# Random Multiplicative Walks on the Integers Modulo $n$

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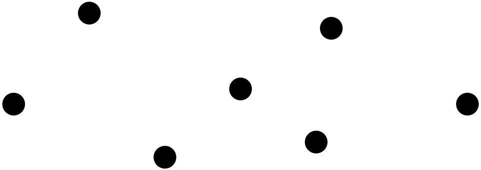

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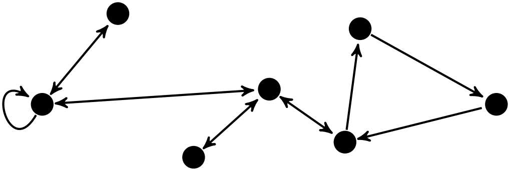

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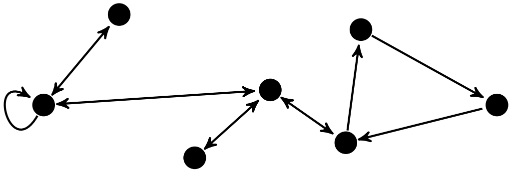

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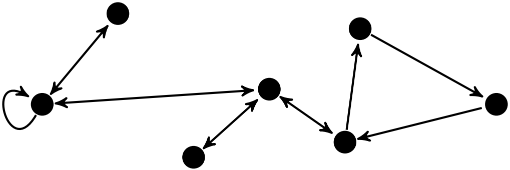

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A state is **absorbing** if it is not possible to leave that state.

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Note that since the elements of $S$ generate $G$, this walk will be transitive, and there are no absorbing elements.

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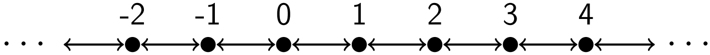

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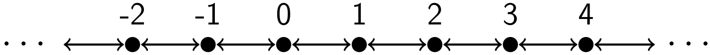


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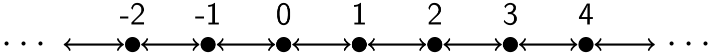


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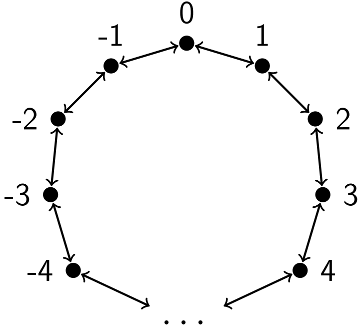

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Recall: A **monoid** is almost a group, but some elements may not have inverses.

Since elements may not have inverses, the walk may not be transitive.

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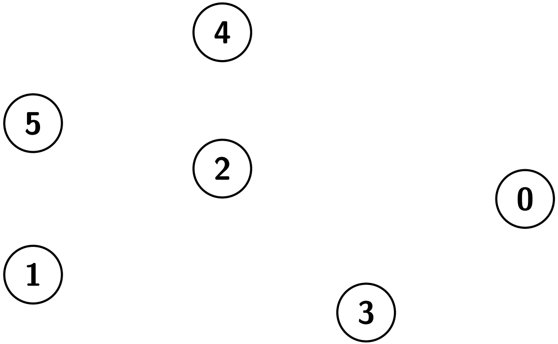

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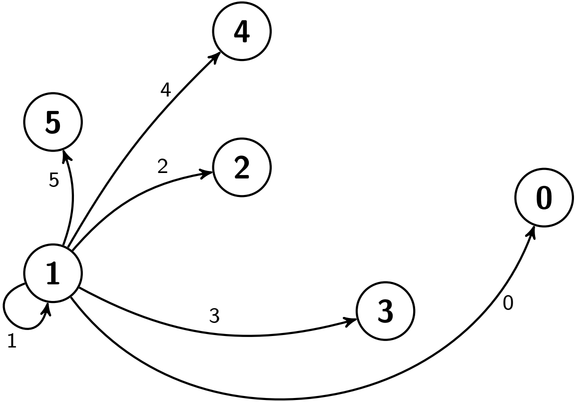

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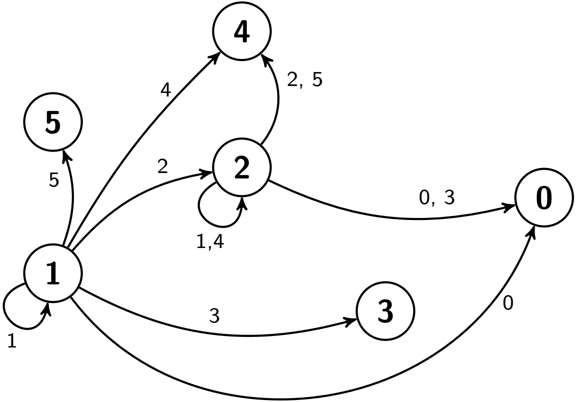

