Quantum Analogs

1. Introduction

In physics, phenomena are usually described with mathematical representations. Interestingly, the same form of the mathematical representation is often associated with completely unrelated phenomenon (i.e., the mathematical expressions are identical, while the parameters in the expressions represent different physical quantities). A good example of this is the mathematics used to describe thermal collective vibrations of the atoms in a solid is the same as a collection of coupled mechanical oscillators. This similarity of the mathematics is even present with the fundamental equations that describe acoustic and quantum mechanical behavior. By exploiting this fact, this lab uses measurements of acoustic parameters to demonstrate quantum mechanical behavior of electrons in a box and an atom. It is important to note, however, that there are some differences, which will be discussed below. We will begin with a primer on acoustic phenomena.

The propagation of sound in air is describable with classical mechanics in the form of three differential equations that relate the air velocity $\boldsymbol{u}$, acoustic pressure p and density ρ. From conservation of momentum, the first equation is the linearized Euler's equation

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla} p \quad . \tag{1}$$

Notice this is Newton's second law applied to a gas element of constant mass. Next, there is the continuity equation. This statement is based on conservation of mass and is given by

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \boldsymbol{u} \quad . \tag{2}$$

Finally, we need an equation of state that describes the medium supporting the sound vibrations. Heat conduction in air negligible over the time scales of the vibrations, so the entropy s of the molecules remains constant (i.e., the dynamics are adiabatic). In this case, the equation of state is

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{B}{\rho} \tag{3}$$

where B is the adiabatic bulk modulus.

These three equations can be combined to give a wave equation for the acoustic pressure of the form

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \quad . \tag{4}$$

Using separation of variables, the solution of Eq. (4) is found to be a traveling wave with a speed given by $c = \sqrt{B/\rho}$. The same equations are found for ρ and $\boldsymbol{u}$ meaning all three of these quantities are represented by travelling waves. For an idea gas, $c = \sqrt{\gamma R T}$ where $\gamma = C_p/C_v$ is the ratio of the specific heats at constant pressure and volume, respectively, R is the gas constant and T is the temperature. Notice that the speed of sound only depends on the temperature. For air $\gamma = 1.4$, and at a temperature of 273 K, $c = 331\ \mathrm{m/s}$.

For this experiment, the sound field will be contained within an enclosed structure. In this case, when the equations used to describe the system involve spatial derivatives, the conditions at the boundary are important. For sound, the perpendicular velocity of the molecules at the container wall is zero. From the first equation above, the derivative of the pressure in the direction perpendicular to the wall must also be zero. This type of boundary condition is called a Neumann boundary condition.

For a quantum mechanical system the equation that describes its behavior is the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\boldsymbol{r},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\boldsymbol{r},t) + V(\boldsymbol{r}) \Psi(\boldsymbol{r},t) \quad . \tag{5}$$

Here, $\Psi(\boldsymbol{r},t)$ is the wavefunction of a particle of mass m in a potential V. Everything known about the system is contained in the wavefunction. Unlike the acoustic pressure, the wave function is complex; however, $|\Psi(\boldsymbol{r},t)|^2$ is real and the probability density of finding the particle at a position $\boldsymbol{r}$. When the potential is time-independent, stationary states are possible. These states are given by the time-independent Schrödinder equation

$$E\psi(\boldsymbol{r}) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\boldsymbol{r}) + V(\boldsymbol{r}) \psi(\boldsymbol{r}) \quad . \tag{6}$$

The solutions of Eq. (6) correspond to a series of eigenfunctions $\psi_n(\boldsymbol{r})$ and their associated eigenvalues E_n, which are the particle's allowed energies. Again the boundary conditions are important; however, unlike with the acoustic case, the amplitude, not the derivative, of the stationary state is zero at the boundary of an infinite potential (Dirichlet boundary conditions). The wavefunction associated with each stationary state is given by $\Psi(\boldsymbol{r},t) = e^{-iE_n t/\hbar}\psi_n(\boldsymbol{r})$. Even though the wavefunction is time-dependent, the probability density and measureable quantities determined from the wavefunction do not depend on time.

Both the acoustic pressure of a sound field in an enclosure and the wavefunction of an electron in a confining potential are solutions of a wave equation describing a delocalized object, and their wave equations, Eqs. (4) and (5), depend on the Laplace operator. As a result, the spatial dependence of these quantities will be similar.

Despite the similarities, there are some differences. Besides the fact that the quantities being described are very different, Eqs. (4) and (5) do not have the same time-dependence. In the quantum-mechanical case, the first-order time derivative multiplied by i leads to complex wave solutions with a quadratic dispersion (i.e., $E(k)$) for the electron. In the classical case, the second-order time derivative yields real wave solutions with a linear dispersion. Also, the Schrödinder equation has a potential energy term that cannot be simulated with an acoustic system. Though, the acoustic reflection at a hard wall simulates an infinitely high potential barrier. A series of eigenstates are found in both systems, but the positions of the nodes for $\psi(\boldsymbol{r})$ and p are not the same because the boundary conditions are different. (With sound, the velocity has the same boundary conditions as the wavefunction, but the velocity is a vector, not a scalar like p and $\psi(\boldsymbol{r})$.) In quantum mechanics, the eigenvalue E from Eq. (6) appears in the phase factor that multiplies $\psi(\boldsymbol{r})$. While with sound, the eigenvalue ω from Eq. (13) below is the frequency of the sound vibration. Finally, the microphone measures the absolute phase of the sound wave, but in quantum mechanics, this is not possible, though differences in phase can be determined when wavefunctions interfere.

