Solving Imperfect-Recall Games: New Representations and Algorithms

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Preliminaries

Example of Normal-Form Game: Rock-Paper-Scissors

- $\mathcal{P} = \{1, 2\}$
- $A = \{A_1, A_2\}$ with $A_1 = \{R, P, S\}$, $A_2 = \{R, P, S\}$

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Extensive-Form Games

Definition

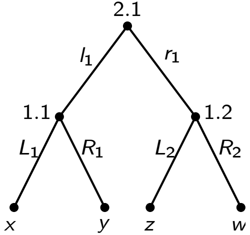
The extensive-form representation of a perfect-information game is a tuple $\langle \mathcal{P}, A, H, Z, P, \pi_c, u \rangle$, where:

- $\mathcal{P} = \{1, 2, ..., n\}$ is the finite set of players
- $A = \{A_1, A_2, ..., A_n\}$, where A_i is a finite set of actions of player i
- H is a finite set of histories (i.e., sequences of actions)
- $Z \subseteq H$ is the set of terminal histories
- $\mathrm{P} : H \rightarrow \mathcal{P}$ is the function returning the player acting at a given decision node
- π_c is the fixed strategy of a chance player
- $u = \{u_1, u_2, ... u_n\}$ is the set of utility functions in which $u_i : Z \rightarrow \mathbb{R}$

Types of Extensive-Form Games

PERFECT INFORMATION

Perfect Recall



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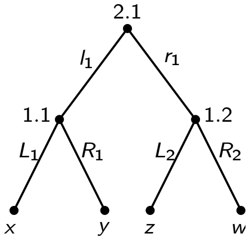

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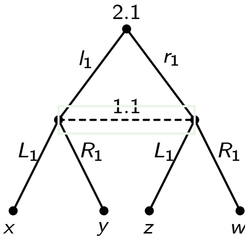

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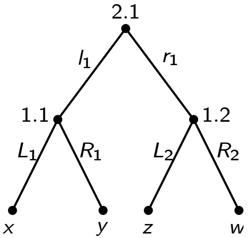

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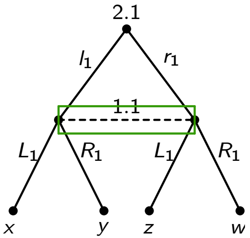

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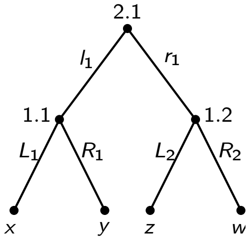

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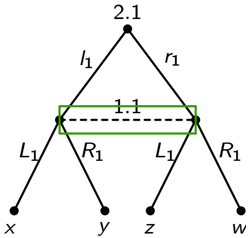

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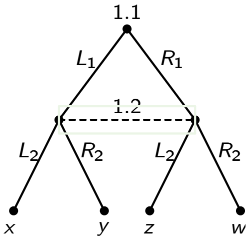

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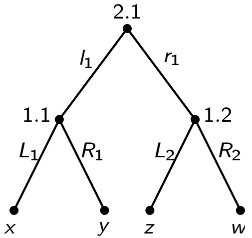

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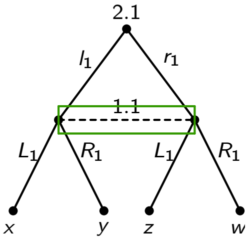

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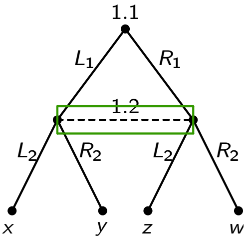
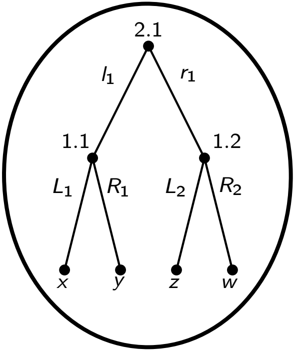
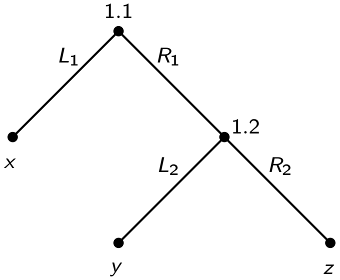


From Extensive Form to Normal Form and Vice Versa



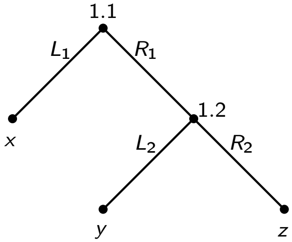
	l_1	r_1
$L_1 L_2$	x	z
$L_1 R_2$	x	w
$R_1 L_2$	y	z
$R_1 R_2$	y	w