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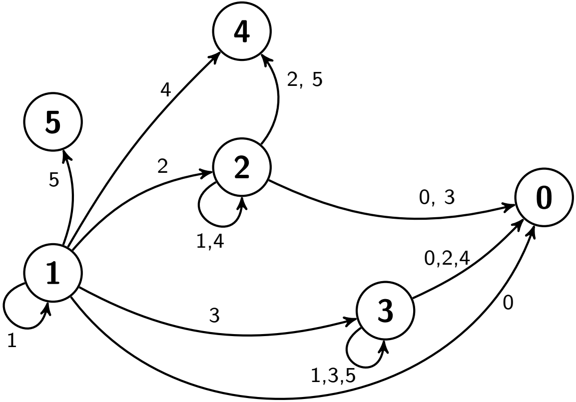

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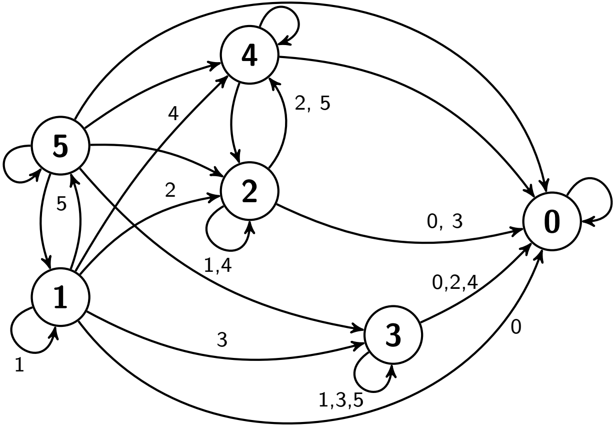

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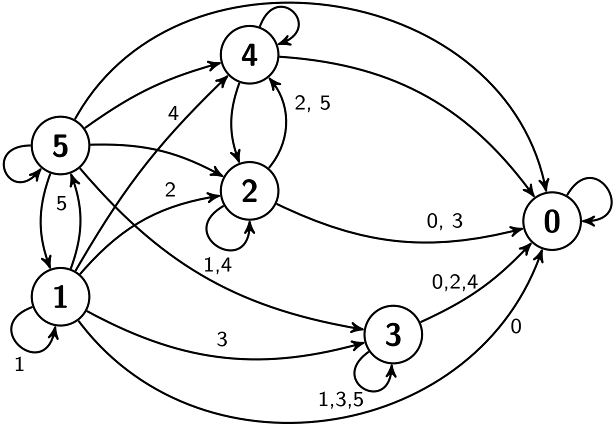


Single absorbing state: 0 (mod $n$).

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$$a(p) = p$$

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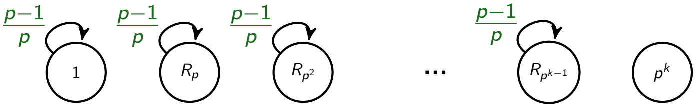


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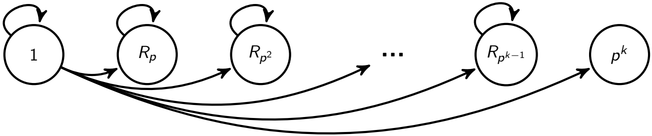


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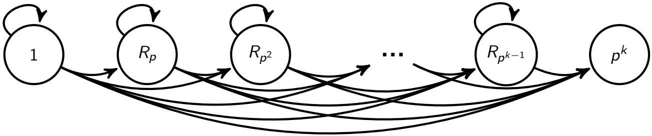


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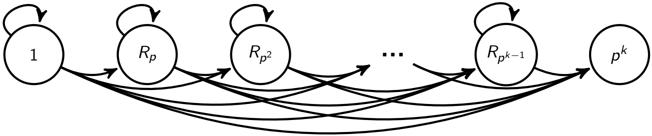


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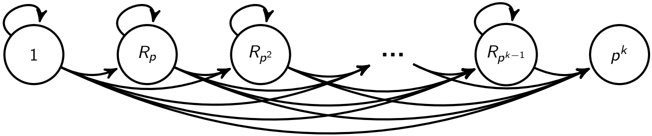


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&= k(p-1)+1
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# Squarefree numbers

## Theorem

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Factor $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1 > p_2 > \ldots > p_k$.

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The asymptotic behavior of $P_i(n)$, $B(n)$ and their friends have been studied by Alladi, De Koninck, Erdős, Ivić, Naslund, Pomerance and others.

# Estimates

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**Theorem (Wheeler, 1990)**

*The integers with $P_1(n) > P_2(n)^2$ have density 0.62432... the Golomb-Dickman constant.*

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**Balasubramanian:** $K_2=\frac{8}{3}\zeta(3/2)$.

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**"On average:"** $a(n) \approx P_1(n) + \left(1 - \frac{\pi}{4}\right) P_2(n)$.

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*Can similar results be obtained about the variance of the time to reach $0 \pmod{n}$?*

Thank you!