Counting primitive subsets and other statistics of the divisor graph of $\{1, 2, \ldots n\}$

Nathan McNew
Towson University

Joint Mathematics Meetings
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- $\mathcal{P}_0 = \{1\}$.

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Theorem (Ahlswede, Khatchatrian, Sárközy, 1999)

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If we restrict to primitive subsets of $\{1, 2, \dots n\}$ then the set $\left\{\left\lfloor \frac{n}{2} \right\rfloor + 1, \dots n\right\}$ is primitive, and has size $\left\lfloor \frac{n}{2} \right\rfloor$.

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How many primitive subsets of $\{1, 2 \ldots n\}$ are there?

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.

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A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

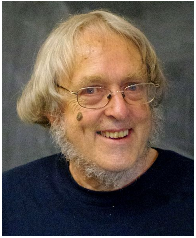
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



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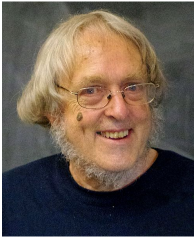

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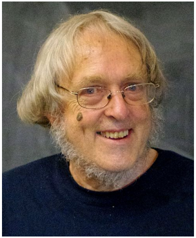

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Every subset of $\left(\frac{n}{2}, n\right]$ is primitive. There are $2^{\lceil \frac{n}{2} \rceil} \geq \sqrt{2}^n$ such subsets.

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$$1.5596^n < Q(n) < 1.60^n$$

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Proof is not effective: Gives no insight on the value of this constant.

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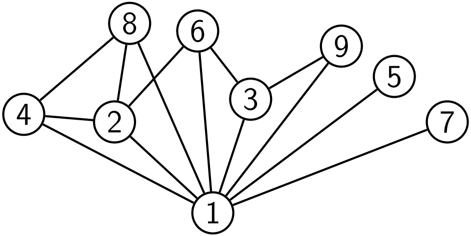

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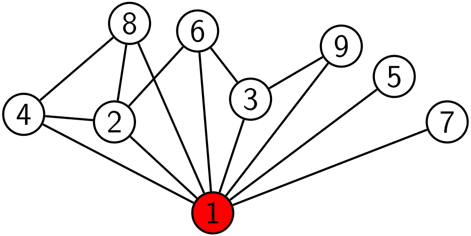

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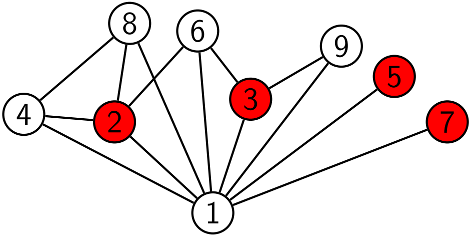

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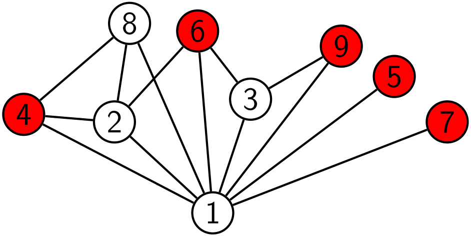

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or

$$\log Q(n) = n\left(\log(c) + O\left(\frac{1}{e^{\sqrt{\left(\frac{1}{6}-\epsilon\right)\log n \log\log n}}}\right)\right).$$

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Proof: Graphs, and smooth number estimates.

Numerical Bound

#24418 closed defect (fixed)

8.2 β4

Opened 5 months ago
Closed 4 months ago

Doctest: bug numerical_approx(2^(450232897/4888643760))

Reported by:	vdelecroix	Owned by:	
Priority:	major	Milestone:	~~sage-8.2~~
Component:	symbolics	Keywords:	bug
Cc:	rws	Merged in:	
Authors:	Ralf Stephan	Reviewers:	Jeroen Demeyer
Report Upstream:	N/A	Work issues:	
Branch:	821f7d9 (Commits)	Commit:	821f7d9f3568316bc0b8b1f5619bce...
Dependencies:		Stopgaps:	

Description (last modified by vdelecroix) Δ

```
sage: numerical_approx(2^(450232897/4888643760))
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
<ipython-input-2-3c4e30ac02c1> in <module>()
----> 1 numerical_approx(Integer(2)**(Integer(450232897)/Integer(4888643760)))

/opt/sage/local/lib/python2.7/site-packages/sage/misc/functional.pyc in numerical_ap
   1406         return numerical_approx_generic(x, prec)
   1407     else:
```

A general theorem

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Theorem (M., 2019)

Fix $\epsilon > 0$, $A \geq 0$. Suppose $f(k,n)$ depends only on the connected component of k in the divisor graph of $[k,n]$ and $|f(k,n)| \leq A$. Then

$$\sum_{a=1}^{n} f(a,n) = nC_f + O_A\left(n \exp\left(-\sqrt{\left(\tfrac{1}{6}-\epsilon\right)\log n \log\log n}\right)\right)$$

where

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id,(i+1)d)} \left(\frac{f(d,t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p}\right).$$

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Theorem (Liu, Pach, Palincza, 2018)

$M(n) = \alpha^{n+o(n)}$ *as* $n \to \infty$ *and* $1.14817 < \alpha < 1.14823.$

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Corollary

Let $V(n)$ *denote the median size of primitive subsets of* $[1, n]$. *Then*

$$0.1681n < V(n) < 0.3918n$$

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Question

Is $V(n) \sim vn$ *for some* v*? If so, is* v *computable?*

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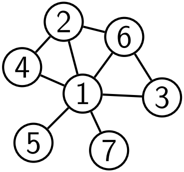
Example: $C(7) = 2$



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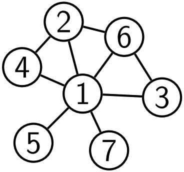
The divisor graph of [1,7] can be covered by $\{7, 1, 5\}$ and $\{3, 6, 2, 4\}$ but it is not possible to use a single path.



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Mazet (2006): $C(n) \sim \rho n$ for some $0.1706 \leq \rho \leq 0.2289$.

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Theorem

$$C(n) = \rho n \left(1 + O\left(\exp\left(-\sqrt{\left(\tfrac{1}{6} - \epsilon\right)\log n \log\log n}\right)\right)\right).$$

The constant ρ *is effectively computable and* $0.1916 < \nu < 0.2143$.

Thank you!