Behavioral Strategies (Agent-Form Strategies)



$$\pi_{1.1} = \begin{cases} \frac{1}{2} & L_1 \\ \frac{1}{2} & R_1 \end{cases} \qquad \pi_{1.2} = \begin{cases} \frac{1}{2} & L_2 \\ \frac{1}{2} & R_2 \end{cases}$$

Normal-Form Strategies



Normal-form plans	Outcome
L_1*	x
$R_1 L_2$	y
$R_1 R_2$	z

Relationship between Strategies Representations

Khun Theorem

Given an extensive-form game and player $i \in \mathcal{P}$, every normal-form strategy has an equivalent behavioral strategy if and only if i has perfect recall.

- PERFECT-RECALL GAMES
  - **Khun Theorem** holds
  - **Mixed Strategies**: the size of their space is exponential
  - **Behavioral Strategies**: the size of their space is linear
- IMPERFECT-RECALL GAMES
  - **Khun Theorem** is not valid
  - **Mixed Strategies**: more expressive
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Original Contributions

Summary of Results

- **BETTER REPRESENTATION**
  - Imperfect-Recall Game $\leftrightarrow$ Team Perfect-Recall Game
  - Inflation
  - Complexity of MIN-P
  - ILP for minimizing the number of personalities
- **ALGORITHMS**
  - Polynomial-time algorithm for finding coarsest outer perfect-recall refinement
  - Column generation approach to compute mixed-strategy Nash Equilibrium in imperfect-recall game
  - New best-response oracle that employs a pruning technique and uses information from the coarsest perfect-recall refinement

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The Notion of Personality

Can we turn an imperfect-recall game into an equivalent perfect-recall one?

- Team: set of players sharing the same objectives
- Personality: given player $i \in \mathcal{P}$ with information partition $\mathcal{I}_i$, a personality $\tilde{\mathcal{I}}_i^k \subseteq \mathcal{I}_i$ is a subset of information sets such that an hypothetical player j with $\mathcal{I}_j = \tilde{\mathcal{I}}_i^k$ would have perfect recall

- Property:
  Given Γ where $i \in \mathcal{P}$ is imperfect recall, there is a bijection between the set of NE of Γ and the set of TMECor of $\tilde{\Gamma}$

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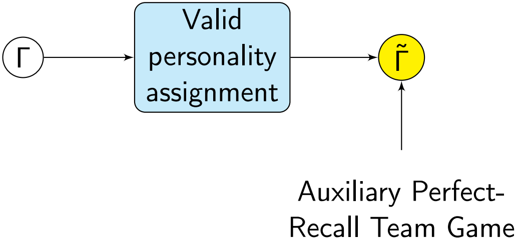

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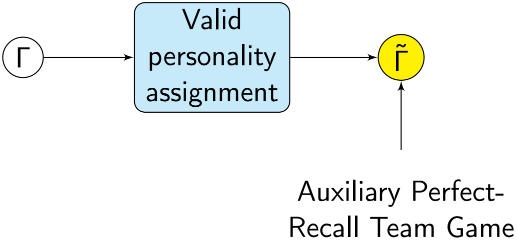

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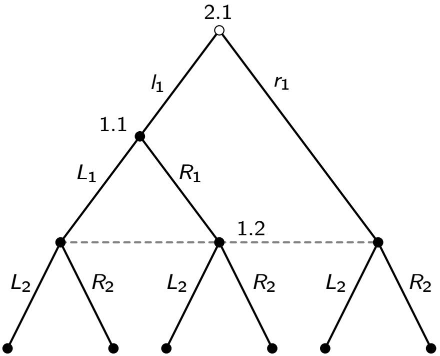
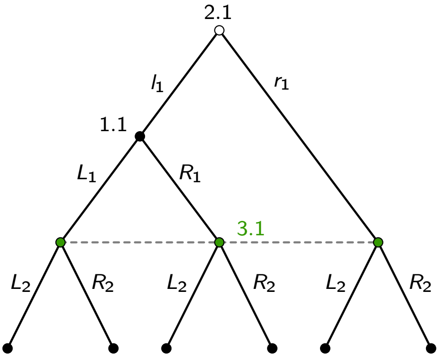
Imperfect-Recall Game

