yc2329 HW0 ECE 4950 Yi Chen

Problem 1 1) f is convex. According to Jensen's inequality $f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2)$

$f(\sum_{i=1}^{k} \lambda_i x_i) = f(\lambda_1 + (1-\lambda_1)\sum_{i=2}^{k} \frac{\lambda_i}{1-\lambda_1} x_i) \le \lambda_1 f(x_1) + (1+\lambda_1) f(\sum_{i=2}^{k} \frac{\lambda_i}{1-\lambda_1} x_i)$

since $\sum_{i=2}^{k} \frac{\lambda_i}{1-\lambda_1} = 1$, we can induce that, $\lambda_1 f(x_1) + \cdots + \lambda_k f(x_k) \ge f(\lambda_1 x_1 + \cdots + \lambda_k x_k)$.

2) $\max\{f,g\} = \begin{cases} f, & f \ge g \\ g, & f < g \end{cases}$ $\because$ f and g are both convex $\therefore \max\{f,g\}$ must be convex.

$\because f \& g$ are convex. $\therefore \begin{cases} \lambda f(x_1) + (1-\lambda) f(x_2) \ge f(\lambda x_1 + (1-\lambda) x_2) \\ \lambda g(x_1) + (1-\lambda) g(x_2) \ge g(\lambda x_1 + (1-\lambda) x_2) \end{cases}$

$f+g$: $\lambda(f(x_1) + g(x_1)) + (1-\lambda)(f(x_2) + g(x_2)) \ge f(\lambda x_1 + (1-\lambda) x_2) + g(\lambda x_1 + (1-\lambda) x_2)$

$\therefore f+g$ is also convex

3) let $f(x) = -\log x$. by AM-GM inequality, we have $\frac{x_1+x_2}{2} \ge \sqrt{x_1 x_2}$

~~$f'(x) = \frac{-1}{x} < 0$. $\therefore f(x)$ decreases $\therefore f(\frac{x_1+x_2}{2}) \le f(\sqrt{x_1 x_2})$~~

~~thus,~~ $\because f'(x) = \frac{-1}{x} < 0$, $\therefore f(x)$ decreases, $\therefore f(\frac{x_1+x_2}{2}) \le f(\sqrt{x_1 x_2})$

$\therefore -\log(\frac{x_1+x_2}{2}) \le -\frac{1}{2}\log x_1 - \frac{1}{2}\log x_2$ $\therefore f(x)$ is convex over $(0, +\infty)$

$\because -f(x) = \log x$. $\therefore \log x$ is concave over $(0, +\infty)$

4) e^x: $\frac{e^{x_1} + e^{x_2}}{2} \ge \sqrt{e^{x_1}} \cdot \sqrt{e^{x_2}} = e^{\frac{x_1+x_2}{2}}$ $\therefore e^x$ is convex.

e^{-x}: $\frac{e^{-x_1} + e^{-x_2}}{2} \ge \sqrt{e^{-x_1}} \cdot \sqrt{e^{-x_2}} = e^{-\frac{x_1+x_2}{2}}$, $\therefore e^{-x}$ is convex

thus, e^x and e^{-x} are both convex over R.

5) from question 3) we know, $-\log x$ is convex

$\therefore -\frac{x_1}{x_1+x_2}\log\frac{y_1}{x_1} - \frac{x_2}{x_1+x_2}\log\frac{y_2}{x_2} \ge -\log(\frac{y_1}{x_1+x_2} + \frac{y_2}{x_1+x_2}) = -\log(\frac{y_1+y_2}{x_1+x_2})$

$\therefore \frac{x_1}{x_1+x_2}\log\frac{x_1}{y_1} + \frac{x_2}{x_1+x_2}\log\frac{x_2}{y_2} \ge \log\frac{x_1+x_2}{y_1+y_2}$. $\therefore x_1\log\frac{x_1}{y_1} + x_2\log\frac{x_2}{y_2} \ge (x_1+x_2)\log\frac{x_1+x_2}{y_1+y_2}$

Problem 2 1) $\because 0 < P_x \le 1$, $H(P) = \sum_x P_x \log\frac{1}{P_x}$. $\therefore H(P) = -\sum_x P_x \log P_x \ge 0$

According to Jensen's inequality, $\sum_x P_x \log\frac{1}{P_x} \le \log\sum_x P_x \cdot \frac{1}{P_x} = \log K$

$\therefore 0 \le H(P) \le \log K$

2) $H(P) = -\sum P_x \log P_x$, according to Jensen's inequality,

$H(P) \ge -\log(\sum_x P_x^2)$.

