Polynomial chaos expansion for acoustic propagation

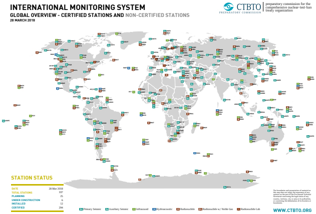
Séminaire des doctorants du CMLA

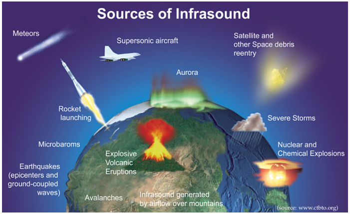
A. Goupy [1,3], D. Lucor [2], C. Millet [1,3]

[1] CMLA, ENS Paris-Saclay - [2] LIMSI, CNRS - [3] CEA, DAM, DIF

Context

- The verification regime of the Comprehensive Nuclear-Test-Ban Treaty (CTBT) is designed to detect any nuclear explosion conducted on Earth – underground, underwater or in the atmosphere.
- Infrasound monitoring is one of the four technologies used by the International Monitoring System (IMS) to verify compliance with the CTBT.

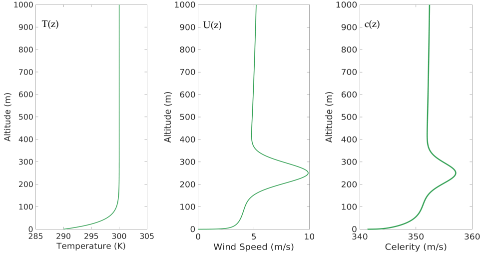




(source: www.ctbto.org)

- Infrasound has the ability to cover long distances with little dissipation.
- Infrasound signals can be severly distorted by the propagation in the atmosphere.

Planetary Boundary Layer



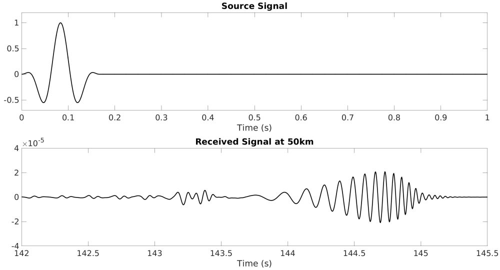
Wave celerity:

$$c(z) = \sqrt{\gamma R T(z)} + U(z)$$

where:

- $T(z)$ is the temperature profile
- $U(z)$ the wind profile.

▶ The celerity profile characterizes the medium for the propagation.



Normal Modes (1/2)

- $(k_i(\omega), \Psi_i(\omega, z))$ eigenvalues and eigenfunctions of H:

$$H\Psi = \frac{\partial^2 \Psi}{\partial z^2} + \frac{\omega^2}{c(z)^2}\Psi = k\Psi$$

- Green function:

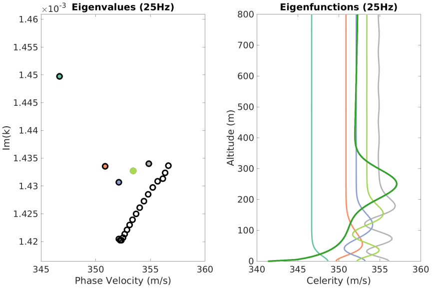
$$G(\omega) = \sum_{i=1}^{N} G_i(\omega) = \sum_{i=1}^{N} \alpha \frac{\Psi_i(\omega, 0)^2}{\sqrt{k_i(\omega)R}} e^{ik_i(\omega)R}$$

with $\alpha = \frac{e^{-i\pi/4}}{\sqrt{8\pi}}$

R: distance source-receiver.

- Signal:

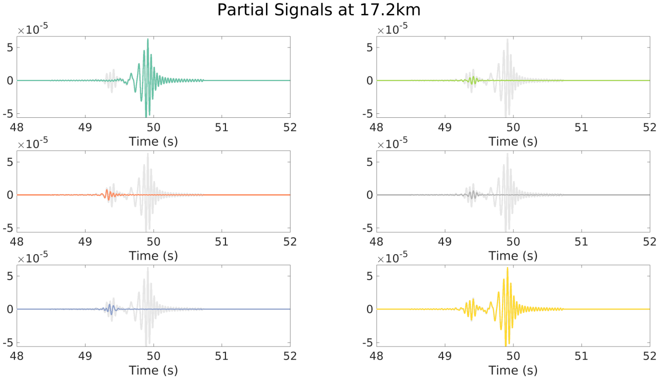
$$p(t) = \mathcal{F}^{-1}[G(\omega)s(\omega)](t)$$



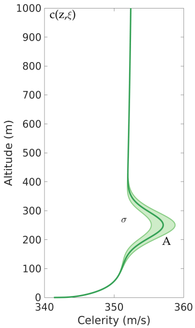
Normal Modes (2/2)

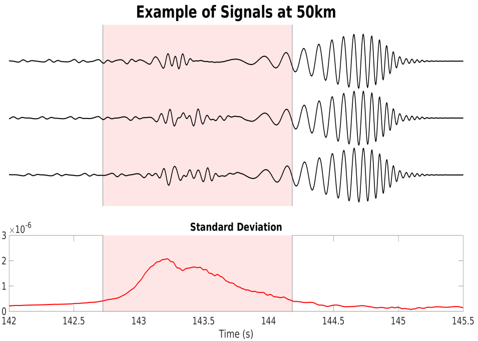
Signal reconstructed with only one mode:

$$p_i(t) = \mathcal{F}^{-1}[G_i(\omega)s(\omega)](t)$$



Random Jet





Impact of the uncertainties on the medium on the acoustic signal received at the ground?

▶ Metamodel of **the eigenpairs of the propagation operator** able to generate such signals.

Polynomial Chaos decomposition (1/2)

Build a metamodel of $X = \mathcal{M}(\boldsymbol{\xi})$ where $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{I}_n)$:

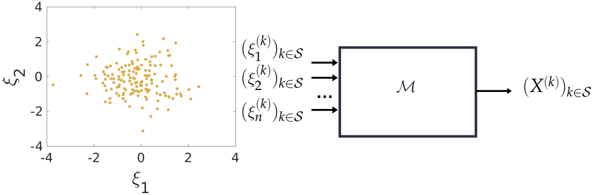
Polynomial chaos decomposition:

- $(H_j)_{j\in J}$ set of orthonormal polynomials for $\langle f, g\rangle = \mathbb{E}[fg]$
- $X(\boldsymbol{\xi}) = \sum_{j\in J} a_j H_j(\boldsymbol{\xi})$ where $a_j = \langle X, H_j\rangle$
- taking polynomials up to degree d, $|J| = \frac{(n+d)!}{n!d!}$

Polynomial Chaos decomposition (2/2)

Coefficients $(a_j)_{j\in J}$ can be computed:

- with Monte-Carlo using a sample $\mathcal{S}$:

