Two problems regarding primitive sets of integers

Nathan McNew
Towson University

George Washington University
Combinatorics Seminar
April 15th, 2019

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors,

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors, $\mathcal{P}_k = \{n \in \mathbb{N} \nmid \Omega(n) = k\}$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors, $\mathcal{P}_k = \{n \in \mathbb{N} \nmid \Omega(n) = k\}$. $\Omega(n)$ is the number of prime factors of n, with multiplicity.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors, $\mathcal{P}_k = \{n \in \mathbb{N} \nmid \Omega(n) = k\}$.
  $\Omega(n)$ is the number of prime factors of n, with multiplicity.
  $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, \ldots\}$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors, $\mathcal{P}_k = \{n \in \mathbb{N} \nmid \Omega(n) = k\}$.
  $\Omega(n)$ is the number of prime factors of n, with multiplicity.
  $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, \ldots\}$.
- $\mathcal{P}_0 = \{1\}$.

Primitive Sets

Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

Examples:

- Prime numbers $\mathcal{P} = \{2, 3, 5, \ldots\}$.
- $\{m, m+1, m+2, \ldots 2m-1\}$.
- Integers with exactly k prime factors, $\mathcal{P}_k = \{n \in \mathbb{N} \nmid \Omega(n) = k\}$.
  $\Omega(n)$ is the number of prime factors of n, with multiplicity.
  $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, \ldots\}$.
- $\mathcal{P}_0 = \{1\}$.
- Primitive abundant numbers.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

If $\sigma(n) > 2n$ then n is called *abundant*, if $\sigma(n) < 2n$ it is *deficient*.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

If $\sigma(n) > 2n$ then n is called *abundant*, if $\sigma(n) < 2n$ it is *deficient*.
$\sigma(20) = 1+2+4+5+10+20 = 42$

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

If $\sigma(n) > 2n$ then n is called *abundant*, if $\sigma(n) < 2n$ it is *deficient*.
$\sigma(20) = 1+2+4+5+10+20 = 42$ $\quad\quad$ $\sigma(21) = 1+3+7+21 = 32$.

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

If $\sigma(n) > 2n$ then n is called *abundant*, if $\sigma(n) < 2n$ it is *deficient*.
$\sigma(20) = 1+2+4+5+10+20 = 42 \qquad \sigma(21) = 1+3+7+21 = 32$.

Any multiple of an abundant number is abundant (σ is multiplicative)

Primitive abundant numbers

Recall $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function.

Ancient Greeks called n *perfect* if $\sigma(n) = 2n$, since the divisors less than n sum to n. For example $1+2+3 = 6$ or $1+2+4+7+14 = 28$.

If $\sigma(n) > 2n$ then n is called *abundant*, if $\sigma(n) < 2n$ it is *deficient*.
$\sigma(20) = 1+2+4+5+10+20 = 42 \qquad \sigma(21) = 1+3+7+21 = 32$.

Any multiple of an abundant number is abundant (σ is multiplicative)

A *primitive abundant number* is an abundant number, all of whose divisors are deficient (20, 70, 88, 104, 272, 304, 368, 464, 550...)

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s\in S}\frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s \in S} \frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

He also showed that the reciprocal sum of primitive abundant numbers converges.

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s \in S} \frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

He also showed that the reciprocal sum of primitive abundant numbers converges. So the abundant numbers have a density.

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s \in S} \frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

He also showed that the reciprocal sum of primitive abundant numbers converges. So the abundant numbers have a density. We now know, due to Kobayashi that this density is about 0.2476...

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s\in S}\frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

He also showed that the reciprocal sum of primitive abundant numbers converges. So the abundant numbers have a density. We now know, due to Kobayashi that this density is about 0.2476...

What about primitive sets where the reciprocal sum diverges?

Primitive abundant numbers

Erdős showed that if S is a set of integers whose reciprocal sum $\sum_{s \in S} \frac{1}{s}$ converges, then the set of multiples $\{ns \mid n \in \mathbb{N}, s \in S\}$ has a density.

He also showed that the reciprocal sum of primitive abundant numbers converges. So the abundant numbers have a density. We now know, due to Kobayashi that this density is about 0.2476...

What about primitive sets where the reciprocal sum diverges? For example, the reciprocal sum of prime numbers $\sum_{p \in \mathcal{P}} \frac{1}{p}$ diverges.

Large primitive sets

In 1935 Erdős defined primitive sets as a generalization of primitive abundant numbers, and showed:

Large primitive sets

In 1935 Erdős defined primitive sets as a generalization of primitive abundant numbers, and showed:

Theorem (Erdős)

If $S \neq \{1\}$ is a primitive set then $$\sum_{n \in S} \frac{1}{n \log n} < \infty.$$

Large primitive sets

In 1935 Erdős defined primitive sets as a generalization of primitive abundant numbers, and showed:

Theorem (Erdős)

If $S \neq \{1\}$ is a primitive set then $$\sum_{n \in S} \frac{1}{n \log n} < \infty.$$

Considering primes: even though $\sum_{p \in \mathcal{P}} \frac{1}{p}$ diverges, $\sum_{p \in \mathcal{P}} \frac{1}{p \log p}$ converges!

Large primitive sets

In 1935 Erdős defined primitive sets as a generalization of primitive abundant numbers, and showed:

Theorem (Erdős)

If $S \neq \{1\}$ is a primitive set then $\displaystyle\sum_{n\in S} \frac{1}{n \log n} < \infty.$

Considering primes: even though $\sum_{p\in\mathcal{P}} \frac{1}{p}$ diverges, $\sum_{p\in\mathcal{P}} \frac{1}{p\log p}$ converges!

Conjecture (Erdős, 1988)

If $S \neq \{1\}$ is a primitive set then $\displaystyle\sum_{n\in S} \frac{1}{n \log n} \le \sum_{p\in\mathcal{P}} \frac{1}{p \log p} = 1.63\ldots$

Finite primitive sets

Finite primitive sets

A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

Finite primitive sets

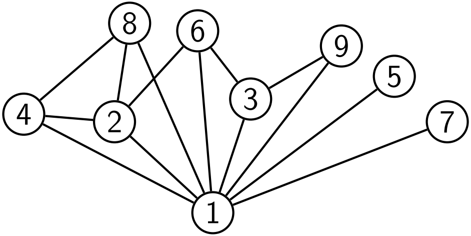
A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

Independent subsets of a divisor graph are primitive sets of integers.

Finite primitive sets

A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

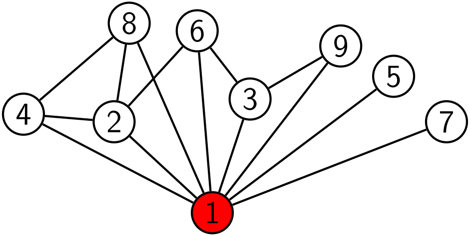
Independent subsets of a divisor graph are primitive sets of integers.



Finite primitive sets

A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

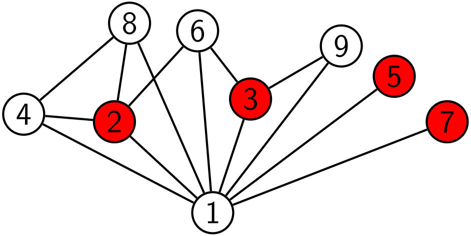
Independent subsets of a divisor graph are primitive sets of integers.



Finite primitive sets

A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

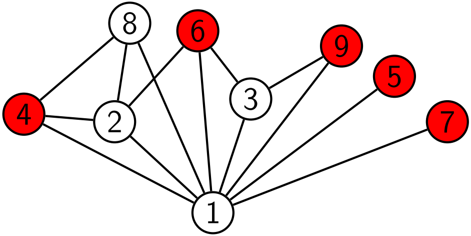
Independent subsets of a divisor graph are primitive sets of integers.



Finite primitive sets

A **divisor graph** is a graph with vertices labeled by integers and edges connecting each pair of integers where one divides another.

