \mathbf{P} \in \{0,1\}^{|\mathcal{I}_1| \times |\mathcal{I}_1|}
\end{aligned}
$$

Nash Equilibrium in Imperfect-Recall Games

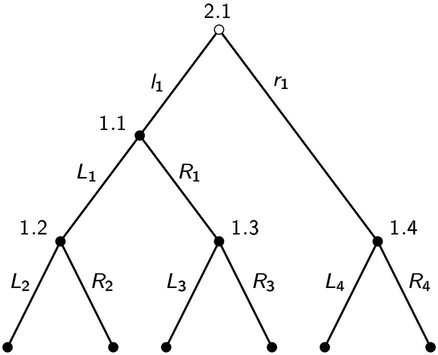
Hybrid Column Generation Approach



From an Imperfect-Recall Game...



...to its Coarsest Outer Perfect-Recall Refinement



Coarsest Outer Perfect-Recall Refinement

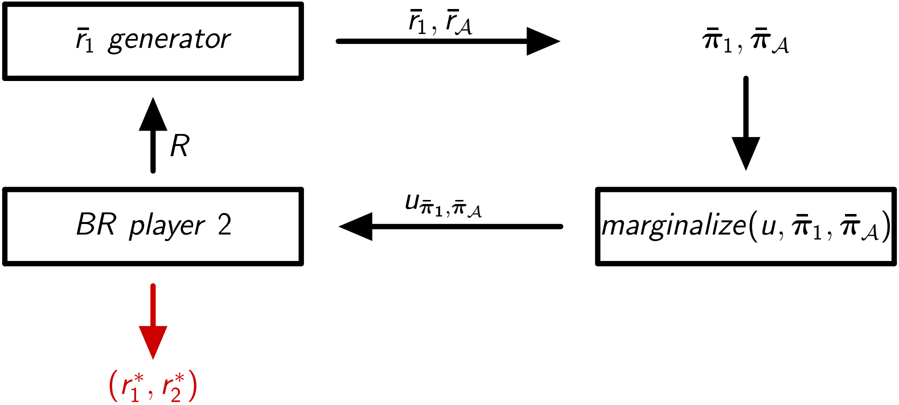
Definition (Coarsest Outer Perfect-Recall Refinement)

The coarsest perfect-recall refinement $\check{\Gamma}^*$ of the imperfect-recall game Γ is a tuple $\langle \mathcal{P}, A', H, Z, P, \pi_c, u, \mathcal{I}' \rangle$ where $\forall i \in \mathcal{P}$, $\forall I_i \in \mathcal{I}_i$, $H(I_i)$ defines the information set partition $\mathcal{I}'$. A' is a modification of A, which guarantees that $\forall I \in \mathcal{I}'$, $\forall h_k, h_l \in I, A'(h_k) = A'(h_l)$, while for all distinct $I^k, I^l \in \mathcal{I}'$, $\forall a^k \in A(I^k)$, $\forall a^l \in A(I^l)$, $a^k \neq a^l$.

Theorem

Given a generic imperfect-recall game Γ, its coarsest perfect-recall refinement $\check{\Gamma}^*$ can be computed in polynomial time in the size of the game tree.

Best-Response Oracle



Conclusions

- The complete inflation of a game can be computed in polynomial time
- MIN-P is NP-hard, and even hard to approximate
- We provided an ILP for minimizing the number of personalities
- We presented an improved column generation approach for finding an optimal mixed-strategy Nash equilibrium in imperfect-recall games
  - new best-response oracle that employs a pruning technique and uses information from the coarsest perfect-recall refinement

Future Developments

- Evaluate, in a practical scenario, the impact of inflation in terms of *split* information sets
- Our negative results on MIN-P suggest that, when working with mixed strategies, one should look for algorithmic techniques for equilibrium computation in team games that are robust with respect to the number of team members.It would be interesting to confirm this idea with an experimental evaluation of state-of-the-art techniques from the team game domain
- Experimentally evaluate our best-response oracle against current known techniques for the problem

Thanks for the Attention!