1 Introduction

- Number of questions: 60
- Time needed: 4 hours (ish)

Instructions:

- Do as many of the questions as you can. There will be many that you will be able to do without any difficulty, and a couple you will have no idea how to begin. That is okay.
- For each question, write out as much working as is necessary. If you are stuck, try a different strategy. Write down all your thoughts and working, complete or incomplete.
- If there are any words, concepts or whole questions you have not seen yet, don't worry, we can talk about them. You should still try these questions, you may be surprised.

2 Questions

Q1. Prove that the square of any odd integer is always 1 more than a multiple of 8.

Q2. Prove that for $n \in \mathbb{Z}$ and $1 \leq n \leq 4$, $n^2 + 1$ is not divisible by 3.

Q3. Prove that the statement "if n^2 is a multiple of 4, then n is a multiple of 4" is false.

Q4. Prove that $\sqrt{2}$ is irrational.

Q5. Given the statement P: "If n is a prime number greater than 2, then n is odd," state the converse of P and determine, with a reason, whether the converse is true or false.

Q6. Solve the equation $2^{2x+1} - 9(2^x) + 4 = 0$.

Q7. Express $\frac{5+\sqrt{3}}{2-\sqrt{3}}$ in the form $a + b\sqrt{3}$, where a and b are integers.

Q8. The quadratic equation $kx^2 + 4x + (k - 3) = 0$ has real roots. Find the range of possible values for the constant k.

Q9. Inequalities Solve the inequality $\frac{2x-5}{x+3} < 1$.

Q10. Let $A = \{x : x^2 - 5x + 6 = 0\}$ and $B = \{x : (x - 1)(x - 2)(x - 3) = 0\}$. List the elements of $A \cap B$ and $A \cup B$.

Q11. The polynomial $p(x) = x^3 + ax^2 + bx + 6$ has a turning point at $x = 1$. Given $p(2) = 0$, find the values of a and b.

Q12. Use polynomial long division to find the quotient and remainder when $3x^4 - 2x^2 + 5x - 1$ is divided by $x^2 + 2$.

Q13. Given that $(x - 3)$ is a factor of $x^3 + kx^2 - 7x + 6$, find the value of k and factorise the polynomial completely.

Q14. Solve the equation $|3x - 2| = |x + 4|$.

Q15. Sketch the graph of $y = \frac{2x+1}{x-1}$, showing clearly the equations of any asymptotes and the coordinates of the points where the graph crosses the axes.

Q16. State the domain and range of the function $f(x) = \ln(x - 2) + 3$.

Q17. Given $f(x) = e^x + 2$ and $g(x) = \ln(x)$, find $f(g(x))$ and find an expression for $f^{-1}(x)$, stating its domain.

Q18. The graph of $y = \sin(x)$ is translated by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and then stretched horizontally by a scale factor of 2. Write down the equation of the resulting graph.

Q19. Express $\frac{5x^2-2x-1}{(x-1)(x+1)^2}$ in partial fractions.

Q20. Find the equation of the line that passes through the point $(4, -1)$ and is perpendicular to the line $3x - 4y + 8 = 0$.

Q21. A circle has center $(3, -2)$ and passes through the point $(7, 1)$. Find the equation of the circle and determine if the point $(0, 2)$ lies inside, on, or outside the circle.

Q22. A particle moves such that its position at time t is given by $x = 2t^2, y = 4t$. Show that the Cartesian equation of the path is $y^2 = 8x$ and sketch the path for $t \geq 0$.

Q23. Find the coefficient of x^3 in the expansion of $(2 - \frac{x}{4})^{10}$.

Q24. A sequence is defined by $u_{n+1} = \frac{1}{3}u_n + 4$, with $u_1 = 9$. Find u_3 and the limit of the sequence as $n \to \infty$.

Q25. The 3rd term of an arithmetic sequence is 18 and the 7th term is 34. Find the sum of the first 20 terms.

Q26. A geometric series has a first term of 10 and a sum to infinity of 30. Find the common ratio and the sum of the first 5 terms.

Q27. A sector of a circle has radius r and an angle of θ radians. Given the perimeter is 20 cm and the area is 24 cm$^{\mathbf{2}}$, find the possible values of r.