Independent subsets of a divisor graph are primitive sets of integers.



Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}, \{2\}$	

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}, \{2\}$	3

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$	

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$	5

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$	5
4	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$	

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$	5
4	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$	7

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$	5
4	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$	7
5	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$,$\{5\}$,$\{2, 5\}$,$\{3, 5\}$, $\{2, 3, 5\}$, $\{4, 5\}$,$\{3, 4, 5\}$	

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \ldots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$	5
4	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$, $\{4\}$,$\{3,4\}$	7
5	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$, $\{4\}$,$\{3,4\}$,$\{5\}$,$\{2,5\}$,$\{3,5\}$, $\{2,3,5\}$, $\{4,5\}$,$\{3,4,5\}$	13

Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

n	primitive subsets of $\{1, 2 \dots n\}$	count
1	$\{\}$, $\{1\}$	2
2	$\{\}$, $\{1\}$, $\{2\}$	3
3	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$	5
4	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$, $\{4\}$,$\{3,4\}$	7
5	$\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2,3\}$, $\{4\}$,$\{3,4\}$,$\{5\}$,$\{2,5\}$,$\{3,5\}$, $\{2,3,5\}$, $\{4,5\}$,$\{3,4,5\}$	13

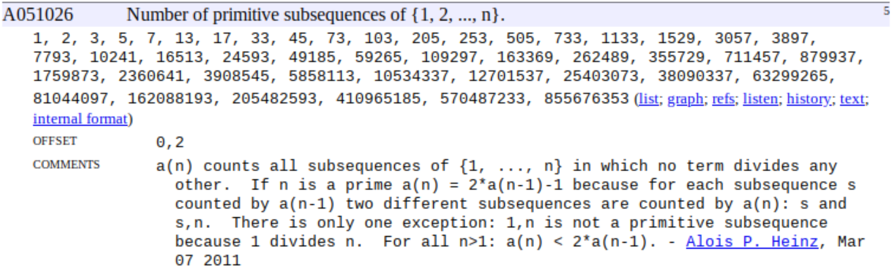
How many primitive subsets of $\{1, 2 \dots n\}$ are there?

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.



A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

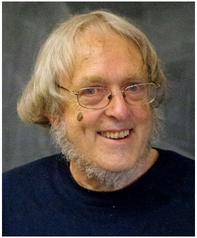
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

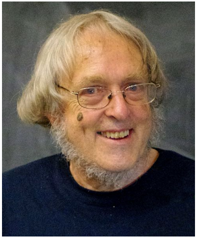
Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.

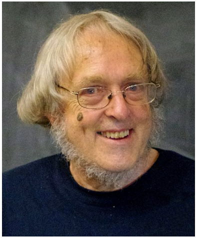


(Almost) trivially:

$$(\sqrt{2})^n \leq Q(n) \leq 2^n$$

Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



(Almost) trivially:

$$(\sqrt{2})^n \leq Q(n) \leq 2^n$$

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive. There are $2^{\lceil \frac{n}{2} \rceil} \geq \sqrt{2}^n$ such subsets.

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Theorem (Angelo, 2017)

The limit $\lim_{n\to\infty} Q(n)^{1/n}$ *exists.*

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Theorem (Angelo, 2017)

The limit $\lim_{n\to\infty} Q(n)^{1/n}$ *exists. Equivalently,* $Q(n) = c^{n+o(1)}$.

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Theorem (Angelo, 2017)

The limit $\lim_{n\to\infty} Q(n)^{1/n}$ *exists. Equivalently,* $Q(n) = c^{n+o(1)}$.

Proof: Study sets without integers having k-smooth integer ratio.

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Theorem (Angelo, 2017)

The limit $\lim_{n\to\infty} Q(n)^{1/n}$ *exists. Equivalently,* $Q(n) = c^{n+o(1)}$.

Proof: Study sets without integers having k-smooth integer ratio.
(n is k-smooth if its largest prime divisor, $P^+(n) \leq k$.)

Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

Theorem (Angelo, 2017)

The limit $\lim_{n\to\infty} Q(n)^{1/n}$ *exists. Equivalently,* $Q(n) = c^{n+o(1)}$.

Proof: Study sets without integers having k-smooth integer ratio. (n is k-smooth if its largest prime divisor, $P^+(n) \le k$.) Let $k\to\infty$.

Proof is not effective: Gives no insight on the value of this constant.

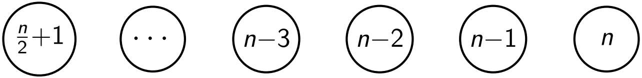
Idea: count primitive sets working backward from n

Idea: count primitive sets working backward from n

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2}$.

Idea: count primitive sets working backward from n

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2}$.



Idea: count primitive sets working backward from n

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2}$.

$\frac{n}{2}+1 \quad \cdots \quad n-3 \quad n-2 \quad n-1 \quad n$

Each integer can either be included or not included (2 possibilities).

Idea: count primitive sets working backward from n

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2}$.

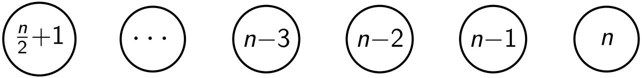
$$\frac{n}{2}+1 \qquad \cdots \qquad n-3 \qquad n-2 \qquad n-1 \qquad n$$

Each integer can either be included or not included (2 possibilities).

Try to make this bigger by considering integers in $\left(\frac{n}{3}, \frac{n}{2}\right]$.

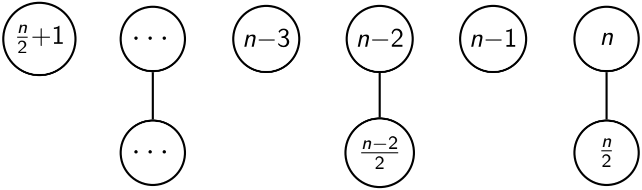
Idea: count primitive sets working backward from n

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2}$.



Each integer can either be included or not included (2 possibilities).

Try to make this bigger by considering integers in $\left(\frac{n}{3}, \frac{n}{2}\right]$.



Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither.

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$

Constructing Primitive Sets

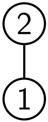
For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

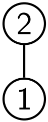
Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:



Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:

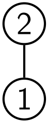


Use this to improve the lower bound for $Q(n)$.

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:

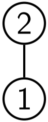


Use this to improve the lower bound for $Q(n)$. For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we multiply by $\frac{3}{2}$, the ratio of primitive subsets of $\{k, 2k\}$, to those of just $\{2k\}$. ($2k$ was in $\left(\frac{n}{2}, n\right]$, so those were already counted.)

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:



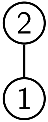
Use this to improve the lower bound for $Q(n)$. For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we multiply by $\frac{3}{2}$, the ratio of primitive subsets of $\{k, 2k\}$, to those of just $\{2k\}$. ($2k$ was in $\left(\frac{n}{2}, n\right]$, so those were already counted.)

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6}$$

Constructing Primitive Sets

For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we can include $k, 2k$ or neither. (3 possibilities)

Each new connected component of the graph is equivalent to counting primitive subsets of $\{1, 2\}$ or independent subsets of:



Use this to improve the lower bound for $Q(n)$. For each $k \in \left(\frac{n}{3}, \frac{n}{2}\right]$ we multiply by $\frac{3}{2}$, the ratio of primitive subsets of $\{k, 2k\}$, to those of just $\{2k\}$. ($2k$ was in $\left(\frac{n}{2}, n\right]$, so those were already counted.)

$$Q(n) \geq 2^{n/2}\left(\frac{3}{2}\right)^{n/6} = 2^{n/3}3^{n/6} \approx 1.5131^n.$$

Constructing Primitive Sets

Pressing on!

Constructing Primitive Sets

Pressing on! Working backward, considering progressively smaller k.

Constructing Primitive Sets

Pressing on! Working backward, considering progressively smaller k.

Let $k \in \left(\frac{n}{4}, \frac{n}{3}\right]$.

