Counting primitive sets and divisor-permutations using divisor graphs

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Loyola Math Department Colloquium
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A *primitive abundant number* is an abundant number, all of whose divisors are deficient (20, 70, 88, 104, 272, 304, 368, 464, 550...)

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What about primitive sets where the reciprocal sum diverges? For example, the reciprocal sum of prime numbers $\sum_{p \in \mathcal{P}} \frac{1}{p}$ diverges.

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The divisor graph

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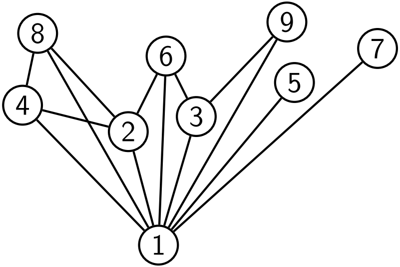

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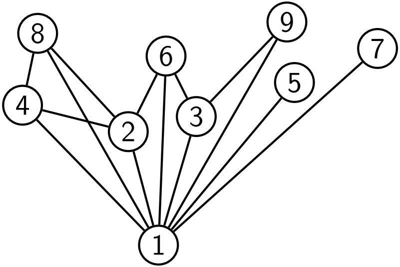

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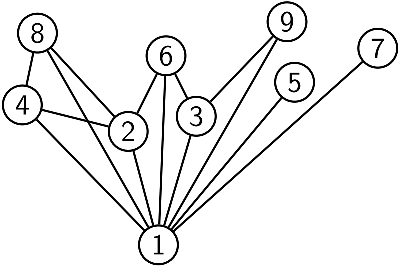

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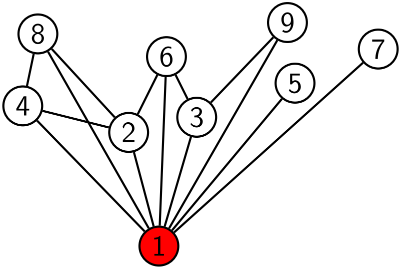

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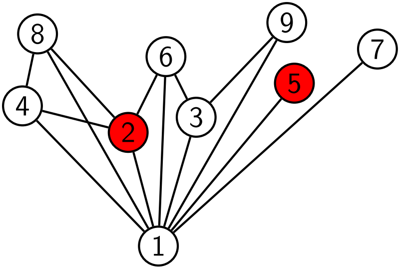

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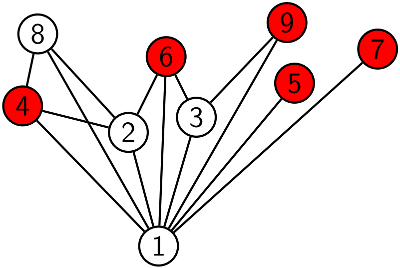

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How many primitive subsets of $\{1, 2 \ldots n\}$ are there?

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.

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A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

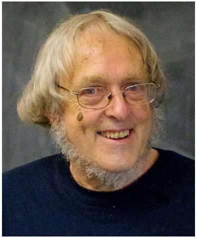
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



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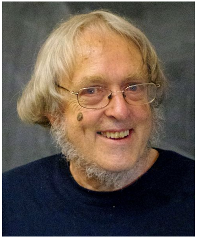

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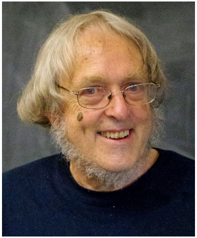

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Every subset of $\left(\frac{n}{2}, n\right]$ is primitive. There are $2^{\lceil \frac{n}{2} \rceil} \geq \sqrt{2}^n$ such subsets.

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His proof gives no insight on the value of this constant.

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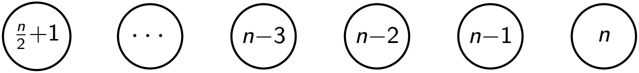

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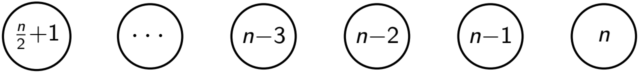

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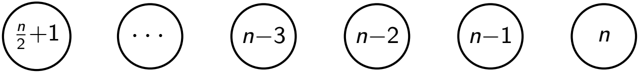

