Primitive and Geometric Progression Free Sets Without Large Gaps

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MASON III
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Gaps between prime numbers

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In practice: At most 7384 (largest known to 10^{100}) 116 on average.
In theory: At most 10^{53}. (Best we can prove, assuming RH).

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$p_n - p_{n-1} \gg \frac{\log p_n \log\log p_n \log\log\log\log p_n}{\log\log\log p_n}$ infinitely often
(Ford, Green, Konyagin, Maynard, Tao, 2016)

Gaps between prime numbers

Summary: The largest gap between primes up to x has size between

$$\frac{\log x \log\log x \log\log\log\log x}{\log\log\log x} \quad \text{and} \quad x^{0.525}$$

but we think the truth is $\log^2 x$.

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Definition

A set $S \subset \mathbb{N}$ is **primitive** if no element of the set divides another: if $m, n \in S$ are distinct then $m \nmid n$.

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  For example, $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, \ldots\}$.

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Conjecture (Erdős)

If S is a primitive set then

$$\sum_{n\in S}\frac{1}{n\log n}\leq\sum_{p\in\mathcal{P}}\frac{1}{p\log p}=1.63\ldots.$$

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It isn't possible to get good bounds for the largest gap in such a set.

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Recall:

$$p_n - p_{n-1} \ll p_n^{0.525} \quad \text{(unconditionally)}$$

$$p_n - p_{n-1} \ll \sqrt{p_n} \log p_n \quad \text{(RH)}$$

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There are even larger geometric progression free sets, but can't say much about their gaps.

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Furthermore, there exists a set $u_1, u_2, \ldots$ avoiding 3-term geometric progressions with **integer** *ratio where*

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Note V avoids 3-term progressions: If (a, ar, ar^2) is a progression, a prime in r appears to different powers in at least two of these terms.

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Proof: The interval has many integers without prime factors greater than $\sqrt{y}$. Such integers have at most $\frac{2\log x}{\log y}$ prime factors. Construct a graph on these numbers, connect vertices with a prime factor in common.

Lemmas

Theorem (Linear Sieve of Rosser and Iwaniec)

There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

Lemma

Any interval $[x - cy, x]$ contains a subset of integers which are pairwise coprime, of size at least $\frac{c'\sqrt{y}}{\log y}$ for some positive constant c'.

Proof: The interval has many integers without prime factors greater than $\sqrt{y}$. Such integers have at most $\frac{2\log x}{\log y}$ prime factors. Construct a graph on these numbers, connect vertices with a prime factor in common. Turan's graph theorem says the graph has a large independent vertex set.

The probabilistic method

Proof sketch:

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Recall $P(X_i = m) = \frac{1}{2^m}$, $V = \{n \geq 2 : p_i | n \to p_i^{X_i} || n\}$.

Let $y = \exp\left(2\sqrt{(\log 2 + \epsilon)\log x}\right)$,

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Suppose $n \in S$, then the probability $P(n \in V)$ that $n \in V$ is

$$\prod_{p_i^\alpha || n} P(X_i = \alpha) = \left(\frac{1}{2}\right)^{\Omega(n)} \geq \left(\frac{1}{2}\right)^{\frac{2\log x}{\log y}} = \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right).$$

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&\quad \le \left(1 - \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)^{\frac{c'\sqrt{y}}{\log y}} \\
&\quad \le \exp\left(-\frac{c'\sqrt{y}}{\log y} \times \exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)
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$$\leq \exp\left(-\frac{c'\sqrt{y}}{\log y}\times\exp\left(-\left(\frac{2\log 2\log x}{\log y}\right)\right)\right)$$

$$= \exp\left(-\exp\left(\frac{1}{2}\log y-\left(\frac{2\log 2\log x}{\log y}\right)-\log\log y+O(1)\right)\right).$$

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Using $y=\exp\left(2\sqrt{(\log 2+\epsilon)\log x}\right)$ the inner exponent is $O_\epsilon(\sqrt{\log x})$.

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Starting the sequence at N and using linearity of expectation, the expected number of intervals of this form avoiding V is at most

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Thus there exists a set, V satisfying the properties of the theorem.

Modifications for primitive sets

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Note Q is primitive: If $a \in Q$, and $a|b$ with $b > a$, then $\Omega(a) < \Omega(b)$, but any prime dividing a also divides b, and so b cannot be in Q.

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Can we find a pairwise coprime set with gaps that are provably smaller than what is known for the primes?

Thank you!