Definitions

A decomposition of n, satisfies

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

For natural a, b.

Note from the example that

$$\left.\begin{aligned} \frac{1}{11} &= \frac{1}{132} + \frac{1}{12} \\ \frac{1}{11} &= \frac{1}{12} + \frac{1}{132} \end{aligned}\right\} \text{2 unique decompositions}$$

Such that n=11 has three unique decompositions:

$$\frac{1}{11} = \frac{1}{12} + \frac{1}{132} = \frac{1}{132} + \frac{1}{12} = \frac{1}{22} + \frac{1}{22}$$

Lemma 1

a and b are greater than n.

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Proof

Assume the contrary, that: a <= n and/or b <= n

As a >0, b >0:

Therefore

$$\frac{1}{a}, \frac{1}{b} > 0$$

As a <= n and/or b <= n

$\frac{1}{a} \geq \frac{1}{n}$ and/or $\frac{1}{b} \geq \frac{1}{n}$

Therefore:

$$\frac{1}{a} + \frac{1}{b} > \frac{1}{n}$$

But

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$$

Therefore, this is a contradiction.

Therefore, $a, b > n$

<u>Note 1</u>:

When considering decompositions for n, as a decreases, b decreases. Therefore, there exists symmetry $a = b$, about which we start swapping unit fractions.

$$\left.\begin{aligned} \frac{1}{11} &= \frac{1}{132} + \frac{1}{12} \\ \frac{1}{11} &= \frac{1}{12} + \frac{1}{132} \end{aligned}\right\} \text{2 unique decompositions}$$

This symmetry exists about a=b=$2n$.

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$$

Therefore, we can optimise the analysis of base cases, whilst considering all a, by double counting every valid decomposition

$$n < a \leq 2n$$

because:

$$(a, b) = (\beta, \varepsilon)$$

$$(a, b) = (\varepsilon, \beta)$$

are two unique decompositions.

Lemma 2:

The upper bound of a is $n(n+1)$

Proof

The upper bound of a can be deduced using the lower bound of b. Both variables have the same bounds by the aforementioned symmetry.

From Lemma 1:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{n+1}$$

$$a(n+1) = n(n+1) + na$$

$$na + a - na = n(n+1)$$

$$a = n(n+1)$$

Therefore, as a corollary of lemmas 1 and 2:

$$n+1 \leq a \leq n(n+1)$$

Furthermore, as a corollary of note 1, lemma 1 and lemma 2:

There exists at least three decompositions:

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n(n+1)} + \frac{1}{n+1}$$

However:

$$2n = n+1 = n(n+1)$$

When n=1

Therefore:

For all $n > 1$ there exists at least 3 decompositions

But, n=1, has 1 decomposition

For n=11: (prime case)

We consider the following region:

$$n + 1 < a < 2n$$

$$12 < a < 22$$

$$a = 13 : \frac{1}{11} - \frac{1}{13} = \frac{13 - 11}{13 \times 11} \times$$

$$a = 14 : \frac{1}{11} - \frac{1}{14} = \frac{14 - 11}{14 \times 11} \times$$

$$a = 15 : \frac{1}{11} - \frac{1}{15} = \frac{15 - 11}{15 \times 11} \times$$

This trend continues

$$a = 21 : \tfrac{1}{11} - \tfrac{1}{21} = \tfrac{21-11}{21 \times 11}$$

These values of a, do not yield integer b, because the numerator is not a divisor of the denominator.

Lemma 3

For prime n, there exists only 3 decompositions:

$$a = n + 1, a = 2n, a = n^2 + n$$

Proof

$$Let, \ 'p', be\ a\ prime\ number, let\ n = p$$

$$\frac{1}{p} = \frac{1}{a} + \frac{1}{b}$$

$$ab = ap + b\rho$$

$$b(a-p) = a\rho$$

$$b = \frac{ap}{a-\rho}$$

We want specific conditions to further analyse a and b:

By using long division with divisor (a -p) and dividend ap

$$\frac{ap}{a-\rho} = p + \frac{p^2}{a-p}$$

As b is an integer and p is an integer, $\frac{p^2}{a-p}$, must be an integer.

Therefore:

$$a - p \mid p^2$$

As p is prime, p^2 has three factors: a, p, p^2Therefore:

$$a - p = 1 \text{ or } p \text{ or } p^2$$

Therefore:

$$a = 1 + p \text{ or } 2p \text{ or } p^2 + p$$

<u>Note 3:</u>

Whilst this method works for composite n, we must use a computational algorithm to find the unique prime decomposition of n.

See appendix for an example I wrote

To brute force the number of factors of n^2, we can first analyse the number of factors of n.

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \ldots$$

Where p_α refers to a unique prime factor for each α

The number of distinct divisors of n is is given by:

$$(a_1+1)(a_2+1)(a_3+1)\ldots$$

$$n^2 = p_1^{2a_1} \cdot \rho_2^{2a_2} \cdot p_3^{2a_3} \ldots$$

The number of factors of n^2 is given by:

$$(2a_1+1)(2a_2+1)(2a_3+1) \ldots where\ z\ is\ the\ number\ of\ a_i\ terms$$

Therefore, we must decompose n into a unique product of prime factors to determine the number of factors of n^2.

<u>n=60: (composite case)</u>

Consider the full range of a:

$$61 \leq a \leq 60(60+1) = 60^2 + 60$$

$$\frac{1}{60} = \frac{1}{a} + \frac{1}{b}$$
$$ab = 60a + 60b$$
$$b = \frac{60a}{a-60}$$

Via long division as for n=11:

$$n = 60 + \frac{60^2}{a-60}$$

As for n=11, as n is an integer, $\frac{60^2}{a-60}$ is an integer.

$$1 \leq a - 60 \leq 60^2$$

Or generally in terms of n:

$$1 \leq a - n \leq n^2$$

Where, a-60 = any factor of 60^2

Or a = any factor of 60^2+60

Therefore, the number of possible, 'a' values (in $n < a \leq 2n$), is the number of factors of n^2
Note that the same symmetry of a, b applies to the factors of n^2

<u>Theorem 1:</u>

Let $f(n)$ be number of factors of n^2 and therefore the number of unique decompositions for n.

$$f(n) = \prod_{i=1}^{z}(2a_i + 1)$$

<u>Verification</u>

We will consider normal n and *extreme* n, the latter being n for which it is less clear if f will output the correct number of decompositions.

The algorithm in the appendix lets us verify f(n) computationally.

An example verification:

n=11: we know $f(11) = 3$

$$11 = 11^1$$

$$\Rightarrow 2a_i + 1 = 2 * 1 + 1 = 3$$

$$\prod_{i=1}^{z}(2a_i + 1) = 3$$

Extreme:

n=1: we know $f(1) = 1$

1 has no prime factors, therefore f(1) isn't defined.

n=2: we know $f(2) = 3$

$$2 = 2^1$$

$$\Rightarrow 2a_i + 1 = 2 * 1 + 1 = 3$$

Normal:

n=60: we know $f(60) = 45$

$$60 = 2^2 * 3^1 * 5^1$$

$$\Rightarrow (2*2+1)(1*2+1)(1*2+1)$$

$$= 5*3*3 = 45$$

$$f(n) = \prod_{i=1}^{z} (2a_i + 1), \quad for\ all\ n \geq 1$$

<u>How could we find the decompositions themselves?</u>

We could alter the prime factorisation algorithm as in the apendix.

<u>What are the bounds of $f(n)$?</u>

Prime factorisations change unpredictably as n grows e.g.

Therefore, we can use the well-defined bounds of d to find the bounds of f. Where a function which returns the number of factors of its input. As f is the number of factors of n^2 and d the number of factors of n we can relate $f\ to\ d$.

$$f(n) = d(n^2) = \prod_{i=1}^{z} (2a_i + 1), \quad for\ all\ n \geq 1$$

As, $d(n) \leq \sqrt{3n}$ source

$$f(n) = d(n^2) \leq \sqrt{3n^2} = n\sqrt{3}$$

$$1 \leq f(n) \leq n\sqrt{3}, for\ all\ n$$

The lower bound is found when $n = 1$.

The upper bound is only found when $n\sqrt{3}$ is an integer. As n is natural, $n\sqrt{3}$ is never an integer, therefore the upper bound is strict.

$$1 \leq f(n) < n\sqrt{3},$$

<u>What if a and b could be negative integers?</u>

Let $i(n)$ be the number of unique decompositions of n.

Note that non-0 integer a, b is a subset of positive a, b. Such that $i(n) \geq f(n)$

For this sub-solution, we can decompose the problem into three cases.

1: both a and b are negative

2: only one is negative

3: both are positive

3 is the main problem, 1 is impossible, because the sum of two negative numbers is negative, whereas $\frac{1}{n} > 0$ as n>0. Therefore, we only consider 2.

As in the principle solution, we will consider a, as, generally, the properties of a hold for b. All possible a values are possible b values when swapped.

Take a to be the positive unit fraction and b the negative.

We will have different bounds for a, b. Via the same method from the principal solution:

<u>Lemma 4</u>

$$1 \leq a \leq n - 1$$

<u>Proof</u>

Assume the contrary of the upper bound

$$a > n - 1$$

$$a \geq n$$

This implies the following inequality

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} \leq \frac{1}{n} + \frac{1}{b}$$

Which can only be true if:

$$\frac{1}{b} \geq 0$$

But this is a contradiction, therefore $a \leq n - 1$

We get the lower bound, from a being positive.

Lemma 5

$$-1 \geq b \geq -n(n-1) = n(1-n)$$

Proof

Upper bound: b must be negative and an integer therefore:

$-1 \geq b$

Lower bound:

Assume the contrary:

$$b < n(1-n)$$

$$b \leq n(1-n) - 1$$

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

This implies that:

$$\frac{1}{n} - \frac{1}{n(1-n)} < \frac{1}{a}$$

$$\frac{1-n-1}{n(1-n)} < \frac{1}{a}$$

$$\frac{-n}{n(1-n)} < \frac{1}{a}$$

$$\frac{-1}{1-n} < \frac{1}{a}$$

n cannot = 1 as dividing by 0 is undefined. Therefore, n must be at least 2 (as n is non negative). Therefore, the reciprocal inequality is:

$$n - 1 < a$$

But this is a contradiction as

$$n - 1 \geq a$$

This proves the lemma.