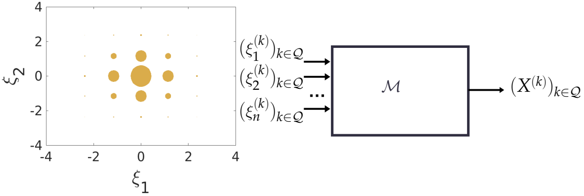


$$a_j = \sum_{k\in\mathcal{S}} X^{(k)} H_j(\boldsymbol{\xi}^{(k)})$$

- $|\mathcal{S}| \approx 3\frac{(n+d)!}{n!d!}$

$n=2, d=10 : |\mathcal{S}| \approx 200$

- with a quadrature $\mathcal{Q}$:



$$a_j = \sum_{k\in\mathcal{Q}} X^{(k)} H_j(\boldsymbol{\xi}^{(k)}) w^{(k)}$$

- $|\mathcal{Q}| = (d+1)^n$

$n=2, d=10 : |\mathcal{Q}| = 121$

gPC decomposition of the acoustic modes

- gPC decomposition of each eigenvalue and eigenvector at the ground

$$\widehat{k_i}(\omega, \boldsymbol{\xi}) = \sum_{j \in J} a_j^{k_i}(\omega) H_j(\boldsymbol{\xi}) \text{ and } \widehat{\Psi_i}(\omega, \boldsymbol{\xi}) = \sum_{j \in J} a_j^{\Psi_i}(\omega) H_j(\boldsymbol{\xi})$$

- Reconstruction of the Green function

$$\widehat{G}(\omega, \boldsymbol{\xi}) = \sum_{i=1}^{N} \widehat{G}_i(\omega, \boldsymbol{\xi}) = \sum_{i=1}^{N} \alpha \frac{\widehat{\Psi_i}(\omega, \boldsymbol{\xi})^2}{\sqrt{\widehat{k_i}(\omega, \boldsymbol{\xi}) R}} e^{i\widehat{k_i}(\omega, \boldsymbol{\xi}) R}$$

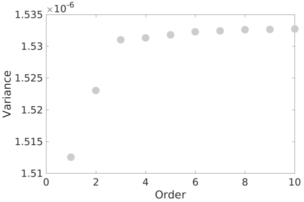
- Simulation of signals using the metamodel:

$$p(t, \boldsymbol{\xi}) = \mathcal{F}^{-1}[\widehat{G}(\omega, \boldsymbol{\xi}) s(\omega)](t)$$

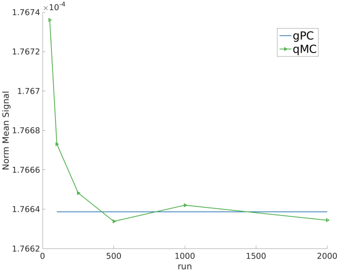
Convergence

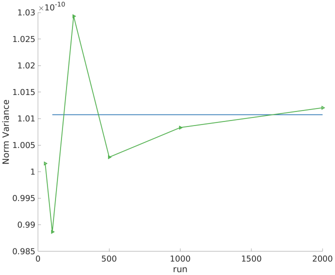
- Variance can be used to control the convergence:

$$Var[k_5] = \sum_{j \in J \setminus \{0\}} (a_j^{k_5})^2$$



- Statistics on the signals can be compared with those obtained by Monte-Carlo simulations:



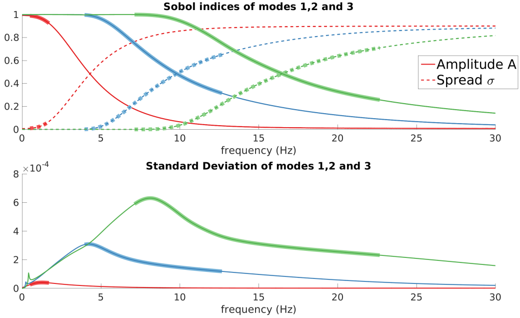


Sensitivity Analysis

- Sobol index $S_j(k_i)$ gives a measure of the sensibility of mode k_i to the parameters ξ_j:

$$S_j(k_i) = \frac{Var(\mathbb{E}[k_i|\xi_j])}{Var(k_i)}$$

- The gPC decomposition allows a quick computation of the Sobol indices: $S_j(k_i) = \sum_{j \in J'} (a_j^{k_i})^2$ where $J' = \{k \in J^* | \exists Q \in \mathbb{R}[X], P_k(\boldsymbol{\xi}) = Q(\xi_j)\}$



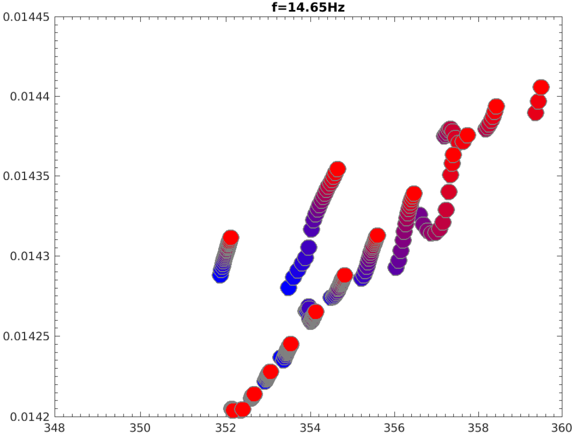
Conclusion:

- gPC representation of the acoustic modes
- Metamodel able to give statistics on the signals
- Sensitivity analysis on the modes

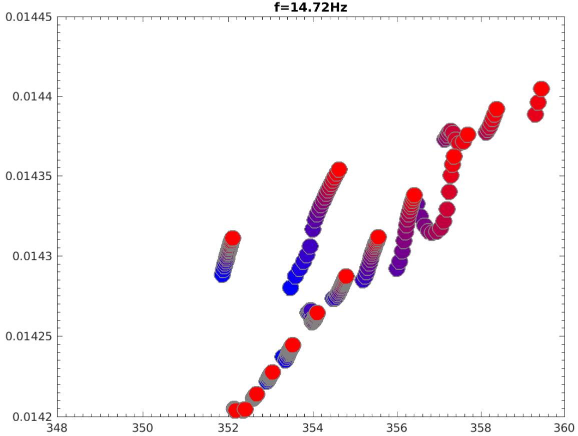
Perspectives:

- Use model reduction to select modes with high acoustical contribution and great sensitivity to the uncertainties.
- Develop an eigenvalue tracking method to generalize to a realistic atmosphere.