Constructing Primitive Sets

Pressing on! Working backward, considering progressively smaller k.

Let $k \in \left(\frac{n}{4}, \frac{n}{3}\right]$. Need to worry about $k, 2k, 3k$.

Constructing Primitive Sets

Pressing on! Working backward, considering progressively smaller k.

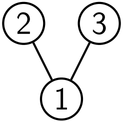
Let $k \in \left(\frac{n}{4}, \frac{n}{3}\right]$. Need to worry about $k, 2k, 3k$. Actually two cases...

Constructing Primitive Sets

Pressing on! Working backward, considering progressively smaller k.

Let $k \in \left(\frac{n}{4}, \frac{n}{3}\right]$. Need to worry about $k, 2k, 3k$. Actually two cases...

For k odd, 5 primitive subsets of $\{k, 2k, 3k\}$ replace 4 of just $\{2k, 3k\}$.



Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

Constructing Primitive Sets

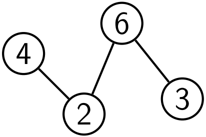
If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.

Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

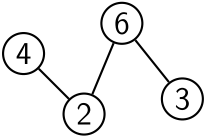
We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of $\{2,3,4,6\}$.



Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of $\{2,3,4,6\}$.



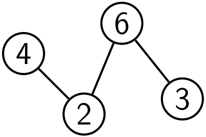
Now we get 8 subsets of $\{2, 3, 4, 6\}$:

$$\{\}, \{2\}, \{3\}, \{4\}, \{6\}, \{2, 3\}, \{3, 4\}, \{4, 6\}$$

Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of{2,3,4,6}.



Now we get 8 subsets of $\{2, 3, 4, 6\}$:

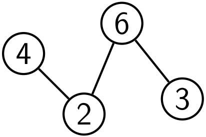
$$\{\}, \{2\}, \{3\}, \{4\}, \{6\}, \{2,3\}, \{3,4\}, \{4,6\}$$

Replacing the 6 primitive subsets of only $\{3, 4, 6\}$.

Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of $\{2,3,4,6\}$.



Now we get 8 subsets of $\{2, 3, 4, 6\}$:

$$\{\}, \{2\}, \{3\}, \{4\}, \{6\}, \{2,3\}, \{3,4\}, \{4,6\}$$

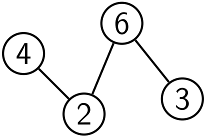
Replacing the 6 primitive subsets of only $\{3, 4, 6\}$.

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} \left(\frac{5}{4}\right)^{n/24} \left(\frac{8}{6}\right)^{n/24}$$

Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of $\{2,3,4,6\}$.



Now we get 8 subsets of $\{2, 3, 4, 6\}$:

$$\{\}, \{2\}, \{3\}, \{4\}, \{6\}, \{2, 3\}, \{3, 4\}, \{4, 6\}$$

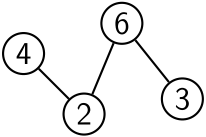
Replacing the 6 primitive subsets of only $\{3, 4, 6\}$.

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} \left(\frac{5}{4}\right)^{n/24} \left(\frac{8}{6}\right)^{n/24} = 2^{5n/24} 3^{n/8} 5^{n/24}$$

Constructing Primitive Sets

If $k = 2\ell$ is even then $\frac{3k}{2} = 3\ell$ is also an integer.

We have to consider all of $2\ell, 3\ell, 4\ell, 6\ell$.
Like counting primitive subsets of $\{2,3,4,6\}$.



Now we get 8 subsets of $\{2, 3, 4, 6\}$:

$$\{\}, \{2\}, \{3\}, \{4\}, \{6\}, \{2, 3\}, \{3, 4\}, \{4, 6\}$$

Replacing the 6 primitive subsets of only $\{3, 4, 6\}$.

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} \left(\frac{5}{4}\right)^{n/24} \left(\frac{8}{6}\right)^{n/24} = 2^{5n/24} 3^{n/8} 5^{n/24} \approx 1.5456^n$$

One more wrinkle

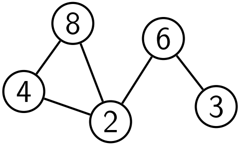
One more wrinkle

Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.

One more wrinkle

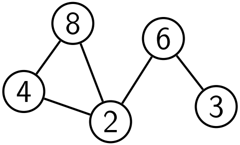
Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.

If $k = 2\ell$ is even we must consider 2ℓ, 4ℓ, $6\ell, 8\ell$, as well as 3ℓ.

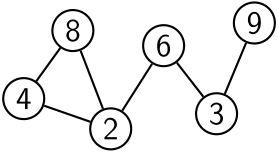


One more wrinkle

Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.



If $k = 2\ell$ is even we must consider 2ℓ, 4ℓ, $6\ell, 8\ell$, as well as 3ℓ.

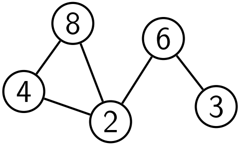


However, if $k \leq \frac{2n}{9}$ then $9\ell \leq n$ must also be considered.

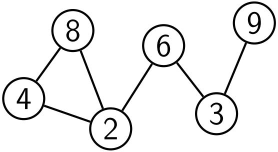
One more wrinkle

Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.

If $k = 2\ell$ is even we must consider 2ℓ, 4ℓ, $6\ell, 8\ell$, as well as 3ℓ.



However, if $k \leq \frac{2n}{9}$ then $9\ell \leq n$ must also be considered.

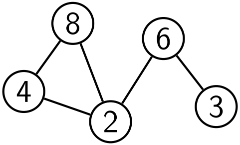


So now we consider $2\ell, 3\ell, 4\ell, 6\ell, 8\ell$ and 9ℓ.

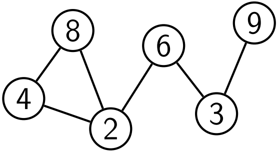
One more wrinkle

Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.

If $k = 2\ell$ is even we must consider 2ℓ, 4ℓ, $6\ell, 8\ell$, as well as 3ℓ.



However, if $k \leq \frac{2n}{9}$ then $9\ell \leq n$ must also be considered.



So now we consider $2\ell, 3\ell, 4\ell, 6\ell, 8\ell$ and 9ℓ.

So $k \in \left(\frac{n}{5}, \frac{2n}{9}\right]$ behave differently than those $k \in \left(\frac{2n}{9}, \frac{n}{4}\right]$

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \leq r(k,n) \leq 2$

Computing $Q(n)$

Let $r(k,n) = \dfrac{\text{\#Primitive subsets of [k,n]}}{\text{\#Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \leq r(k,n) \leq 2$
- $r(k,n)$ only depends on the structure of the connected component of k in the divisibility graph of $[k,n]$.

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \le r(k,n) \le 2$
- $r(k,n)$ only depends on the structure of the connected component of k in the divisibility graph of $[k,n]$.
- $Q(n) = \prod_{i=1}^{n} r(i,n)$

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \leq r(k,n) \leq 2$
- $r(k,n)$ only depends on the structure of the connected component of k in the divisibility graph of $[k,n]$.
- $Q(n) = \prod_{i=1}^{n} r(i,n)$ (Telescoping product)

Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \le r(k,n) \le 2$
- $r(k,n)$ only depends on the structure of the connected component of k in the divisibility graph of $[k,n]$.
- $Q(n) = \displaystyle\prod_{i=1}^{n} r(i,n)$ (Telescoping product)
- Goal: group together equal terms in this product.

Key Observation

Fix k, n.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are k, $2k$, $\ldots ik$.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are k, $2k$, $\ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are k, $2k$, $\ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e | k : P^{+}(e) \leq i\}$, the largest i-smooth divisor of k.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are $k, 2k, \ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e|k : P^+(e) \leq i\}$, the largest i-smooth divisor of k.

Let $\ell = \frac{k}{d}$ the "rough" part of k.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are k, $2k$, $\ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e | k : P^+(e) \leq i\}$, the largest i-smooth divisor of k.