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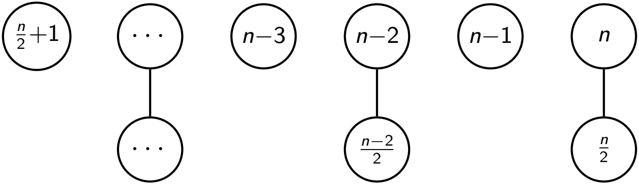

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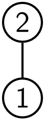

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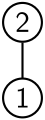

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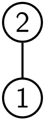

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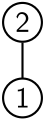

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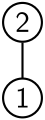

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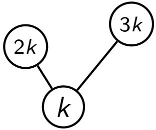

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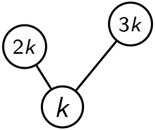

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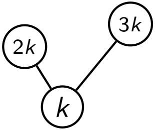

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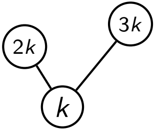

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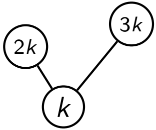

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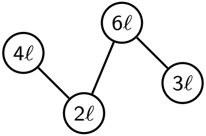

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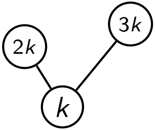

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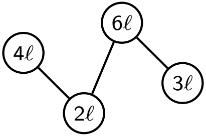

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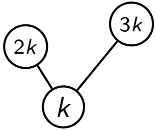

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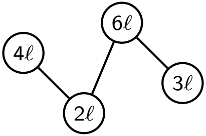

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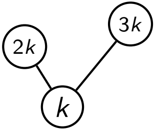

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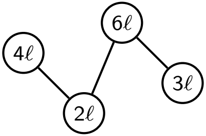

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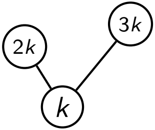

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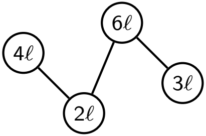

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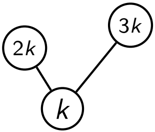

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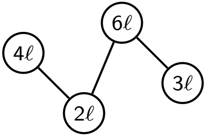

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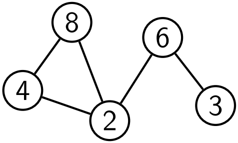

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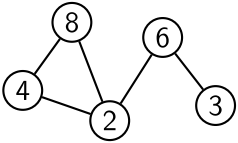

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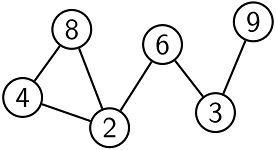

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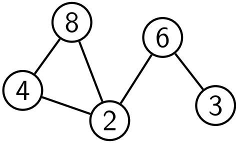

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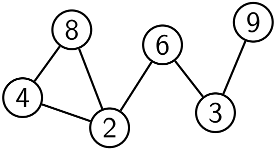

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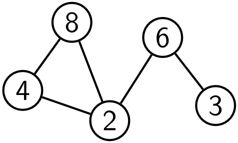

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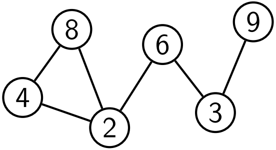

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As $n \to \infty$, the number of $k \leq n$ sharing the same "d" and "t" is

$$\frac{n + o(n)}{t(t+1)} \prod_{p \leq i} \left(1 - \frac{1}{p}\right).$$

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Set $c = \displaystyle\prod_{i=1}^{\infty} \prod_{\substack{d \\ d \text{ is } i\text{-smooth}}} \prod_{t=id}^{(i+1)d-1} r(d,t)^{\frac{1}{t(t+1)} \prod_{p<i} \frac{p-1}{p}}.$

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We find that $Q(n) = c^{n+o(n)}$.

Approximating $Q(n)$

Theorem

As $n \to \infty$ the number of primitive subsets of $[1, n]$ is

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The constant $c = \prod_{i=1}^{\infty} \prod_{\substack{d \\ d \text{ is } i\text{-smooth}}} \prod_{t \in [id, (i+1)d)} r(d,t)^{\frac{1}{t(t+1)} \prod_{p<i} \frac{p-1}{p}}$

is bounded between $1.5729 < c < 1.5745$.

A general theorem

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Theorem (M.)

