8.02.22

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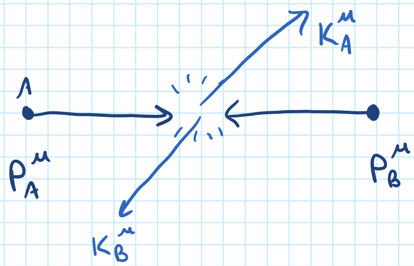
Collisions, Conserved Quantities & Invariants.

Collisions

(In) elastic Collisions

The only rule is: 4-momentum is conserved

$p^\mu = (E, \vec{p})$ "before = after"



$$(p_A^\mu + p_B^\mu) = (K_A^\mu + K_B^\mu)$$

$$(p_A + p_B)^\mu = (K_A + K_B)^\mu$$

$\mu = 0$

$\underbrace{E_A + E_B}_{\text{Before}} = \underbrace{E_A + E_B}_{\text{After}}$, Total energy conserved

$\mu = i$

$\vec{p}_A + \vec{p}_B = \vec{K}_A + \vec{K}_B$, 3-momentum conserved

Invariants - Independent of reference frame.

Mass is an invariant. It does not matter how fast we observe a particle to be moving, it will always have the same mass.

$\boxed{\text{Invariant quantities must not have any free (unpaired) indicies.}}$

p^μ Free index

Lorentz transformations move us to different frames

↳ Can be written as a 4x4 matrix $\Lambda^\mu{}_\nu$

Acting a Lorentz transformation on an invariant will do nothing.

Is p^μ an invariant? ☆

$$\Lambda^\mu{}_\nu \, p^\nu = \underbrace{K^\mu}$$

$\mu = 0$

$$K^0 = \Lambda^0{}_0 \, p^0 + \Lambda^0{}_1 \, p^1 + \Lambda^0{}_2 \, p^2 + \Lambda^0{}_3 \, p^3$$

The transformed quantity is a combination of the untransformed quantities

New reference frame = Sum of things in old reference frame.

☆ How do we get mass from p^μ?

Start with relativistic energy : $E^2 = (mc^2)^2 + (\vec{p}c)^2$

$\downarrow$ natural units

$E^2 = m^2 + \vec{p}^{\,2}$, $p^\mu = (E, \vec{p})$

$m^2 = E^2 - \vec{p}^{\,2}$

$= E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2$, define p_1^μ & p_2^μ

$= p_1^0 p_2^0 - \vec{p}_1 \cdot \vec{p}_2$, "index is the same for each term"

$\sim p_1^\mu p_2^\mu$, not a valid mathematical operation

$$\begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix} \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$$

One index MUST be lowered

$p^\mu \rightarrow p_\mu$, the way to do this is with the Metric ($g_{\mu\nu}$, $\eta_{\mu\nu}$)

Metric : tells us how to raise & lower indicies

(required for index contraction)

Contraction : Sum over repeated indicies

$M^i{}_j V^j = B^i$, Sum over the values of $j = 1, 2$

$$B^1 = \underbrace{M^1{}_1 V^1 + M^1{}_2 V^2}_{\text{Contraction}}$$

$[\,\backslash\backslash\,]_\mu \, [\,\backslash\backslash\,]^{\mu\beta} = [\,\backslash\backslash\,]^\beta$

Contraction : 1 upper , 1 lower AND they match

$[\,\backslash\backslash\,]_\mu \, [\,\backslash\backslash\,]^\alpha = [\,\backslash\backslash\,]_\mu{}^\alpha$

Not a contraction

3 - Vectors live in Euclidean space.

$$(g_{\text{Euclid}})^{ij} = (g_{\text{Euclid}})_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Upper-ness & lower ness mean different things

Matrix: $(M)^i{}_j$ converts an object with upper index into a different object with an upper index

upper → upper

$M^i{}_j V^j = B^i$

Metric $(g)_{ij}$ converts an upper index into a lower index or vice-versa.

upper → lower

$g_{ij} V^j = B_i$

$g^{ij} V_j = B^i$

lower → upper

4-Vectors live in Minkowski space

$$\bigstar\ (g_{\text{Minkowski}})_{\mu\nu} = (g_{\text{Minkowski}})^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \text{diag}(+1, -1, -1, -1)$$

$$= \text{diag}(+, ---)$$

We left off at:

$p^\mu p^\mu$, but we want to contract the indicies.

↓ what we want, but we are only given p^μ

$$p_\mu p^\mu = \underbrace{g_{\mu\nu} p^\nu p^\mu}_{\text{Hopefully is } E^2 - \vec{p}^{\,2}}, \quad p^\mu \equiv (E, \vec{p}), \quad g_{\mu\nu} = \text{diag}(+, ---)$$

$$= \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$$

Observe that $g_{\mu\nu}$ has only 4 nonzero indicies along the diagonal

$$= g_{00}\, p^0 p^0 + g_{11}\, p^1 p^1 + g_{22}\, p^2 p^2 + g_{33}\, p^3 p^3$$

$$= (+1) E^2 - p_x^2 - p_y^2 - p_z^2$$

$$= E^2 - \vec{p}^{\,2}$$

$p_\mu p^\mu = E^2 - \vec{p}^{\,2} = m^2$, all of these quantities are invariant

Is m^2 an invariant?

Act Λ on mass but written as 4-vectors

$$\text{“}\Lambda m^2\text{”} \rightarrow m^2 = P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu$$, each P gets its own Λ

$$\text{“}\Lambda m^2\text{”} = g_{\mu\nu} \Lambda^\mu{}_\alpha P^\alpha \Lambda^\nu{}_\beta P^\beta$$

$$= \underbrace{\Lambda^{\mu}{}_\alpha \, g_{\mu\nu} \, \Lambda^{\nu}{}_\beta}_{} P^\alpha P^\beta$$

$[g]_{\alpha\beta}$. Metric is an invariant.

$$= g_{\alpha\beta} P^\alpha P^\beta = m^2$$ so m^2 is invariant

Conserved: Total amount is the same

Invariant: Individual amount is the same

We will revise

Homework: Transverse Mass. (See wiki)

Transverse mass is invariant along boosts in the z direction.

↓

$$M_T^2 = m^2 + P_x^2 + P_y^2$$ (1) = write in terms of E & P_z

$$\left(\Lambda_{z\text{-Boost}}\right) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & & & 0 \\ 0 & & 0 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$, (2) Show M_T^2 is invariant under $\Lambda_{z\text{-boost}}$

steps: (1) write M_T^2 in terms of contracted P^μ's.

(2) Write both P^μ's as upper index using the metric

(3) Act $\Lambda_{z\text{-boost}}$ on each P^μ