In this lab, you will model the hydrogen atom with a single electron in the Coulomb potential of the nucleus. The spherical symmetry of the three-dimensional problem makes it possible to separate the angular and radial variables when the time-independent Schrödinder is expressed using spherical coordinates. In terms of the radial coordinate r and the angular coordinates θ and φ, Eq. (6) becomes

$$E\psi = \frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\hbar^2}{2mr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\hbar^2}{2mr^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{e^2}{r}\psi \quad . \tag{7}$$

The last term is the Coulomb potential. A separation of variables is possible with Eq. (7) using a stationary state of the form

$$\psi(r,\theta,\varphi) = Y_l^m(\theta,\varphi)\chi_l(r) \quad . \tag{8}$$

Here the functions $Y_l^m(\theta,\varphi)$ are spherical harmonics, and l and m are the angular and magnetic quantum numbers, respectively. These functions are solutions of the equation

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m(\theta,\varphi) = l(l+1)Y_l^m(\theta,\varphi) \quad , \tag{9}$$

and they can be expressed as

$$Y_l^m(\theta,\varphi) \propto P_l^m(\cos\ \theta)e^{im\varphi} \tag{10}$$

where the terms $P_l^m(cos\ \theta)$ are the Legendre polynomials. The radial component of the stationary state $\chi_l(r)$ is a solution of the radial differential equation

$$-\left[\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)\hbar^2}{2mr^2} + \frac{e^2}{r}\right]\chi_l(r) = E\chi_l(r) \quad . \tag{11}$$

Now let's examine the spherical acoustic resonator. We begin by trying a solution of the form

$$p(\boldsymbol{r},t) = p(\boldsymbol{r})\cos(\omega t) \tag{12}$$

in the acoustic wave equation, Eq. (4). The result is something called the time-independent Helmholtz equation

$$-\omega^2 p(\boldsymbol{r}) = \nabla^2 p(\boldsymbol{r}) \quad . \tag{13}$$

As was done with the Schrödinder equation, the Helmholtz equation is expressed in spherical coordinates to take advantage of the spherical cavity that encloses the sound field. When this is done, the result is

$$\frac{\omega^2}{c^2}p = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\varphi^2} \quad . \tag{14}$$

Notice that this equation has the same form as Eq. (7) for the hydrogen atom without the Coulomb potential term at the end. Again it is possible to separate Eq. (14) into two differential equations that isolate the radial and angular coordinates with the solution

$$p(r,\theta,\varphi) = Y_l^m(\theta,\varphi) f(r) \quad . \tag{15}$$

The angular functions in Eq. (15) the same spherical harmonics that were found with the Schrödinder equation because the angular component of Eq. (14) when separation of variables is used is identical to Eq. (9). However, the differential equation for the radial coordinate that results from Eq. (14) is

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right] f(r) = \frac{\omega^2}{c^2} f(r) \tag{16}$$

which is not the same as Eq. (11) for the hydrogen atom because the Coulomb potential term is absent.

Since the Coulomb potential does not appear in Eq. (9) for the atomic system, it does not affect the spherical harmonics. In both the atomic and acoustic systems, the spherical harmonics are identical and identified with two integers: l (azimuthal quantum number) and m (magnetic quantum number) where $l \geq 0$ and $-l \leq m \leq l$. The eigenvalues or energy levels $E_{n'l}$ of the hydrogen atom and the eigenfrequencies $\omega_{n'l}$ of the spherical acoustic resonator are also identified by an integer $n' \geq 0$. Unlike the spherical harmonics, these eigenvalues and eigenfunctions for the two systems are quantitatively different because their respective radial equations are not the same.

For the hydrogen atom, the energy eigenvalues, which are determined from Eq. (11), are given by

$$E_{n'l} = -\left(\frac{e^2}{\hbar c}\right)^2 \frac{mc^2}{2(l+1+n')} \quad . \tag{17}$$

All the levels with the same value for $(l+1+n)$ are degenerate. Therefore it is possible to define what is called the principle quantum number $n = (l+1+n')$, and then for a given *n*, the azimuthal quantum number is restricted to the following values $0 \leq l \leq n-1$. The degeneracy of levels with the same principle quantum number does not have an analog in the spherical acoustic resonator.