$\mathcal{I}_1 = \{1.1, 1.2\} \quad \mathcal{I}_2 = \{2.1\}$



Auxiliary Team Game

$\tilde{\mathcal{I}}_1^1 = \{1.1\}$ $\quad \tilde{\mathcal{I}}_1^2 = \{3.1\}$ $\quad \mathcal{I}_2 = \{2.1\}$



Inflation

Realization equivalence: Two strategies of player $i \in \mathcal{P}$ are realization equivalent if, for every strategy of the opponents, they induce the same probability distribution over the outcomes of the game

Definition (Immediate Inflation)

Let $\mathcal{I}_i$ and $\mathcal{I}'_i$ be two possible information partitions of player $i \in \mathcal{P}$. We say that $\mathcal{I}'_i$ is an immediate inflation of $\mathcal{I}_i$ iff there exist $I \in \mathcal{I}_i$ and $I_1, I_2 \in \mathcal{I}'_i$ such that:

- $I = I_1 \cup I_2$ and $\mathcal{I}_i \setminus \{I\} = \mathcal{I}'_i \setminus \{I_1, I_2\}$;
- $\forall h_1 \in I_1, h_2 \in I_2$, there exists $\bar{I} \in \mathcal{I}_i \cap \mathcal{I}'_i$ such that $(\bar{I}, a) \in X_i(h_1)$, $(\bar{I}, b) \in X_i(h_2)$ and $a \neq b$.

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Application of Inflation Operation

Σ-equivalence: Two extensive-form games Γ and Γ' are Σ-equivalent if, $\forall i \in \mathcal{P}$, $\forall \sigma_i \in \Sigma_i$ there exists a realization equivalent mixed strategy $\sigma'_i \in \Sigma'_i$, and vice versa

Example 1: we obtain a perfect-recall game after applying inflation

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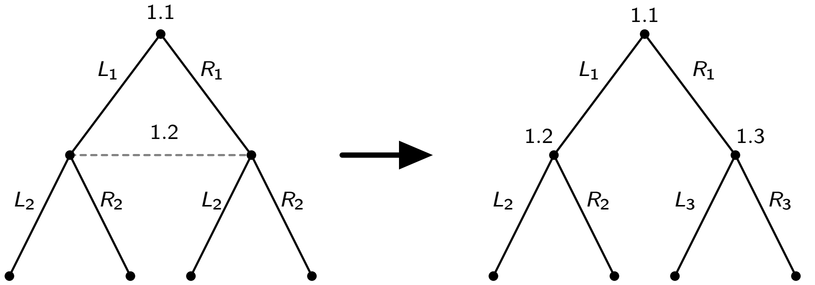
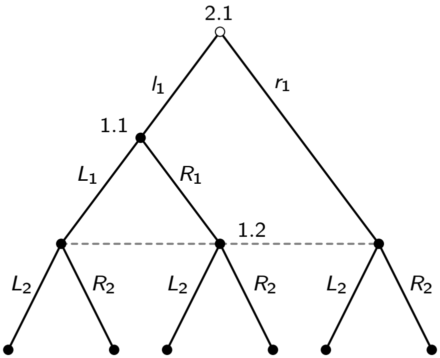


Completely-Inflated Tree

Example 2: inflation cannot transform this imperfect-recall game in a perfect-recall one



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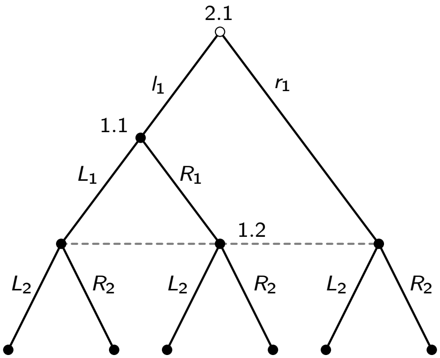

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New Bound on Behavioral Strategies' Inefficiency

The loss incurred by a player when employing behavioral strategies in an **imperfect-recall game** may be linearly large in the size of the game

Previous Inefficiency Bound

Given a two-player game Γ with player 1 having imperfect recall, take two strategy profiles $\sigma \in \times_{i \in \mathcal{P}} \Sigma_i$ and $\pi \in \times_{i \in \mathcal{P}} \Pi_i$ identifying two NE of the game. In the worst case, the ratio u_1^σ / u_1^π is $|Z|/4$.

