OCR Further Mathematics A (2018)

4.01 Proof by Induction

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1 Preface

1.1 Introduction

Proof by induction is a type of mathematical proof often used to prove an infinite number of related statements using a carefully constructed chain reaction. It can be used to prove statements about sums, divisibility rules and inequalities, to name a few. A proof by induction has three main parts:

1. Base case - this is the trivial example that you prove first to set up the rest of the chain reaction
2. Assumption step - this is an assumption that the statement is true for an arbitrary value, in preparation for the inductive step
3. Inductive step - the inductive step uses the assumption to deduce 'if one domino falls, the next will always fall too'

1.2 Worked Example

Prove that:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \qquad (*)$$

for all $n \in \mathbb{N}$.

Proof. (Base case) Let $n = 1$. Then,

$$\text{LHS} = 1, \quad \text{RHS} = (1)^2 = 1.$$

Hence LHS = RHS and $(*)$ is true for $n = 1$. *(Assumption step)* Assume $(*)$ is true for $n = k$, i.e.

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

(Inductive step) Now consider $n = k + 1$,

$$\text{LHS} = 1 + 3 + 5 + \cdots + (2k-1) + (2k+1).$$

Using the assumption,

$$= k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2.$$

Hence LHS = RHS and $(*)$ is true for $n = k + 1$. By the principle of mathematical induction, $(*)$ is true for all $n \in \mathbb{N}$. $\square$

Explanation

We begin the proof by verifying the base case. The question wants us to prove $(*)$ is true for all natural numbers $n \in \mathbb{N}$ i.e. $1, 2, 3, \ldots$ and so on. 1 is the first natural number and so we use it as our base case. We can verify the base case by checking that $(*)$ is satisfied when $n = 1$:

(Base case) Let $n = 1$. Then,

$$\text{LHS} = 1, \quad \text{RHS} = (1)^2 = 1.$$

Hence LHS = RHS and $(*)$ is true for $n = 1$.

We then assume that $(*)$ is true for an arbitrary value of $n = k$. This is to prepare for our inductive step.

(Assumption step) Assume $(*)$ is true for $n = k$, i.e.

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

The inductive step aims to prove that $(*)$ is true for $n = k + 1$, if $(*)$ is true for $n = k$. This is helpful because we have already proven the base case i.e. $(*)$ is true for $n = 1$. If we manage to prove the inductive step then $n = 1$ is true $\implies$ $n = 2$ is true $\implies$ $n = 3$ is true, and so on, and then we have proved $(*)$ for all $n \in \mathbb{N}$ as required.

(Inductive step) Now consider $n = k + 1$,

$$\text{LHS} = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1).$$

We need to show that the LHS = RHS = $(n+1)^2$. We need to use the assumption to prove this. Sometimes, it might not be immediately obvious how to use the assumption to advance the proof, but in this case it is a simple substitution:

Using the assumption,

$$= k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.$$

Hence LHS = RHS and $(*)$ is true for $n = k + 1$.

We have shown that the LHS = RHS using our assumption and so we have demonstrated the following:

1. Base case ($(*)$ is true for $n = 1$)
2. Inductive step (if $(*)$ is true for $n = k$ then $(*)$ is true for $n = k + 1$)

Now we are able to conclude our proof:

By the principle of mathematical induction, $(*)$ is true for all $n \in \mathbb{N}$.

2 Examples

2.1 Sums and Series

Prove that:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{$*$}$$

for all $n \in \mathbb{N}$.

Proof. Let $n = 1$. Then,

$$\text{LHS} = (1)^2 = 1, \quad \text{RHS} = \frac{(1)((1)+1)(2(1)+1)}{6} = \frac{(1)(2)(3)}{6} = 1.$$

Hence LHS = RHS and $(*)$ is true for $n = 1$. Assume $(*)$ is true for $n = k$, i.e.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Now consider $n = k + 1$,

$$\text{LHS} = 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2.$$

Using the assumption,

$$\begin{aligned} &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(2k^2 + k + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \end{aligned}$$

Hence LHS = RHS and $(*)$ is true for $n = k + 1$. By the principle of mathematical induction, $(*)$ is true for all $n \in \mathbb{N}$. □

2.2 Divisibility Rules

Prove that:

$$5^n - 1 \text{ is divisible by } 4. \tag{$*$}$$

for all $n \in \mathbb{N}$.

Proof. Let $n = 1$. Then,

$$5^1 - 1 = 4 \text{ which is divisible by } 4.$$

Hence $(*)$ is true for $n = 1$. Assume $(*)$ is true for $n = k$, i.e.

$$5^k - 1 \text{ is divisible by } 4.$$

Now consider $n = k+1$; we want to show that $5^{k+1} - 1$ is divisible by 4. Using the assumption,

$$5^k - 1 \text{ is divisible by } 4 \quad \text{i.e.} \quad 5^k - 1 = 4a \text{ for some } a \in \mathbb{Z}$$
$$\implies 5(5^k - 1) = 4(5a)$$
$$\implies 5^{k+1} - 5 = 4(5a)$$
$$\implies 5^{k+1} - 1 = 4(5a+1) \text{ which is divisible by 4, as required}$$

Hence $(*)$ is true for $n = k+1$. By the principle of mathematical induction, $(*)$ is true for all $n \in \mathbb{N}$. $\square$

2.3 Inequalities

Prove that:
$$2^n > n^2 \qquad (*)$$
for all $n \geq 5, n \in \mathbb{Z}$.

Proof. Let $n = 5$. Then,
$$2^5 = 32 > 25 = 5^2$$
Hence $(*)$ is true for $n = 5$. Assume $(*)$ is true for $n = k$, i.e.
$$2^k > k^2$$
Now consider $n = k+1$,
$$\text{LHS} = 2^{k+1}$$
Using the assumption,
$$2 \times 2^k > 2k^2$$
$$\implies 2^{k+1} > 2k^2$$
$$\implies 2^{k+1} > (k\sqrt{2})^2 > (k+1)^2 \text{ for } k \geq 5$$
Hence $(*)$ is true for $n = k+1$. By the principle of mathematical induction, $(*)$ is true for all $n \geq 5, n \in \mathbb{Z}$. $\square$

2.4 Matrices

Let
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Prove that:
$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \qquad (*)$$
for all $n \in \mathbb{N}$

Proof. Let $n = 1$. Then,

$$\text{LHS} = A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{RHS} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hence LHS = RHS and $(*)$ is true for $n = 1$. Assume $(*)$ is true for $n = k$, i.e.

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Now consider $n = k + 1$,

$$\text{LHS} = A^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1}.$$

Using the assumption,

$$\begin{aligned} A^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence LHS = RHS and $(*)$ is true for $n = k + 1$. By the principle of mathematical induction, $(*)$ is true for all $n \in \mathbb{N}$. □

3 Problems

1. Prove that for all $n \in \mathbb{N}$:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2} \right)^2 \qquad (*)$$

2. Prove that for all $n \in \mathbb{N}$:

$$2^{3n} - 1 \text{ is divisible by } 7. \qquad (*)$$

3. Prove that for all $n \in \mathbb{N}$:

$$n! > 2^n \quad \text{for all } n \geq 4 \qquad (*)$$

4. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

   Prove by induction that for all $n \in \mathbb{N}$,

$$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \qquad (*)$$