When spherical symmetry exists, the eigenvalues for different magnetic quantum numbers *m* are degenerate for any form of the radial equation, both classically and quantum mechanically. In general, states specified by the quantum numbers (n', l) are $(2l+1)$ degenerate. This degeneracy is lifted when the spherical symmetry is broken. For the atomic system, this can be done with the application of a magnetic field in what is known as the Zeeman Effect. For the acoustic spherical resonator, the same effect is achieved by extending the spherical cavity along one direction.

2. Equipment

In this experiment, you will use four pieces of equipment: an oscilloscope, a function generator, the Quantum Analogs controller and a metallic cavity. The metallic cavity is spherical and can be separated into two hemispheres. One hemisphere has small speaker for exciting the sound vibrations, and the other hemisphere has a microphone for measuring the acoustic pressure associated with those sound vibrations. There are also two spacer rings with thicknesses of 3 mm and 6 mm, which can be inserted between the two hemispheres to break the spherical symmetry.

The Quantum Analogs controller provides some additional functionality that is needed to perform these experiments. A compete description of this unit can be found in the Appendix 1. A diagram of the controller's front panel is shown in Figure 1. The controller has a *Sine Wave Input* and a *Speaker Output* on the right-hand side of its front panel. This is where an AC signal from either the function generator or the computer is provided for the speaker on the resonant cavity. On the left-hand side of the front panel is the *Microphone Input*. This port provides a 5 V bias for the microphone in the cavity and also detects the signal from the microphone that represents the acoustic pressure at its location in the cavity. This single is first amplified with a fixed gain of ~100 and then attenuated to the appropriate level. The amount of attenuation is indicated on the metal rings around the attenuator's knob. The number on the outer ring gives the first digit of the attenuation; the numbers on the inner ring give the second and third digits. For example, if the outer ring indicates 9 and the inner ring indicates 2.3, then it means that the signal is attenuated by a factor of 0.923. At that setting, the fraction of the signal that remains is 0.077. A higher setting produces a smaller signal. The amplified AC single at the attenuator's output can be viewed at the *AC Monitor*, which is the output that you will be using in these experiments. The *Envelope Detector* and *Detector Output* provide a way of viewing the rectified signal (i.e., its amplitude).

Figure 1. A diagram of the Quantum Analogs controller front panel showing various controls, inputs and outputs.

A computer with a sound card will also be used to measure the resonant behavior. The sound card is controlled with a software package called SpectrumSLC, and this software will also be used to display your data.

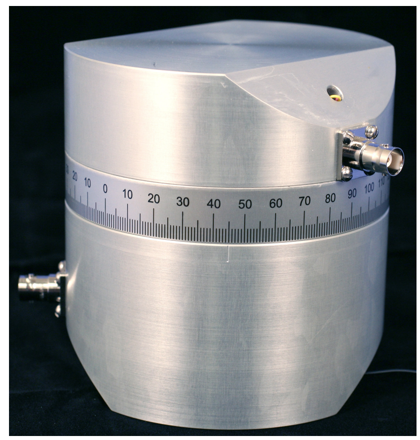


Figure 2. Aluminum spherical cavity showing the microphone and speaker BNC connectors. The scale gives the angle $\boldsymbol{\alpha}$.

The spherical cavity (see Figure 2) is composed of an upper hemisphere equipped with a microphone that is stacked on top of a lower hemisphere equipped with a speaker. The upper hemisphere can rotate with respect to the lower hemisphere to allow you to measure the acoustic pressure at different positions respect to the drive speaker. The angle of rotation between the two hemispheres is indicated on a scale at the rim of the upper half. This angle of rotation α is not to be confused with either θ or φ. The axis of symmetry of the resonant sound wave is determined by the position of the speaker, which is in the wall of the lower hemisphere at a position that is 45° from the vertical rotational axis of the upper hemisphere. The microphone is in the upper hemisphere also at an angle of 45° from the vertical. The angle between the speaker and microphone corresponds to θ in the spherical harmonics. When $\alpha = 180°$, the speaker and microphone are at opposite end of the cavity and $\theta = 180°$. Changing $\alpha = 0°$ moves the microphone to a position that is vertically above the speaker, and $\theta = 90°$. See Figure 3. In general the relationship between θ and α is

$$\cos\theta = \frac{1}{2}\cos\alpha - \frac{1}{2} \quad . \tag{18}$$

A derivation of this expression can be found in the Appendix 2. **It is very important to not damage the cavity. Imperfections in the shape and surface can disrupt the spherical symmetry and change the resonances. Be careful to avoid dropping the cavity or impacts with another object. Do not touch the inside surface. If it is dirty, ask the lab technical staff to clean it.**

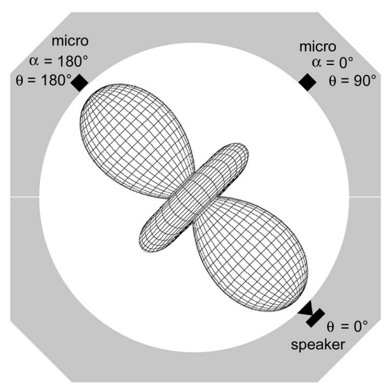


Figure 3. Cut-away of the cavity from the side showing the microphone in the top hemisphere at two angles and the speaker in the bottom hemisphere.