From the principle solution which is algebraically identical:

$\frac{n^2}{a-n}$ is an integer, therefore (a-n) is a factor of n^2.

If we subtract n from the full range of a:

$$-1 \leq a \leq n - 1$$

We get:

$$-(1+n) \leq a-n \leq -1$$

Therefore, $a-n$ is any divisor of n^2 between -1 and –(n+1)

As divisors are either factors or negative factors, we can use a similar function to f.

Only the lower bound separates the range from being -1 to $-n$, so we find n for which the lower bound is a factor of n^2 to make our function more efficient.

$$-\frac{n^2}{1+n}$$

By long division

$$-n+\frac{n}{1+n}$$

1+ n is never a divisor of n, as 1+ n> n as n is positive. So we can narrow our range for $a-n$

$$-n \leq a-n \leq -1$$

Note that the number of factors of a number between -1 and $-n$ = the number of factors between 1 and n.

<u>Lemma 6:</u>

There exists symmetry about n in the factors of n^2

<u>Proof:</u>

If $x < n \; then \frac{n^2}{x} > n$

This lets us exploit the symmetry about n of the factors of n^2

For instance: n=8

$$1,2,4,8,16,32,64$$

There are 3 factors to the left and right of n=8. Where those to the left of 8 are less than n, and those to the right are greater than n.

<u>Theorem 2</u>

Let $D(n)$ be the number of factors of n^2 less than or equal to n.

Let $d(n)$ be the number of factors of n.

Via lemma 6

$$D(n) = \frac{d(n^2) - 1}{2} + 1 = \frac{d(n^2) + 1}{2} = \frac{\prod_{i=1}^{z}(2a_i + 1) + 1}{2} \ for\ all\ n \geq 1$$

$$Where\ z\ is\ the\ number\ of\ a_i\ terms\ as\ defined\ in\ the\ principle\ solution$$

The +1 comes from n, which is always a factor of n^2.

As we've assumed that a is negative and b is positive, due to the symmetry of a and b, b could be the negative and a the positive.

$$i(n) = 2D(n) + f(n), for\ all\ n \geq 1$$

$$= \prod_{i=1}^{z}(2a_i + 1) + \prod_{i=1}^{z}(2a_i + 1) + 1\ ,\ \ for\ all\ n \geq 1$$

$$= 2\prod_{i=1}^{z}(2a_i + 1) + 1,\ \ for\ all\ n \geq 1$$

Note that $i(n)$= $2f(n) + 1$therefore, we have <u>bounds</u> for $i(n)$

As

$$f(n) \leq n\sqrt{3}$$

$$1 * 2 + 1 = 3 \leq 2f(n) + 1 \leq 2n\sqrt{3} + 1$$

$$3 \leq i(n) \leq 2n\sqrt{3} + 1$$

<u>What if we split one of the unit fractions into 2 others?</u>

$$\frac{1}{n} = \frac{1}{2a} + \frac{1}{2a} + \frac{1}{b}$$

The symmetry changes.

The same three-unit fractions can be arranged in 3 ways based on which term of the RHS the unit fraction of b is. Therefore, for every decomposition with at least 1 unique unit fraction (that is, each combination, not permutation), we multiply by 3 to account for this and subtract 1 when

$$\frac{1}{n} = \frac{1}{2a} + \frac{1}{2a} + \frac{1}{b} = \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n}$$

For the following solution, we will work with $\frac{1}{a}$ $instead\ of\ \frac{1}{2a}$

$$\frac{1}{n}=\frac{1}{a}+\frac{1}{a}+\frac{1}{b}$$

$$\frac{1}{n}=\frac{2}{a}+\frac{1}{b}$$

$$ab=2nb+na$$

$$b(a-2n)=na$$

$$b=\frac{na}{a-2n}$$

Via long division

$$=n+\frac{2n^2}{a-2n}$$

As b is an integer and n is an integer, $\frac{2n^2}{a-2n}$ must be an integer

We want bounds on a, as for the principal problem, for an expression for the number of decompositions.

From the principal solution

<u>Lemma 7</u>

The lower bound of a, b is $2n$+1

<u>Proof</u>

Assume the contrary is true:

$$a<2n+1$$
$$a\leq 2n+1$$

Therefore:

$$\frac{1}{n}=\frac{2}{a}+\frac{1}{b}\geq\frac{1}{n}+\frac{1}{b}$$

Therefore:

$\frac{1}{b}\leq 0$ which is a contradiction, therefore the lower bound of a, b is $2n$+1, as the same argument follows for b.

Upper bound:

$$\frac{1}{n} - \frac{1}{b} = \frac{2}{a}$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{2}{a}$$

$$a(n+1) - na = 2n(n+1)$$

$$a = 2n(n+1) \ is\ the\ upper\ bound$$

$$2n+1 \leq a \leq 2n(n+1)$$

$$1 \leq a - 2n \leq 2n^2$$

$a - 2n$ is any factor of $2n^2$

For the number of factors of $2n^2$:

$2n^2$ is the same as n^2 but with extra power 2.

<u>Theorem 3</u>

$$t(n) = (2a_1 + 2)\prod_{i=2}^{z-1}(2a_i + 1),\ for\ n \geq 1$$

$$assuming\ that\ without\ loss\ of\ generality, a_1 is\ the\ exponent\ of\ 2,\ \ for\ n \geq 1$$

<u>Appendix</u>

For the brute forcing

```python
from fractions import Fraction as convert_to_fraction
import math
n=int(input('integer'))
total_decompositions=0
upper_bound_a=(1/n)-(1/(n+1))
upper_bound_a=1/(upper_bound_a)
upper_bound_a=math.ceil(upper_bound_a)
all_decompositions=[]
for possible_a in range(n+1, upper_bound_a+1): #upper bound of a is given by 1/n - lower bound of b, (lb of b= lb of a)
    b = convert_to_fraction(1, n) - convert_to_fraction(1, possible_a)
    b=convert_to_fraction(b)
    if b.numerator==1:
        total_decompositions+=1
        all_decompositions.append([convert_to_fraction(1, possible_a),b])

print(total_decompositions)
for each_unique_decomposition in all_decompositions:
```

For the prime factorisation

```python
n=int(input('n'))
def d(n):
    factors=[]
    import math
    n=n*n #really n^2
    counter=0 #this gets the index of each prime factor
    while n % 2 == 0:
        counter+=1
        n = n / 2
    factors.append(['2', counter])


    for i in range(3, int(math.sqrt(n)) + 1, 2): #UB, bcs every composite nuimber has a factorless less than or equal to its sqrt
        counter=0

        # while i divides n , print i ad divide n
        while n % i == 0:
            counter+=1
            n = n / i
        i=str(i)
        if counter!=0:
            factors.append([i, counter])
    for factor in factors:
        print(factor[1]) #prints the index of each prime factor


d(n)
```

2)

<u>General definition</u>

$$g_1 = a, a > 0$$

$$g_2 = b, b < 0$$

$$g_{n+1} = g_n + g_{n-1}$$

<u>Idea: what happens if you go backwards</u>

Define a sequence, u_n of this nature:

$$u_1 = -1$$

$$u_2 = 1$$

$$u_{n+1} + u_n = u_{n-1} \; or \; u_{n+1} = u_{n-1} - u_n$$

By observing u : -1,1,-2,3,-5

We observe the Fibonacci sequence (1,1,2,3…), F

<u>Lemma 1</u>

$$u_n = (-1)^n F_n$$

<u>Proof (strong induction)</u>

<u>Base case:</u>

n=1:

$$u_1 = -1$$

$$F_1 = 1$$

$$-F_1 = -1 = u_1$$

<u>Induction</u>

Suppose that

$$u_n = (-1)^n F_n \; holds \; for \; all \; n \leq k, \epsilon\mathbb{Z}$$

Thus:

$$u_{k-1} = (-1)^{k-1} F_{k-1}$$

And

$$u_k = (-1)^k F_k$$

Via the definition of u_n:

$$u_{n+1} = u_{n-1} - u_n$$

$$u_{k+1} = (-1)^{k-1} F_{k-1} - (-1)^k F_k$$

As k is an integer, k is either even or odd:

If k is even:

$$-(F_{k-1} + F_k), via \; the \; definition \; of \; F = -F_{k+1}$$

If k is odd:

$$F_{k-1} + F_k, via\ the\ definition\ of\ F = F_{k+1}$$

Therefore,

$$u_{k+1} = (-1)^{k+1} F_{k+1}$$

And by going back one term:

$$u_k = (-1)^k F_k, as\ required.$$

Therefore, as the term before the Fibonacci sequence is 0, for a sequence that alternates for n elements:

$$g_1 = F_k$$

$$g_2 = -F_{k-1}$$

5 elements:

5, -3, 2, -1, 1, 0

10 elements:

55, -34, 21, -13, 8, -5, 3, -2, 1, -1, 0

Lemma 2

Let $n \geq 2$

Define $sgn(x)$ as

$$sgn(x) = \begin{cases} x > 0: 1 \\ x = 0: 0 \\ x < 0: -1 \end{cases}$$

Suppose $sgn(g_{n+1}) \neq sgn(g_{n+2})$

Further suppose that $sgn(g_{n+1}), sgn(g_{n+2}) \neq 0$

Then

$$|g_n| > |g_{n+1}|$$

What this means is: while in the alternating region of a sequence, if term n+1 has a greater absolute value than n, the sequence will stop alternating at term n+2.

Note that when the sign of term n is 0, the term stops alternating at term n.

Also, this means that g is strictly decreasing as, even terms are negative and odd terms are positive as proven below.

$$g_1 > g_3 > g_5 \ldots$$

$$g_2 < g_4 < g_6 \ldots$$

An inductive proof seems unsuitable, as, the claim for all n, whereas an inductive step would cause the claim to be true for all n, thus we opt for a direct proof.

