Let (A, d_A) be a metric space. Let $B \subseteq A$. Then B is said to be open in A if

$$\forall x \in B, \ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq B$$

$$B_\varepsilon(x) = \{ y \in A \mid d_A(y, x) < \varepsilon \}$$

"every point in B has a neighborhood in B"

eg.: $\underline{A} = \mathbb{R}$, $\underline{d_A}(x, y) = |x-y|$, $B \doteq \mathbb{R} \setminus \mathbb{Q}$

Is B open?

Take arbitrary x For some $\varepsilon > 0$

Does $B_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq \mathbb{Q}$? $\frac{1}{\pi^x} < \varepsilon$

let x be s.t.

$A = \mathbb{R}$, $d_A(x,y) = |x-y|$.

x, $x+\varepsilon$

$$B = \bigcap_{x \in \mathbb{Q}} B_x \qquad B_x := \mathbb{R} \setminus \{x\}$$

$x + \varepsilon - \frac{1}{\pi^x}$

→ Arbitrary Union of open subsets is open.

→ Finite intersection of open substs is open.

A is any set, $d_A : A \times A \to \mathbb{R}$ is a dist. f^n

$B = \phi \subseteq A$. Is B open in (A, d_A)?

$\rightarrow \phi$ is trivially open.

$\rightarrow A$ is " " in (A, d_A)

Let (A, d_A) be a metric space, let $B \subseteq A$.

Then B is said to be closed in (A, d_A)

if B^c is open in (A, d_A)

$$B^c = A \setminus B$$

OR

If for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in B that converges to x in A, $x \in B$

eg:- $A = \mathbb{R}$, $d_A = |x-y|$, $B = \mathbb{N}$

Is B closed?   HW: Prove it!

eg:- $A = \mathbb{R}$, $d_A = |x-y|$, $B = \emptyset \; \mathbb{R}$

Is B closed?

→ Finite union of closed subsets is closed.

→ Arbitrary intersection of closed subsets is closed

Let (A, d_A) be a metric space. Let $B \subseteq A$.
Then x is a limit point of B if
for every $\varepsilon > 0$ $B_\varepsilon(x) \cap B$ ~~contain~~ is infinite.

OR $B_\varepsilon(x) \cap B$ has atleast one point other than x.

eg:- $A = \mathbb{R}^2$, $d_A(x,y) =$ ~~|x-y|~~ std metric, $B = \{(x,0) \mid x \in \mathbb{R}\}$ "x-axis"

Is B open? closed? limit points?

Proof of closed:

$$B = \{(x,0) \mid x \in \mathbb{R}\}$$

$$B^c = \mathbb{R}^2 \setminus B = \{(x,y) \mid y \neq 0\}$$

Take (x,y) s.t. $y \neq 0$. Let $|y| = \delta$

choose $\varepsilon = \delta/2$.

Then $B_\varepsilon((x,y)) = \{(x',y') \mid \sqrt{(x'-x)^2+(y'-y)^2} < \varepsilon\}$

$\Rightarrow$ For $(x',y') \in B_\varepsilon(x,y)$, $|y'-y| < \sqrt{(x'-x)^2+(y'-y)^2}$

$\Rightarrow y' \neq 0$ $\Leftarrow$ $\Rightarrow y - \frac{\delta}{2} < y' < y + \frac{\delta}{2}$ $< \varepsilon = \frac{\delta}{2}$

Let $A \subseteq \mathbb{R}$. x is said to ~~have~~ be an upper bound of A

if $x \geqslant y \quad \forall y \in A$

The least upper bound of A is called supremum.

Key property of $\mathbb{R}$: Every ~~set~~ subset of $\mathbb{R}$ that has an upper bound has a least upper bound.

$\longrightarrow$ Supremum of ϕ is $-\infty$, infinum is ∞

Let X, Y be sets. Then $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$

cart. prod.

A binary relⁿ on X and Y R is a subset of $X \times Y$

R on $(X \times Y)$

→ A relⁿ is said to be transitive if

$\forall\ (x, y) \in R, (y, z) \in R$, we

have $(x, z) \in R$

→ Identity relⁿ on $X \times X = \{(x, x) \mid x \in X\} =: I$

R is a relⁿ on $X \times X$ ...

→ Reflexive relⁿ: R is reflexive if $I \subseteq R$

→ Symmetric relⁿ: R is symmetric if $\forall (x, y) \in R$, we have $(y, x) \in R$

→ Anti-symm. relⁿ: If $(x, y) \in R$, $(y, x) \in R$, then $x = y$

→ Irreflexive: $\forall x$, $(x, x) \notin R$

→ Equivalence: Reflexive + Symm. + Transitive

→ Partial Order: Reflexive + Anti-symmetric + Transitive