While performing these experiments, it is important that the temperature of the cavity and the air inside remain as stable as possible. Recall that the speed of sound depends on temperature because changes in the sound velocity will affect the resonances of the cavity. Try to avoid warming the cavity by touching it too much.

3. Acoustic Analog of the Hydrogen Atom

3.1 Equipment Setup

Start by using the function generator as a drive for the speaker. Plug the output from the function generator into the *Sine Wave Input* on the Quantum Analogs controller. The speaker from the spherical cavity should then be connected to the controller's *Speaker Output*. To view the cavity's response, connect the microphone on the cavity to the *Microphone Input* on the controller. An oscilloscope can be used to view the amplified output at the *AC Monitor* or the rectified signal at the *Detector Output*. It is also possible to use a sound card in a computer to generate and detect the sound field in the cavity. It this case, the cables from the function generator and oscilloscope are replaced with the speaker and microphone jacks from the sound card, respectively.

3.2 Experiments

3.2.1 Acoustic Resonances of a Spherical Cavity

The objective here is to measure the resonant frequencies of the spherical cavity and observe the angular dependence of the acoustic pressure amplitude at some of these resonances. Begin by connecting the function generator to the *Sine Wave Input* on the controller and Channel 1 of the oscilloscope. Use this signal to trigger the scope. Connect the speaker to the *Speaker Output* and the Microphone to the *Microphone Input* on the controller. Finally the *AC Monitor* should be connected to Channel 2 of the oscilloscope. The amplitude on the signal at the *AC Monitor* can be measured with a DC voltmeter connected to the *Detector Output* on the controller. The attenuator can be used to reduce the signal if necessary.

Set the cavity angle α to 180° (i.e., the speaker and microphone are at opposite ends of the diameter). Starting at a low frequency, sweep the frequency up to ~8 kHz. As the frequency is changed, you will see the amplitude of acoustic pressure change. At certain frequencies it will reach a maximum, which is a resonance.

Determine the frequencies for several resonances. At each resonance, map out the angular dependence of the acoustic pressure. To do this, fine-tune the frequency until it is as close as possible to the resonant peak. Then measure the amplitude as angle between the hemispheres is changed from 180° to 0°. It is important that the frequency remain on the resonant peak during this process.

Your report should include a table of the measured resonant frequencies with uncertainties. Include the angles of the nodes and the azimuthal quantum number l for each resonance. The speaker creates waves with cylindrical symmetry about the speaker axis, so only spherical harmonics with no φ dependence or $m = 0$ are present. In this case, $Y_l^0(\theta, \varphi) \propto P_l^0(\cos\theta)$. (Modes with $m \neq 0$ have nodes at $\theta = 0$, so these modes do not couple strongly to the speaker located on the z-axis.) From the number of nodes, it is possible to determine *l*. Plot the acoustic pressure as a function of θ for several resonances. Compare your data to the Legendre polynomials. Some of the resonances are close enough to each other that their peaks overlap. This will result in a superposition of two eigenfunctions with different quantum numbers. In this case, the measured angular dependence does not match a single Legendre polynomial.

3.2.2 Measuring Spherical Cavity Resonances with the Computer

For the next part of this experiment, a computer with a sound card will be used to generate and measure the resonances in the spherical cavity. The Spectrum SLC software program will be used to control the sound card. This software can also display a spectrum of the response, which is the amplitude of the signal from the microphone as a function of the sound frequency.

Begin by setting the controller's attenuator to its highest value (10). This will protect the sound card from saturating the input. Connect the output of the sound card to the *Sine Wave Input* on the controller and Channel 1 on the oscilloscope. The speaker on the cavity should still be connected to the *Speaker Output* port. With the microphone still connected to the *Microphone Input*, connect the *AC Monitor* on the controller to both the microphone input on the sound card and Channel 2 on the oscilloscope. BNC-to-3.5 mm adapters may be needed for the cables from the sound card. **Never exceed 5 V peak-to-peak on either port of the sound card.** Optimal performance is achieved with a peak-to-peak voltage from 0.5 V to 2 V, depending on the sound card.

Run the program SpectrumSLC.exe. Go to the menu at the top and choose *Configure* > *Input Channel/Volume*, and choose *Line In* if it is available, if not, choose *Microphone*. On this screen, set the microphone volume to the middle of the range. The speaker volume is set using the *Amplitude Output Signal* on the lower half of the screen. This should also be in the middle range.

First take a spectrum over a wide frequency range. Select a portion of the spectrum that includes the highest peak and a smaller one next to it. Readjust the scan to cover just this portion. Set the program to allow successive spectra to be viewed, and for each sweep, reduce the attenuator by a fixed small amount. The signal should increase in a linear fashion. Eventually the computer program will indicate it has reached saturation. (See Appendix 3 for information about saturation and cross talk.) Once saturation is reached, reduce the signal to the linear range. At this point, the signal should be proportional to the amplitude of the sound wave.