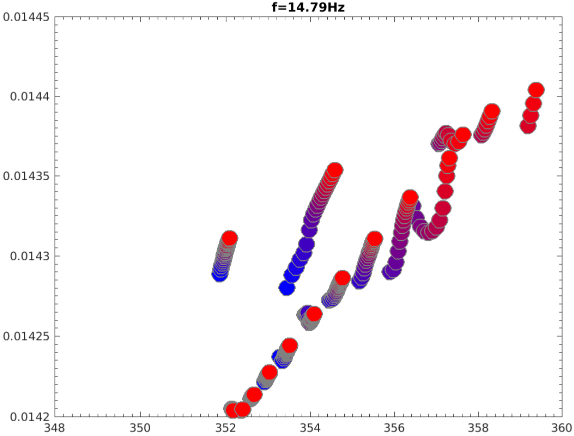
Eigenvalue tracking



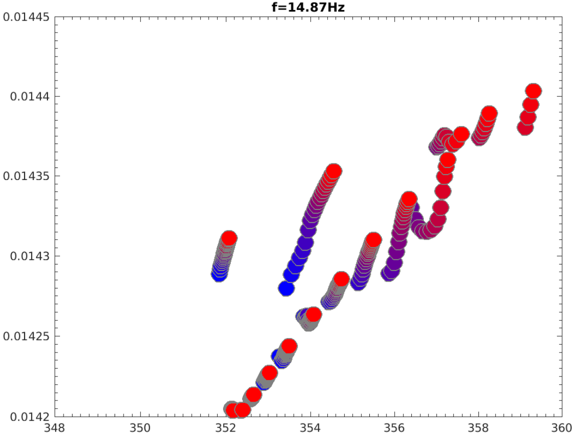
EIGENVALUE TRACKING



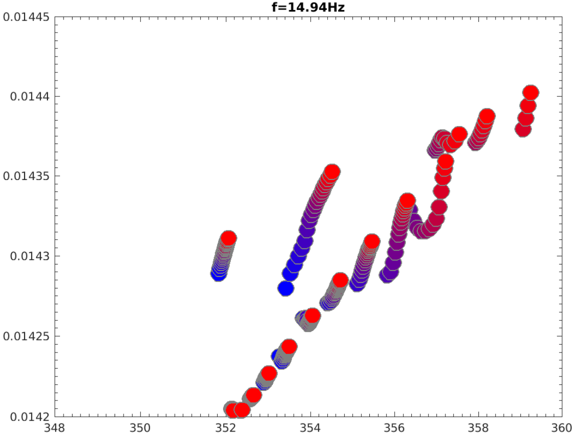
Eigenvalue tracking



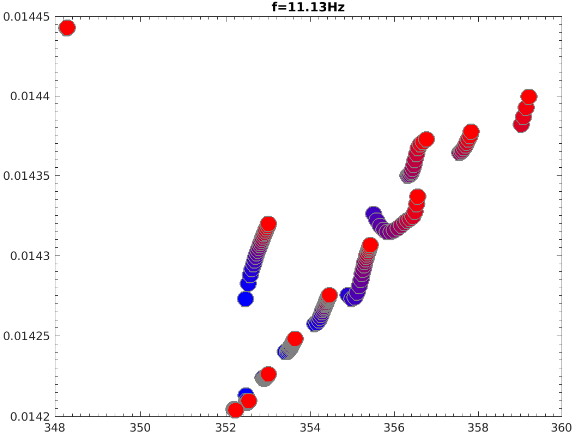
EIGENVALUE TRACKING



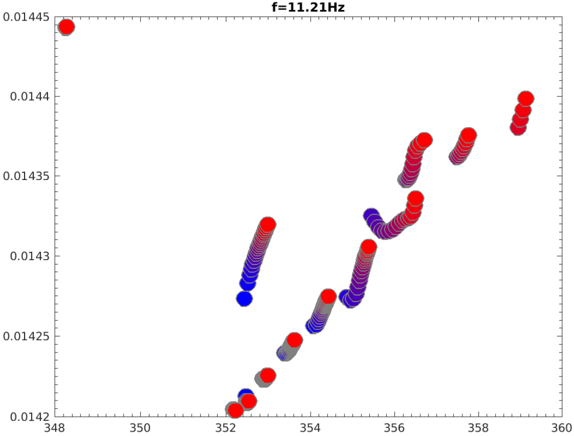
Eigenvalue tracking



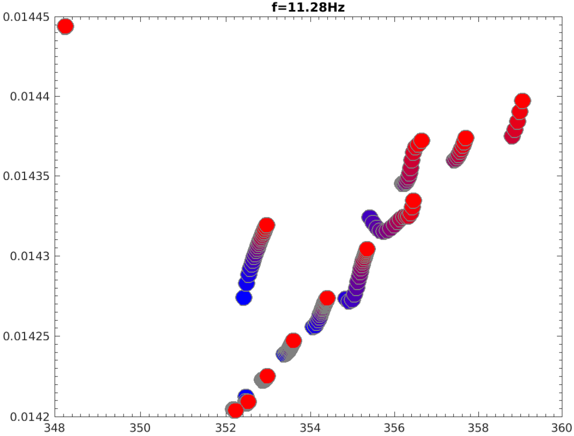
Eigenvalue tracking



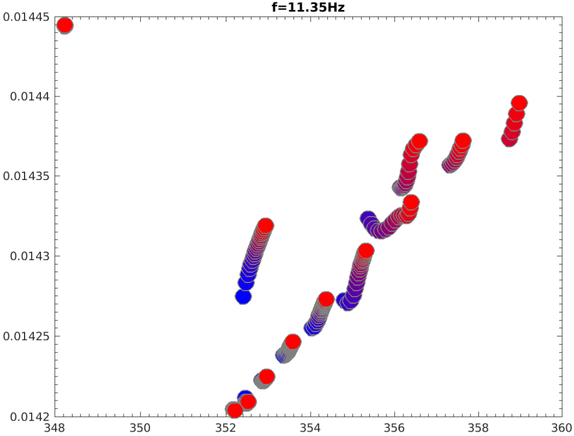
Eigenvalue tracking



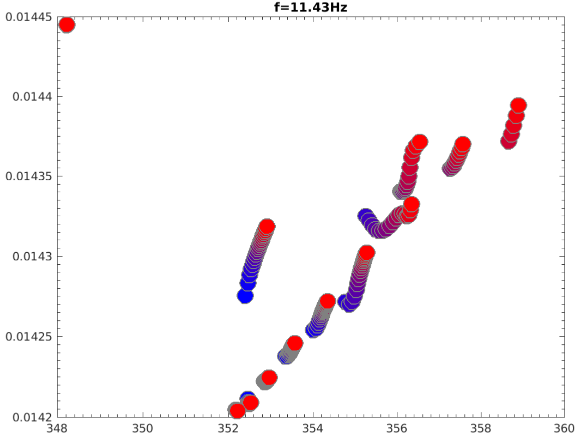
Eigenvalue tracking



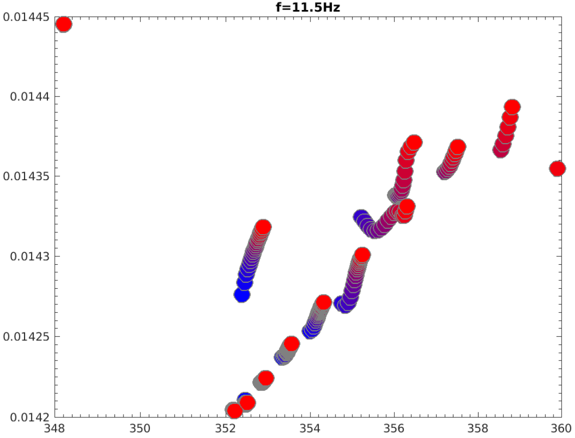
