
Note: This document is a part of the lectures given to students of IIT Guwahati during the Jan-May 2018 Semester.

Box-Muller Method:

Perhaps the simplest method to implement (though not the fastest or necessarily the most convenient) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal, each component of which is thus a univariate standard normal. The algorithm is based on the following two properties of the bivariate normal:
If $Z \sim \mathcal{N}(0, I_2)$ then,

1. $R = Z_1^2 + Z_2^2$ is exponentially distributed with mean 2, that is, $P(R \leq x)1 = 1 - e^{-x/2}$.

2. Given R, the point (Z_1, Z_2) is uniformly distributed on the circle of radius $\sqrt{R}$ centered at the origin.

Thus, to generate (Z_1, Z_2), we first generate R and then choose a point uniformly from the circle of radius $\sqrt{R}$. To sample from the exponential distribution we may set $R = -2\log(U_1)$ with $U_1 \sim \mathcal{U}[0,1]$. To generate a random point on a circle, we may generate a random angle uniformly between 0 and 2π and then map the angle to a point on the circle. The random angle may be generated as $V = 2\pi U_2$ with $U_2 \sim \mathcal{U}[0,1]$. The corresponding point on the circle has coordinates $(\sqrt{R}\cos(V), \sqrt{R}\sin(V))$. The complete algorithm is:

1. Generate U_1, U_2 independent on $\mathcal{U}[0,1]$.

2. $R = -2\log(U_1)$.

3. $V = 2\pi U_2$.

4. $Z_1 = \sqrt{R}\cos(V)$ and $Z_2 = \sqrt{R}\sin(V)$.

5. Return Z_1 and Z_2.

Marsaglia and Bray Method:

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by avoiding evaluation of the $\cos$ and $\sin$ functions.. The Marsaglia Bray method instead uses acceptance rejection method to sample paths uniformly in the unit disc and transforms the points to normal variates. The algorithm is as follows:

1. (While $X > 1$)
   Generate $U_1, U_2 \sim \mathcal{U}[0,1]$
   $U_1 = 2U_1 - 1$
   $U_2 = 2U_2 - 1$
   $X = U_1^2 + U_2^2$.

2. $Y = \sqrt{-2\log(X)/X}$.

3. $Z_1 = U_1 Y$ and $Z_2 = U_2 Y$.

4. Return Z_1 and Z_2.

The transformation $U_i \rightarrow 2U_i - 1\ ,\ i = 1 : 2$ makes (U_1, U_2) uniformly distributed on the square $[-1, 1] \times [-1, 1]$. Accepting only those pairs for which $X = U_1^2 + U_2^2$ is less than or equal to 1 produces points uniformly distributed over the disc of radius 1 centered at the origin. Conditional on acceptance, X is uniformly distributed between 0 and 1 so that $\log(X)$ has the same affect as $\log(U_1)$ for Box-Muller.