With $\alpha = 180°$, measure the spectrum over a large range of frequencies. Start with a course step size of 10 Hz and a short time per step of 50 ms. Change the angle between the upper and lower hemispheres several times, and observe how the spectrum changes. Examine the spectra with $\alpha = 0°$ carefully, especially near the peak at 5 kHz. Scan this range slowly and with sufficiently small steps to resolve the detail. Also measure the spectra at $\alpha = 20°$ and $\alpha = 40°$. Do you understand why the amplitudes vary between the angles?

Finally, you will use the program to measure the angular dependence of the acoustic pressure for a series of resonances, and from these measurements, identify the azimuthal quantum number and spherical harmonic of each resonance. The program displays the amplitude on a polar using the angle θ, which it calculates from the angle α. Start by taking a detailed spectrum covering a range from 2 kHz to 7 kHz with $\alpha = 180°$. If you left-click on a peak, the program automatically configures the sound card to send a signal at that specific frequency. On the menu bar, select *Windows > Measure Wave Function* to enable the polar plot feature. Measure the amplitude from $\alpha = 0°$ to 180° in increments of 10° by clicking *Measure* on the computer after each increment. Use the function *Complete by Symmetry* to complete the figure. Create polar-plots for the prominent resonance peaks. Compare the polar plots to the spherical harmonics.

3.2.3 Cavity Resonances with Broken Symmetry

For this last part of the experiment, the spherical symmetry of the cavity will by broken. As a result some of the resonance peaks split. This is analogue to applying a magnetic field to an atom in what is known as the Zeeman Effect.

In the accessories, you should find two spacer rings, one 3 mm think and another 6 mm think, that can be inserted between the two hemispheres, elongating the cavity. With the system configured as described in Section 3.2.2, measure a spectrum that contains the three lowest resonances on the spherical cavity. Now insert the 3 mm spacer ring and measure the spectrum again on the modified cavity. Repeat this procedure using the 6 mm spacer ring and with both rings to create a 9 mm separation between the hemispheres. Your report should describe in detail what happens. Include a graph showing the peak splitting as a function of the spacer thickness for each peck and discuss the relationship between these quantities.

Recall that each resonance in the spherical cavity with a given l is $(2l+1)$-fold degenerate, and these states are associated with the quantum number $m = -l, -l+1, \ldots, l$. In the spherical cavity, the quantization axis (z-axis) is determined by the speaker's position, and only modes with non-zero amplitude on the z-axis (i.e., $m = 0$) are excited. When the spacer ring is introduced, the spherical symmetry is broken and some of the degeneracy of the eigenstates is lifted. The quantization axis is now defined by the symmetry axis of the cavity, which is vertical. The speaker, which is positioned at a 45° with respect to the vertical axis, can now excite all the states with different quantum numbers m. This is illustrated in Figure 4. Notice that with the shift of the quantization axis, the angle between the two hemispheres α is now equal to the angle φ in the spherical harmonic. With respect to the angle θ, the position of the microphone is fixed at 45°.

The degeneracy of these states is not completely lifted. States with $+m$ and $-m$ are still degenerate. These states have waves circulating around the z-axis of the cavity in the right-handed and left-handed directions with the same amplitude. A superposition of these two waves results in a standing wave with respect to the angle φ with an amplitude that is modulated by

$$A(\varphi) = e^{im\varphi} + e^{-im\varphi} = 2\cos(m\varphi) \quad . \tag{19}$$

In quantum chemistry, this type of superposition is used to form orbitals. For example, when $m = \pm 1$, the orbitals are p_x, p_y, d_{xz}, d_{yz}. When $m = \pm 2$, the orbitals are d_{xy} and $d_{x^2-y^2}$. As long as the perturbation that breaks the symmetry is small, the eigenfunctions are very similar to the spherical harmonics.

Focus on the $l = 1$ resonance. Acquire high-resolution spectra of this resonance using the 3 mm, 6 mm and 9 mm spacer rings. You should be able to resolve the $m = 0$ and $m = \pm 1$ resonant peaks. Next, measure the amplitude as a function of the angle $\varphi = \alpha$ for each resonance. Select the resonant frequency by left-clicking on the desired peak. Then open the polar plot feature by selecting *Windows > Measure Wave Function*, and select the box labeled *Lifted Degeneracy*, which tells the program that the quantization axis in now vertical and to use α for the graphs. In this mode, the data is displayed in green.

With the oscilloscope, you should be able to see how the sign of the signal changes with angle. Set $\alpha = 180°$. For the $m = 0$ resonance, measure the phase of the signal from the microphone in the upper hemisphere. Then connect the cable to the microphone in the lower hemisphere and measure the phase again. Repeat this measure with the $m = \pm 1$ resonance.

If time permits, you may want to explore the $l = 2$ resonances, which will split into three peaks with $m = 0$, $m = \pm 1$ and $m = \pm 2$. With increasing l, the overlap of the resonances for the different m's becomes more of a problem. One way to overcome this is to measure the spectrum at each angle and use a peak fitting routine to determine the amplitude of each resonance.