Eigenvalue tracking



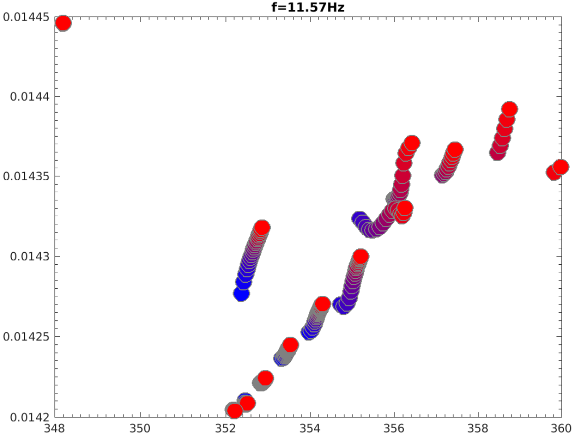
Eigenvalue tracking



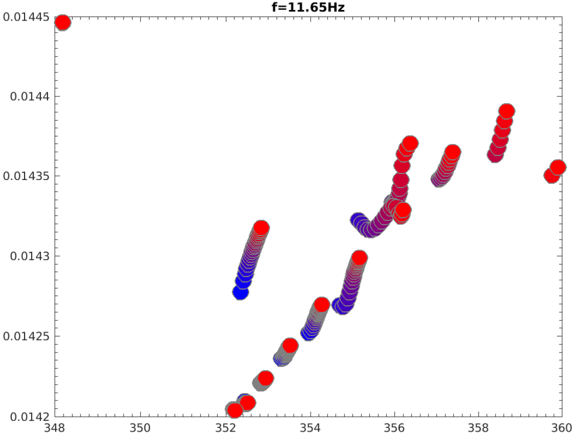
Eigenvalue tracking



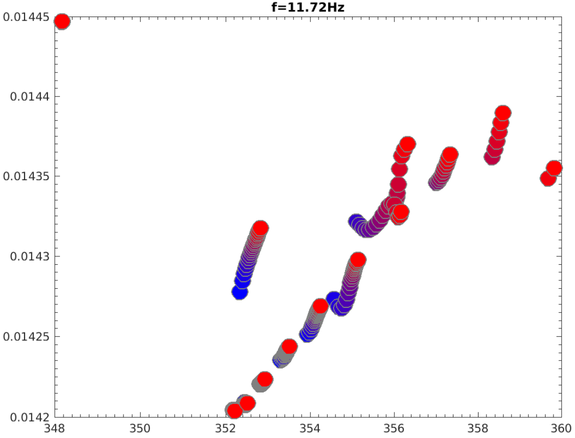
Eigenvalue tracking



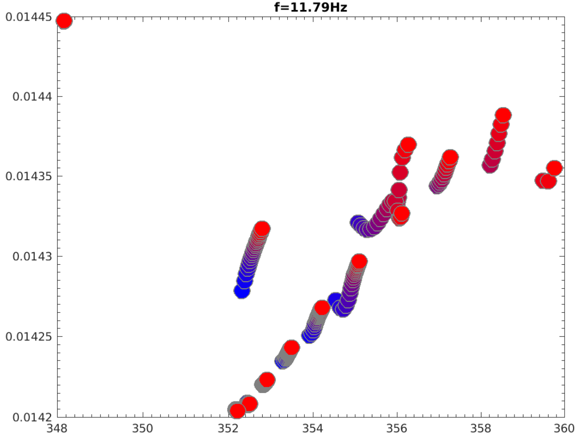
Eigenvalue tracking



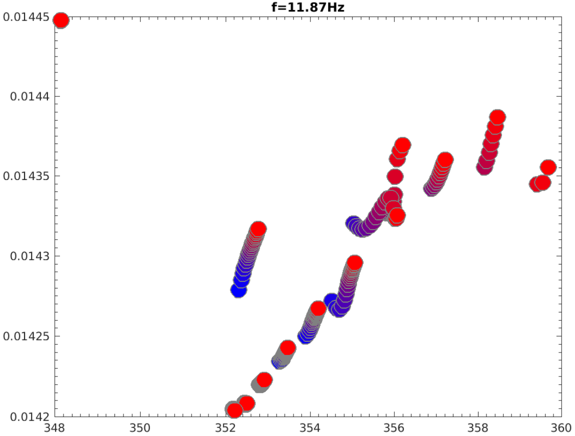
Eigenvalue tracking



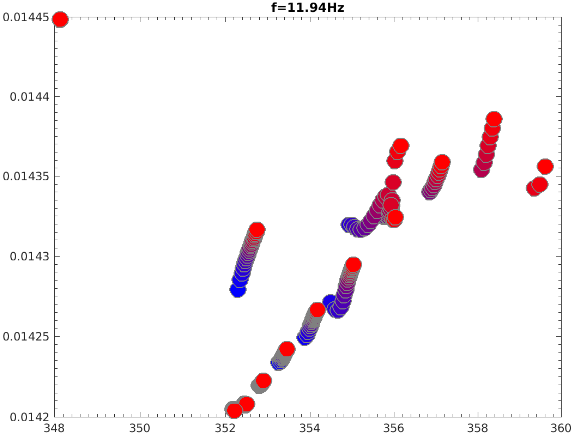
Eigenvalue tracking



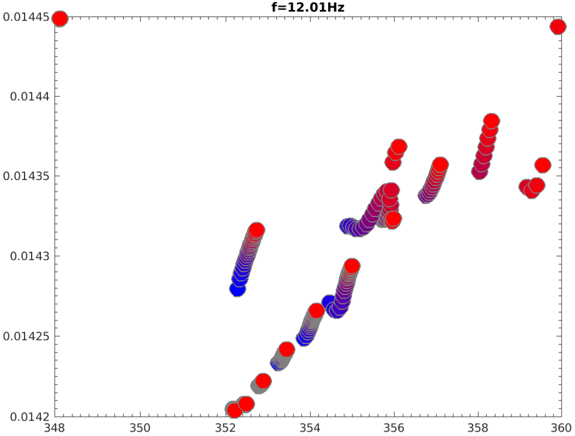
Eigenvalue tracking



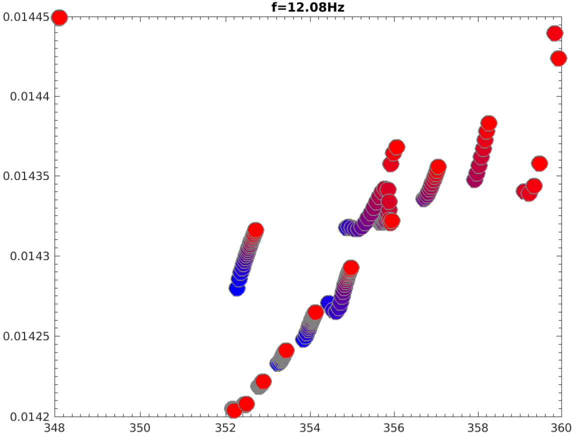
Eigenvalue tracking