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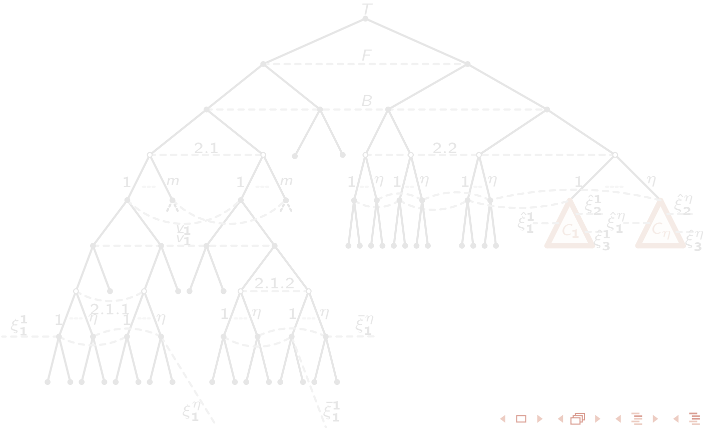
Complexity of MIN-P

- We call MIN-P the problem of finding a valid personality assignment with the minimum possible number of personalities
- Sub-problem: 3-P
- **Reduction from 3-SAT**



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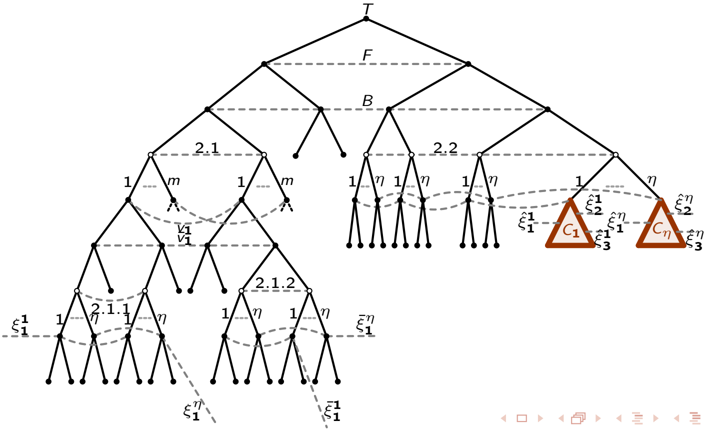

Complexity of MIN-P

3-P over Γ_ϕ is satisfiable if and only if ϕ is satisfiable

Theorem

MIN-P is NP-hard.

Corollary

MIN-P is NP-hard even for the class of completely inflated games.

Theorem

MIN-P is APX-hard, and no better approximation than 4/3 is possible in polynomial time, unless $P = NP$.

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ILP for Minimizing the Number of Personalities

In this section we describe an ILP in order to find a valid minimum personality assignment

$$
\begin{aligned}
\min_{w,P}\ & \mathbb{1}^T \mathbf{w} \\
\text{s.t. } & \mathbb{1}^T P_{\cdot j} = 1 && \forall j \in \{1, \dots, |\mathcal{I}_1|\} \\
& w_i \geq \frac{1}{|\mathcal{I}_1|} P_{i \cdot} \mathbb{1} && \forall i \in \{1, \dots, |\mathcal{I}_1|\} \\
& && \forall i \in \{1, \dots, |\mathcal{I}_1|\}, \\
& P_{i,j} + P_{i,k} - 1 \leq M_{j,k} && \forall (j,k) \in \{1, \dots, |\mathcal{I}_1|\} \times \{1, \dots, |\mathcal{I}_1|\} \\
& && \text{s.t. } k > j \\
& \mathbf{w} \in \{0,1\}^{|\mathcal{I}_1|} \\
& \mathbf{P} \in \{0,1\}^{|\mathcal{I}_1| \times |\mathcal{I}_1|}
\end{aligned}
$$

Conclusions

- The complete inflation of a game can be computed in polynomial time
- MIN-P is NP-hard, and even hard to approximate
- We provided an ILP for minimizing the number of personalities
- We presented an improved column generation approach for finding an optimal mixed-strategy Nash equilibrium in imperfect-recall games
    - new best-response oracle that employs a pruning technique and uses information from the coarsest perfect-recall refinement

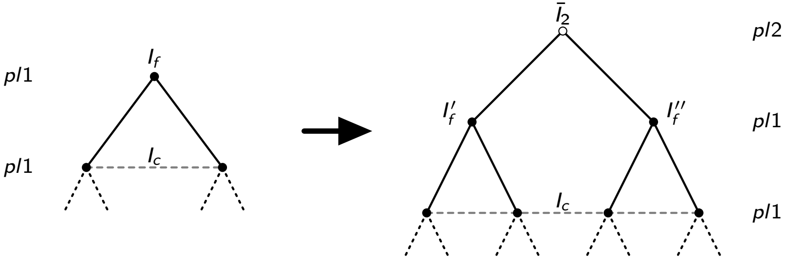
Future Developments

- Evaluate, in a practical scenario, the impact of inflation in terms of *split* information sets
- Our negative results on MIN-P suggest that it would be interesting to study algorithmic techniques for equilibrium computation in team games that are *robust* with respect to the number of team members
- Experimentally evaluate our best-response oracle against current known techniques for the problem

Thanks for the Attention!