In your report, identify the magnetic quantum number for each resonant peak, and compare the angular dependence of the measured acoustic pressure to Eq. (19). Describe in detail the phase relationships of the states.

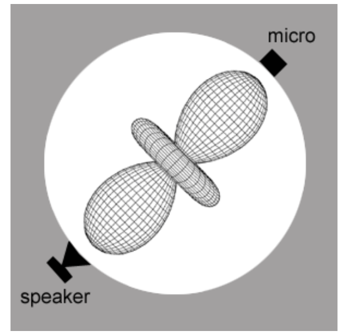
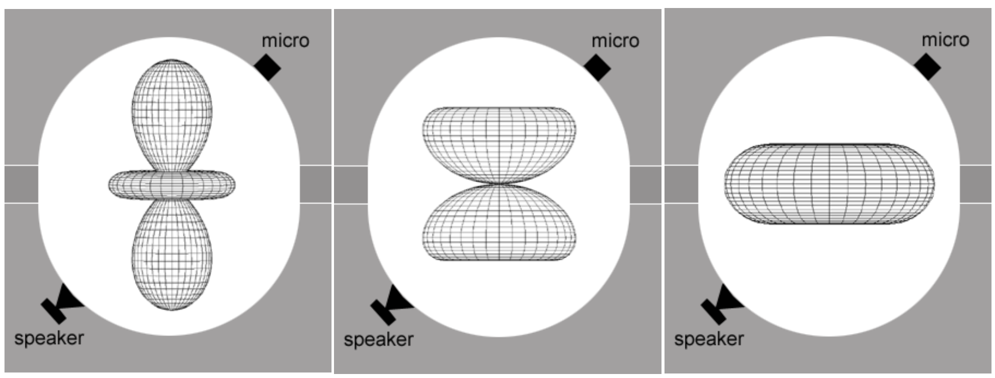


Figure 4a. In the spherical resonator, the quantization axis is determined by the location of the speaker.



$\boldsymbol{m = 0}$ $\boldsymbol{m = 1}$ $\boldsymbol{m = 2}$

Figure 4b. This diagram shows a cut-away from the side for three modes. When the cavity is elongated by the spacer rings, the quantization axis is shifted to the vertical direction. The degeneracy of the states with different m is lifted. Notice that rotation of the upper hemisphere changes the angle φ of the microphone, leaving the angle θ fixed at 45°.

4. References

1. Any textbooks on quantum mechanics should have a discussion of the Schrödinger equation in three dimensions and its solutions for the hydrogen atom. Two examples are Stephen Gasiorowicz, *Quantum Physics* (Wiley, 2003) and David McIntyre, Corinne Manogue and Janet Tate, *Quantum Mechanics* (Pearson, 2013).

2. Two standard textbooks on acoustics are John Kinsler, Austin R. Frey, Alan B. Coopens and James F. Sanders, *Fundamentals of Acoustics*, 4th ed. (Wiley, 1999) and Allan D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications* (Acoustical Society of Amer, 1989).

Appendix 1 – Technical Description of TeachSpin's Quantum Analogs Controller Box

The following chart provides a description of the role of each component on the Controller Box.

Controller Label	Function
Microphone Input	Provides a source of +5 V DC (for biasing of capacitor microphones), and accepts the AC signal placed atop that bias by the microphone.
AC Amplifier	Provides a fixed gain, of order 100, from about 20 Hz to 20 kHz; AC-coupled at the input.
Attenuator	10-turn scale, provides *attenuation* of amplified AC signal, by a factor given by the (dial setting)/10. Example – a dial setting of 9.5 turns implies an *attenuation* of 9.5/10 = .95 or 95%. Only 5 % of the signal is transmitted. NOTE: A higher reading means less signal.
AC Monitor	Provides a direct view of the amplified AC signal at the attenuator's output.
Envelope Detector	A rectifier system, giving the amplitude of the sine-wave signal present at the AC Monitor output, on a cycle-by-cycle basis.
Detector Output	A DC-coupled positive voltage, which is the output of the envelope detector.
Sine Wave Input	Provides the entry point for AC signals from signal generator or computer sound-card.
Speaker Output	Directly coupled to the Sine Wave Input below it on the panel; provides the point of attachment for 3.5-mm speaker plug.
Frequency-to-Voltage Converter	When toggled to On, this module derives a signal from Sine Wave Input and converts its frequency to a voltage, at a conversion ratio of 1 Volt per kHz.
DC Offset	10-turn dial, allowing the addition of a 0 to −10 V offset to the output of the Frequency-to-Voltage converter.
DC Output	The output voltage from the Frequency-to-Voltage converter module.