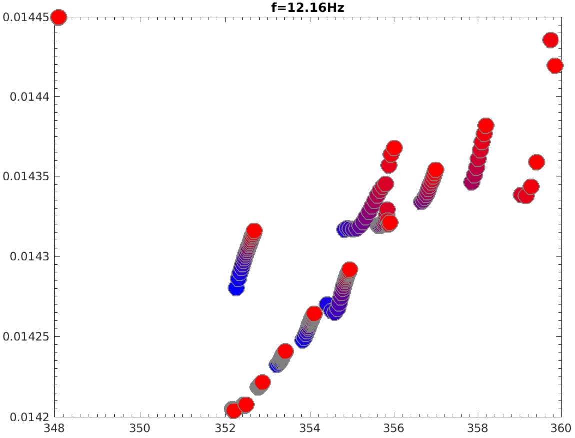
Eigenvalue tracking



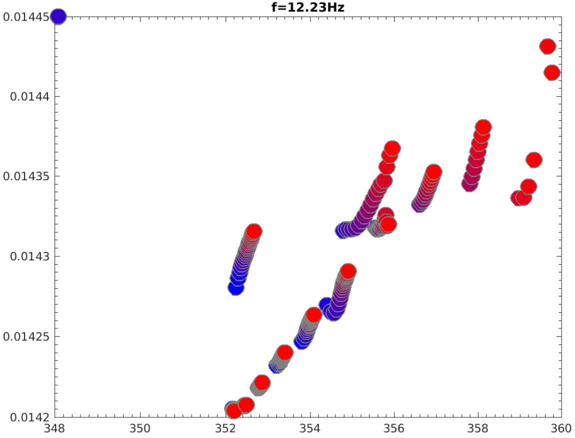
EIGENVALUE TRACKING



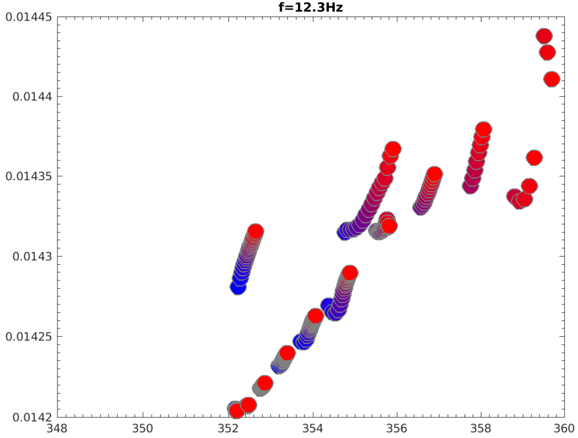
Eigenvalue tracking



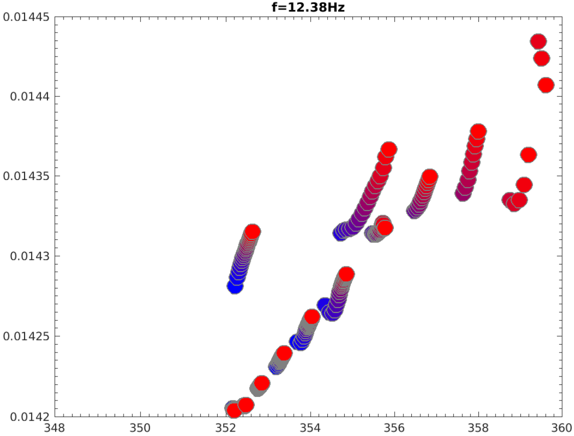
Eigenvalue tracking



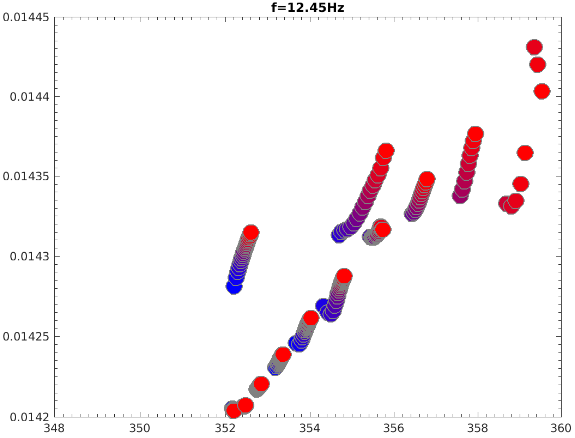
EIGENVALUE TRACKING



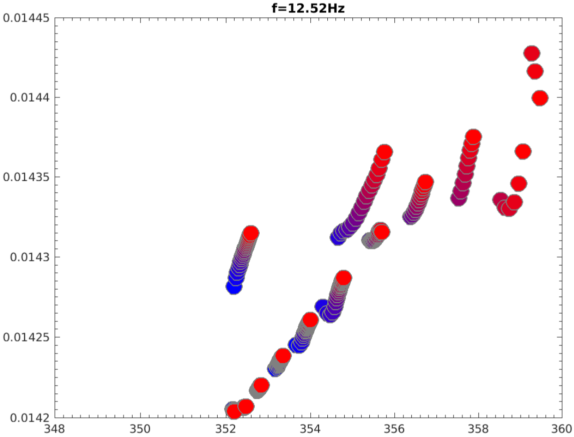
Eigenvalue tracking



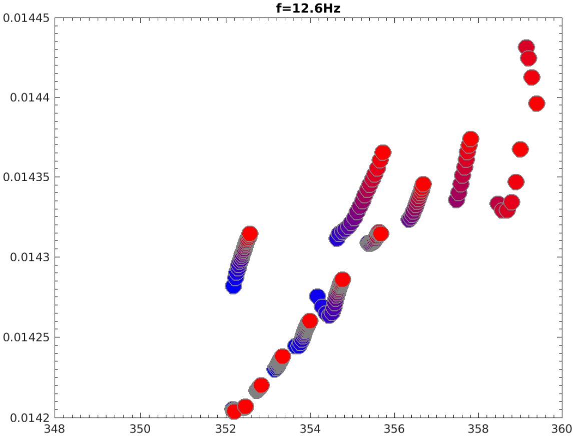
Eigenvalue tracking



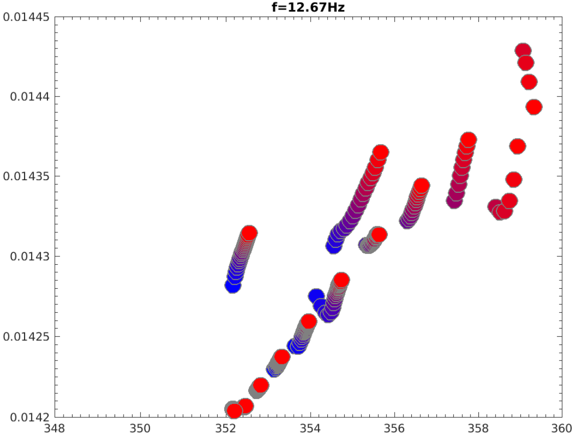
Eigenvalue tracking



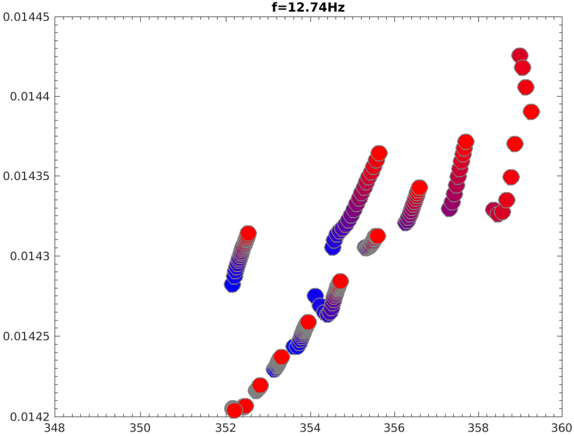