Proof by contradiction

Assume that

$$|g_n| \leq |g_{n+1}|$$

From the definition of g:

$$g_{n+2} = g_{n+1} + g_n$$

Because $g_1 > 0$ and $g_2 < 0$ from the definition of g, and when the sequence is alternating, $sgn(n) = sgn(n+2) \neq 0$. Therefore, even terms are positive and odd terms are negative.

n can either be even or odd

When n is odd:

$$g_n \text{ is positive}$$

$$g_{n+1} \text{ is negative}$$

Since

$|g_n| \leq |g_{n+1}|$:

$$g_{n+2} \leq 0$$

$But\ if\ g_{n+2} = 0$, this contradicts our supposition

If $g_{n+2} < 0$,

This contradicts the fact that odd terms are negative.

When n is even:

$$g_n\ is\ negative$$

$$g_{n+1}\ is\ positive$$

Since
$|g_n| \leq |g_{n+1}|$:

$$g_{n+2} \geq 0$$

$But\ if\ g_{n+2} = 0$, this contradicts our supposition

If $g_{n+2} > 0$,

This contradicts the fact that odd terms are positive.

Therefore,

$$|g_n| > |g_{n+1}|$$

When the sequence alternates at least till term n+2.

<u>Lemma 3</u>

The minimum of $|g_n - g_{n-1}|$ (the difference between consecutive terms) is 2 if the g alternates for at least n+1 terms.

<u>Proof</u>

The elements of g are integers only.

Therefore the difference between terms is an integer.

Therefore, if $|g_n - g_{n-1}|$ is less than 2, it can only be 1 or 0.

If the difference is 0, g_{n+1}=0, therefore there are n alternating terms.

If the difference is 1, g cannot alternate for more than n terms, because, to alternate between two terms, the sgn function must change from -1 to 1 or 1 to -1. For which, $|g_n - g_{n-1}|$must be at least 2. As the greatest negative number is -1, the least positive number is 1, 1-(-1)=2. Additionally, this chain of logic holds for a difference of 0.

Therefore, if there exists an example of g where a difference between consecutive terms is 2, we've proved that the minimum of $|g_n - g_{n-1}|$is 2.

5, -3, 2, -1, 1.

1-(-1)=2

<u>Lemma 4</u>

No sequence can alternate infinitely.

<u>Proof</u>

As a corollary of lemma 2, the alternating part of a sequence is strictly decreasing (a decreasing monovariant). As a corollary of lemma 3, the difference between alternating terms is at least 2.

As a corollary of lemma 2, the alternating part of a sequence cannot = 0.

As the sequence is decreasing from positive g_1, the sequence will eventually have al element 0, or skip 0 and start increasing in absolute value, such that via corollary 2, the sequence will stop alternating.

<u>Lemma 5</u>

All sequences will either approach ∞ or $-\infty$

<u>Proof</u>

When a sequence stops alternating, we either get:

0, positive, positive

0, negative, negative

Positive, negative, negative

Negative, positive, positive

Two consecutive positive terms sum to a larger positive term, this continues till ∞. Similarly, two consecutive negative terms sum to $-\infty$.

This links to concept of program termination, or if a measure decreases for every step of an algorithm, because an algorithm cannot infinitely descend.

What can be said about the rate at which all sequences diverge?

Via the general definition:

$$g_3 = a + b$$

$$g_4 = a + 2b$$

$$g_5 = 2a + 3b$$

$$g_6 = 3a + 5b$$

$$g_7 = 5a + 8b$$

By observing the coefficients of a and b:

Lemma 6

$$g_n = aF_{n-2} + bF_{n-1}, n \geq 3$$

Proof (strong induction)

Base case: n=3

$$g_3 = a + b = aF_{3-2} + bF_{3-1} = a(1) + b(1)$$

Induction

Suppose the claim is true for all $n \leq k$

$$g_k = aF_{k-2} + bF_{k-1}$$

$$g_{k-1} = aF_{k-3} + bF_{k-2}$$

As we know that $g_{k+1} = g_k + g_{k-1}$

$$g_{k+1} = aF_{k-2} + bF_{k-1} + af_{k-3} + bF_{k-2}$$

$$= a(F_{k-2} + F_{k-3}) + b(F_{k-1} + +bF_{k-2})$$

As $F_{n+1} = F_n + F_{n-1}$

$$g_{k+1} = aF_{k-1} + bF_k$$

Therefore, by going one term backwards

$$g_k = aF_{k-2} + bF_{k-1}, as\ required.$$

Via Binet's formula , we can find what is likely a scale factor that transforms $g_1 \; to \; g_2$ to maximise the number of alternating terms.

From

$$g_n = aF_{n-2} + bF_{n-1}, n \geq 3$$

and Binet's formula

$$g_n = \frac{a}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right) + \frac{b}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)\right)$$

Note that the b term is expanded like this to isolate the dominant term

$$= \frac{a+b\left(\frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \frac{a+b\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

This lets us analyse the growth of g by studying the exponential terms, namely the first, as $\left(\frac{1+\sqrt{5}}{2}\right) > 1, whereas \left(\frac{1-\sqrt{5}}{2}\right) < 1$, therefore, the first term is dominant.

g grows to ∞ if $a + b\left(\frac{1+\sqrt{5}}{2}\right) > 0$

g grows to $-\infty$ if $a + b\left(\frac{1+\sqrt{5}}{2}\right) < 0$

Therefore, the closer, $\frac{a+b\left(\frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}}$, is to zero, (which happens when $a + b\left(\frac{1+\sqrt{5}}{2}\right)$ is close to 0), the slower $g_n$will grow to ∞ or $-\infty$ and therefore, longer the series will alternate.

To find the relationship a and b:

10000	10000	10000		8	8	8		10	10	10
-6180.34	-6180	-6181		-4.94427191	-5	-4		-6.18034	-6	-7
3819.66	3820	3819		3.055728094	3	4		3.81966	4	3
-2360.68	-2360	-2362		-1.88854381	-2	0		-2.36068	-2	-4
1458.98	1460	1457		1.167184281	1			1.45898	2	-1
-901.699	-900	-905		-0.72135953	-1			-0.9017	0	
557.2809	560	552		0.445824751	0			0.557281		
-344.418	-340	-353		-0.27553478				-0.34442		
212.8625	220	199		0.17028997				0.212862		
-131.556	-120	-154		-0.10524481				-0.13156		
81.30645	100	45		0.06504516				0.081306		
-50.2496	-20	-109		-0.04019965				-0.05025		
31.05689	80	-64		0.02484551				0.031057		
-19.1927	60			-0.01535414				-0.01919		
11.86421				0.00949137				0.011864		
-7.32846				-0.00586277				-0.00733		
4.535751				0.003628601				0.004536		
-2.79271				-0.00223417				-0.00279		
1.743041				0.001394433				0.001743		
-1.04967				-0.00083974				-0.00105		
0.693372				0.000554698				0.000693		
-0.3563				-0.00028504				-0.00036		
0.337076				0.000269661				0.000337		
-0.01922				-1.5377E-05						
0.317855				0.000254284						
0.298634				0.000238907						

$$a + b\left(\frac{1+\sqrt{5}}{2}\right) = 0$$

$$\frac{a}{b} = -\left(\frac{1+\sqrt{5}}{2}\right) = -\phi$$

$$b = a * -\frac{1}{\phi}$$

Verification

We can test this using excel. In the table below, we investigate a=1000, 20 and 3.

As ϕ is irrational, there exists no integer pair (a,b), where

$$b = a * -\frac{1}{\phi}$$

Hence, we must round b. The table above shows, that the nearest integer from the irrational b, causes g to have as many alternating elements as possible.

$$b = a * -\frac{1}{\phi}$$

3)

For the following solution, I will work in cm.

Define sequences, g_n and f_n which respectively define the displacement of the hare and flea from the stake after the n^{th} jump.

$$g_n = 100{,}000n$$

Is it okay that I

$f_0 = 0$ (defining f for n=0, will help us define recursive formulae for n=1)

$$f_1 = 1$$

answer the base

$$f_2 = \frac{2}{1} * f_1 + 1 = \frac{2}{1} * 1 + 1 = 3 \; (stretch\ first\ then\ the\ flea\ jumps\ 1cm)$$

$$f_3 = \frac{3}{2} * 3 + 1 = 5.5$$

<u>Lemma 1:</u> cases afterwards?

$$f_n = \frac{n}{n-1} f_{n-1} + 1$$

From lemma 1, perhaps we can get an explicit formula:

$$(n-1)f_n = nf_{n-1} + 1(n-1), n \geq 2$$

$$f_1 = 1$$

$$f_2 = 2f_1 + 1$$

$$2f_3 = 3f_2 + 2$$

$$3f_4 = 4f_3 + 3$$

$$4f_5 = 5f_4 + 4$$

This can be written as:

$$f_2 = 2 * 1 + 1 = 2 * \left(1 + \frac{1}{2}\right)$$
$$f_3 = \frac{3}{2} * \left(2 * \left(1 + \frac{1}{2}\right)\right) + 1 = 3\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

Crucially, this resembles a harmonic series.

<u>Lemma 2:</u>

$$f_n = n \sum_{z=1}^{n} \frac{1}{z}$$

<u>Proof by strong induction</u>

Base case: n=1

$$f_1 = 1 \sum_{z=1}^{1} \frac{1}{z} = 1 \; as \; required$$

Inductive step:

Assume the claim holds for $n - 1$

$$f_{n-1} = (n-1) \sum_{z=1}^{n-1} \frac{1}{z}$$

$$(n-1)\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n+1}\right)$$
$$\Rightarrow f_n = \frac{n}{n+1} f_{n-1} + 1 \;\; (from \; recursive \; definition)$$
$$= (n-1) * \frac{n}{n-1}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n+1}\right) + 1$$
$$= n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n+1}\right) + 1$$

$$= n\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n+1}+\frac{1}{n}\right)$$

$$f_n = n\sum_{z=1}^{n}\frac{1}{z}, as\ required$$

We can generalise this for all i, where i is the distance travelled by the flea independent of the rubber band stretch in cm.

$$f_n = ni\sum_{z=1}^{n}\frac{1}{z}$$

<u>Lemma 3:</u>

There exists k such that $f_k - f_{k-1} > 10^5$

Meaning that eventually the flea's displacement will exceed the hare's per jump.