Suppose $f(k,n)$ depends only on the connected component of k in $\mathcal{D}_{[k,n]}$ and $|f(k,n)| \leq A$ for some fixed A. Then

$$\sum_{a=1}^{n} f(a,n) = nC_f + o\left(An\exp\left(-\sqrt{\tfrac{1}{2}\log n\log\log n}\right)\right)$$

where

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ d \text{ is } i\text{-smooth}}} \sum_{t\in[id,(i+1)d)} \left(\frac{f(d,t)}{t(t+1)} \prod_{p\leq i} \frac{p-1}{p}\right).$$

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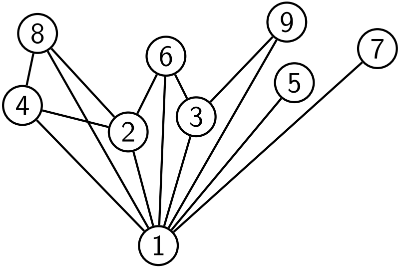
Example: $C(9) = 2$



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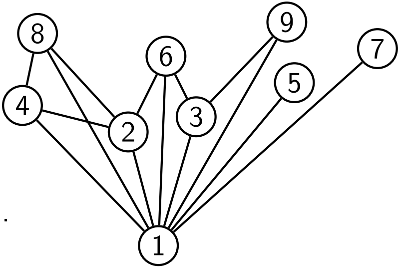
The divisor graph $\mathcal{D}_{[1,9]}$ can be covered by $\{7, 1, 5\}$ and $\{9, 3, 6, 2, 4, 8\}$ but it is not possible to use a single path.



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In a subsequent paper he proves

$$1.93^n \leq D(n) \leq 13.6^n$$

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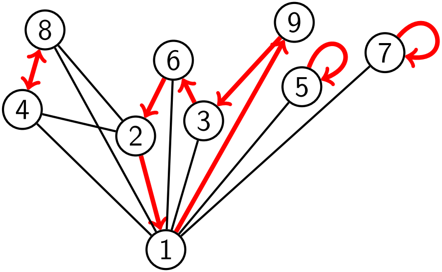

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In fact it is unbounded, $d(k,n) > \pi\left(\frac{n}{k}\right) - \pi\left(\frac{n}{2k}\right)$.

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Let $W = \{w_1, w_2, \ldots, w_{d-1}\}$ be the neighbors of v in G.

Proof outline

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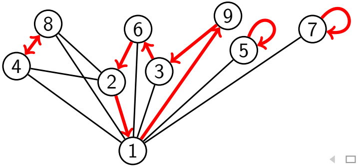

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Write $C_v = X + Y_i + Y_j + Z$

- X counts the cases where v, w_i, and w_j are all part of 1-cycles
- Y_i counts those where only v and w_i are fixed (not w_j).
- Z counts those where v is fixed but neither w_i nor w_j is fixed.

Then write

$$(C_v)^2 = (X+Y_i+Y_j+Z)^2 \ge (X+Z)^2 + (Y_i+Y_j)^2$$

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by the AM-GM inequality.

Proof outline

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by the AM-GM inequality. It remains to show $XZ + Y_iY_j \ge (C_{w_i v w_j})^2$.

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Find an injection from objects counted by $(C_{w_i v w_j})^2$ to objects counted by $XZ + Y_i Y_j$.

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Each of XZ, $Y_i Y_j$ and $(C_{w_i v w_j})^2$ count pairs of directed vertex-disjoint cycle covers. In each pair, color the first one blue, and the second red.

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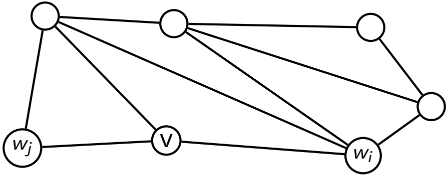

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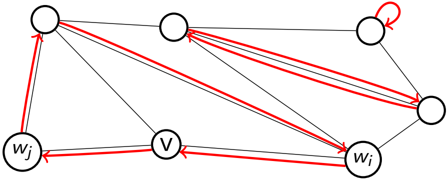

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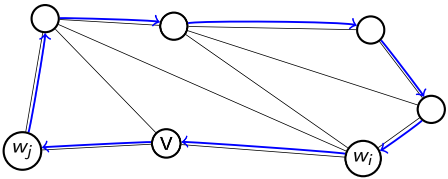

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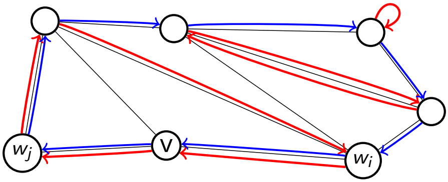
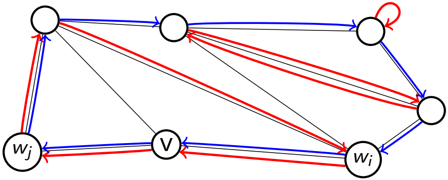
Draw both on the same graph. Get a colored, directed multigraph, every vertex has one inward and outward pointing edge of each color.