Let $\ell = \frac{k}{d}$ the "rough" part of k.
Every integer in the connected component of k is divisible by ℓ.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are $k, 2k, \ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e | k : P^+(e) \le i\}$, the largest i-smooth divisor of k.

Let $\ell = \frac{k}{d}$ the "rough" part of k.
Every integer in the connected component of k is divisible by ℓ.

Let $t = \left\lfloor \frac{n}{\ell} \right\rfloor$, so $t\ell$ is the largest integer up to n divisible by ℓ.

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are $k, 2k, \ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e|k : P^+(e) \leq i\}$, the largest i-smooth divisor of k.

Let $\ell = \frac{k}{d}$ the "rough" part of k.
Every integer in the connected component of k is divisible by ℓ.

Let $t = \left\lfloor \frac{n}{\ell} \right\rfloor$, so $t\ell$ is the largest integer up to n divisible by ℓ.

Then $r(k, n) = r(d\ell, t\ell)$

Key Observation

Fix k, n. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of k up to n are $k, 2k, \ldots ik$.

If one integer from $[k, n]$ divides another the ratio must be i-smooth.

Let $d = \max\{e | k : P^+(e) \leq i\}$, the largest i-smooth divisor of k.

Let $\ell = \frac{k}{d}$ the "rough" part of k.
Every integer in the connected component of k is divisible by ℓ.

Let $t = \left\lfloor \frac{n}{\ell} \right\rfloor$, so $t\ell$ is the largest integer up to n divisible by ℓ.

Then $r(k, n) = r(d\ell, t\ell) = r(d, t)$.

Approximating $Q(n)$

Theorem

For any $\epsilon > 0$*, the number of primitive subsets of* $[1, n]$ *is*

$$Q(n) = c^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$$

Approximating $Q(n)$

Theorem

For any $\epsilon > 0$, the number of primitive subsets of $[1, n]$ is

$$Q(n) = c^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$$

or

$$\log Q(n) = n\log(c) + O\left(\frac{n}{e^{\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n\log\log n}}}\right).$$

Approximating $Q(n)$

Theorem

For any $\epsilon > 0$, the number of primitive subsets of $[1, n]$ is

$$Q(n) = c^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$$

or

$$\log Q(n) = n\log(c) + O\left(\frac{n}{e^{\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}}}\right).$$

The constant $c = \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d)<i}} \prod_{t\in[id,(i+1)d)} r(d,t)^{\frac{1}{t(t+1)}\prod_{p<i}\frac{p-1}{p}}$

is effectively computable and $1.5729 < c < 1.5745$.

Numerical Bound

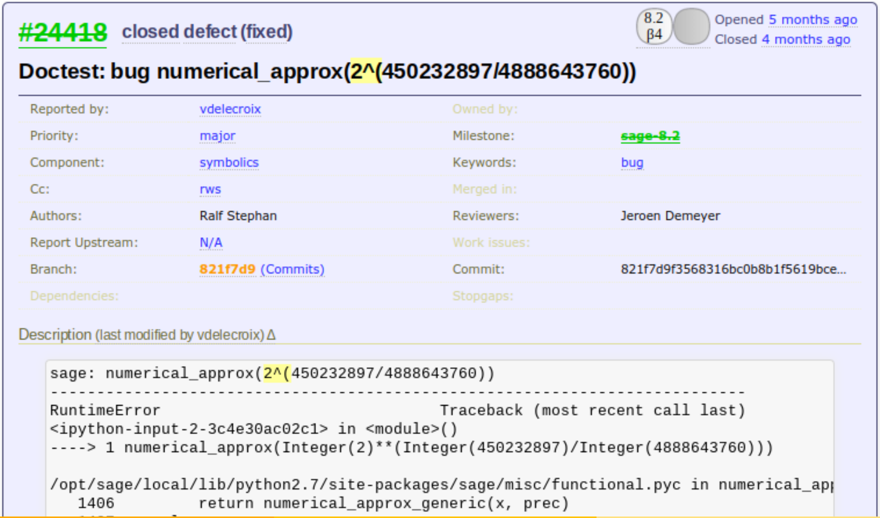
Get a lower bound for c by computing terms in this product.

Numerical Bound

Get a lower bound for c by computing terms in this product. Using Sage to compute $r(d,t)$ for all i,d,t with $di^5 \leq 10^8$ gives $1.5729 < c$.

Numerical Bound

Get a lower bound for c by computing terms in this product. Using Sage to compute $r(d,t)$ for all i,d,t with $di^5 \leq 10^8$ gives $1.5729 < c$.



~~#24418~~ closed defect (fixed)

8.2 β4

Opened 5 months ago
Closed 4 months ago

Doctest: bug numerical_approx(2^(450232897/4888643760))

Reported by:	vdelecroix	Owned by:	
Priority:	major	Milestone:	~~sage-8.2~~
Component:	symbolics	Keywords:	bug
Cc:	rws	Merged in:	
Authors:	Ralf Stephan	Reviewers:	Jeroen Demeyer
Report Upstream:	N/A	Work issues:	
Branch:	821f7d9 (Commits)	Commit:	821f7d9f3568316bc0b8b1f5619bce...
Dependencies:		Stopgaps:	

Description (last modified by vdelecroix) Δ

```
sage: numerical_approx(2^(450232897/4888643760))
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
<ipython-input-2-3c4e30ac02c1> in <module>()
----> 1 numerical_approx(Integer(2)**(Integer(450232897)/Integer(4888643760)))

/opt/sage/local/lib/python2.7/site-packages/sage/misc/functional.pyc in numerical_ap
   1406         return numerical_approx_generic(x, prec)
```

Numerical Bound

Get a lower bound for c by computing terms in this product. Using Sage to compute $r(d,t)$ for all i,d,t with $di^5 \leq 10^8$ gives $1.5729 < c$.

This technique doesn't give very good upper bounds since any unestimated terms need to be replaced by 2 (or 1.5).

Numerical Bound

Get a lower bound for c by computing terms in this product. Using Sage to compute $r(d,t)$ for all i,d,t with $di^5 \leq 10^8$ gives $1.5729 < c$.

This technique doesn't give very good upper bounds since any unestimated terms need to be replaced by 2 (or 1.5). Gives $c < 1.58$.

Numerical Bound

Get a lower bound for c by computing terms in this product. Using Sage to compute $r(d,t)$ for all i,d,t with $di^5 \leq 10^8$ gives $1.5729 < c$.

This technique doesn't give very good upper bounds since any unestimated terms need to be replaced by 2 (or 1.5). Gives $c < 1.58$.

The bound $c < 1.5745$ was recently obtained by Liu, Pach and Palincza (2018) who also prove that c is effectively computable.

A general theorem

A general theorem

Theorem (M.)

Fix $\epsilon > 0$, $A \geq 0$. Suppose $|f(k,n)| \leq A$ and $f(k,n)$ depends only on the connected component of k in the divisor graph of $[k,n]$. Then

$$\sum_{a=1}^{n} f(a,n) = nC_f + O_A\left(n \exp\left(-\sqrt{\left(\tfrac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)$$

where

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id,(i+1)d)} \left(\frac{f(d,t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p}\right).$$

Other Applications: Maximum Primitive Subsets

Other Applications: Maximum Primitive Subsets

The largest primitive subsets of the interval $[1, n]$ have size $\left\lceil \frac{n}{2} \right\rceil$.

Other Applications: Maximum Primitive Subsets

The largest primitive subsets of the interval $[1, n]$ have size $\left\lceil \frac{n}{2} \right\rceil$.

Let $M(n)$ count the primitive subsets of $\{1, \ldots n\}$ of size $\left\lceil \frac{n}{2} \right\rceil$.

Other Applications: Maximum Primitive Subsets

The largest primitive subsets of the interval $[1, n]$ have size $\left\lceil \frac{n}{2} \right\rceil$.