Problem 2/3 from problem 1.5 we know, $x_1 \log \frac{x_1}{y_1} + x_2 \log \frac{x_2}{y_2} \geq (x_1+x_2)\log(\frac{x_1+x_2}{y_1+y_2})$

let $x_1 = p_x$, $x_2 = q_x$, $y_1 = y_2 = 1$, we have $p_x \log p_x + q_x \log q_x \geq (p_x+q_x)\log(\frac{p_x+q_x}{2})$

thus, $\sum_x p_x \log p_x + \sum_x q_x \log q_x \geq \sum_x (p_x+q_x) \log(\frac{p_x+q_x}{2})$

thus, $-\frac{1}{2}\sum_x (p_x+q_x)\log(\frac{p_x+q_x}{2}) \geq -\frac{1}{2}(\sum_x p_x \log p_x + \sum_x q_x \log q_x)$

we have $H(\frac{p+q}{2}) \geq \frac{1}{2}(H(p)+H(q))$.

Problem 3/ $N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, we know $N(\mu,\sigma^2) = 1 \therefore \int e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}$ (*)

differentiate (*) on μ, we have: $\mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

that is, $\int_{-\infty}^{+\infty} x \cdot N(\mu,\sigma^2) dx = \mu \cdot 1 = \mu$. and this is the def. of expectation

$\therefore N(\mu,\sigma^2)$'s expectation is μ.

Problem 3 2) $E(X+Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x+y)\, f_{XY}(x+y)\, dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{XY}(x+y)dx + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f(x+y)dy$

$= E(X)+E(Y).$

$var(X+Y)= E((X+Y-\mu)^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (X+Y-\mu)^2 f_{XY}(X+Y-\mu)^2 dxdy = \sigma_X^2+\sigma_Y^2$

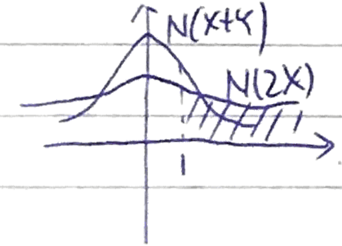
$E(\alpha X) = \int_{-\infty}^{+\infty} a x f_X(x)\, dx = aE(X)$

$var(\alpha X) = \int_{-\infty}^{+\infty} (\alpha x - E(\alpha x))^2 f(x)\, dx = \int_{-\infty}^{+\infty} (\alpha x - \alpha\mu)^2 f(x)\, dx = \int_{-\infty}^{+\infty} \alpha^2 (x-\mu)^2 f(x)\, dx$

$= \alpha^2 \int_{-\infty}^{+\infty} (x-E(x))^2 f(x)\, dx = \alpha^2 \sigma^2$

$\therefore$ Distribution of $X+Y$: $N(X+Y) \sim (0, 2)$

- - - - - - - - $2X$: $N(2X) \sim (0, 4)$

3) 

I'll choose 2X, because N(2X) variance is bigger, and its expected value is 0, which means it has more chance to get number larger than 1.

problem 4 1/ $f(x,y) = x^2+y^2$ (distance function) constraint: $x+2y-10=0$

we get $(x,y)=\lambda(1,2)$. therefore, $x=\frac{y}{2}=\lambda$. and $x+4y-10=0$

$\therefore \lambda=2$. closest point: $(2,4)$

2/ $\varphi(x,y) = x^2y+y^2-xy$. $\nabla\varphi = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}) = (2xy-y, x^2+2y-x)$.

at $(1,1)$, $\nabla\varphi = 1+2=3$

Problem 4 3

constraint : $(x-3)^2+(y-2)^2=C$ $(0\le C\le 1)$

let $f(x,y,\lambda)=x^2+y^2+\lambda((x-3)^2+(y-2)^2-C)$

$$\begin{cases}\frac{\partial f}{\partial \lambda}=0\\ \frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial z}=0\end{cases} \Rightarrow \begin{cases}(x-3)^2+(y-2)^2=C\\ 2x+2\lambda(x-3)=0\\ 2y+2\lambda(y-2)=0\end{cases}$$

$$\therefore \begin{cases}x_1=3(1+\sqrt{\frac{C}{13}})\\ x_2=2(1+\sqrt{\frac{C}{13}})\\ \frac{1}{1+\lambda_1}=1+\sqrt{\frac{C}{13}}\end{cases} \quad \& \quad \begin{cases}x_2=3(1-\sqrt{\frac{C}{13}})\\ y_2=2(1-\sqrt{\frac{C}{13}})\\ \frac{1}{1+\lambda_2}=1-\sqrt{\frac{C}{13}}\end{cases}$$

$f(x_1,y_1,\lambda_1)=(\sqrt{C}+\sqrt{13})^2$. min $(C=0)$, $f_{min}=13$.

$C=1$, $f_{max}=14+2\sqrt{3}$

$f(x_2,y_2,\lambda_2)=(\sqrt{C}-\sqrt{13})^2$. $C=0$ $f_{max}=13$

$C=1$ $f_{min}=14-2\sqrt{13}$

$\therefore$ min: $14-2\sqrt{13}$.

max: $14+2\sqrt{13}$