Appendix 2 – Derivation of the Relationship Between the Angles α and θ

Assume that the speaker is located in the x-z-plane and the vertical axis is the z-axis. The position of the speaker in a sphere with unit-radius is given by the vector $\boldsymbol{s} = \left(\sqrt{1/2}, 0, -\sqrt{1/2}\right)$. The objective is to calculate the angle between speaker and microphone θ. To do this two rotation matrices are used. In the first step, rotate the vector $\boldsymbol{s}$ from the position of the speaker by 90° around the y-axis arriving at the position $\left(\sqrt{1/2}, 0, \sqrt{1/2}\right)$. The next step is to rotate by the angle α around the z-axis to arrive at the position of the microphone. This vector, which is now at the position of the microphone, will be called $\boldsymbol{m}$. To find the angle between the vectors at the speaker and microphone, use the dot product. For the first rotation,

$$\begin{bmatrix} \cos 90° & 0 & -\sin 90° \\ 0 & 1 & 0 \\ \sin 90° & 0 & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} \sqrt{1/2} \\ 0 \\ -\sqrt{1/2} \end{bmatrix} = \begin{bmatrix} \sqrt{1/2} \\ 0 \\ \sqrt{1/2} \end{bmatrix} .$$

For the second rotation,

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{1/2} \\ 0 \\ \sqrt{1/2} \end{bmatrix} = \begin{bmatrix} \sqrt{1/2}\cos\alpha \\ \sqrt{1/2}\sin\alpha \\ \sqrt{1/2} \end{bmatrix} .$$

The final step is to take the dot-product between $\boldsymbol{s}$ and $\boldsymbol{m}$ to find the angle between these vectors.

$$\boldsymbol{m} \cdot \boldsymbol{s} = |\boldsymbol{m}||\boldsymbol{s}| \cos\theta = \cos\theta = \begin{bmatrix} \sqrt{1/2}\cos\alpha \\ \sqrt{1/2}\sin\alpha \\ \sqrt{1/2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{1/2} \\ 0 \\ -\sqrt{1/2} \end{bmatrix} = \tfrac{1}{2}\cos\alpha - \tfrac{1}{2} .$$

The final result, which is the same as Eq. (18), is

$$\boxed{\cos\theta = \tfrac{1}{2}\cos\alpha - \tfrac{1}{2}} .$$

Appendix 3 – Identifying and Correcting Signal Problems

When using a computer sound card for both the source the drives the speaker and the detector of the resulting signal from the microphone, check to make sure that the signals are not saturating. Because the specifications for computer soundcards vary widely, it is difficult to specify a level that will be sufficient.

In the Quantum Analogs computer-based experiments, the amplitude of the microphone signal on the computer screen is an arbitrary measure of the intensity of the sound the microphone is receiving. For these relative measurements to be accurate, the system must be operating in a region where the relationship of the signal to the response is linear. This means that any change the input signal from the microphone must result in proportional changes in the heights of the peaks.

1. Problems with the Microphone Mode of the Sound Card

Whenever possible, the *Line-In* mode should be used. If the sound card does not provide this mode, it may be possible to use the microphone mode. However, there may be problems with saturation and/or distortion if the signal is not adjusted to the optimum range. In the *Microphone* mode, the soundcard sends a DC-voltage to bias the microphone. This DC-voltage can significantly offset the input signal. In addition, the input in the *Microphone* mode is much more sensitive (by a factor of about 100) compared to the *Line-In* mode. The input signal can be saturated at a level of about 10 to 100 mV rms. To avoid saturation, you will need to attenuate the signal much more than in the *Line-In* mode. The attenuator may need to be high as 9.9 turns to prevent saturation.

2. Adjusting Input and Output Levels with the Computer

There are four different places where input and output levels can be controlled.

- The output level of the speaker can be set within Windows in the same way it is do when you listen to music from the computer. You may open *System-Control* > *Sounds and Audio* > *Audio* > *Volume* to adjust the output volume. On some computers, there is an output-volume slider provided for easier control.
- The program SpectrumSLC.exe has a slider in the lower left corner labeled *Amplitude Output Signal*. It determines the amplitude of the sine wave sent to the output by the program. Be aware that this and the previous method are independent ways for adjusting the output level.
- The input level in either the *Line In* mode or the *Microphone mode* can be adjusted with a slider in the program SpectrumSLC.exe under the menu *Configure* > *Input Channel/Volume*.
- The attenuator knob on the Quantum Analogs Controller also determines the level of the microphone signal going to the sound card.

3. Cross-Talk Between Channels on the Sound Card

Depending on the quality of the sound card and the actual settings, there might be a problem with cross-talk between the output and input channels. If some of the speaker output signal is detected by the input channel internally on the sound card, a flat background is created in the measurement, which interferes with the detected signal. This can be a serious problem because the line-shapes of the resonances are modified significantly. See Figures 1-3 for examples of cross-talk. If you detect cross-talk in your experiment, it may help to reduce the amplitude of the speaker output signal. In less expensive sound cards, feedback of the cross-talk increases the interfering signal. In this case, a fixed frequency is observed in the resonator, which does not change when sweeping the frequency. The only way of avoiding this is to use a better soundcard.

