Assignment Zero
ECE 4950

Due: 2/10/17

- This assignment will not be graded.
- You are expected to turn it in.
- Provide credit to **any sources** other than the course staff that helped you solve the problems.
- You can look up definitions/basics online (eg wikipedia, stackexchange, etc)
- Assignments will be uploaded on CMS.

Problem 1. A function f over a space $\mathcal{X}$ is convex if and only if for $0 \leq \lambda \leq 1$, and $x_1, x_2 \in \mathcal{X}$,

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2).$$

f is called mid-point convex if

$$\frac{f(x_1) + f(x_2)}{2} \geq f(\frac{x_1 + x_2}{2}).$$

All functions we will deal in this course are convex if and only if they are mid-point convex. Throughout this course, assume mid-point convexity to be equivalent to convexity.

A function f is concave if $-f$ is convex.

1. Let f be convex. Suppose $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\lambda_1 + \ldots + \lambda_k = 1$. Prove that
$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + \lambda_k f(x_k) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_k x_k).$$
2. If f and g are two convex functions, show that $\max\{f, g\}$, and $f + g$ are both convex.
3. Show that $\log(x)$ is concave over $(0, \infty)$.
4. Show that e^x and e^{-x} are both convex over $x \in \mathbb{R}$.
5. Let $x_1, x_2, y_1, y_2 > 0$. Then show that
$$x_1 \log \frac{x_1}{y_1} + x_2 \log \frac{x_2}{y_2} \geq (x_1 + x_2) \log \frac{x_1 + x_2}{y_1 + y_2}.$$

Problem 2. A discrete distribution p over $\{1, \dots, k\}$ is a map $p : \{1, \dots, k\} \to [0, 1]$, such that $\sum_x p_x = 1$ (p_x is the probability of x under p). The entropy of p is $H(p) = \sum_x p_x \log \frac{1}{p_x}$.

1. Show that $0 \le H(p) \le \log k$. (assume $0 \log 0 = 0$)

2. For any p, show that $H(p) \ge -\log(\sum_x p_x^2)$.

3. For distributions p, q over $\{1, \dots, k\}$, $(p+q)/2$ be the distribution that assigns probability $(p_x + q_x)/2$ to the symbol x. Show that $H((p+q)/2) \ge \frac{1}{2}(H(p) + H(q))$, namely entropy is concave.

Problem 3. The most ubiquitous continuous distributions are Gaussian distributions. They are parametrized by $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$, and are denoted $N(\mu, \sigma^2)$. Their probability density function is given by:

$$N(\mu, \sigma^2)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

1. Show that the expectation of $N(\mu, \sigma^2)$ is μ.

2. Your friend generates X, Y independently from $N(0, 1)$, and says you can select one of the following two numbers: $2X$ or $X + Y$. You get a candy if the number you select is larger than 1. What do you choose to maximize your chances of getting the candy?

3. What is the distribution of $X + Y$? Hint: It is a Gaussian too.

Problem 4.

1. Find the point closest to the origin, which is also a solution to:

$$x + y + 2z = 10$$
$$x - y + z = 20.$$

2. The gradient is of a function is its direction of highest increase. What is the gradient of $x^2y + y^2 - xy$ at the point $(1, 1)$?

3. What is the smallest and largest value of $x^2 + y^2$ within $(x-3)^2 + (y-2)^2 \le 1$?

Problem 5. Please do the following. Some links are on the class website, and search online for more.

1. Install Python3.x, and Jupyter on your computer.

2. Install numpy and scipy. We may need more things later.

After this try and play around with python to get a handle of things such as printing, and doing mathematical computations. Feel free to use the internet as much as you like. Watch scipy, and numpy tutorials, and any other videos you like.

Then do the following:

1. Load the file heights-age-test.csv. The file has four test examples, one in each row. Each example has four attributes, $NAME, AGE, BOY?$, and $Height > 55$".

2. You have to implement the decision tree that we did in class in python. In other words, you simply have to do the following:
   (a) Check if the example is a boy.
   (b) If Yes, check if $age > 8$, if No check if $age > 11$.
   (c) Output 1 if the answer to the previous point was yes, otherwise 0.
3. Match the answers you get with the last column of the file (Which you should not have touched before!).
4. Submit the code with a read me file that we can use to run the code.
5. The output should be simply one number, the number of test cases that are correctly classified.

The goal of this exercise is more for gaining familiarity with python rather than implementing machine learning algorithms, which we will do in the next assignment.