Eigenvalue tracking



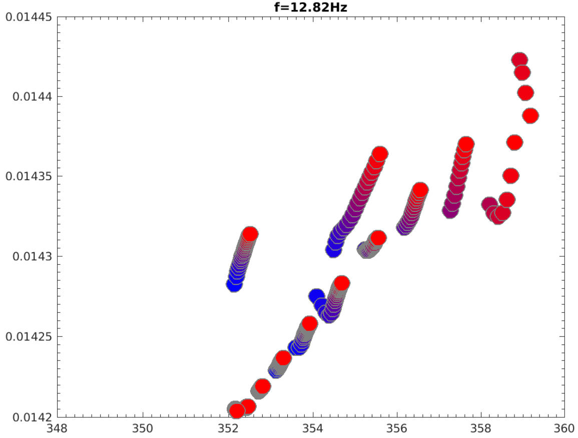
Eigenvalue tracking



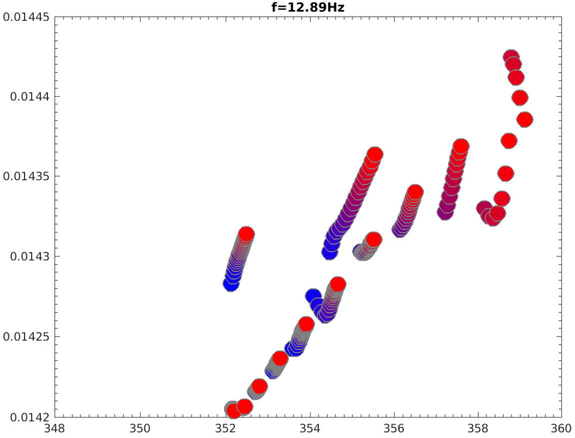
Eigenvalue tracking



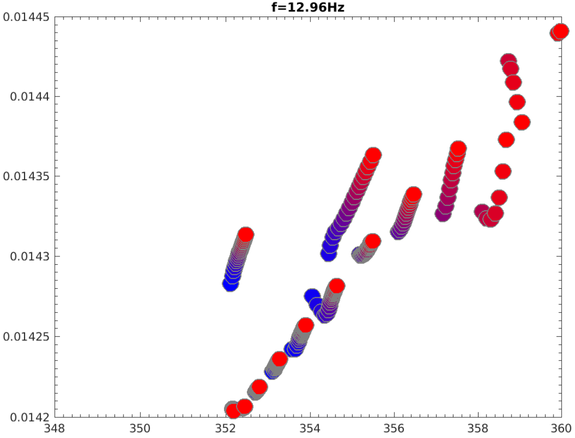
Eigenvalue tracking



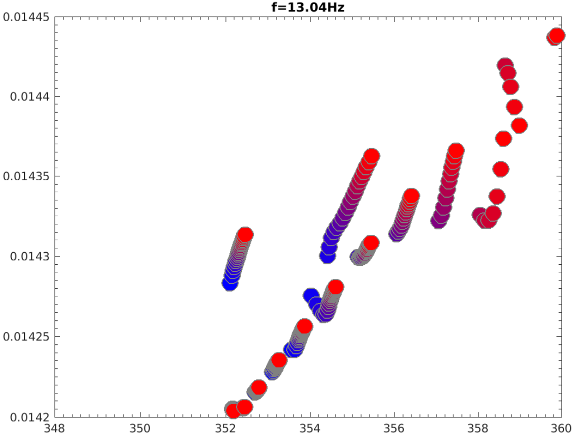
Eigenvalue tracking



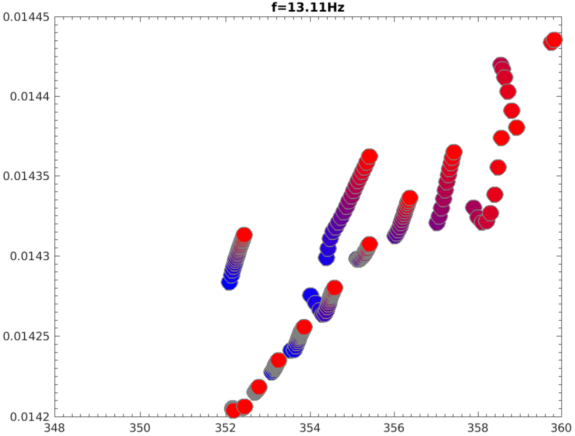
Eigenvalue tracking



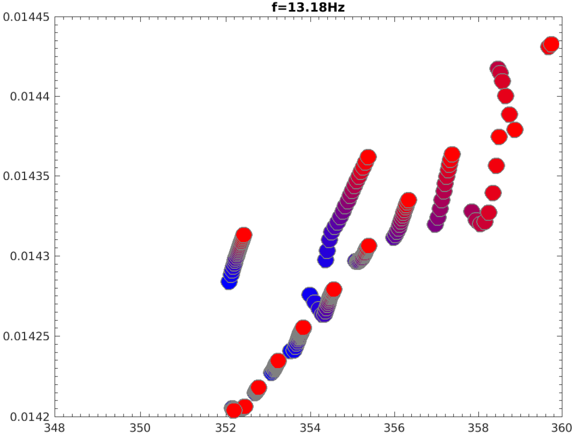
Eigenvalue tracking



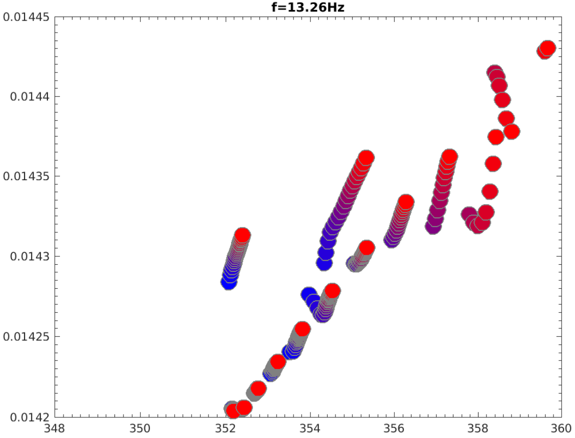
Eigenvalue tracking



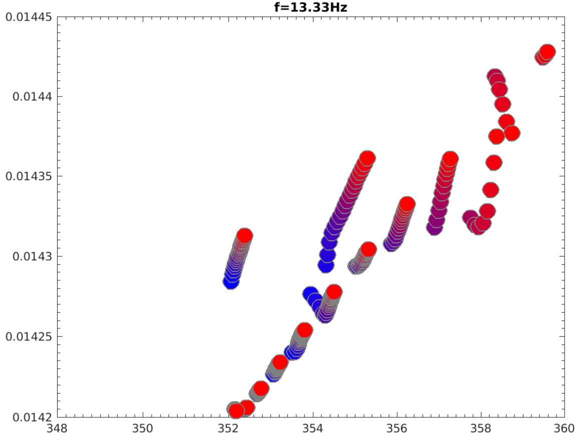
Eigenvalue tracking



