Q1. Prove that for any integer n, the expression $n^3 - n$ is always divisible by 6.

Q2. Prove by contradiction that $\log_2(3)$ is an irrational number.

Q3. Prove that for any integer n, the square n^2 leaves a remainder of either 0 or 1 when divided by 3

Q4. Prove that the number 23 cannot be written as the sum of three distinct square numbers.

Q5. For any three non-zero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then it must follow that $\mathbf{b} = \mathbf{c}$. Prove this fact, else find a counterexample to disprove it.

Q6. Prove that the sum of any rational number and any irrational number is always an irrational number.

Q7. Prove by mathematical induction that the sum of the squares of the first n positive integers is given by the formula:

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Q8. Prove by mathematical induction that for all positive integers n, the expression $3^{2n} - 1$ is divisible by 8.

Q9. A sequence is defined by the recurrence relation $u_{n+1} = 4u_n - 3$, with the first term $u_1 = 2$. Prove by mathematical induction that the n^{th} term of the sequence is given by the formula:

$$u_n = 4^{n-1} + 1$$

Q10. Given the matrix $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, prove by mathematical induction that for all positive integers n:

$$A^n = \begin{pmatrix} 1 & 3n \\ 0 & 1 \end{pmatrix}$$

Using basis vectors, work out what transformation matrix A represents, and thus what matrix A^n represents. Does this agree with the result from your proof by induction?

Q11. Prove that for any real numbers x, y, and z:

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

Q12. Prove that for all $x > 0$, the inequality $e^x > 1 + x$ holds true.

Q13. For every integer $n \geq 0$, the expression $n^2 + n + 41$ produces a prime number. Prove this fact, else find a counterexample to disprove it.

Q14. Prove that there are no integer solutions (x, y) to the equation $x^2 - 4y = 2$.

Q15. Prove by mathematical induction that for all integers $n \geq 5$, the following inequality holds:

$$2^n > n^2$$