Let $M(n)$ count the primitive subsets of $\{1, \ldots n\}$ of size $\left\lceil \frac{n}{2} \right\rceil$.

Theorem (Vijay, 2018)

$$1.141^n < M(n) < 1.187^n$$

Other Applications: Maximum Primitive Subsets

The largest primitive subsets of the interval $[1, n]$ have size $\left\lceil \frac{n}{2} \right\rceil$.

Let $M(n)$ count the primitive subsets of $\{1, \ldots n\}$ of size $\left\lceil \frac{n}{2} \right\rceil$.

Theorem (Vijay, 2018)

$$1.141^n < M(n) < 1.187^n$$

Theorem (Liu, Pach, Palincza, 2018)

$$M(n) = \alpha^{n+o(n)} \textit{ as } n \to \infty \textit{ and } 1.14817 < \alpha < 1.14823.$$

Other Applications: Maximum Primitive Subsets

The largest primitive subsets of the interval $[1, n]$ have size $\left\lceil \frac{n}{2} \right\rceil$.

Let $M(n)$ count the primitive subsets of $\{1, \ldots n\}$ of size $\left\lceil \frac{n}{2} \right\rceil$.

Theorem (Vijay, 2018)

$$1.141^n < M(n) < 1.187^n$$

Theorem (Liu, Pach, Palincza, 2018)

$$M(n) = \alpha^{n+o(n)} \textit{ as } n \to \infty \textit{ and } 1.14817 < \alpha < 1.14823.$$

Theorem (M., 2018)

$$M(n) = \alpha^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)} \textit{ and } 1.14819 < \alpha.$$

Other Applications: Maximal Primitive Sets

A primitive subset of $[1, n]$ is maximal if it is not contained in another primitive subset. Let $m(n)$ count maximal primitive subsets of $[1, n]$.

Theorem

$m(n) = \beta^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$ *and* $1.2125 < \beta < 1.2409$.

Other Applications: Maximal Primitive Sets

A primitive subset of $[1, n]$ is maximal if it is not contained in another primitive subset. Let $m(n)$ count maximal primitive subsets of $[1, n]$.

Theorem

$m(n) = \beta^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$ *and* $1.2125 < \beta < 1.2409$.

Corollary

Let $V(n)$ denote the median size of primitive subsets of $[1, n]$. Then

$$0.1681n < V(n) < 0.3918n$$

Other Applications: Maximal Primitive Sets

A primitive subset of $[1, n]$ is maximal if it is not contained in another primitive subset. Let $m(n)$ count maximal primitive subsets of $[1, n]$.

Theorem

$m(n) = \beta^{n\left(1+O\left(\exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)\right)}$ *and* $1.2125 < \beta < 1.2409$.

Corollary

Let $V(n)$ denote the median size of primitive subsets of $[1, n]$. Then

$$0.1681n < V(n) < 0.3918n$$

Question

Is $V(n) \sim vn$ for some v? If so, is v computable?

Path covers of the divisor graph

Path covers of the divisor graph

Pomerance, Erdős, Saias and others study the the length of the longest path in the divisor graph of $[1, n]$, showing it is $\asymp \frac{n}{\log n}$.

Path covers of the divisor graph

Pomerance, Erdős, Saias and others study the the length of the longest path in the divisor graph of $[1, n]$, showing it is $\asymp \frac{n}{\log n}$.

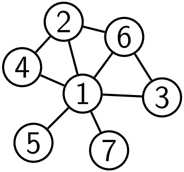
Let $C(n)$ be the least number of disjoint paths that contain all the vertices of this graph.

Path covers of the divisor graph

Pomerance, Erdős, Saias and others study the the length of the longest path in the divisor graph of $[1, n]$, showing it is $\asymp \frac{n}{\log n}$.

Let $C(n)$ be the least number of disjoint paths that contain all the vertices of this graph.

Example: $C(7) = 2$



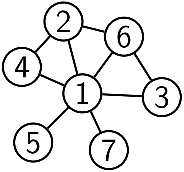
Path covers of the divisor graph

Pomerance, Erdős, Saias and others study the the length of the longest path in the divisor graph of $[1, n]$, showing it is $\asymp \frac{n}{\log n}$.

Let $C(n)$ be the least number of disjoint paths that contain all the vertices of this graph.

Example: $C(7) = 2$

The divisor graph of [1,7] can be covered by $\{7, 1, 5\}$ and $\{3, 6, 2, 4\}$ but it is not possible to use a single path.



Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of this divisor graph have improved over time.

Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of this divisor graph have improved over time.

Saias (2003): $\frac{n}{6} \le C(n) \le \frac{n}{4}$ for sufficiently large n.

Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of this divisor graph have improved over time.

Saias (2003): $\frac{n}{6} \leq C(n) \leq \frac{n}{4}$ for sufficiently large n.

Mazet (2006): $C(n) \sim \nu n$ for some $0.1706 \leq \nu \leq 0.2289$.

Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of this divisor graph have improved over time.

Saias (2003): $\frac{n}{6} \leq C(n) \leq \frac{n}{4}$ for sufficiently large n.

Mazet (2006): $C(n) \sim \nu n$ for some $0.1706 \leq \nu \leq 0.2289$.

Chadozeau (2008): $C(n) = \nu n \left(1 + O\left(\frac{1}{\log\log n \log\log\log n}\right)\right)$.

Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of this divisor graph have improved over time.

Saias (2003): $\frac{n}{6} \leq C(n) \leq \frac{n}{4}$ for sufficiently large n.

Mazet (2006): $C(n) \sim \nu n$ for some $0.1706 \leq \nu \leq 0.2289$.

Chadozeau (2008): $C(n) = \nu n \left(1 + O\left(\frac{1}{\log\log n \log\log\log n}\right)\right)$.

Theorem

$$C(n) = \nu n \left(1 + O\left(\exp\left(-\sqrt{\left(\tfrac{1}{6} - \epsilon\right) \log n \log\log n}\right)\right)\right).$$

The constant ν *is effectively computable and* $0.1843 < \nu < 0.2229$.

Changing Gears

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Worst case scenario: How far do we have to go to find one?

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Worst case scenario: How far do we have to go to find one?

In practice: At most 7384 (largest known to 10^{100})

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Worst case scenario: How far do we have to go to find one?

In practice: At most 7384 (largest known to 10^{100}) 116 on average.

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Worst case scenario: How far do we have to go to find one?

In practice: At most 7384 (largest known to 10^{100}) 116 on average.
In theory:

Gaps between prime numbers

Problem (Cryptography)

Need a random 100 digit prime number.

Solution: Pick a random 100 digit integer, check if it is prime. If not, add 1 and repeat until you find one.

Worst case scenario: How far do we have to go to find one?

In practice: At most 7384 (largest known to 10^{100}) 116 on average.
In theory: At most 10^{53}. (Best we can prove, assuming RH).

Gaps between prime numbers

How far apart can prime numbers be?

Gaps between prime numbers

How far apart can prime numbers be?

Let p_n be the nth prime number.

Gaps between prime numbers

How far apart can prime numbers be?

Let p_n be the nth prime number. $p_n - p_{n-1}$ is the nth prime gap.

Gaps between prime numbers

How far apart can prime numbers be?

Let p_n be the nth prime number. $p_n - p_{n-1}$ is the nth prime gap.

Theorem (Baker, Harman, Pintz)

$$p_n - p_{n-1} \ll p_n^{0.525}$$

Gaps between prime numbers

How far apart can prime numbers be?

Let p_n be the nth prime number. $p_n - p_{n-1}$ is the nth prime gap.

Theorem (Baker, Harman, Pintz)

$$p_n - p_{n-1} \ll p_n^{0.525}$$

Theorem (Cramér)

Assuming the Riemann Hypothesis

$$p_n - p_{n-1} \ll \sqrt{p_n} \log p_n$$

Gaps between prime numbers

How far apart can prime numbers be?

Let p_n be the nth prime number. $p_n - p_{n-1}$ is the nth prime gap.