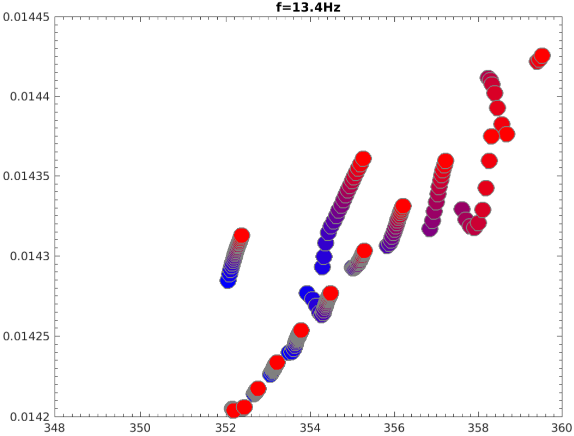
Eigenvalue tracking



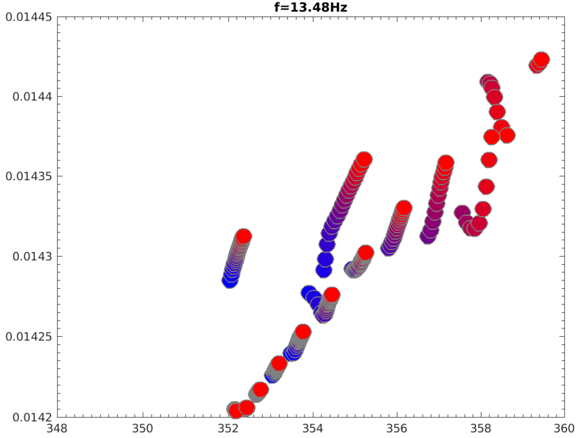
Eigenvalue tracking



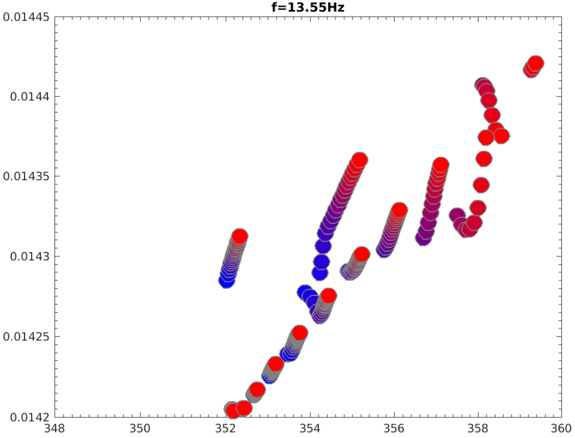
Eigenvalue tracking



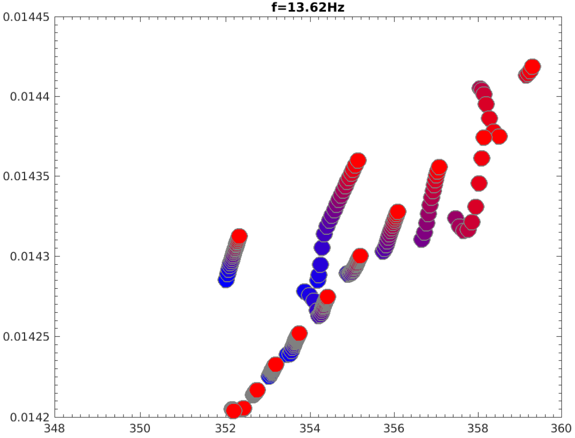
Eigenvalue tracking



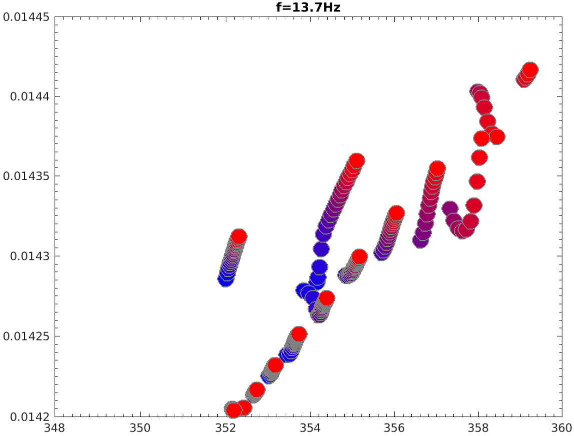
Eigenvalue tracking



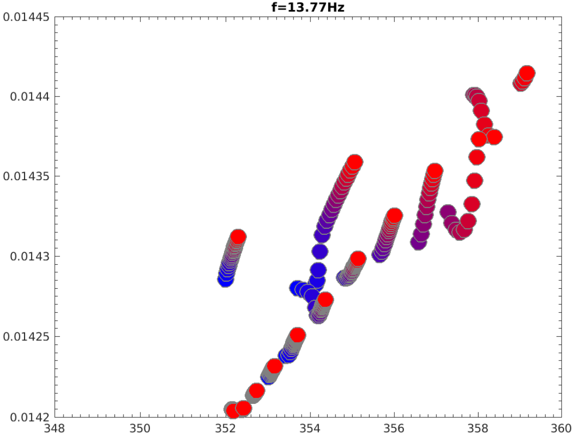
Eigenvalue tracking



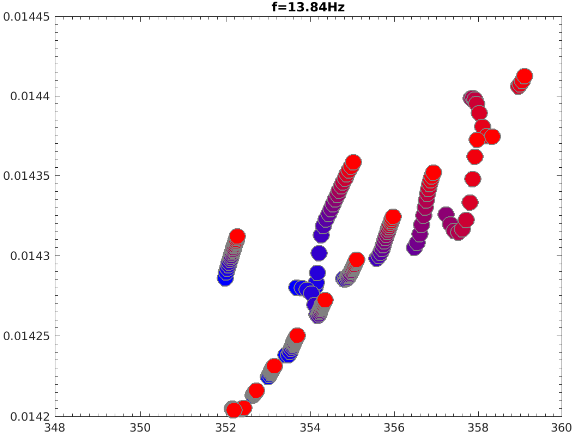
Eigenvalue tracking



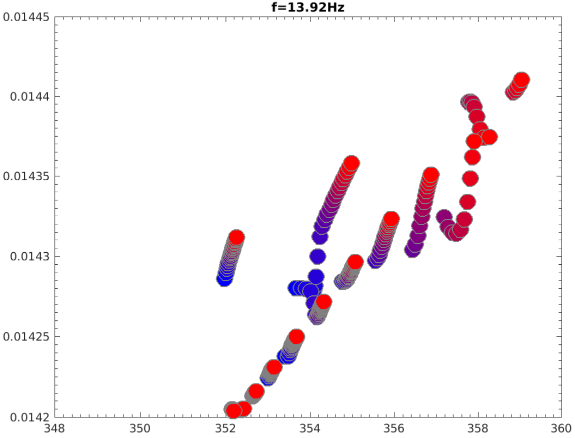
Eigenvalue tracking



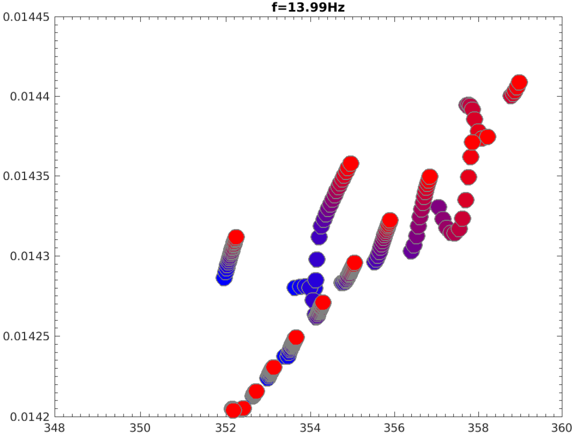