Q28. Given that θ is small, give an approximation for $\frac{1-\cos 2\theta}{\theta \sin 3\theta}$

Q29. Solve $\arcsin(2x) = \frac{\pi}{3}$ for x.

Q30. Prove that $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$.

Q31. The mass M of a radioactive substance at time t years is given by $M = M_0 e^{-kt}$. If the mass halves every 50 years, find the value of k.

Q32. The relationship between variables P and t is believed to be $P = at^b$. A plot of $\log_{10} P$ against $\log_{10} t$ gives a straight line passing through $(0, 1)$ and $(2, 5)$. Find the values of a and b.

Q33. Prove, from first principles, that the derivative of x^3 is $3x^2$.

Q34. Find the coordinates and nature of the stationary points of the curve $y = x^3 - 6x^2 + 9x + 5$.

Q35. Find the coordinates of the point of inflection on the curve $y = e^{-x^2}$.

Q36. Differentiate $y = \frac{\sin(3x)}{x^2+1}$ with respect to x.

Q37. Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Q38. A boat travels with constant velocity $(3\mathbf{i} + 4\mathbf{j})$ km/h. At $t = 0$ the boat is at position $(-2\mathbf{i} + \mathbf{j})$. Find the position of the boat after 3 hours and its distance from the origin.

Q39. Prove by induction that for all $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$.

Q40. Given that $z_1 = a + 2i$ and $z_2 = 3 - bi$, where $a, b \in \mathbb{R}$, find the values of a and b such that $\frac{z_1}{z_2} = 1 + i$.

Q41. Express the complex number $z = -2\sqrt{3} + 2i$ in modulus-argument form.

Q42. The complex numbers z_1, z_2, z_3 are represented by the points A, B, C on an Argand diagram. If $z_1 = 2 + i$, $z_2 = 5 + i$ and $z_3 = 2 + 5i$, find the area of the triangle ABC.

Q43. Sketch the locus of points z on an Argand diagram satisfying the equation $|z-4i| = |z-2|$.

Q44. Find the 2×2 matrix $\mathbf{M}$ that represents a rotation of $135°$ anticlockwise about the origin, followed by a reflection in the y-axis.

Q45. A transformation is represented by the matrix $\mathbf{T} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. Find the equations of the two invariant lines through the origin for this transformation.

Q46. Given that $\mathbf{M} = \begin{pmatrix} k & 2 & 1 \\ 0 & 1 & k \\ 3 & -1 & 2 \end{pmatrix}$, find the values of k for which the matrix $\mathbf{M}$ is singular.

Q47. Find the inverse of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

Q48. A shop sells three types of fruit baskets. Basket A contains 2 apples, 3 oranges, and 1 pear for £4.40. Basket B contains 3 apples, 1 orange, and 2 pears for £4.10. Basket C contains 1 apple, 2 oranges, and 4 pears for £5.30. Write this as a matrix equation and find the cost of each individual fruit.

Q49. Determine whether the following system of equations is consistent, and if so, whether there is a unique solution or infinitely many solutions: $x + 2y - z = 3$ $2x - y + 3z = 5$ $4x + 3y + z = 11$

Q50. Three planes are defined by the equations $x+y+z = 6$, $2x-y+z = 3$, and $x+2y-2z = 2$. Find the coordinates of the point where these three planes intersect.

Q51. A plane has the vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$. Find the Cartesian equation of this plane.

Q52. Find the equation of the plane that contains the point $(3, 1, 4)$ and is perpendicular to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Q53. Find the angle between the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Q54. Find the coordinates of the point of intersection of the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{1}$ and the plane $3x - y + 2z = 12$.

Q55. Given $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, calculate $\mathbf{a} \times \mathbf{b}$ and hence find the area of the triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$.

Q56. Find the shortest distance between the skew lines $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Q57. The roots of the cubic equation $2x^3 - 5x^2 + 3x - 4 = 0$ are α, β, and γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

Q58. The equation $x^3 + 2x^2 - x + 5 = 0$ has roots α, β, γ. Find a cubic equation with integer coefficients whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Q59. Show that $\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(n^2+n+1)$. You may use the following identities: $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Q60. Find the sum of the series $\sum_{r=1}^{n} \frac{2}{r(r+2)}$.