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The real and apparent convergence of N-body simulations of the dark matter structures: Is the Navarro–Frenk–White profile real?

A.N. Baushev *

DESY, 15738 Zeuthen, Germany
Institut für Physik und Astronomie, Universität Potsdam, 14476 Potsdam-Golm, Germany

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ABSTRACT

While N-body simulations suggest a cuspy profile in the centra of the dark matter halos of galaxies, the majority of astronomical observations favor a relatively soft cored density distribution of these regions. The routine method of testing the convergence of N-body simulations (in particular, the negligibility of two-body scattering effect) is to find the conditions under which formed structures is insensitive to numerical parameters. The results obtained with this approach suggest a surprisingly minor role of the particle collisions: the central density profile remains untouched and close to the Navarro–Frenk–White shape, even if the simulation time significantly exceeds the collisional relaxation time τ_r. In order to check the influence of the unphysical test body collisions we use the Fokker–Planck equation. It turns out that a profile $\rho \propto r^{-\beta}$ where $\beta \simeq 1$ is an attractor: the Fokker–Planck diffusion transforms any reasonable initial distribution into it in a time shorter than τ_r, and then the cuspy profile should survive much longer than τ_r, since the Fokker–Planck diffusion is self-compensated if $\beta \simeq 1$. Thus the purely numerical effect of test body scattering may create a stable NFW-like pseudosolution. Moreover, its stability may be mistaken for the simulation convergence. We present analytical estimations for this potential bias effect and call for numerical tests. For that purpose, we suggest a simple test that can be performed as the simulation progresses and would indicate the magnitude of the collisional influence and the veracity of the simulation results.

1. Introduction

1.1. The 'cusp vs. core' problem

N-body simulations is one of the most prominent methods of investigation of the universe structure formation: the complexness of the task makes any detailed analytical consideration almost hopeless. The rapid growth of computational capability, as well as the algorithm improvement, have made the progress in this field really impressive. Despite of it, there are still difficulties to be solved: the problem of the density profile in the central region of formed dark matter (DM) haloes is probably the most important and intriguing among them. N-body simulations always suggest a very steep, cuspy profile in the center (Navarro–Frenk–White (hereafter NFW) or Einasto with high power index [1–4]). The NFW profile, for instance, has an infinite density in the center

$$\rho = \frac{\rho_s r_s^3}{r(r+r_s)^2} \tag{1}$$

Meanwhile, observations strongly favor a cored profile, i.e. a respectively shallow density distribution at $r \to 0$ [5–10].

The cusp disappearance might be accounted for by the influence of the baryon component. However, observations suggest that the cusp is absent in dwarf satellites of the Local Group galaxies having only a very minor fraction of baryons [11–13]. The conflict between the simulations and the observations calls for additional verification of the simulation accuracy and convergence.

1.2. N-body simulations and relaxation processes

The idea of N-body simulations is to substitute real dark matter particles by heavy test bodies, so that the average density remains the same. It allows to decrease the number of particles and make the task calculable. In order to avoid unphysical close encounters, the Newtonian potential of the bodies is cut on short distances: potential is set to be constant inside some radius ϱ. The initial conditions of the cosmological simulations are usually chosen as

* Address: DESY, 15738 Zeuthen, Germany.
E-mail address: baushev@gmail.com

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a random Gaussian field, so that they model the real initial cosmological perturbations. Then the system freely evolves forming structures, including haloes of various masses. It is important to mention that, though the total number of test particles exceeds $\sim 10^9$ at present, a single formed halo rarely contains more than $\sim 10^6$ simulation particles in this type of simulations.

It was shown in [14,15] that the energy evolution of the system plays the key role in the cusp formation: a core inevitably occurs in the center, if the energy relaxation of the system is moderate; a significant part of the halo particles should change their energies many ($\sim c_{vir}$) times with respect to the initial values, in order to form a cusp. This requirement is rather weak in the case of galaxy clusters with low NFW concentrations ($c_{vir} \simeq 3-6$), but implies the energy change by at least an order of magnitude in the case of galactic haloes ($c_{vir} \simeq 12-20$).

A dark matter halo has several ways of relaxation. First of all, the baryon matter can affect this process, though dwarf galaxies have only a tiny fraction of the baryons and show no cusps [11]. However, even a purely collision-less DM halo may relax very effectively via the so-called violent relaxation [16]. The essence of this mechanism is simple: when the halo collapses, strong density inhomogeneities (caustics etc.) should appear. The inhomogeneities generate a small-scale gravitational field, that is a mediator of the energy exchange between particles. Analytical calculations show that the mechanism can be very effective in the center of the halo. However, the violent relaxation 'works' only during the halo collapse: the formed halo has a stationary gravitational field. Moreover, the efficiency of the violent relaxation rapidly drops with radius [16].