<u>Proof</u>

$$f_n - f_{n-1} = n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - (n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)$$
$$= n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - n\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)$$
$$= 1 + \left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) = 1 + \sum_{z=1}^{n-1}\frac{1}{z}$$

As harmonic series are divergent, this not only proves that there exists k, for which the flea's displacement increases by 10^5 per jump, but as iterations pass, the flea's displacement increases by a greater amount per iteration.

Therefore, for any linear increase in the hare's displacement per iteration, and for any jump distance of the flea (that is, the +1cm, not including the stretch), the flea will catch the hare.

<u>Lemma 4:</u>

Define H_n as $\sum_{z=1}^{n}\frac{1}{z}$

$ln(n) + \frac{1}{2} < H_n$

Proof

Via the trapezoidal rule.

$$\ln(n) - \ln(1) = \ln(n) = \int_{1}^{n} \frac{1}{z} dz$$

$$\int_{1}^{n} \frac{1}{z} dz < \frac{1}{2}\left(1 + 2\sum_{z=2}^{n-1} \frac{1}{z} + \frac{1}{n}\right) = \frac{1}{2} + H_{n-1} - 1 + \frac{1}{2n}$$

$$= \frac{1}{2} + H_n - \frac{1}{n} - 1 + \frac{1}{2n}$$

$$= H_n - \frac{1}{2n} - \frac{1}{2}$$

As $-\frac{1}{2n} - \frac{1}{2}$ is strictly negative:

$$H_n > \ln(n) + \frac{1}{2} + \frac{1}{2n}$$

As n is non-negative, $\frac{1}{2n}$ is nonnegative.

$$H_n > ln(n) + \frac{1}{2}$$

Subtracting $\frac{1}{2n}$ will let us get bounds on n.

This lemma is key to finding the lower bounds of f_n and $f_k - f_{k-1}$.

Bounds for $f_k - f_{k-1}, k,\ f_n$ $and\ n\ via\ Riemann\ Sums$ and the trapezoidal rule

Since $f_k - f_{k-1}$ is strictly increasing, we can use Riemann Sums.

We can approximate $\sum_{z=1}^{k-1} \frac{1}{z}$ as $\int_1^k \frac{1}{z} dz$, where $\int_1^k \frac{1}{z} dz > \sum_{z=1}^{k-1} \frac{1}{z} > \int_1^{k-1} \frac{1}{z} dz$

Let $f_k - f_{k-1} \equiv\ x$

$$x \equiv 1 + \sum_{z=1}^{k-1} \frac{1}{z}$$

$$x < 1 + \int_{1}^{k} \frac{1}{z} dz = 1 + [\ln z]_1^k = 1 + \ln(k) - 0 = 1 + \ln(k)$$
$$x < 1 + \ln(k)$$

Applying lemma 4 gives a lower bound. Therefore:

$$\frac{3}{2} + \ln(k-1) < x < 1 + \ln(k)$$

For this specific problem, we want k such that.

$$\frac{3}{2} + \ln(k-1) > 10^5$$

$$\mathrm{e}^{10^5 - \frac{3}{2}} + 1 < \mathrm{k}$$

We can generalise this for all hare jump distances, let j = the hare's jump distance

$$k > \mathrm{e}^{\mathrm{j} - \frac{3}{2}} + 1$$

For $f_n = n \sum_{z=1}^{n} \frac{1}{z}$:

Using the fact that:

$$f_n = nH_n$$

Lower bound:

Let us first consider the specific case:

$$f_n > g_n$$

$$f_n > 10^5 n$$

$$nH_n > 10^5 n$$
$$H_n > 10^5$$

As the lower bound of H_n is $ln(n) + \frac{1}{2}$

The lower bound of $nH_n = f_n = is \ n(ln(n) + \frac{1}{2})$

$$f_n > n(ln(n) + \frac{1}{2})$$

Via lemma 4, we can bound n:

$$\ln(n) + \frac{1}{2} = 10^5$$

$$n > e^{10^5 - \frac{1}{2}}$$

Generalising for all j:

$$n > e^{j - \frac{1}{2}}$$

4)

Definitions

A point is 'good' if you can have any integer number of lattice points inside of a circle centred at that point. Otherwise, the point is 'bad'.

The problem can be recomposed. We are being asked to find a point on the xy plane, such that we can construct a circle of all radii (imagine an enlarging radius) where there are never 2 lattice points on its circumference.

Idea:

We can narrow our search for good points by first excluding not good points.

If we consider 2 arbitrary distinct lattice points (countably infinite number of pairs as the set has cardinality Z^2). We can construct a straight line, such that any points on the line are equidistant from the 2 lattice points. Therefore, if we construct a circle with a centre on any point on this line. As we enlarge its radius, both lattice points will eventually be on the circle's circumference simultaneously. Therefore, there is a countably infinite number of lines as the set of lines has cardinality Q^2 and therefore countably points where circles are not good.

Perhaps centre coordinates being irrational and not half an integer will not only prevent the required symmetry needed for 2 lattice points to be on a circle's symmetry for a given radius, but enable a proof by contradiction.

Lemma 1

A point is good if an only if one of the following holds.

- a is irrational and b is rational such that $2b$ is not an integer.
- b is irrational and a is rational such that $2a$ is not an integer.
- a is irrational and b is irrational and $a \neq \frac{n}{m} + b\left(\frac{k}{m}\right)$ for any integers n, k, m

Forwards proof

Consider the contrapositive:

If **not**

(a is irrational **and** b is rational and $2b$ is **not** an integer) **or** ($2a$ is not an integer and a is rational **and** b is irrational) **or** (a is irrational and b is irrational and $a \neq p + qb$ for any rationals p, q.

Then (a, b) is not good.

By De Morgan's laws:

$$\neg(A \ or \ B \ or \ C) \equiv \neg A \ and \ \neg B \ and \ \neg C$$
$$\neg(A \ and \ B) = \neg A \ or \ \neg B$$

Therefore

If (**not** (a is irrational **and** $2b$ is **not** an integer) **or** ($2a$ is not an integer **and** b is irrational) **or** (a is irrational and b is irrational and $a \neq p + qb$ where $q \ and \ b$ are rational)

is equivalent to:

If
($a \ is$ rational $or \ 2b \ is \ an \ integer \ or \ b \ is \ irrational$) **and**
($2a$ is an integer or a is irrational b is rational) **and**
(a is rational or b is rational or there exists rational , $q \ a \neq p + qb$):

then (a, b) is not good.

For this statement to be true, it is sufficient to prove that:

If $a \ and \ b$ are rational and $a = p + qr$, then (a, b) is bad.

By transforming the circle by the same non-0 stretch of qs in $x \; and \; y$ about the origin. We can work with integers and therefore the Euclidean algorithm.

The lattice point (n, m) is on the circle $(x-a)^2+(y-b)^2=R^2$
where $a=\frac{p}{q} and \; b=\frac{r}{s} \; where \; p, q, r, s \; are \; integers.$

If and only if the point $(qsn, qsm))$ is on the circle

$$(x-ps)^2+(y-qr)^2=(Rqs)^2$$

We want distinct points (qsn_1, qsm_1) and (qsn_2, qsm_2) on the circle's circumference.

$$(qsn_1-ps)^2+(qsm_1-qr)^2=(qsn_2-ps)^2+(qsm_2-qr)^2$$

Via the difference of two squares

$$\begin{aligned}((qsn_1-ps)-(qsn_2-ps))((qsn_1-ps)+(qsn_2-ps))\\ =((qsm_2-qr)-(qsm_2-qr))((qsm_2-qr)+(qsm_1-qr))\end{aligned}$$

$$qs(n_1-n_2)(qs(n_1+n_2)-2ps)=qs(m_2-m_1)(qs(m_2+m_1)-2qr)$$

We can take:

(1) $n_1-n_2=qs(m_2+m_1)-2qr$
(2) $qs(n_1+n_2)-2ps=m_2-m_1$

We can rearrange to use the Euclidean algorithm:

(1) $qs(m_2+m_1)+n_2- \; n_1=2qr$

(2) $qs(n_1+n_2)+m_1-m_2=2ps$

The Euclidean algorithm guarantees infinite integer solutions $(\alpha-k, \beta+qsk)$, for any integer k, to:

$$qsx+y=1$$

We take $k=0$ for simplicity:

By multiplying by $2qr$:

$$(2qr)(qs\alpha) + (2qr)\beta = 2qr$$

$$(2qs)(qs\alpha) + (2qs)\beta = 2qs$$

Take:

$$m_1 + m_2 = 2qr\alpha$$
$$n_2 - n_1 = 2qr\beta$$
$$n_2 + n_1 = 2qs\alpha$$
$$m_1 - m_2 = 2qs\beta$$

Then:

$$m_1 = qr\alpha + qs\beta$$
$$m_2 = qr\alpha - qs\beta$$

$$n_1 = qs\alpha + qr\beta$$
$$n_2 = qs\alpha - qr\beta$$

Lastly: we can directly prove that $n_1 \neq n_2$ $(or\ that\ m_1 \neq m_2)$

If $n_1 = n_2$ then $2qr\beta = 0$, as our transformations are non 0, $\beta = 0$ therefore $qs\alpha = 1$ and finally $|q| = 1$.

If $|q| = 1$, then by earlier work the point is bad.

