Exercise
03 - Differential Geometry and Mesh Data Structure
Digital 3D Geometry Processing

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1. Theory Exercise

King Archimedes wants to renovate his palace. The most striking structure is a spherical half-dome of 20m in diameter that covers the great hall. The king wants to cover this dome in a layer of pure gold. He has decided to split the work into two parts, each one covering a vertical slice of the dome of the same height (see Figure 1). For each part he hires different people and gives them 700kg of gold. The task is to cover the surface of one vertical slice with a layer of gold of 0.1mm thickness. The amount of gold that is left over is the salary for doing the job. Which slice should you pick if you want to make the most profit? Explain your answer.
How does your answer change when you have n slices instead of just two?

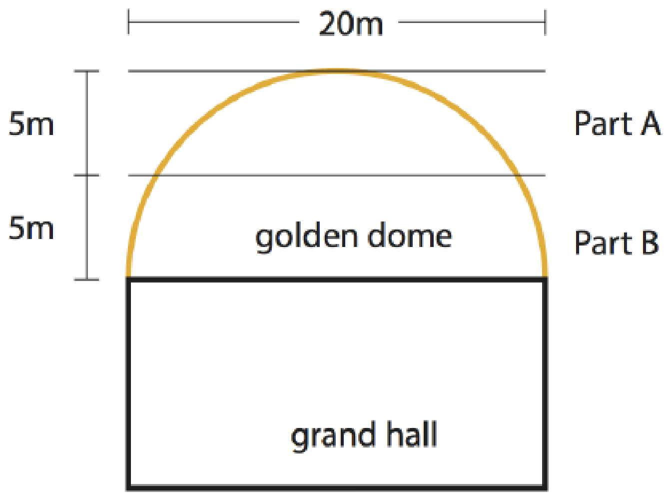

Figure 1: Sketch of King Archimedes dome.

Solution:

We know that the dome is a hemisphere (half sphere) so we know its surface area is given by the following integral :

$$R^2 \int_0^{2\pi} \int_0^{\pi/2} \sin(\theta)\, \mathrm{d}\sigma \mathrm{d}\theta$$

$$= R^2 2\pi \int_0^{\pi/2} \sin(\theta)\, \mathrm{d}\theta$$

$$= R^2 2\pi * -\cos(\theta)|_0^{\pi/2}$$

$$= R^2 2\pi * -(0-1)$$

$$= R^2 2\pi$$

$$= 200\pi$$

Now we want to compute the area surface of different part of the hemisphere. In the first part of the exercise, it is asked to cut the hemisphere in 2 equal parts, as seen on Figure 1. To compute these surface areas we only have to change the limits to which the integrals will be calculated. To find the limit at half height of the hemisphere we can use the trigonometric circle and we want to find x such that $\sin(x) = 1/2$. It implies that $x = \arcsin(1/2) = \pi/6$

Part A :

$$R^2 \int_0^{2\pi} \int_0^{\pi/6} \sin(\theta)\, \mathrm{d}\sigma \mathrm{d}\theta$$

$$= R^2 2\pi * -\cos(\theta)|_0^{\pi/6}$$

$$= R^2 2\pi * -(\sqrt{3}/2 - 1)$$

$$= R^2 2\pi(-\sqrt{3}/2 + 1)$$

$$= R^2 2\pi(-\sqrt{3}/2 + 1)$$

$$= 200(-\sqrt{3}/2 + 1)\pi$$

$$\approx 27\pi$$

Part B :

$$R^2 \int_0^{2\pi} \int_{\pi/6}^{\pi/2} sin(\theta)\, \mathrm{d}\sigma \mathrm{d}\theta$$

$$= R^2 2\pi * -\cos(\theta)|_{\pi/6}^{\pi/2}$$

$$= R^2 2\pi * -(0 - \sqrt{(}3)/2$$

$$= R^2 \pi \sqrt{3}$$

$$= 100\sqrt{3}\pi$$

$$\approx 173\pi$$

It would mean that having to cover the Part A would be much more profitable.

In the second part of the exercise, we want to know how the answer will differ if we change the number of slices to n. If we slice the dome in n different slices,it means that the θ limits for the first slice will be 0 (as always) and some smaller x always greater than 0. And as we know the smaller the θ the bigger $cos(\theta)$ will be (always between 0 and 1) and then it means that if we slice our dome by a big n the first slice will get bigger and bigger.

Example of first split for n = 10:

$$R^2 \int_0^{2\pi} \int_0^{\arcsin(1/10)} \sin(\theta)\, \mathrm{d}\sigma \mathrm{d}\theta$$

$$= R^2 2\pi * -\cos(\theta)|_0^{\arcsin(1/10)}$$

with $\arcsin(1/10) \approx 0.1$ and then $\cos(0.1) \approx 0.995$

$$= R^2 2\pi * -(0.995 - 1)$$
$$= R^2 2\pi * (1 - 0.995)$$
$$= 200 * 0.005\pi$$
$$= 1\pi$$

Then the first part is a really little part to cover. And if we compute the last one :

$$R^2 \int_0^{2\pi} \int_{\arcsin(9/10)}^{\arcsin(1)} \sin(\theta)\, \mathrm{d}\sigma \mathrm{d}\theta$$
$$= R^2 2\pi * -\cos(\theta)|_{\arcsin(9/10)}^{\arcsin(1)}$$

with $\arcsin(9/10) \approx 1.12$ and then $\cos(1.12) \approx 0.436$

$$= R^2 2\pi * -(0 - 0.436)$$
$$= R^2 2\pi * 0.436$$
$$= 200 * 0.436\pi$$
$$\approx 87.2\pi$$

Then it means that the upper you are on the hemisphere the bigger the parts are to cover. Still the same as in the first part of the exercise.