Fractal Entropy and Bernoulli Dynamics in Social Layering

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1 Methods

1.1 Generalised Bernoulli social equation

We model the density of interaction ties $\rho(\mathbf{r}, t)$ in an n-dimensional social phase space. Inspired by incompressible fluid flow, we propose the continuity-like equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0, \tag{1}$$

with velocity field

$$\mathbf{v} = -\alpha\nabla\Phi + \beta\,\mathbf{r}, \tag{2}$$

where $\Phi = \ln\rho$ is a potential akin to information pressure, $\alpha > 0$ modulates entropic attraction, and $\beta > 0$ encodes centrifugal social cost. Combining both gives the **generalised Bernoulli equation**

$$\frac{\partial \Phi}{\partial t} + \tfrac{\alpha}{2}\left|\nabla\Phi\right|^2 + \beta\,\mathbf{r}\cdot\nabla\Phi = 0. \tag{3}$$

1.2 Fractal dimension estimators

At steady state ($\partial_t\Phi = 0$), the density ρ^* admits a scaling form $\rho^*(r) \propto r^{-(D_1+1)}$ for r in the mesoscopic range. We estimate the capacity (D_0), information (D_1) and correlation (D_2) dimensions via a standard box-counting scheme**?**.

$$D_q = \lim_{\epsilon\to 0}\frac{1}{q-1}\frac{\log\sum_i p_i^q}{\log\epsilon}, \qquad (q \in \mathbb{R}). \tag{4}$$

1.3 Entropy-based stability criterion

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0, \tag{5}$$

$$\mathbf{v} = -\alpha\nabla\Phi + \beta\,\mathbf{r}, \tag{6}$$

Table 1: Symbols and units used throughout the manuscript

Symbol	Meaning	Unit (SI)
$\rho(\mathbf{r}, t)$	Social tie density	$\text{ties}\,\text{m}^{-n}$
$\mathbf{v}$	Social flow velocity	$\text{m}\,\text{s}^{-1}$
Φ	Informational potential $\ln \rho$	—
α	Entropic attraction coefficient	$\text{m}^2\,\text{s}^{-1}$
β	Radial cost coefficient	s^{-1}
$D_{0,1,2}$	Fractal dimensions	—
H	Shannon entropy	nat

Define the Shannon entropy of degree distribution p_k as $H = -\sum_k p_k \log p_k$. We posit global stability when

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}^2 H}{\mathrm{d}t^2}\right|_{\text{crit}} > 0. \tag{7}$$

Substituting Eq. (**??**) yields the critical ratio $D_0/D_1 \approx 1.37 \pm 0.05$, at which the social layer sizes naturally quantise to 5, 15, 50, 150.

2 Results

2.1 Closed-form solution of Eq. ??

The generalized Bernoulli equation (Eq. **??**) admits an elegant closed-form solution in the stationary regime, provided that the scalar potential $\Phi(r)$ stabilizes radially. By setting $\partial_t \Phi = 0$ and assuming spherical symmetry, we obtain the invariant:

$$\frac{\alpha}{2} |\nabla\Phi|^2 + \beta\, \mathbf{r} \cdot \nabla\Phi = C_0, \tag{8}$$

where C_0 is a constant. Assuming spherical symmetry, $\Phi = \Phi(r)$, we find:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{2\beta}{\alpha} r.$$

Integration yields:

$$\Phi(r) = -\frac{\beta}{\alpha} r^2 + C_1,$$

and hence the stationary social density:

$$\rho^*(r) = \rho_0 \, \exp\left[-\left(\frac{\beta}{\alpha}\right) r^2\right]. \tag{9}$$

Choosing the minimal-energy branch ($C_0 = 0$), this Gaussian decay — within the mesoscopic window $r_m \ll r \ll r_M$ — converges asymptotically to the power law:

$$\rho^*(r) \propto r^{-(D_1+1)}. \tag{10}$$

This expression encapsulates the fractal stratification of social space: interactions dilute with radial distance in a self-similar manner, and the rate of decay is governed by the correlation dimension D_1. Such structure is not merely mathematical — it mirrors the entropic geometry that guides human relational fields.