Here we investigate an another, quite unphysical, relaxation mechanism: the pair collisions of the test bodies. In the case of real systems this process is completely absent: the particle mass of real dark matter is so small that their gravitational collisions play no role, as we will see below. On the contrary, the test bodies are quite massive ($\sim 10^{-6}$ of the total halo mass) and may effectively collide redistributing their energy and momentum. In reality, where the dark matter particles carry $<10^{-60}$ halo masses, this process is irrelevant. So, do N-body collisions contribute to cusp formation in N-body simulations?

An exact investigation of the collision influence is a very complex task: it requires careful consideration of the particle velocity and spatial distributions. However, a simple (but quite reliable) and commonly-used method to estimate the collisional relaxation time is to use 'characteristic', averaged values of the particle velocities v and radii r, instead of real distributions [17, Eq. (1.32)]. Substituting the orbital time r/v into this equation, we obtain:

$$\frac{\langle \Delta v \rangle}{\delta t} \simeq 0 \quad \frac{\langle \Delta v^2 \rangle}{\delta t} \simeq \frac{8 v^2 \ln\Lambda}{N(r)} \cdot \frac{v}{r} \tag{2}$$

Here $N(r)$ is the number of particles inside radius r, $\ln\Lambda$ is the Coulomb logarithm. Generally speaking, $\ln\Lambda$ depends on radius. According to [17, chapter 1.2.1], $\ln\Lambda = b_{max}/b_{min}$, where b_{max} and b_{min} are the characteristic maximum and the minimum values of the impact parameter. b_{max} is the maximum radius where the surface density $N(r)/\pi r^2$ of the test bodies can be approximately considered as constant. b_{min} is defined by the radius where either the assumption of a strait-line trajectory breaks, or the newtonian potential is no longer valid. Since $\ln\Lambda$ depends on b_{max} and b_{min} only logarithmically, it is usually enough to make rough estimations of these quantities. For a stellar system [17, chapter 1.2.1] estimated $b_{max} \simeq R_{vir}$ and $b_{min} \simeq b_{90}$, where b_{90} is the 90 degree deflection impact parameter. Then $\Lambda = R_{vir}/b_{90} \simeq R_{vir} v^2/(Gm) \simeq N$, and this is the estimation of Λ applied by Power et al. [18].

In the case of N-body simulations, b_{90} should not be smaller than the smoothing radius ϱ. Second, the surface density rapidly drops at the halo edge. Let us consider the center of NFW profile (1), where $\rho \propto r^{-1}$: then the surface density $N(r)/\pi r^2$ remains constant up to $r = r_s$, and $b_{max} \simeq r_s$. So $\Lambda \sim r_s/\varrho$ in this important instance; Klypin et al. [19] obtained $\Lambda = 3r_s/\varrho$, i.e. Λ in the center of an NFW halo depends almost not at all on radius. The dependence may be more significant for other profiles; however, $\ln\Lambda$ cannot be a strong function of radius in the central region: it depends on b_{max} and b_{min} only logarithmically. Moreover, the gravitational friction force acting on a particle, which is proportional to $\ln\Lambda$, is apparently neither zero nor infinite in the halo center. Consequently, $\ln\Lambda$ is finite, but not equal to 0 at $r = 0$. For reasons of symmetry, $\ln\Lambda$ has an extremum at $r = 0$ and so cannot be a sharp function near this point in the very general case. In the important instance of NFW profile $\ln\Lambda$ is quite constant, as we could see. For a halo of concentration $c_{vir} = 10$ and a typical value $\varrho \simeq 10^{-3} R_{vir}$ we obtain $\ln\Lambda \simeq 6$.

The relaxation time τ_r is roughly defined by the moment when $\Delta v^2 \simeq v^2$, and we obtain from (2)

$$\tau_r = \frac{N(r)}{8 \ln\Lambda} \cdot \frac{r}{v} \tag{3}$$

Since real DM haloes contain $\sim 10^{65}$ particles, the collisional relaxation plays no role there. On the contrary, the number of test bodies in a separate halo is relatively small in simulations, and the unphysical relaxation may be important on the simulation time t_0.

