Given a Square Matrix $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^{n \times 1}$, $x^T A x$ is called quadratic form.

A Symmetric matrix is called positive definite, if for all non-zero $x \in \mathbb{R}^{n \times 1}$

$$x^T A x > 0$$

A Symmetric matrix is called positive semidefinite if

$$x^T A x \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^{n \times 1}$$

The Gradient :

Suppose $f : \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}$ function, i.e. input a real vector, output a real scaler, Then gradient is defined as, let $x \in \mathbb{R}^{n \times 1}$

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

The Hessian :

$$\nabla_x^2 f(x) \triangleq \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \frac{\partial^2 f(x)}{\partial x_2^2} & & \\ & \vdots & \ddots & \\ \frac{\partial^2 f(x)}{\partial^2 x_n x_1} & \vdots & & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Example :

i) if $f(x) = a^T x$, then

$$\nabla_x^2 f(x) = 0$$

ii) $f(x) = x^T A x$

$$\frac{\partial f(x)}{\partial x_k} = 2 \sum_{i=1}^{n} A_{ki} x_i$$

$$\frac{\partial^2 f(x)}{\partial x_l \partial x_k} = 2 A_{kl}$$

$$\Rightarrow \nabla_x^2 f(x) = 2A$$

$$= \sum_{i=1}^{n} A_{ik} x_i + \sum_{j=1}^{n} A_{kj} x_j$$

$$= 2 \sum_{i=1}^{n} A_{ki} x_i \quad (A_{ik} = A_{ki})$$

$$\nabla_x f(x) = 2 \begin{bmatrix} \sum_{i=1}^{n} A_{1i} x_i \\ \vdots \\ \sum_{i=1}^{n} A_{ni} x_i \end{bmatrix} = 2Ax$$