4. Saturation

Problems with saturation of the analog-to-digital converter in the sound card are particularly likely when using the *Microphone* mode instead of the recommended *Line-In* mode. To prevent saturation, the input-signal needs to be reduced to an optimum range. In the newest version of the SpectrumSLC.exe software (versions starting with 7.1), there is a blue bar in the lower left corner of the main window, which indicates saturation. If the blue bar blinks while you are passing through the top of a peak, the signal is saturating the analog-to-digital conversion of the sound card. In this case, you need to reduce the level of the signal by increasing the reading on the attenuator knob. Saturation is observed at amplitudes above 100 units on the computer display.

5. Detecting Problems with the Signal

A reliable way to detect problems with the signal is to examine the Fourier-transformation of the input signal from the microphone in real time. It is possible to display this in an extra window with the program by selecting *Menu* > *Windows* > *Live FFT*. An ideal spectrum will have a series of isolated peaks as displayed in Figure A1. If the peaks significantly overlap or are distorted, there is a problem with the signal. See the signals in Figures A2 and A3. The existence of additional peaks at higher harmonics (double or triple the frequency), as shown in Figure A4, indicates distortion or saturation of the signal. Peaks that do not change in frequency when the resonant structure is modified (e.g., the length of the resonant tube is changed) may indicate a problem with cross-talk or that another external signal, either acoustic or electrical, is somehow coupling to the experiment. Additional peaks can also be created by software filters or sound-effect features associated with the sound card. Make sure these features are turned off.

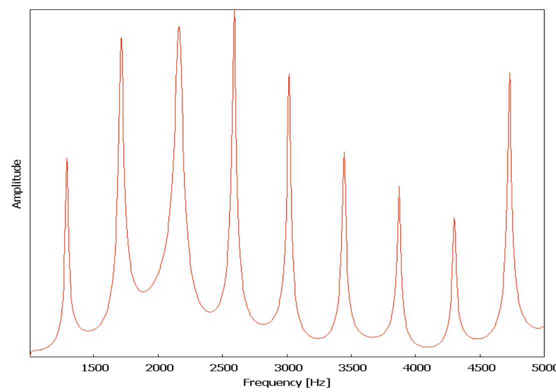


Figure A1. Spectrum without cross-talk.

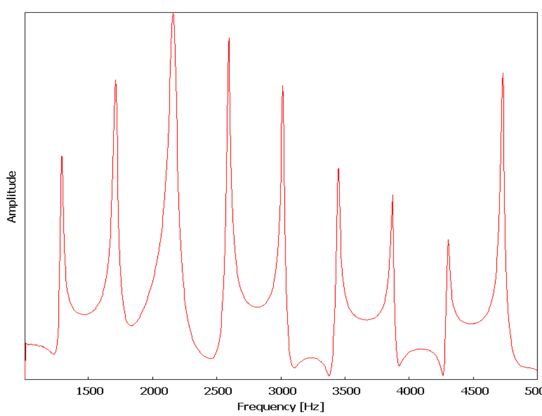


Figure A2. Spectrum with cross-talk.

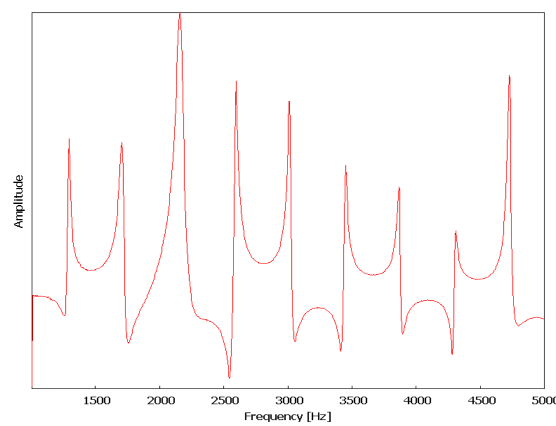


Figure A3. Spectrum with severe cross-talk.

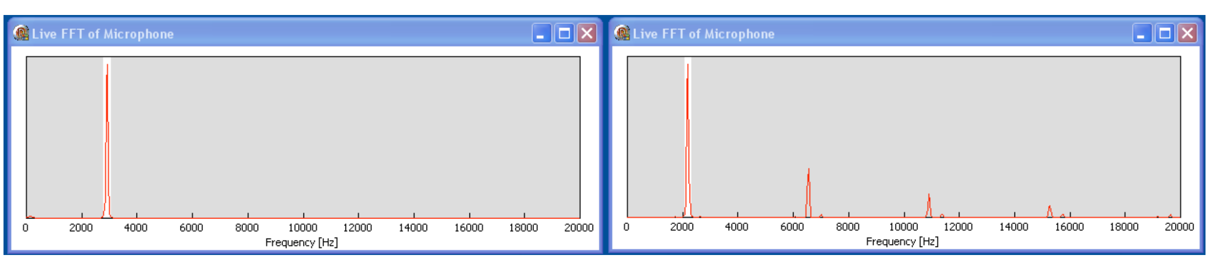


Figure A4. The panel on the left shows a Fourier-transform of a perfect signal – a single peak with a low background. The single in the panel on the right contains higher harmonics due to distortion or saturation.