Colorings!



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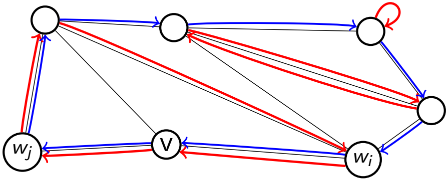

Take a colored multigraph obtained from $(C_{w_i v w_j})^2$. (It has both blue and red edges $w_i \to v \to w_j$.)

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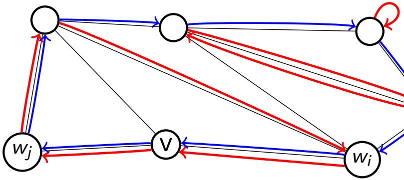

Take a colored multigraph obtained from $(C_{w_i v w_j})^2$. (It has both blue and red edges $w_i \to v \to w_j$.) Remove all four of these edges.

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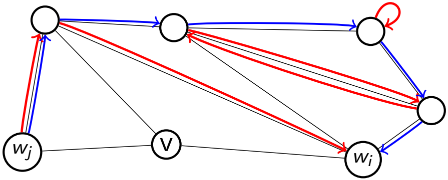

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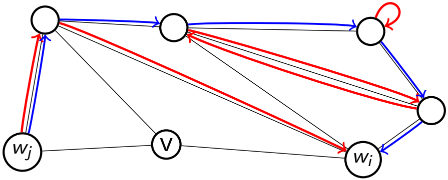

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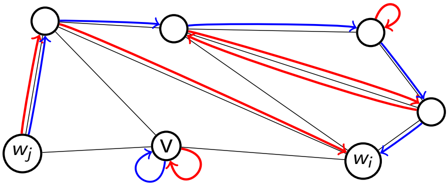
Add two loops to v one of each color.

Colorings!



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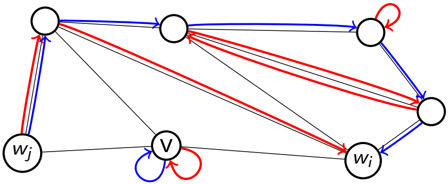

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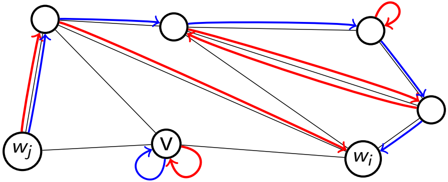
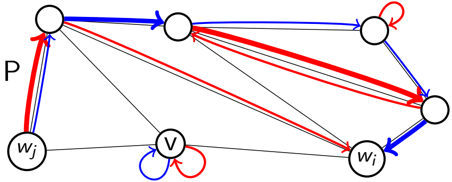
Add two loops to v one of each color.

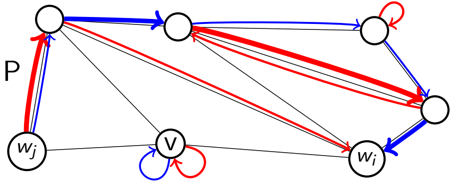
Now every vertex (except w_i, w_j) has in- and out-edges of each color. w_i has in-edges of each color, and w_j has out-edges of each color.



The colored multigraph consists of alternating-color cycles, plus two alternating-color paths $w_j \to w_i$.



The colored multigraph consists of alternating-color cycles, plus two alternating-color paths $w_j \to w_i$. Call the path ending in a blue edge P.



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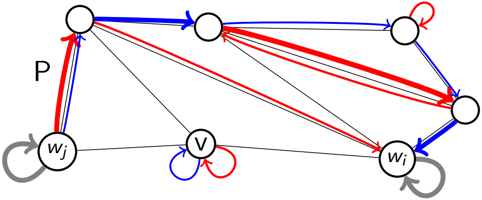
Now add a (initially uncolored) loop to each of the vertices w_i and w_j.