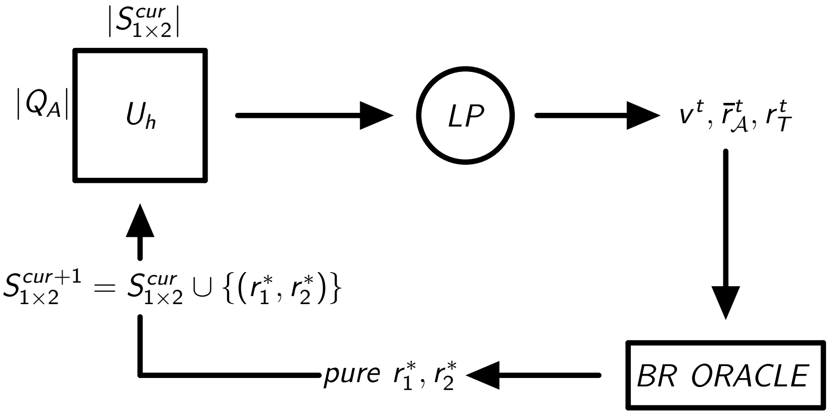
Appendix

Making a Game-Tree Robust with respect to Inflation

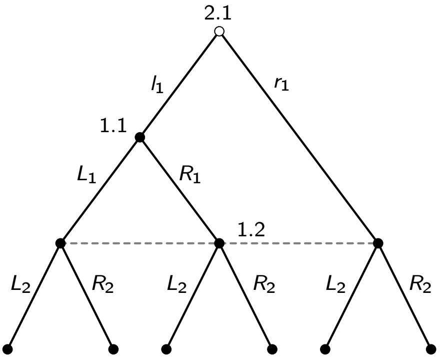


Nash Equilibrium in Imperfect-Recall Games

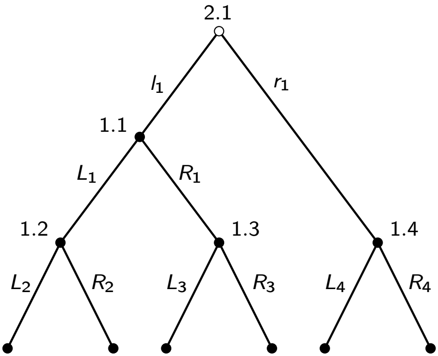
Hybrid Column Generation Approach



From an Imperfect-Recall Game...



...to its Coarsest Outer Perfect-Recall Refinement



Coarsest Outer Perfect-Recall Refinement

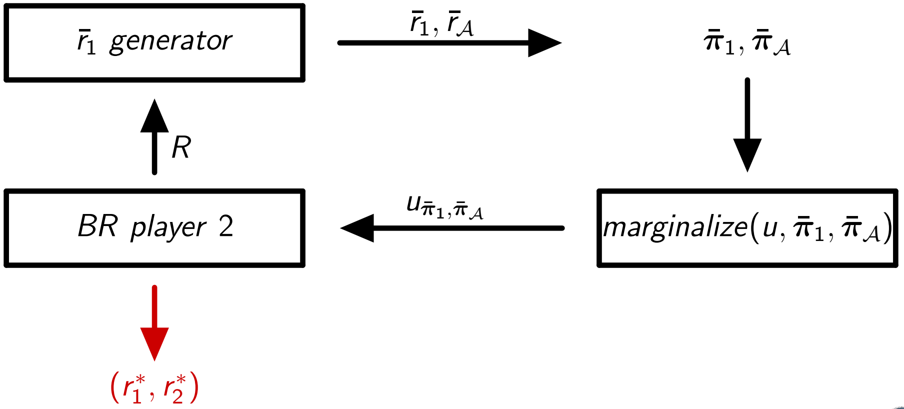
Definition (Coarsest Outer Perfect-Recall Refinement)