Now suppose $a\ and\ b$ are irrational and let $p = \frac{n}{m}\ and\ q = \frac{k}{m}$ for any non-0 integers n, k, m

As in cases 1 and 2: we assume that the point is not good.

$$(x_1 - p - qb)^2 + (y_1 - b)^2 = (x_2 - p - qb)^2 + (y_2 - b)^2$$
$$\Rightarrow x_1^2 - 2x_1(p + qb) + y_1^2 = x_2^2 - 2x_2(p + qb) + y_2^2 - 2y_2 b$$

$$\Rightarrow b(2x_1 q - 2x_2 q + 2y_1 - 2y_2) = x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_1 p + 2x_2 p$$

The right hand side is rational, therefore to avoid contradiction:

$2x_1 q - 2x_2 q + 2y_1 - 2y_2 = 0$

$$\Rightarrow q(x_1 - x_2) = y_2 - y_1 = 0$$

We label the above equation {1}

Also:

$$0 = x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_1 p + 2x_2 p$$

$$\Rightarrow 2p(x_1 - x_2) = (x_1 - x_2)(x_1 + x_2) + (y_1 - y_2)(y_1 + y_2)$$

$$\Rightarrow \frac{y_2 - y_1}{q}\left(2p - (x_1 + x_2)\right) = (y_1 - y_2)(y_1 + y_2)$$

$$\Rightarrow (y_2 - y_1)\left(\left[\frac{x_1 + x_2}{q} - \frac{2p}{q}\right] - (y_1 + y_2)\right) = 0$$

Therefore either

$y_1 = y_2$ or $y_1 + y_2 = \frac{x_1 + x_2 - 2p}{q}$

If $y_1 = y_2$

As

$2p(x_1 - x_2) = (x_1 - x_2)(x_1 + x_2) + (y_1 - y_2)(y_1 + y_2)$, this implies:

$$x_1 = x_2$$

This contradicts the distinctiveness of the two lattice points.

If $y_1 + y_2 = \frac{x_1 + x_2 - 2p}{q}$

We label the above equation {2}

Recall that $p = \frac{n}{m}$ and $q = \frac{k}{m}$. Suppose that q is in reduced form, i.e. there are no common divisors of k and m besides 1.

Then {1} becomes $k(x_1 - x_2) - m(y_2 - y_1) = 0$

One solution to this is

$$x_1 - x_2 = 2m\ \{A\}$$
$$and\ y_2 - y_1 = 2k\ \{B\}$$

This lets us work with a Diophantine equation where the right hand side is 1.

{2} becomes $k(y_1 + y\_2) = m\left(x_1 + x_2 - \frac{2n}{m}\right)$

We rearrange for $m(x_1 + x_2) - k(y_1 + y_2) = 2n$

Since $\frac{k}{m}$ is in reduced form, the greatest common divisor of m and k is 1.

We can use the Euclidean algorithm to guarantee integers (γ, δ) such that:

$$m\gamma - k\delta = 1$$
$$\Rightarrow m(2n\gamma) - k(2n\delta) = 2n$$

So we can take

$$x_1 + x_2 = 2n\gamma \, \{C\}$$
$$and \; y_1 + y_2 = 2n\delta \, \{D\}$$

{A} + {C} gives $x_1 = m + n\gamma$

{B} + {D} gives $y_2 = k + n\delta$

Via {C} and {A} + {C}

$$x_2 = n\gamma - m \; and \; y_1 = n\delta - k$$

If $x_1 = x_2, then \; x_1 - x_2 = 2m = 0, thus \; m = 0$

This is contradictory as $q = \frac{k}{m}$ is rational

Therefore, (a, b) is a bad centre.

<u>Backwards Proof:</u>

Suppose that a is irrational and $2b$ is not an integer.

Assume the contrary: that the point is not good, meaning that there exists two distinct lattice points (x_1, y_1), (x_2, y_2) on the circumference of a circle centred (a, b), with radius r:

Case 1:

Via the equation of a circle:

$$r^2 = (x_1 - a)^2 + \; (y_1 - b)^2 = (x_2 - a)^2 + \; (y_2 - b)^2$$
$$x_1^2 - x_2^2 + \; y_1^2 - y_2^2 = -2(a(x_2 - x_1) + b(y_2 - y_1))$$
$$a = \frac{x_1^2 - x_2^2 + \; y_1^2 - y_2^2}{-2(x_2 - x_1)} - \frac{b(y_2 - y_1)}{(x_2 - x_1)}$$
$$= \frac{-2b(y_2 - y_1) - (x_1^2 - x_2^2 + \; y_1^2 - y_2^2)}{2(x_2 - x_1)}$$

Since b is rational and a is irrational, $x_1 = x_2$ to avoid contradiction

If $x_1 = x_2$, $y_1 - b = \pm(y_2 - b)$

Therefore either:

$y_1 = y_2$

Or $y_1 + y_2 = 2b$

In the first case, we reach a contradiction that (x_1, y_1) and (x_2, y_2) are distinct. In the second case, we contradict $2b$ not being an integer.

Case two follows the same proof.

Case three:

Suppose a and b are irrational and $a \neq \frac{n}{m} + b\left(\frac{k}{m}\right)$ for any integers n, k, m

As before:

$$r^2 = (x_1 - a)^2 + (y_1 - b)^2 = (x_2 - a)^2 + (y_2 - b)^2$$

$$x_1^2 - 2ax_1 + y_1^2 - 2by_1 = x_2^2 - 2ax_2 + y_2^2 - 2by_2$$

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 = -2(a(x_2 - x_1) + b(y_2 - y_1))$$

$$a = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{-2(x_2 - x_1)} - b\frac{(y_2 - y_1)}{(x_2 - x_1)}$$

This contradicts $a \neq \frac{n}{m} + b\left(\frac{k}{m}\right)$ as $x_1^2 - x_2^2 + y_1^2 - y_2^2$

Therefore when a and b are irrational and $a \neq p + qb$ where q and b are rational, (a, b) is good.

As a corollary of this lemma, there are countably many points which are good and uncountably many points which are not good, as there are countably many rational points uncountably many irrational points. An example of a set that contains an uncountable number of not good points (rational) is [r, r] for all real r.

Does this extend to n dimensions?

First start with 3 dimensions:

We aim to extend the backwards direction of lemma 1.

I suspect that similarly to lemma 1, all coordinates must not be half an integer, and one must be irrational. If this is not true, we will narrow our coordinate restrictions to two irrational coordinates.

Lemma 2

If the following holds then (a, b, c) is good:

- Two of a, b, c are irrational e.g. a, b where $a \neq pb + q$ for any rationals p, q, and the remaining coordinate is not half an integer.

Proof

Suppose $a \text{ and } b$ are irrational and $a \neq p + qb$ for any rational p, q, and $2c$ is not an integer.

Suppose for contradiction that $(x_1, y_1, z_1), (x_2, y_2, z_2)$ lie on the sphere

$$\Rightarrow (x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = (x_2 - a)^2 + (y_2 - b)^2 + (z_2 - c)^2$$

$$\Rightarrow -2ax_1 - 2by_1 - 2cz_1 + x_1^2 + y_1^2 + z_1^2 = -2ax_2 - 2by_2 - 2cz_2 + x_2^2 + y_2^2 + z_2^2$$

$$\Rightarrow a = \frac{2b(y_1 - y_2) + 2c(z_1 - z_2) + x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2}{2(x_1 - x_2)}$$

If $x_1 \neq x_2, we\ contradict\ a \neq p + qb$

Hence $x_1 = x_2$

Then

$$2b(y_1 - y_2) + 2c(z_1 - z_2) + y_2^2 - y_1^2 + z_2^2 - z_1^2 = 0$$
$$\Rightarrow b = \frac{2c(z_1 - z_2) + y_2^2 - y_1^2 + z_2^2 - z_1^2}{2(y_2 - y_1)}$$

If $y_1 \neq y_2$, we contradict b being irrational.

Hence

$$y_1 = y_2$$

Then

$$2c(z_1 - z_2) = z_2^2 - z_1^2$$

$$\Rightarrow -2c = z_1 + z_2$$

This is a contradiction as $2c$ is not an integer.

It is insufficient for only one coordinate to be irrational and the other two e.g. $a, b,$ to be such that $a \neq \pm b, 2a \ is \ not \ an \ integer \ and \ 2b \ is \ not \ an \ integer$.

Consider $(a, b, c) = \left(\sqrt{2}, \frac{1}{3}, \frac{2}{3}\right)$

Suppose that $(x_1, y_1, z_1) \ and \ (x_2, y_2, z_2)$ are distinct lattice points on a sphere with centre (a, b, c)

By earlier work, we know that to avoid contradiction with the irrationality of $\sqrt{2}$, we must have $x_1 = x_2$.

Our problem reduces to 2 dimensions.

$$\left(y_1 - \frac{1}{3}\right)^2 + \left(z_1 - \frac{2}{3}\right)^2 = \left(y_2 - \frac{1}{3}\right)^2 + \left(z_2 - \frac{2}{3}\right)^2$$

We know that by the first lemma this has distinct solutions as the centre coordinates are both rational.

To prove this explicitly:

$$y_1^2 - \frac{2}{3}y_1 + z_1^2 - \frac{4}{3}z_1 = y_2^2 - \frac{2}{3}y_2 + z_2^2 - \frac{4}{3}z_2$$
$$\Rightarrow (y_1 + y_2)(y_1 - y_2) + (z_1 + z_2)(z_1 - z_2) = \frac{2}{3}(y_1 - y_2) + \frac{4}{3}(z_1 - z_2)$$
$$\Rightarrow (3y_1 + 3y_2 - 2)(y_1 - y_2) = (3z_1 + 3z_2 - 4)(z_1 - z_2) = 0$$

Let $y_1 = 0, y_2 = -1$

Then $(3z_1 + 3z_2 - 4)(z_1 - z_2) = 5$

We can take

$$3z_1 + 3z_2 - 4 = 5$$

And

$$z_1 - z_2 = 1$$

We can take $x_1 = x_2 = 0$ for instance

So (0, 0, 2) and (0, -1, 1) lie on the sphere of radius $2^{\frac{1}{2}}$ centred at $\left(2^{\frac{1}{2}}, \frac{1}{3}, \frac{2}{3}\right)$

Generalising to n dimensions:

Lemma 3 (backward only)

If the following holds then a point $(a_1, a_2, a_3 \dots, a_n)$ is good.

- n-1 of $a_1, a_2, a_3 \dots, a_n$ are irrational

  where:

$$\sum_{i=1}^{n-1} p_i a_i \neq q, for\ any\ rationals\ p_i, q$$

Proof idea

By repeating the contradiction argument in the 3D proof, each time the problem recurses the dimension reduces by 1 until we arrive at the 2D case.