Eigenvalue tracking



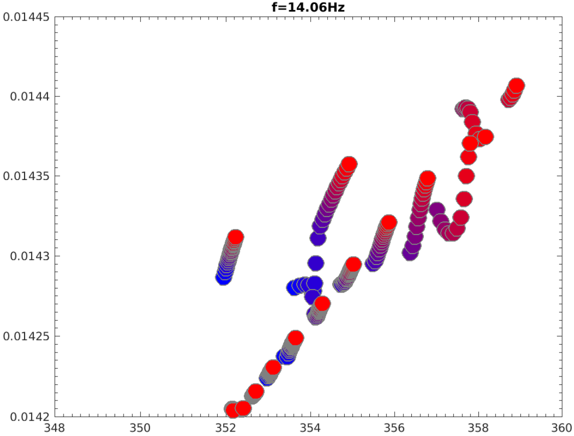
Eigenvalue tracking



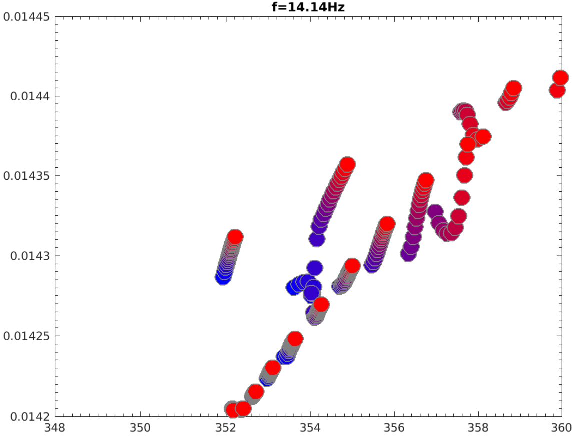
Eigenvalue tracking



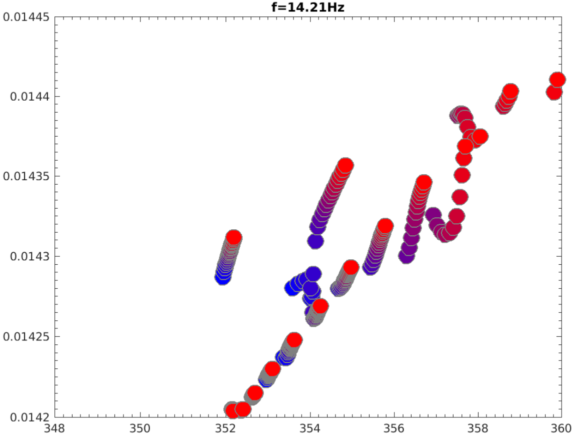
Eigenvalue tracking



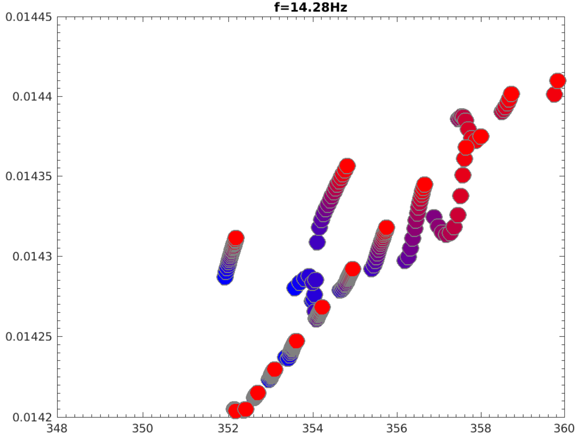
Eigenvalue tracking



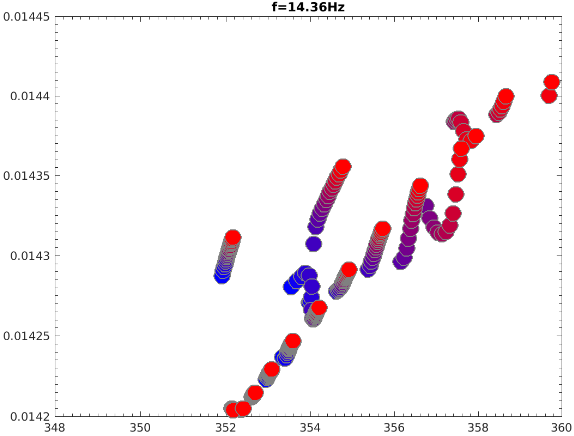
Eigenvalue tracking



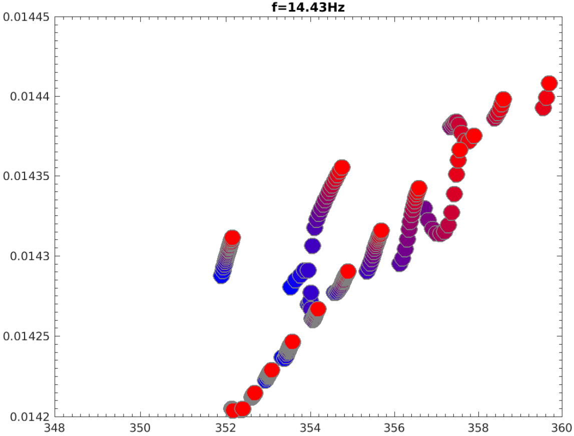
Eigenvalue tracking



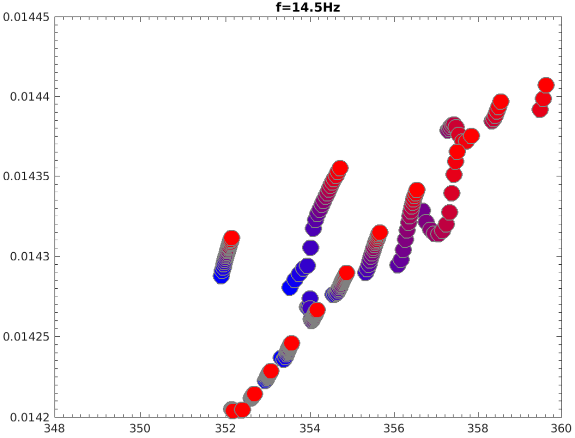
Eigenvalue tracking



Eigenvalue tracking



Eigenvalue tracking



EIGENVALUE TRACKING