The coarsest perfect-recall refinement $\breve{\Gamma}^*$ of the imperfect-recall game Γ is a tuple $\langle \mathcal{P}, A', H, Z, P, \pi_c, u, \mathcal{I}' \rangle$ where $\forall i \in \mathcal{P}$, $\forall I_i \in \mathcal{I}_i$, $H(I_i)$ defines the information set partition $\mathcal{I}'$. A' is a modification of A, which guarantees that $\forall I \in \mathcal{I}'$, $\forall h_k, h_l \in I, A'(h_k) = A'(h_l)$, while for all distinct $I^k, I^l \in \mathcal{I}'$, $\forall a^k \in A(I^k)$, $\forall a^l \in A(I^l)$, $a^k \neq a^l$.

Theorem

Given a generic imperfect-recall game Γ, its coarsest perfect-recall refinement $\breve{\Gamma}^*$ can be computed in polynomial time in the size of the game tree.

Best-Response Oracle



Algorithm 3 Coarsest Outer Perfect-Recall Refinement

function cOPRR(Γ, i) $\triangleright$ $i \in \mathcal{P}$ has imperfect recall
 $O \leftarrow \{I \in \mathcal{I}_i \,|\, \exists h \in I, X_i(h) = \emptyset\}$
 $\mathcal{I}_i^* \leftarrow \mathcal{I}_i \setminus O$
 $\mathcal{L}_i \leftarrow \emptyset$
 $L \leftarrow \emptyset$
 for $I \in \mathcal{I}_i^*$ **do**
  $L \leftarrow I$
  for $h \in I$ **do**
   if $h \in L$ **then**
    $I_h \leftarrow \text{CHECK}(\Gamma, I, h)$
    $L \leftarrow L \setminus \{I_h\}$
    $\mathcal{L}_i \leftarrow \mathcal{L}_i \cup I_h$
   end if
  end for
 end for
 $\bar{\mathcal{I}}_i \leftarrow O \cup \mathcal{L}_i$
 return $\bar{\mathcal{I}}_i$
end function
function CHECK(Γ, I, h)
 $I_h \leftarrow \{h\}$
 $P \leftarrow \{I_i \in \mathcal{I}_i \,|\, \exists a \in A(I_i), (I_i, a) \in X_i(h)\}$
 for $\bar{h} \in I \setminus \{h\}$ **do**
  $\bar{P} \leftarrow \{p \in P | X_i(p, h) \neq \emptyset \wedge X_i(p, \bar{h}) \neq \emptyset\}$
  for $\bar{p} \in \bar{P}$ **do**
   if $X_i(\bar{p}, h) = X_i(\bar{p}, \bar{h})$ **then**
    $I_h \leftarrow I_h \cup \{\bar{h}\}$
   end if
  end for
 end for
 return I_h
end function

Algorithm 5 Player 2's Best Response with Pruning

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function BRP(Γ, v̆, h, u_{π1πA}, LB*, R)                ▷ i ∈ P has perfect recall
    if (h ∈ Z) then
        return u_{π1πA}(h)
    end if
    if (h ∈ I | P(I) = 2) then
        I_2 ← I(h)
        for i = 1 : |A_o(h)| do
            a ← A_o(h)_i
            flag ← 0
            for h̄ ∈ I_2 do
                if (LB*(I_2) > v̆(h̄a)) then
                    flag ← flag + 1
                end if
            end for
            if (flag = |I_2|) then
                R ← R ∪ PRUNE(Γ, I_2, a)
                continue;
            end if
            if (I_2, a) ∉ E then
                LB'(I_2) ← 0
                for h̄ ∈ I_2 do
                    LB'(I_2) ← LB'(I_2)+BRP(Γ, v̆, h̄a, u_{π1πA})
                end for
                if (LB'(I_2) > LB*(I_2)) then
                    LB*(I_2) ← LB'(I_2)
                end if
                E ← E ∪ {(I_2, a)}
            end if
        end for
    end if
    return LB*(I_2)
end function
function PRUNE(Γ, Ī, a)
    I_{h,a} ← {I(ha)}_{h∈Ī}
    Ĥ ← ⋃_{I∈I_{h,a}} {h' ∈ H_1 | ∃b, (I, b) ∈ X_{1,2}(h')}
    R̄ ← ∅
    for I_1 ∈ 𝓘_1 do
        if (∀h ∈ I_1, h ∈ Ĥ) then
            R̄ ← R̄ ∪ {I_1}
        end if
    end for
    return R̄
end function
```