Theorem (Baker, Harman, Pintz)

$$p_n - p_{n-1} \ll p_n^{0.525}$$

Theorem (Cramér)

Assuming the Riemann Hypothesis

$$p_n - p_{n-1} \ll \sqrt{p_n} \log p_n$$

Conjecture (Cramér)

$$p_n - p_{n-1} \ll \log^2 p_n$$

Gaps between prime numbers

How far apart can prime numbers be?

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Gaps between prime numbers

How far apart can prime numbers be?
"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)

$c = e^{\gamma}$ (Rankin, 1968)

Gaps between prime numbers

How far apart can prime numbers be?
"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!
$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)
$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:
$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)
$c = e^\gamma$ (Rankin, 1968) $c = 2.3$ (Maier, Pomerance, 1989)

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)

$c = e^\gamma$ (Rankin, 1968) $c = 2.3$ (Maier, Pomerance, 1989) $c = 2e^\gamma$ (Pintz 1999)

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)

$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)

$c = e^\gamma$ (Rankin, 1968) $c = 2.3$ (Maier, Pomerance, 1989) $c = 2e^\gamma$ (Pintz 1999)

Erdős \$10,000 prize: Show c can be arbitrarily large.

Gaps between prime numbers

How far apart can prime numbers be?

"On average" $p_n - p_{n-1}$ is $\log p_n$ (Prime number theorem)

Recent Developments!

$p_n - p_{n-1} \leq 70,000,000$ infinitely often (Zhang 2013)
$p_n - p_{n-1} \leq 246$ infinitely often (Maynard, Tao, Polymath 2015)

Large gaps:

$p_n - p_{n-1} > c\frac{\log p_n \log\log p_n \log\log\log\log p_n}{(\log\log\log p_n)^2}$ Infinitely often (Rankin, 1931)
$c = e^\gamma$ (Rankin, 1968) $c = 2.3$ (Maier, Pomerance, 1989) $c = 2e^\gamma$ (Pintz 1999)
Erdős \$10,000 prize: Show c can be arbitrarily large.

$p_n - p_{n-1} \gg \frac{\log p_n \log\log p_n \log\log\log\log p_n}{\log\log\log p_n}$ infinitely often
(Ford, Green, Konyagin, Maynard, Tao, 2016)

Gaps between prime numbers

Summary: The largest gap between primes up to x has size between

$$\frac{\log x \log\log x \log\log\log\log x}{\log\log\log x} \quad \text{and} \quad x^{0.525}$$

but we think the truth is $\log^2 x$.

Large primitive sets

Nevertheless, there exist primitive sets much “larger” than the primes.

Large primitive sets

Nevertheless, there exist primitive sets much "larger" than the primes.

Theorem (Ahlswede, Khatchatrian, Sárközy, 1999)

For each $\epsilon > 0$ there exist primitive sets with a counting function asymptotic to $\frac{x}{(\log\log x)^{1+\epsilon}}$.

Large primitive sets

Nevertheless, there exist primitive sets much "larger" than the primes.

Theorem (Ahlswede, Khatchatrian, Sárközy, 1999)

For each $\epsilon > 0$ there exist primitive sets with a counting function asymptotic to $\frac{x}{(\log\log x)^{1+\epsilon}}$.

Note: If S has counting function at least $\frac{x}{\log\log x}$, then $\displaystyle\sum_{n\in S}\frac{1}{n\log n} \to \infty$.

Large primitive sets

Nevertheless, there exist primitive sets much "larger" than the primes.

Theorem (Ahlswede, Khatchatrian, Sárközy, 1999)

For each $\epsilon>0$ there exist primitive sets with a counting function asymptotic to $\frac{x}{(\log\log x)^{1+\epsilon}}$.

Note: If S has counting function at least $\frac{x}{\log\log x}$, then $\sum_{n\in S}\frac{1}{n\log n}\to\infty$.

The average gap size in a primitive set can be $(\log\log x)^{1+\epsilon}$, much smaller than the primes $(\log x)$.

Large primitive sets

Nevertheless, there exist primitive sets much "larger" than the primes.

Theorem (Ahlswede, Khatchatrian, Sárközy, 1999)

For each $\epsilon > 0$ there exist primitive sets with a counting function asymptotic to $\frac{x}{(\log\log x)^{1+\epsilon}}$.

Note: If S has counting function at least $\frac{x}{\log\log x}$, then $\displaystyle\sum_{n\in S}\frac{1}{n\log n} \to \infty$.

The average gap size in a primitive set can be $(\log\log x)^{1+\epsilon}$, much smaller than the primes $(\log x)$.

It isn't possible to get good bounds for the largest gap in such a set.

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Theorem (He, 2015)

There exists a primitive set of integers $s_1, s_2, s_3 \ldots$ *such that*

$$s_n - s_{n-1} < 2\sqrt{s_n}.$$

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Theorem (He, 2015)

There exists a primitive set of integers $s_1, s_2, s_3 \dots$ such that

$$s_n - s_{n-1} < 2\sqrt{s_n}.$$

Recall:

$$p_n - p_{n-1} \ll p_n^{0.525} \quad \text{(unconditionally)}$$

$$p_n - p_{n-1} \ll \sqrt{p_n} \log p_n \quad \text{(RH)}$$

Geometric Progression Free Sets

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Question (Bergelson)

Do there exist syndetic sets (bounded gaps) that do not contain k-term geometric progressions?

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Question (Bergelson)

Do there exist syndetic sets (bounded gaps) that do not contain k-term geometric progressions?

The squarefree integers avoid 3-term-geometric progressions (with density $\approx .6079$)

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Question (Bergelson)

Do there exist syndetic sets (bounded gaps) that do not contain k-term geometric progressions?

The squarefree integers avoid 3-term-geometric progressions (with density $\approx .6079$) but have unbounded gaps.

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Question (Bergelson)

Do there exist syndetic sets (bounded gaps) that do not contain k-term geometric progressions?

The squarefree integers avoid 3-term-geometric progressions (with density $\approx .6079$) but have unbounded gaps.

Theorem (Filaseta, Trifonov, 1992)

The squarefree integers $s_1, s_2, \ldots$ satisfy $s_n - s_{n-1} \ll s_n^{1/5} \log s_n$.

Geometric Progression Free Sets

A k-term geometric progression is a sequence $a, ar, ar^2, \ldots ar^{k-1}$.

Question (Bergelson)

Do there exist syndetic sets (bounded gaps) that do not contain k-term geometric progressions?

The squarefree integers avoid 3-term-geometric progressions (with density $\approx .6079$) but have unbounded gaps.

Theorem (Filaseta, Trifonov, 1992)

The squarefree integers $s_1, s_2, \ldots$ satisfy $\quad s_n - s_{n-1} \ll s_n^{1/5} \log s_n$.

There are even larger geometric progression free sets, but can't say much about their gaps.

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ there exists a set $t_1, t_2, \ldots$ avoiding 6-term geometric progressions with gaps satisfying

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ there exists a set $t_1, t_2, \ldots$ avoiding 6-term geometric progressions with gaps satisfying

$$t_n - t_{n-1} \ll_\epsilon \exp\left(\left(\frac{5}{6}\log 2 + \epsilon\right)\frac{\log t_n}{\log\log t_n}\right).$$

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ there exists a set $t_1, t_2, \ldots$ avoiding 6-term geometric progressions with gaps satisfying

$$t_n - t_{n-1} \ll_\epsilon \exp\left(\left(\frac{5}{6}\log 2 + \epsilon\right)\frac{\log t_n}{\log\log t_n}\right).$$

Furthermore, there exists a set $u_1, u_2, \ldots$ avoiding 3-term geometric progressions with **integer** *ratio where*

$$u_n - u_{n-1} \ll_\epsilon u_n^\epsilon.$$

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Theorem (He, 2015)

There exists a primitive set $q_1, q_2, q_3 \ldots$ *where* $q_n - q_{n-1} < 2\sqrt{q_n}$.