5)

Definitions:

We count rule one when counting the number of rules needed to get each element of S.

All elements of S are distinct – we do not double count repeats

Lemma 1

If d is a divisor of $a\ and\ b$, d is a divisor of $a - b$

Proof

$$\frac{a-b}{d}$$

$$= \frac{a}{d} - \frac{b}{d}$$

Since d is a divisor of $a \; and \; b$, $\frac{a}{d}$ and $\frac{b}{d}$ are integers, therefore, $\frac{a}{d} - \frac{b}{d}$ and $\frac{a-b}{d}$ is an integer. Therefore, d is a divisor of $a - b$.

<u>Lemma 2</u>

S contains all $\frac{n}{n+1}$, for all natural n

<u>Proof</u>

$$\frac{1}{1} \epsilon \; s$$

By rule (ii):

$$\frac{1}{2} \epsilon \; s$$

By rule (iii)

$$\frac{1+1}{1+2} = \frac{2}{3} \epsilon \; s$$

We can continue to use rule (iii) for all n such that:

$$\frac{n}{n+1} \epsilon \; s$$

As $\frac{n}{n+1}$ never has a common factor via lemma 1.

<u>Lemma 3</u>

All elements of S are rational.

<u>Proof</u>

Rule (i): $\frac{1}{1}$ is rational.

Rule (ii): $\frac{b}{2a}$ is rational as $a \; and \; b$ are integers.

Rule (iii): $\frac{a+c}{b+d}$ is rational as $a + c \; and \; b + d$ are integers as the sum of two integers is an integer.

Lemma 4

Via rule (iii): given two elements of S: $\frac{a}{b}$ and $\frac{c}{d}$

WLOG assume $\frac{a}{b} < \frac{c}{d}$ as one element must be greater than the other.

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Proof

$$\begin{aligned}
&\frac{a}{b} < \frac{c}{d} \\
&ad < bc \\
&ab + ad < ab + bc \\
&a(b+d) < b(a+c) \\
&\frac{a}{b} < \frac{a+c}{b+d}
\end{aligned}$$

Also:

$$\begin{aligned}
&\frac{a}{b} < \frac{c}{d} \\
&ad < bc \\
&ab + cd < bc + cd \\
&d(a+c) < c(b+d) \\
&\frac{a+c}{b+d} < \frac{c}{d}
\end{aligned}$$

By combining the two:

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Lemma 5

Building on lemma 2

All rationals in the interval $[0.5, 1\ ]$ are elements of S.

Proof by double inclusion

Forward inclusion: all elements of S are in the rational interval [0.5, 1]

Strong induction

Let rank(q) = the minimum number of applications needed to generate an element of S, q.

Base case: rank$(q) = 1$. The only element of S that you generate in one step is 1 using rule 1. 1 is rational and in the interval [0.5, 1].

Inductive step

Suppose rank$(q) \leq k$ for some natural constant k and that q is a rational in the interval [0.5, 1]

We are trying to show that for all rationals, r, if rank$(r) = k + 1$, then r is a rational in the interval [0.5, 1].

r must be generated by either rule (ii) or rule (iii).

Suppose that r is generated by rule (ii)

Therefore, $r = \frac{1}{2q}$ for some element of S, q.

Therefore, rank $(q) \leq k$, so q is a rational in the interval [0.5, 1]

$$\frac{1}{2} \leq q \leq 1$$

$$q \geq \frac{1}{2} and\ q \leq 1$$

$$\frac{1}{2q} \leq \frac{1}{2\left(\frac{1}{2}\right)} = 1\ and \frac{1}{2q} \geq \frac{1}{2(1)} = \frac{1}{2}$$

$$\frac{1}{2} \leq \frac{1}{2q} \leq 1$$

$$\frac{1}{2} \leq r \leq 1$$

Therefore, r is a rational in the interval [0.5, 1] as $\frac{2}{rational_1} = rational_2$.

Therefore, all elements of S, generated by rule (ii) are elements of the set of all rationals in the interval [0.5, 1].

For (iii):

Via lemma 4, as there exists two elements of $S, \frac{1}{2}\ and\ 1$:

$$\frac{1}{2} \leq \frac{a+c}{b+d} \leq 1$$

Therefore, all elements of S generated by rule (iii) are elements of the set of all rationals in the interval [0.5, 1].

This completes the forward inclusion.

<u>Backwards inclusion:</u>

We can describe a procedure which shows that all rationals in the interval [0.5, 1] are elements of S.

Take $\frac{p}{q}$ to be the goal element of S (not yet an element), where $0.5 \leq \frac{p}{q} \leq 1$

Take two previously generated elements, $\frac{a}{b}$ and $\frac{c}{d}$, where WLOG $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ as one element must be greater than the other. Via lemma 4, by applying rule (iii) to $\frac{a}{b}$ and $\frac{c}{d}$:

If $\frac{a+c}{b+d} = \frac{p}{q}$, we've found the desired element

If $\frac{a+c}{b+d} < \frac{p}{q}$ then we apply (iii) to $\frac{a+c}{b+d}$ and $\frac{c}{d}$

Otherwise, $\frac{a+c}{b+d} > \frac{p}{q}$ then we apply (iii) to $\frac{a}{b}$ and $\frac{a+c}{b+d}$

We repeat the above three lines by replacing $\frac{a+c}{b+d}$ with the result of rule (iii).

Therefore, if the sequence terminates, we prove that $\frac{p}{q}$ is an element of S.

<u>Proof that the sequence terminates</u>

Let $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$

Suppose $bc - ad = 1$

$$\frac{a+c}{b+d} - \frac{a}{b}$$

$$\Rightarrow \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{bc - ad}{b(b+d)} = \frac{1}{b(b+d)}$$

$$and\ \frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d) - d(a+c)}{d(b+d)} = \frac{1}{d(b+d)}$$

$$If\,\frac{a}{b} < \frac{p}{q} < \frac{a+c}{b+d}\ then$$

$$\frac{p}{q} - \frac{a}{b} < \frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)}$$

$$\frac{pb-aq}{qb} = \frac{p}{q} - \frac{a}{b} < \frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)}$$

$$\frac{1}{qb} < \frac{1}{b(b+d)} \Rightarrow q > b+d$$

Else if $\frac{a+c}{b+d} < \frac{p}{q} < \frac{c}{d}\, then$

$$\frac{1}{dq} \leq \frac{cq-pd}{dq} = \frac{c}{d} - \frac{p}{q} < \frac{\mathrm{c}}{\mathrm{d}} - \frac{\mathrm{a}+\mathrm{c}}{\mathrm{b}+\mathrm{d}} = \frac{\mathrm{cb}-\mathrm{da}}{\mathrm{d}(\mathrm{b}+\mathrm{d})} = \frac{1}{\mathrm{d}(\mathrm{b}+\mathrm{d})}$$

Which similarly implies that

$$\mathrm{q} > \mathrm{b}+\mathrm{d}$$

q is a constant, $b+d$ is an increasing monovariant, therefore, similarly to question 5, the procedure terminates.

Then as both:

The set of all rationals in the interval [0.5, 1] is a subset of the set of all elements of S.

And

The set of all elements of S is a subset of the set of all rationals in the interval [0.5, 1].

The set of all rationals in the interval [0.5, 1] must be the same as the set of all elements of S.

Note that: since the backwards inclusion doesn't utilise rule (ii), rule (ii) is only necessary to get the second element of S: $\frac{1}{2}$. If we modify rule (i):

If rule one says that if $\frac{a}{b}$ $and\ \frac{c}{d}\ are\ elements\ of\ S\ \left(WLOG\ \frac{a}{b} < \frac{c}{d}\right)$

Then (iii) gives all rationals in $\left[\frac{a}{b}, \frac{c}{d}\right], that\ is\ S = \mathbb{Q} \cap \left[\frac{a}{b}, \frac{c}{d}\right]$

Furthermore, as the forward inclusion shows that rule (iii) only generates terms between $\frac{a}{b}\ and\ \frac{c}{d}$, we can modify rule (i) to get any interval of rationals, by controlling the first two (boundary) elements.

6)

<u>Definitions</u>

We define the nth iteration as the procedure being applied to P_{n-1} to P_n

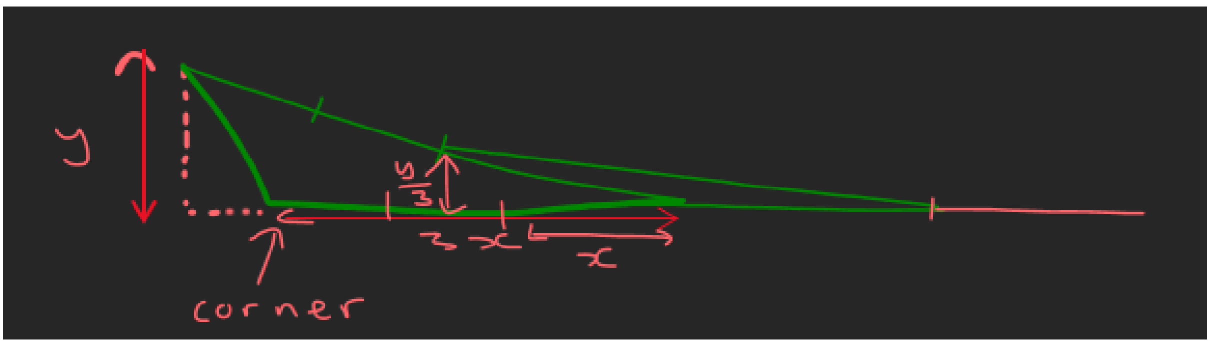
<u>Note</u>

On the nth iteration of the procedure, the shape loses twice as many triangles as in the previous iteration. This is because each iteration doubles the number of sides that the shape has. On the nth iteration we lose $2 * 3^n$ triangles.

<u>Idea</u>

By considering the fraction of the shape lost per iteration, I suspect we can derive a recursive formula.

