Counting Primitive Sets
and other statistics of the divisor graph of $\{1, 2, \ldots n\}$

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Independent subsets of a divisor graph are primitive sets of integers.

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How many primitive subsets of $\{1, 2 \dots n\}$ are there?

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.

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A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

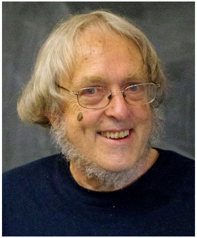
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



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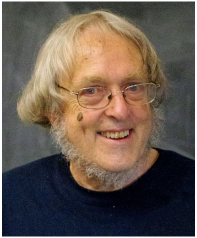

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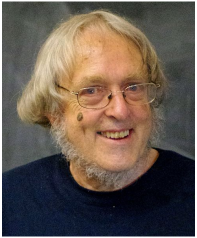

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Every subset of $\left(\frac{n}{2}, n\right]$ is primitive. There are $2^{\lceil \frac{n}{2} \rceil} > \sqrt{2}^n$ such subsets.

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Proof is not effective: Gives no insight on the value of this constant.

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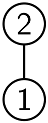

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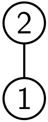

An **independent** set of vertices in a graph has no adjacent vertices.

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Thus

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} = 2^{n/3} 3^{n/6} \approx 1.5131^n.$$

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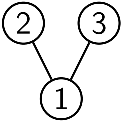
For k odd, 5 primitive subsets of $\{k, 2k, 3k\}$ replace 4 of just $\{2k, 3k\}$.



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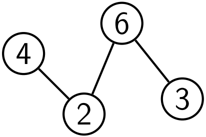

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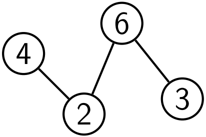

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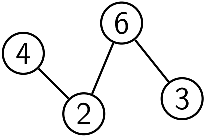

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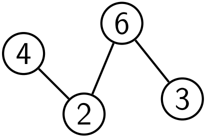

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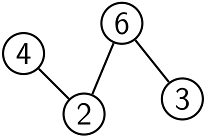

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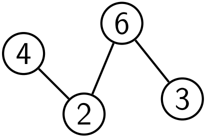

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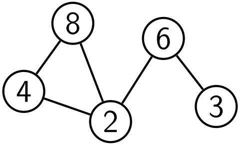

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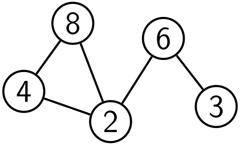

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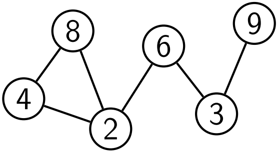

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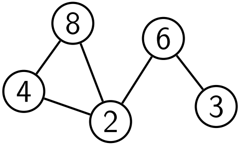

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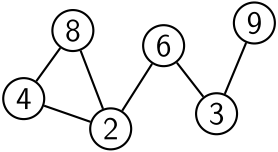

So now we consider $2\ell, 3\ell, 4\ell, 6\ell, 8\ell$ and 9ℓ.

Computing $Q(n)$

Let $r(k, n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

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- Goal: group together equal terms in this product.

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Approximating $Q(n)$

Theorem

For any $\epsilon > 0$, the number of primitive subsets of $[1, n]$ is

$$Q(n) = c^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$$

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The constant $c = \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d)<i}} \prod_{t\in[id,(i+1)d)} r(d,t)^{\frac{1}{t(t+1)}\prod_{p<i}\frac{p-1}{p}}$

is effectively computable and $1.5729 < c < 1.5745$.

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~~#24418~~ closed defect (fixed)

8.2 β4

Opened 5 months ago
Closed 4 months ago

Doctest: bug numerical_approx(2^(450232897/4888643760))

Reported by:	vdelecroix	Owned by:	
Priority:	major	Milestone:	~~sage-8.2~~
Component:	symbolics	Keywords:	bug
Cc:	rws	Merged in:	
Authors:	Ralf Stephan	Reviewers:	Jeroen Demeyer
Report Upstream:	N/A	Work issues:	
Branch:	821f7d9 (Commits)	Commit:	821f7d9f3568316bc0b8b1f5619bce...
Dependencies:		Stopgaps:	

Description (last modified by vdelecroix) Δ

```
sage: numerical_approx(2^(450232897/4888643760))
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
<ipython-input-2-3c4e30ac02c1> in <module>()
----> 1 numerical_approx(Integer(2)**(Integer(450232897)/Integer(4888643760)))

/opt/sage/local/lib/python2.7/site-packages/sage/misc/functional.pyc in numerical_ap
   1406         return numerical_approx_generic(x, prec)
```

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The bound $c < 1.5745$ was recently obtained by Liu, Pach and Palincza (2018) who also prove that c is effectively computable.

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As before,

$$Q(n) = \prod_{k=1}^{n} (g(k) + 1)$$

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Assuming the conjecture, and using values of $g(n)$, $n < 899$ gives

$$1.5729 < c < 1.5735.$$

A general theorem

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Theorem (M.)

Fix $\epsilon > 0$, $A \geq 0$. Suppose $|f(k,n)| \leq A$ and $f(k,n)$ depends only on the connected component of k in the divisor graph of $[k,n]$. Then

$$\sum_{a=1}^{n} f(a,n) = nC_f + O_A\left(n \exp\left(-\sqrt{\left(\tfrac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)$$

where

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id,(i+1)d)} \left(\frac{f(d,t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p}\right).$$

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Theorem (Liu, Pach, Palincza, 2018)

$$M(n) = \alpha^{n+o(n)} \textit{ as } n \to \infty \textit{ and } 1.14817 < \alpha < 1.14823.$$

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$$M(n) = \alpha^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)} \textit{ and } 1.14819 < \alpha.$$

Other Applications: Maximal Primitive Sets

A primitive subset of $[1, n]$ is maximal if it is not contained in another primitive subset. Let $m(n)$ count maximal primitive subsets of $[1, n]$.

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$m(n) = \beta^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$ *and* $1.2125 < \beta < 1.2409$.

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Let $V(n)$ *denote the median size of primitive subsets of* $[1, n]$*. Then*

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Question

Is $V(n) \sim vn$ *for some* v*? If so, is* v *computable?*

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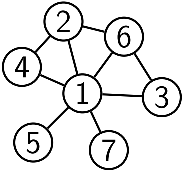
Let $C(n)$ be the least number of disjoint paths that contain all the vertices of this graph.

Example: $C(7) = 2$



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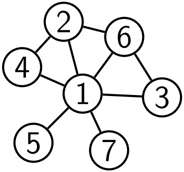
The divisor graph of [1,7] can be covered by $\{7, 1, 5\}$ and $\{3, 6, 2, 4\}$ but it is not possible to use a single path.



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The constant ν is effectively computable and $0.17644 < \nu$.

THANK YOU!

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Take $M = n^{1/2-\epsilon}$, $N = \exp\left((\log n)^{1/2-\epsilon}\right)$.