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Theorem (He, 2015)

There exists a primitive set $q_1, q_2, q_3 \ldots$ *where* $q_n - q_{n-1} < 2\sqrt{q_n}$.

Theorem (M.)

For every $\epsilon > 0$ *there exists a primitive set* $a_1, a_2, a_3, \ldots$, *where*

Primitive sets without large gaps

Question

Do there exist primitive sets which (provably) have no large gaps?

Theorem (He, 2015)

There exists a primitive set $q_1, q_2, q_3 \ldots$ *where* $q_n - q_{n-1} < 2\sqrt{q_n}$.

Theorem (M.)

For every $\epsilon > 0$ *there exists a primitive set* $a_1, a_2, a_3, \ldots$, *where*

$$a_n - a_{n-1} \leq \exp\left(\sqrt{(2+\epsilon)\log a_n \log\log a_n}\right).$$

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ *there exists a set* $\{t_1, t_2, \ldots\}$ *avoiding 6-term geometric progressions (rational ratio) with gaps satisfying*

$$t_n - t_{n-1} \ll_\epsilon \exp\left(\left(\tfrac{5}{6}\log 2 + \epsilon\right)\frac{\log t_n}{\log\log t_n}\right).$$

and a set $\{u_1, u_2, \ldots\}$ *avoiding 3-term geometric progressions with* **integer** *ratio with* $u_n - u_{n-1} \ll_\epsilon u_n^\epsilon$*.*

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ there exists a set $\{t_1, t_2, \ldots\}$ avoiding 6-term geometric progressions (rational ratio) with gaps satisfying

$$t_n - t_{n-1} \ll_\epsilon \exp\left(\left(\tfrac{5}{6}\log 2 + \epsilon\right)\frac{\log t_n}{\log\log t_n}\right).$$

and a set $\{u_1, u_2, \ldots\}$ avoiding 3-term geometric progressions with **integer** *ratio with $u_n - u_{n-1} \ll_\epsilon u_n^\epsilon$.*

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

Geometric Progression Free Sets

Theorem (He, 2015)

For each $\epsilon > 0$ there exists a set $\{t_1, t_2, \ldots\}$ avoiding 6-term geometric progressions (rational ratio) with gaps satisfying

$$t_n - t_{n-1} \ll_\epsilon \exp\left(\left(\tfrac{5}{6}\log 2 + \epsilon\right)\frac{\log t_n}{\log\log t_n}\right).$$

and a set $\{u_1, u_2, \ldots\}$ avoiding 3-term geometric progressions with **integer** *ratio with $u_n - u_{n-1} \ll_\epsilon u_n^\epsilon$.*

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction:

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction: A probabilistic analogue of the squarefree numbers.

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction: A probabilistic analogue of the squarefree numbers.

For each prime p_i choose an integer valued random variable X_i with

$$P(X_i = m) = \frac{1}{2^m}.$$

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction: A probabilistic analogue of the squarefree numbers.

For each prime p_i choose an integer valued random variable X_i with

$$P(X_i = m) = \frac{1}{2^m}.$$

Let $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction: A probabilistic analogue of the squarefree numbers.

For each prime p_i choose an integer valued random variable X_i with

$$P(X_i = m) = \frac{1}{2^m}.$$

Let $V = \{n \geq 2 : p_i | n \rightarrow p_i^{X_i} || n\}$. V contains the integers divisible only by primes p_i appearing to the corresponding power X_i.

Geometric Progression Free Sets

Theorem (M. 2017)

For each $\epsilon > 0$ there exists a set $\{v_1, v_2, \ldots\}$ avoiding 3-term geometric progressions (rational ratio) with gaps satisfying

$$v_n - v_{n-1} \ll_\epsilon \exp\left(2\sqrt{(\log 2 + \epsilon)\log v_n}\right).$$

Construction: A probabilistic analogue of the squarefree numbers.

For each prime p_i choose an integer valued random variable X_i with

$$P(X_i = m) = \frac{1}{2^m}.$$

Let $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$. V contains the integers divisible only by primes p_i appearing to the corresponding power X_i.

Note V avoids 3-term progressions: If (a, ar, ar^2) is a progression, a prime in r appears to different powers in at least two of these terms.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Proof:

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Proof: By theorem, the interval has a large subset without prime factors greater than $\sqrt{y}$.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ *contains at least* $\frac{y}{\log^2 y}$ *integers free of prime factors smaller than* $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ *contains a subset of integers which are pairwise coprime, of size at least* $\frac{c'\sqrt{y}}{\log y}$ *for some positive constant* c'.

Proof: By theorem, the interval has a large subset without prime factors greater than $\sqrt{y}$. Each such integer can have at most $\frac{2\log x}{\log y}$ prime factors.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Proof: By theorem, the interval has a large subset without prime factors greater than $\sqrt{y}$. Each such integer can have at most $\frac{2\log x}{\log y}$ prime factors. Construct a graph on these numbers, connecting those vertices with a prime factor in common.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Proof: By theorem, the interval has a large subset without prime factors greater than $\sqrt{y}$. Each such integer can have at most $\frac{2\log x}{\log y}$ prime factors. Construct a graph on these numbers, connecting those vertices with a prime factor in common. Turan's graph theorem implies this graph has a large independent vertex set.

The probabilistic method

Proof sketch:

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon) \log x}\right)$.

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

Suppose $n \in S$, then the probability, $P(n \in V)$, that $n \in V$ is

$$\prod_{p_i^{\alpha_i} || n} P(X_i = \alpha_i)$$

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

Suppose $n \in S$, then the probability, $P(n \in V)$, that $n \in V$ is

$$\prod_{p_i^{\alpha_i} || n} P(X_i = \alpha_i) = \prod_{p_i^{\alpha_i} || n} \left(\frac{1}{2}\right)^{\alpha_i}$$

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

Suppose $n \in S$, then the probability, $P(n \in V)$, that $n \in V$ is

$$\prod_{p_i^{\alpha_i}||n} P(X_i{=}\alpha_i) = \prod_{p_i^{\alpha_i}||n} \left(\frac{1}{2}\right)^{\alpha_i} = \left(\frac{1}{2}\right)^{\Omega(n)}$$

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

Suppose $n \in S$, then the probability, $P(n \in V)$, that $n \in V$ is

$$\prod_{p_i^{\alpha_i}||n} P(X_i{=}\alpha_i) = \prod_{p_i^{\alpha_i}||n} \left(\frac{1}{2}\right)^{\alpha_i} = \left(\frac{1}{2}\right)^{\Omega(n)} \geq \left(\frac{1}{2}\right)^{\frac{2\log x}{\log y}}$$

The probabilistic method

Proof sketch: Construct a geometric progression free set probabilistically, show it almost surely has no large gaps.

Pick $P(X_i = m) = \frac{1}{2^m}$. Then $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$.

We expect every interval $I_x = [x - cy, x]$ to contain an element of V:

By lemma I_x has a subset S of size at least $\frac{c'\sqrt{y}}{\log y}$ which is pairwise coprime and the integers in S have at most $\frac{2\log x}{\log y}$ prime factors.