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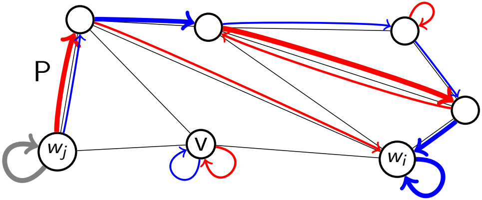

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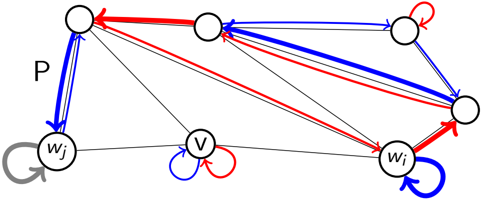

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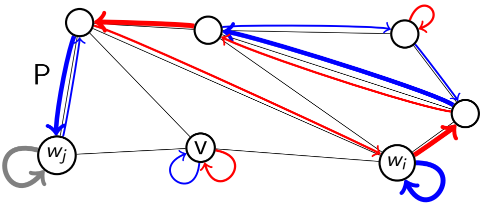

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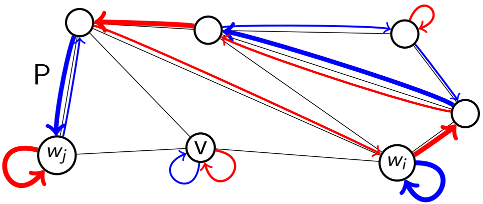

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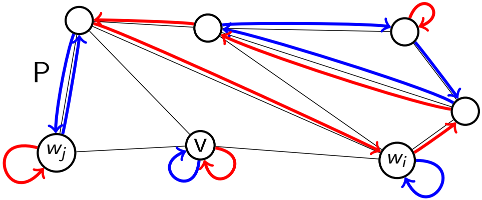

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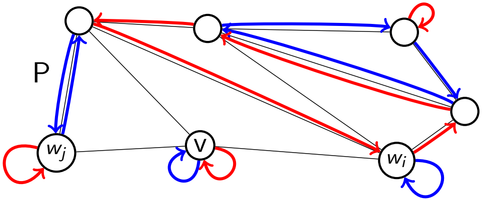
Wrapping up



The final result is a coloring in which v is fixed by both colors, every vertex has exactly one in-edge and out-edge of each color.

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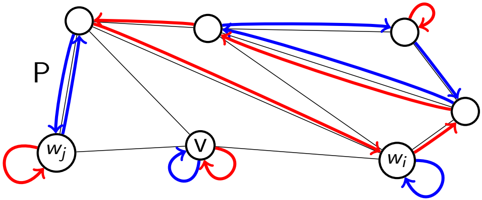

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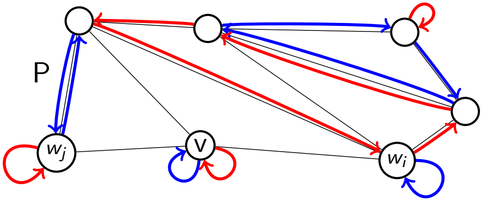

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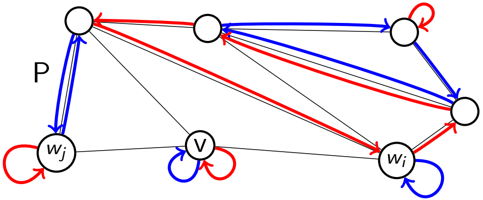

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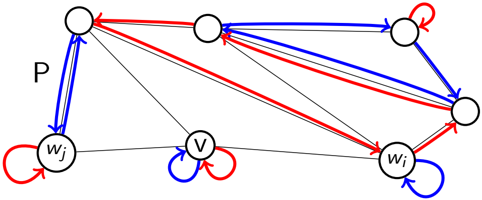

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Using this, we find that

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$$D(n) = \prod_{k=1}^{n} d(k, n).$$

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Theorem

As $n \to \infty$, $D(n) = c_d^{\,n+o\left(n\exp\left(-\sqrt{\frac{1}{2}\log n \log\log n}\right)\right)}$.

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Using numerical computation we improve this to $2.069 < c_d < 2.694$.

Open questions

What is the best constant C, $R(G, v) \leq (C + o(1))d^2$ as $d \to \infty$?

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On the other hand, looking at complete bipartite graphs $K_{\frac{n}{2},n}$ we find

$$R(K_{\frac{d}{2},d}, v) \geq \left(\frac{1}{4} + o(1)\right) d^2.$$

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THANK YOU!