Brownian Motion

Arabin Kumar Dey

Assistant Professor

Department of Mathematics
Indian Institute of Technology Guwahati

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Outline

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By a standard one-dimensional Brownian Motion on $[0, T]$ we mean a stochastic process $\{W(t) : 0 \leq t \leq T\}$ with the following properties :

1. $W(0) = 0$
2. $W(t)$ is continuous with respect to t.
3. The increments $\{W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots, W(t_k) - W(t_{k-1})\}$ are independent for any k and any $0 \leq t_0 < t_1 < \cdots < t_k \leq T$.
4. $W(t) - W(s) \sim N(0, t - s)$.

$W(t) \sim N(0, t)$ for $0 < t \leq T$.
For constants μ and $\sigma > 0$, we call a process $X(t)$, a Brownian motion with drift μ and diffusion coefficient σ^2 (abbreviated $X \sim BM(\mu, \sigma^2)$) if $\frac{X(t)-\mu t}{\sigma}$ is a standard Brownian motion.

$$X(t) = \mu t + \sigma W(t).$$

It follows that $X(t) \sim N(\mu t, \sigma^2 t)$.

Moreover X solves the SDE : $dX(t) = \mu dt + \sigma dW(t)$.
we may define a Brownian Motion with time dependent drift and diffusion coefficient through the SDE :

$$dX(t) = \mu(t)dt + \sigma(t)dW(t).$$

i.e. through

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

with $X(0)$ an arbitrary constant. The process X has continuous sample paths and independent increments. Each increment $X(t) - X(s)$ is normally distributed with mean,

$$E(X(t) - X(s)) = \int_s^t \mu(u)du$$

$$V(X(t) - X(s)) = \int_s^t \sigma^2(u)dW(u)$$

Let $Z_1, Z_2, Z_3, \cdots, Z_n$ be independent standard normal variables generated using any of the standard methods. For a standard Brownian Motion we set : $t_0 = 0$ and $W(0) = 0$. Subsequent values can be generated using :

$$W(t_{i+1}) = W(t_i) + \sqrt{(t_{i+1} - t_i)} \cdot Z_{i+1}, i = 0, 1, \cdots, n-1.$$

For $X \sim BM(\mu, \sigma^2)$ with constants μ and σ and given $X(0)$ we set,

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{(t_{i+1} - t_i)} \cdot Z_{i+1}, i = 0, 1, \cdots, n-1.$$

With time dependent coefficients, the recursion becomes,

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(s)ds + \sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(u)du} \cdot Z_{i+1}, i = 0, 1, 2, \cdots, n-$$

Replacing the integral with Euler approximation we get :

$$X(t_{i+1}) = X(t_i) + (t_{i+1} - t_i)\mu(t_i) + \sigma(t_i)\sqrt{(t_{i+1} - t_i)} \cdot Z_{i+1};\; i = 0, 1, \cdots, n$$