Suppose $n \in S$, then the probability, $P(n \in V)$, that $n \in V$ is

$$\prod_{p_i^{\alpha_i}||n} P(X_i{=}\alpha_i) = \prod_{p_i^{\alpha_i}||n} \left(\frac{1}{2}\right)^{\alpha_i} = \left(\frac{1}{2}\right)^{\Omega(n)} \geq \left(\frac{1}{2}\right)^{\frac{2\log x}{\log y}} = \exp\left(-\frac{2\log 2\log x}{\log y}\right)$$

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.
Thus the probability that no element of S is included in Q is

$$\prod_{n \in S} (1 - P(n \in V))$$

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.
Thus the probability that no element of S is included in Q is

$$\prod_{n\in S}(1-P(n\in V))$$

$$\leq \prod_{n\in S}\left(1-\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)$$

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.
Thus the probability that no element of S is included in Q is

$$\prod_{n\in S}(1 - P(n \in V))$$

$$\leq \prod_{n\in S}\left(1 - \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)$$

$$\leq \left(1 - \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)^{\frac{c'\sqrt{y}}{\log y}}$$

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.
Thus the probability that no element of S is included in Q is

$$\prod_{n\in S}(1-P(n\in V))$$

$$\leq \prod_{n\in S}\left(1-\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)$$

$$\leq \left(1-\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)^{\frac{c'\sqrt{y}}{\log y}}$$

$$\leq \exp\left(-\frac{c'\sqrt{y}}{\log y}\times \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)$$

The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in V are independent.
Thus the probability that no element of S is included in Q is

$$
\begin{aligned}
&\prod_{n\in S}(1-P(n\in V))\\
&\quad\leq \prod_{n\in S}\left(1-\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)\\
&\quad\leq \left(1-\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)^{\frac{c'\sqrt{y}}{\log y}}\\
&\quad\leq \exp\left(-\frac{c'\sqrt{y}}{\log y}\times\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)\\
&\quad= \exp\left(-\exp\left(\frac{1}{2}\log y-\left(\frac{2\log 2\log x}{\log y}\right)-\log\log y+O(1)\right)\right).
\end{aligned}
$$

The probabalistic method

$$\prod_{n\in S}(1-P(n\in V))$$

$$\le= \exp\left(-\exp\left(\tfrac{1}{2}\log y-\left(\frac{2\log 2\log x}{\log y}\right)+O(\log\log y)\right)\right).$$

The probabalistic method

$$\prod_{n \in S} (1 - P(n \in V))$$

$$\leq = \exp\left(-\exp\left(\tfrac{1}{2}\log y - \left(\frac{2\log 2 \log x}{\log y}\right) + O(\log\log y)\right)\right).$$

Using $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$ the innermost exponent simplifies to $\epsilon\sqrt{\log x} + O(\log\log x)$.

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

so there exists an N such that

$$\sum_{x=N}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < 1.$$

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

so there exists an N such that

$$\sum_{x=N}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < 1.$$

Starting the sequence at N and using linearity of expectation, the expected number of intervals of this form avoiding V is at most

$$\sum_{x>N} P([x - cy, x] \cap V = \emptyset)$$

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

so there exists an N such that

$$\sum_{x=N}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < 1.$$

Starting the sequence at N and using linearity of expectation, the expected number of intervals of this form avoiding V is at most

$$\sum_{x>N} P([x - cy, x] \cap V = \emptyset) \leq \sum_{x>N} \exp\left(-\exp\left(\epsilon\sqrt{\log x}\right)\right)$$

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

so there exists an N such that

$$\sum_{x=N}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < 1.$$

Starting the sequence at N and using linearity of expectation, the expected number of intervals of this form avoiding V is at most

$$\sum_{x>N} P([x - cy, x] \cap V = \emptyset) \leq \sum_{x>N} \exp\left(-\exp\left(\epsilon\sqrt{\log x}\right)\right) < 1.$$

The probabilistic method

The probability that no integer in $[x - cy, x]$ is included in V is at most $\exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right)$.

$$\sum_{x=1}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < \infty,$$

so there exists an N such that

$$\sum_{x=N}^{\infty} \exp\left(-\exp\left(C_\epsilon\sqrt{\log x}\right)\right) < 1.$$

Starting the sequence at N and using linearity of expectation, the expected number of intervals of this form avoiding V is at most

$$\sum_{x>N} P([x - cy, x] \cap V = \emptyset) \leq \sum_{x>N} \exp\left(-\exp\left(\epsilon\sqrt{\log x}\right)\right) < 1.$$

Thus there exists a set, V satisfying the properties of the theorem.

Modifications for primitive sets

Fix $\epsilon > 0$.

Modifications for primitive sets

Fix $\epsilon > 0$. For each prime number p_i choose an integer valued random variable X_i with distribution

$$P(X_i = m) = \frac{C_\epsilon}{m^{1+\epsilon}}$$

with $C_\epsilon = \frac{1}{\zeta(1+\epsilon)}$ chosen to normalize the distribution.

Modifications for primitive sets

Fix $\epsilon > 0$. For each prime number p_i choose an integer valued random variable X_i with distribution

$$P(X_i = m) = \frac{C_\epsilon}{m^{1+\epsilon}}$$

with $C_\epsilon = \frac{1}{\zeta(1+\epsilon)}$ chosen to normalize the distribution. Let

$$Q = \{n \geq 2 \mid p_i | n \to \Omega(n) = X_i\}$$

Modifications for primitive sets

Fix $\epsilon > 0$. For each prime number p_i choose an integer valued random variable X_i with distribution

$$P(X_i = m) = \frac{C_\epsilon}{m^{1+\epsilon}}$$

with $C_\epsilon = \frac{1}{\zeta(1+\epsilon)}$ chosen to normalize the distribution. Let

$$Q = \{n \geq 2 \mid p_i | n \to \Omega(n) = X_i\}$$

consist of precisely those integers, n, where the total number of prime factors of n agrees with the variable X_i for **all** its prime divisors, p_i.

Modifications for primitive sets

Fix $\epsilon > 0$. For each prime number p_i choose an integer valued random variable X_i with distribution

$$P(X_i = m) = \frac{C_\epsilon}{m^{1+\epsilon}}$$

with $C_\epsilon = \frac{1}{\zeta(1+\epsilon)}$ chosen to normalize the distribution. Let

$$Q = \{n \geq 2 \mid p_i | n \to \Omega(n) = X_i\}$$

consist of precisely those integers, n, where the total number of prime factors of n agrees with the variable X_i for **all** its prime divisors, p_i.

Note Q is primitive: If $a \in Q$, and $a|b$ with $b > a$, then $\Omega(a) < \Omega(b)$, but any prime dividing a also divides b, and so b cannot be in Q.

Summary

Summary

Asymptotically the best possible upper bound for the gaps between consecutive terms in a primitive set lies between

$$\log\log x \quad \text{and} \quad \exp\left(\sqrt{(2+\epsilon)\log x \log\log x}\right).$$

Summary

Asymptotically the best possible upper bound for the gaps between consecutive terms in a primitive set lies between

$$\log\log x \quad \text{and} \quad \exp\left(\sqrt{(2+\epsilon)\log x \log\log x}\right).$$

Asymptotically the best upper bound for gaps between consecutive terms in a 3-term geometric progression free set is between

Summary

Asymptotically the best possible upper bound for the gaps between consecutive terms in a primitive set lies between

$$\log\log x \quad \text{and} \quad \exp\left(\sqrt{(2+\epsilon)\log x \log\log x}\right).$$

Asymptotically the best upper bound for gaps between consecutive terms in a 3-term geometric progression free set is between

$$2 \quad \text{and} \quad \exp\left(2\sqrt{(\log 2+\epsilon)\log x}\right).$$

Summary

Asymptotically the best possible upper bound for the gaps between consecutive terms in a primitive set lies between

$$\log\log x \quad \text{and} \quad \exp\left(\sqrt{(2+\epsilon)\log x \log\log x}\right).$$

Asymptotically the best upper bound for gaps between consecutive terms in a 3-term geometric progression free set is between

$$2 \quad \text{and} \quad \exp\left(2\sqrt{(\log 2+\epsilon)\log x}\right).$$

Recently He and Patil make partial progress to increasing the 2 to a 3.

Pairwise coprime sets

Pairwise coprime sets

Note the set we constructed is primitive, but not pairwise coprime.

Pairwise coprime sets

Note the set we constructed is primitive, but not pairwise coprime.

Clearly the primes are pairwise coprime, any pairwise coprime set is primitive.

Pairwise coprime sets

Note the set we constructed is primitive, but not pairwise coprime.

Clearly the primes are pairwise coprime, any pairwise coprime set is primitive.

Question

Can we find a pairwise coprime set with gaps that are provably smaller than what is known for the primes?

Thank you!