The following diagram shows corners cut off the shape in consecutive iterations. This helps us investigate the fraction of the shape lost per iteration.



Area of the big triangle = $\frac{3xy}{2}$

Area of the small triangle = $\frac{\frac{1}{3}xy}{2}$

Therefore, ratio of small to big triangle is

$$\frac{\left(\frac{\frac{1}{3}xy}{2}\right)}{\left(\frac{3xy}{2}\right)} = \frac{1}{9}$$

Therefore, the area of each triangle removed on the n^th^ iteration is $\frac{1}{9} *$ the area of each triangle removed on the n-1^th^ iteration.

<u>Lemma 1</u>

The area of p_n is given by

<u>Proof</u>

The area of p_1 is given by:

$$10 - 3 * \frac{10}{9}$$

Where 3 is the number of triangles and $\frac{10}{9}$ is the area of each.

Therefore, the area of p_n is

$$10 - \left(3 * \frac{10}{9}\right) - \left(6 * \frac{10}{9^2}\right) - \left(12 * \frac{10}{9^3}\right) \ldots - \left(3 * 2^n * \frac{10}{9^n}\right)$$

We can rewrite this as the following, making it easy to generalise the area of p_n for all areas of p_1.

$$= 10(1 - \left(3 * \frac{1}{9}\right) - \left(6 * \frac{1}{9^2}\right) - \left(12 * \frac{1}{9^3}\right) \ldots - \left(3 * 2^n * \frac{1}{9^n}\right)$$

$$= 10 - 10\left(\frac{1}{3} - \frac{2}{27} - \frac{4}{243} - \cdots\right)$$

$$= 10 - 10 \sum_{r=0}^{n} \frac{1}{3}\left(\frac{2}{9}\right)^n$$

$$= 10\left(1 - \sum_{r=0}^{n} \frac{1}{3}\left(\frac{2}{9}\right)^n\right)$$

To generalise for all areas of p_0, A:

$$= A\left(1 - \sum_{r=0}^{n} \frac{1}{3}\left(\frac{2}{9}\right)^{n}\right)$$

Lemma 2

The area of p_∞ is $\frac{40}{7}$

Proof

From lemma 1, as r is less than 1, the sum converges:

$$10 - 10\left(\frac{\frac{1}{3}}{1 - \frac{2}{9}}\right) = \frac{4}{7} * 10$$

We can generalise the area of p_∞ for all areas of p_1, A:

$$\frac{4}{7}A$$

But what is the shape of p_∞?

At first, I thought that it would be circular, however after p_3 the shape is irregular.

Lemma 3

P_∞ is not a circle

Proof

We analyse the distance from the centre of p to the midpoint of each side.

Via the rotational symmetry of a triangle, this distance is constant.

Another way of thinking about it is that during each iteration, we are trimming either end of a side by the same amount, such that the midpoint is always the same.

The distance from $p's$ centre to a midpoint, x, is given by:

$$h = \left(x^2 - \frac{x^2}{4}\right)^{\frac{1}{2}} = \frac{3^{\frac{1}{2}}}{2}x$$

$$\Rightarrow 10 = \frac{\frac{3^{\frac{1}{2}}}{2}x^2}{2}$$

$$x^2 = 40\left(\frac{3^{\frac{1}{2}}}{3}\right)$$

$$x = \left(\frac{40}{3^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

Therefore, we can prove the lemma by contradiction: if we assume that p_∞ is a circle.

The area of the circle $p_\infty = \pi\left(\left(\frac{40}{3^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right)^2$

But from lemma 2, the area of $p_\infty = \frac{40}{7}$

$\frac{40}{7} \neq \pi\left(\frac{40}{3^{\frac{1}{2}}}\right)$ so we've reached a contradiction.

Generalisation

We can generalise the above to smaller/bigger divisions e.g. bisection.

Bisection is unique: the total area reduces by a factor of 4 infinitely, such that the area of $p_\infty = 0$.

If we divide each side into n sides, the area of p_∞ for some arbitrary area of $p_0 = A$ is given by:

$$A - \left(3 * \frac{A}{(n^2)}\right) - \left(6 * \frac{A}{(n^2)^2}\right) - \left(12 * \frac{A}{(n^2)^3}\right) \dots - \left(3 * 2^\infty * \frac{A}{(n^2)^n}\right)$$

$$= A - A\left(\frac{3}{(n^2)} - \frac{6}{(n^2)^2} - \frac{12}{(n^2)^3} - \cdots\right)$$

$$= A\left(1 - \sum_{r=0}^{\infty} \frac{3}{n^2}\left(\frac{2}{n^2}\right)^r\right)$$

$$= A\left(1 - \frac{\left(\frac{3}{n^2}\right)}{1 - \left(\frac{2}{n^2}\right)}\right)$$

7)

Definitions

A cell is a location on the grid which holds one of the integers $1\ to\ n^2$ on an n x n grid for all natural n.

A grid that satisfies the question conditions is ‘good’, one that does not is ‘bad’.

The main diagonal is the diagonal from the top left cell to the bottom right cell.

Lemma 1

Let n be natural. Then the number of good grids for an n by n grid is a multiple of $2(n!)^2$ (including 0)

For an n x n grid, with a working grid, there are $2(n!)$ good grids.

Proof

All rotations, permutations of rows and permutations of columns on a good grid produce good grids. This boils down to the commutative nature of multiplication.

Note that for all n x n grids, there is only one rotation that we cannot achieve via permutations of rows and columns: a rotation of 90 degrees.

Therefore, for an n x n grid, there is a multiple of $2(n!)^2$ good grids.

3^2	$2*3$	2^2
3	7	2
2^3	1	5

As a corollary, there are 72 good grids when n=3.

We can define a unique grid as a good grid that we cannot generate by permutations of rows, permutations of columns and rotations of an existing good grid.

Furthermore, we can assume that the product of row n = the product of column n.

Lemma 2

Via the above assumption: unique prime factors must be placed in cells along the main diagonal.

Proof

As prime factorisation is unique, placing a unique prime factor elsewhere doesn't follow our assumption at the end of lemma 1.

Lemmas 1 and 2 lets us reduce the number of cases from $2(n!)$ to one.

For a 5 x 5 grid:

We first place the unique primes (and 25) along the main diagonal.

Then we fill in remaining cells with descending rarity from the top left – we do this because having a small number of a certain factor e.g. 7, limits our options, working from the top left gives us the most information to deduce where certain numbers must go.

17	$7*2$	$3*2$	$5*2$	5
2	5^2	$7*3$	$11*2$	2^3*3
7	3^2	13	3^2*2	2^2
$5*3$	11	3	23	2^3
$5*2^2$	16	$6*2$	1	19

Idea

Perhaps when considering conditions to check if an n x n grid is good, we can first consider conditions where this isn't true.

One way this can happen is if there are more than n unique prime factors, as, unique prime factors must be placed along the main diagonal, and there are n cells along the main diagonal.

The algorithm in the appendix checks if this is the case for some n. Our optimisations are described in the comments but to summarise:

Unique primes must be greater than $\frac{n^2}{2}$ because any prime p, such that $p \leq \frac{n^2}{2}$ is a factor of $2p$ which is a number in the grid, therefore p is not unique.

When n=11:

The prime factors between 121 and 61 are:

61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113.

There are 13 unique primes. Therefore, there exists no good grid for n=11.

Lemma 3

Proof

$\pi(n)$ =the number of primes less than or equal to n.

We want n such that:

$$\pi(n^2) - \pi\left(\left\lceil \frac{n^2}{2} \right\rceil\right) > n$$

We can use bounds on $\pi(n)$ to find the minimum n where grids are guaranteed to not exist:

Whilst the following holds for $n \geq 17$, similarly to 11, this shows that there are no good grids for [11, 17], as the number of unique prime factors greater than half of n^2 > n when n ranges from 11 to 17.

For $n \geq 17$:

We use the lower bound for $\pi(n^2)$ and the upper bound for $\pi\left(\left\lceil \frac{n^2}{2} \right\rceil\right)$ to consider small n:

When n is even:

$$\frac{n^2}{2logn} - 1.25507\frac{\left(\frac{n^2}{2}\right)}{\log\left(\frac{n^2}{2}\right)} > n$$

When n is odd:

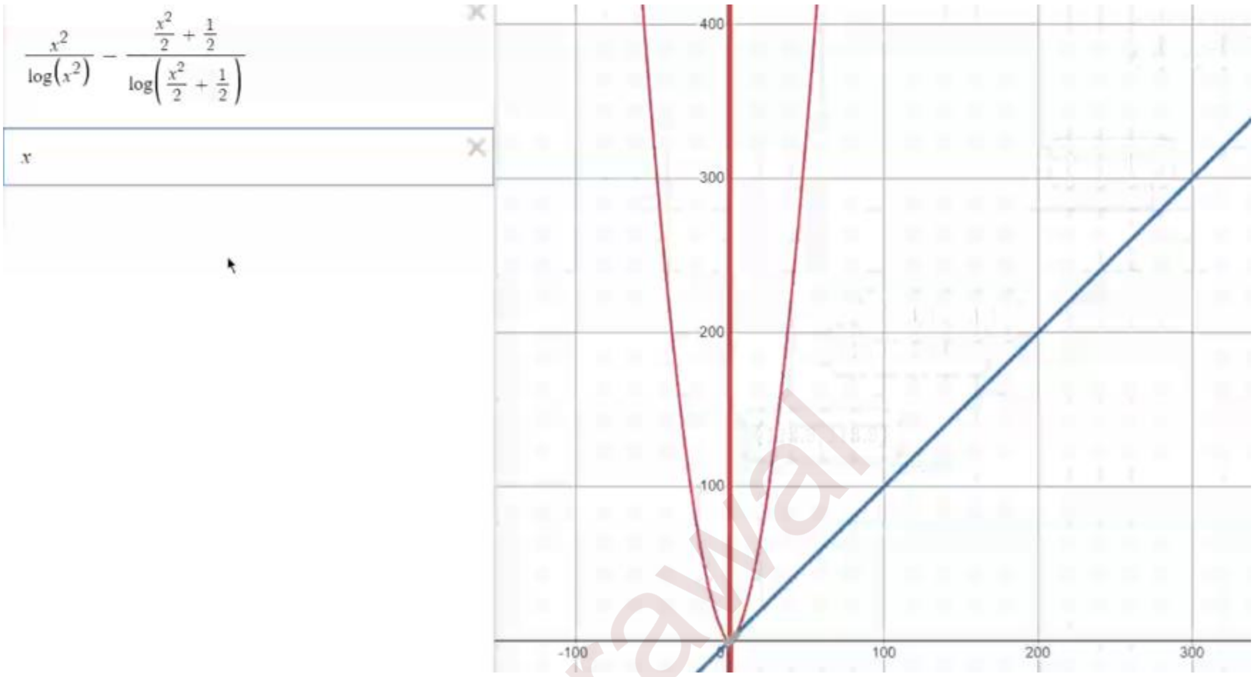
$$\frac{n^2}{2logn} - 1.25507\frac{\left(\frac{n^2}{2}\right) + \frac{1}{2}}{\log\left(\frac{n^2}{2}\right)} > n$$