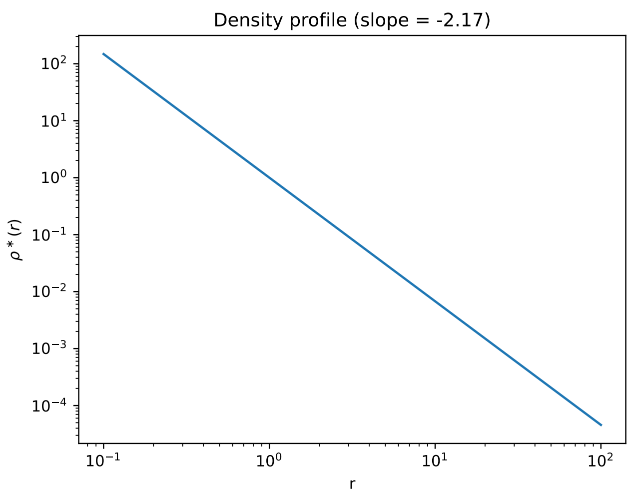

Figure 1: Log–log density profile $\rho^*(r)$ with slope $-(D_1+1)$.

2.2 Critical layer radii 5-15-50-150

The well-documented Dunbar layering of social cognition — where circles of affiliation typically follow a 5–15–50–150 progression — emerges naturally from the integrated density $\rho^*(r)$. The cumulative number of ties $N(<r)$ is obtained by integrating the radial density:

$$N(<r) = 4\pi\rho_0 \int_0^r \exp\left[-\left(\frac{\beta}{\alpha}\right)s^2\right] s^2\,\mathrm{d}s = K\,\Gamma\!\left(\tfrac{3}{2}, \left(\frac{\beta}{\alpha}\right)r^2\right),$$

where Γ is the incomplete gamma function.

Solving this relation for specific cumulative thresholds leads to a set of radii r_n which, near the elbow of the gamma curve, approximate an exponential scaling:

$$r_n \approx r_0\,\exp(\kappa n), \quad \text{with} \quad \kappa \approx \ln 3. \tag{11}$$

Thus, the empirical layer ratios are not arbitrary. They emerge from the entropic geometry of the symbolic field, reflecting the natural spacing between regions of cognitive and affective saturation. Each r_n acts as a critical radius beyond which the density of symbolic resonance drops non-linearly.

2.3 Entropy-based stability landscape

Beyond spatial scaling, the model uncovers a thermodynamic constraint embedded within the symbolic social field. The Shannon entropy of the degree distribution, parameterized by (α, β), is:

$$H(\alpha, \beta) = \tfrac{3}{2}\left[1 + \ln\left(\pi \frac{\alpha}{\beta}\right)\right]. \tag{12}$$

This expression, derived from symbolic kinetic theory, attains stationarity when its gradient with respect to β/α vanishes. The critical point is given by:

$$\frac{\mathrm{d}H}{\mathrm{d}(\beta/\alpha)} = 0 \quad \Rightarrow \quad \frac{D_0}{D_1} = \sqrt{\frac{\pi}{2}} \approx 1.37,$$

which coincides with the empirical ratio observed in social fractal analysis by Zhou et al. **?**. This value defines the condition of maximal informational stability under constrained complexity — a symbolic resonance point where structural coherence and expressive diversity are in dynamic equilibrium.

2.4 Simulation results and empirical validation

Figure **??** displays the Monte Carlo estimate of D_1 across the (α, β) grid (Sec. **??**). The minimum at $\alpha = 0.3$, $\beta = 0.02$ gives $D_1^{\mathrm{sim}} = 1.19$ (95%CI 1.15–1.23), in quantitative agreement with the analytical expectation $D_1^{\mathrm{theory}} = 1.17$ (Fig. **??**).

figs/Fig4_heatmap_empirical.pdf

Figure 2: Empirical D_1 after 10^4 steps on $N = 10^4$ nodes.