Proof for even n

$$\frac{n^2}{2logn} - 1.25507\frac{\left(\frac{n^2}{2}\right)}{\log\left(\frac{n^2}{2}\right)} = \frac{n^2}{2logn} - 1.25507\frac{n^2}{4logn - log4}$$

$$\frac{n^2}{2logn} - \frac{n^2}{4logn - log4} \geq \frac{n^2}{2logn} - 1.25507\frac{n^2}{3logn} \ (for\ n \geq 4, which\ is\ satisfied)$$

$$= \frac{(3 - 2 * 1.25507)n^2}{6logn} > n$$



Is there a most efficient algorithm for finding an initial good grid?

Appendix

```
n=int(input())
grid=[]
for row_to_construct in range(n):
    grid.append([])
#Our grid looks like
#[ []
# " "
# " "
# " "
# " "
# " "
# " " n times (each row, arranged in a column)     ]
# We optimise brute forcing by using len() to control the length of 2d
lists instead of fillers like 'x'

#Our first step is to look for unique prime factors.
#Primes p<=n^2/2 will not be unique as 2p<=n^2
#So we search for primes in the interval [(n^2/2)+1, n^2] inclusive.
```

```
def it_is_prime(number):
    for possible_divisor in range(2, int(number*0.5) +1):
        if number%possible_divisor==0:
            return False
        return True

list_of_unique_primes=[]
for possible_prime in range(  int((n^2)/2 +1), n^2 +1): #checks all ints less than or equal to the prime
    if it_is_prime(possible_prime):
        list_of_unique_primes.append(possible_prime)

print(list_of_unique_primes)
```

8)

<u>Definitions</u>

We locate letters using a traditional 1 quadrant (x, y) coordinate system e.g. (0, 0) in the 3 x 3 example is A. This lets us work with vectors of integer components.

We describe an n x n grid as a repeating grid where the repeating unit/bold square has dimensions n by n

We are interested in combinations of letters. {A, H, F} $\equiv \{F, A, H\}$

A valid vector movement $\binom{a}{b}$ is such that it goes from one lattice point to another in a straight line without touching another lattice point resembles a reduced fraction as $a\ and\ b$ have no common divisors.

<u>Idea</u>

The repeating grid makes me think that we can utilise modular arithmetic.

If instead of thinking of a straight line, we think of a vector being repeated. This lets us use modular arithmetic such that we can reduce the scope of the problem e.g. to the main 3 x 3 grid in the first case.

Lemma 1

The sequence of coordinates in an n x n grid are periodic with fundamental period n.

Where the fundamental period is the number of unique lattice points a ‘loop’ (repeated vector movement) goes through.

Proof

Suppose that

$$k\binom{a}{b} \equiv \binom{0}{0} \, mod\ n$$

Further suppose that $a\ and\ b$ have no common divisors other than 1, i.e. $\binom{a}{b}$ is the integer ratio of the $x\!:y$ translation.

Then the greatest common divisor of $a\ and\ b$ is 1.

And

$$n|k$$

Because via the Euclidean algorithm, there exists integers r, s such that

$$ra + sb = 1$$

As a corollary of lemma 1, repeating a vector creates a loop e.g.

$\binom{1}{2}$ on a 3 x 3 grid looks like

Therefore, we can consider all possible vectors in a repeating n by n grid in just the bold/main n by n grid.

This makes more sense if we write each point as a position vector in modulo 3:

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\2\end{pmatrix} \longrightarrow \begin{pmatrix}2\\1\end{pmatrix} \longrightarrow \begin{pmatrix}0\\0\end{pmatrix}$$

And therefore, all lines pass through exactly n letters.

This also motivates the following:

.

<u>Lemma 2</u>

Vectors $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}-a\\-b\end{pmatrix}$ give us the same set of letters.

<u>Proof</u>

As in the diagram below, we are moving in reverse

Lemma 2 will be useful in avoiding double counting vectors, such that each vector is a line that goes through a distinct combination of n letters.

To find the four sets of letters in the 3 x 3 case, we can consider all possibilities which will help optimise our search for larger grids.

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}$$

Since $\binom{2a}{2b}$ forms the same line as $\binom{a}{b}$ because they have the same component ratios, we can eliminate any vectors $\binom{a}{b}$ where $a \; and \; b$ have a common divisor other than 1. Exceptions to this rule are vectors where there is a 0 component, in which case we take the least vector magnitude possible and $\binom{0}{0}$ which has no magnitude thus forming no line.

So for the 3 x 3 case we get:

As a corollary, we may only consider vectors with positive components.

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$$

Lemma 3

The loop of position vectors as defined at the end of lemma 1 is n+1 position vectors long.

Proof

As in the proof of lemma 3, the components of our vector $\binom{a}{b}$ have no common divisors.

$$n\binom{a}{b} \; mod \; n = \binom{a}{b}$$

Thus, it is sufficient to prove that there exists no k_2 *such that* $k_2 \binom{a}{b}$ *mod* $n = \binom{a}{b}$ where $|k_2| \leq n$

As the only common divisor of k_2a *and* k_2b *is* k_2, the remainder of $\frac{k_2a}{n} \neq$ *that of* $\frac{k_2b}{n}$

The +1 is because we start at $\binom{0}{0}$

Similarly in the 5 x 5 case, we can build on this idea for an n x n grid.

Since $\binom{k_1a}{k_1b}$ forms the same line as $\binom{a}{b}$ for any integer k_1 and we are working in modulo n, we must verify that $k_2 \binom{a}{b}$ *mod* $n \neq$ some existing vector $\binom{c}{d}$ *mod* n where k_2 is a non-negative integer less than or equal to n via lemma 4. E.g.

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrow \begin{pmatrix}2\\1\end{pmatrix} \rightarrow \begin{pmatrix}4\\2\end{pmatrix} \rightarrow \begin{pmatrix}1\\3\end{pmatrix} \rightarrow \begin{pmatrix}3\\4\end{pmatrix} \rightarrow \begin{pmatrix}0\\0\end{pmatrix}$$

Note that we can start anywhere in this loop and get the same elements in a different order e.g.:

$$\begin{pmatrix}4\\2\end{pmatrix} \rightarrow \begin{pmatrix}3\\4\end{pmatrix} \rightarrow \begin{pmatrix}2\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\3\end{pmatrix} \rightarrow \begin{pmatrix}0\\0\end{pmatrix} \rightarrow \begin{pmatrix}4\\2\end{pmatrix}$$

So for the 5 x 5 case, the possible vectors are:

Therefore, the four sets are:

{ADG}, {AHF}, {AEI} and {ABC}

For the 5 x 5 case:

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}0\\4\end{pmatrix},$$

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix},$$

$$\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix},$$

$$\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix},$$

$$\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}$$

Via our logic from the 3 x 3 case we reduce this to:

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}$$

Via lemma 2 we reduce this to:

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}$$

Via our logic about loops above in conjunction with lemma 2 (note that whilst we could use this logic first, it is less efficient as lemma 2 is simpler to implement) we reduce this to:

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}$$

Therefore, there are 5 distinct combinations (sets) of letters in a 5 x 5 grid.

The 7 x 7 case is similar.

The 6 x 6 case is different however, as 6 is not prime:

<u>Lemma 4</u>

Let the grid be p x p where p is a prime.

Let $P \neq \begin{pmatrix}0\\0\end{pmatrix}$ be a point in the main grid.

Then P belongs to exactly one set.

<u>Proof</u>

Note that P belongs to at least one set; just draw a line segment connecting $\begin{pmatrix}0\\0\end{pmatrix}$ $and\ P$.

Suppose that P belongs to two sets: $S_A\colon \{0, A, 2A, \ldots (p-1)A\}\ mod(p)$

$$S_C\colon \{0, C, 2C, \ldots., (p-1)C\}\ mod(p)$$

Where A and C are Valid vector movements which generate the sets.

Write $A = \begin{pmatrix}a\\b\end{pmatrix}, C = \begin{pmatrix}c\\d\end{pmatrix}$

Then $P = k\begin{pmatrix}a\\b\end{pmatrix} = l\begin{pmatrix}c\\d\end{pmatrix}$

$$for\ some\ interger\ k, l$$

$$S_A and\ S_C\ are\ cyclic\ additive\ groups\ of\ prime\ order\ p.$$

Hence every element of S_A generates S_A which is proven by Bezout's Lemma, likewise for S_C

As $k\begin{pmatrix}a\\b\end{pmatrix} = l\begin{pmatrix}c\\d\end{pmatrix}, by\ the\ above,$ we know that $k\begin{pmatrix}a\\b\end{pmatrix}\ generates\ S_C$.

Thus $nk\binom{a}{b} = \binom{c}{d}, for\ some\ integer\ n.$

So $\binom{a}{b}$ generates $\binom{c}{d}$ and thus all of S_C.

Likewise, $\binom{c}{d}$ generates all of S_A. Thus, $S_A = S_C$.

Hence P belongs to exactly one set.

Note: this is not true if the grid is not p x p e.g. 6 x 6 we have sequences