1.3. Convergence

The main purpose of the convergence test is to find the maximum ratio t_0/τ_r whereby the density profile is still not corrupted by the collisions. τ_r rapidly grows with radius: the influence of the collision relaxation may still be negligible on the halo outskirts, but already significant in the center. So we may introduce the convergence radius r_{conv} of a halo at given t_0, so that the collisions are already significant inside r_{conv}, but the simulated density profile is still reliable for $r > r_{conv}$.

Of course, this aspect of the N-body simulation convergence has been explicitly explored; we mention here only a few works of the vast literature. Because of the lack of reliable analytical predictions of dark matter distribution near the halo center, the main method of the tests is to find the conditions under which the structure of simulated halos is independent of numerical parameters. Moore et al. [20] found that $r_{conv} \sim R_{vir}/\sqrt[3]{N_{vir}}$ (R_{vir} and N_{vir} are the virial radius of the halo and the number of the test particles in it), and so r_{conv} should contain thousands test particles. Ghigna et al. [21] gave a similar estimation of the particle number, but emphasized that r_{conv} should exceed the smoothing radius ϱ by no less than a factor of three. On the contrary, Klypin et al. [22] suggested that it could be enough to have ~ 200 simulative particles inside r_{conv} if we properly chose other simulation parameters. Klypin et al. [19] considered the density profiles of subhaloes and found that even ~ 100 test particles can be enough in this case, because of the relatively small number of crossing-times of dark matter in the subhaloes. Diemand et al. [23] argued for the opposite: the two-body collisions in the subhaloes can be so effective that it affects the whole hierarchical structure formation.

However, Power et al. [18] remains the fundamental investigation of the N-body convergence; the criterion offered by this paper is routinely used in modern simulations to determine r_{conv} (see, for instance, Navarro etal. [4]). In order to determine r_{conv}, Power et al. [18] considered the time dependence of the overdensity at some radius r (i.e., the ratio of the average density inside radius r from the halo center to the average universe density). It was found that a cuspy profile (close to $\rho \propto r^{-1}$) forms fairly rapidly in the center, and then the density contrast remains almost constant up to, at least, $t_0 \simeq 1.7\tau_r$ ([18] used relaxation time, based on (3)). So the

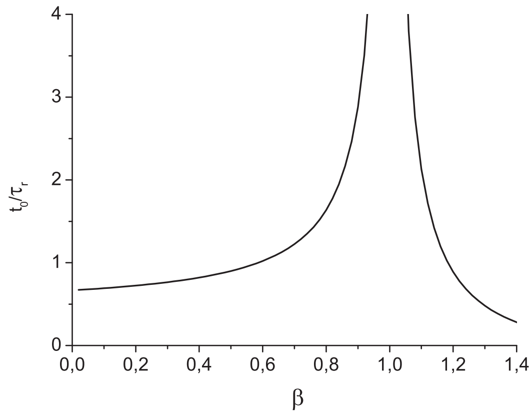

Fig. 1. The time t_0 the central power-law profile $\rho \propto r^{-\beta}$ takes to be significantly changed by Fokker–Planck diffusion (in units of relaxation time τ_r, see Eq. (15)). As we can see, if β significantly differs from 1, the profile evolves towards $\rho \propto r^{-1}$ in a time shorter than τ_r. On the contrary, profile $\rho \propto r^{-1}$ survives for a much longer time interval ($t_0 \gg \tau_r$).

criterion of r_{conv} offered by Power et al. [18] is $\tau_r \geqslant 0.6t_0$. Such a large ratio ($t_0/\tau_r \simeq 2$) seems surprisingly high: actually, it means that test particle collisions have little effect on the simulations even at a time interval exceeding the relaxation time. However, it was found in further convergence tests that the criterion might be even softer: the NFW-like profile in the center survives almost unchanged many times longer than the analytical estimations of τ_r [24,19]. If one accepts stability of the profile as a proof of convergence, it leads to a very optimistic appraisal of the central resolution of N-body simulations.

The aim of this paper is to show a potential pitfall of the above-mentioned numerical methods of the convergence tests. The very fact of the simulation results converging to some 'stationary' (or quasi-stationary) solution does not guarantee that the results have converged to a physical solution. Indeed, we will show that test body collisions may result in the persistent occurrence of the steep NFW-like central profile in numerous N-body simulations. If this is confirmed to be true, the above-mentioned criteria of convergence (like $\tau_r \geqslant 0.6t_0$), routinely used in literature, would significantly overestimate the central resolution of N-body simulations.

2. Calculations

2.1. The momentum distribution of the test bodies

Let us consider test body collisions in the center of formed halo. We suppose a power-law density distribution in the center $\rho \sim r^{-\beta}$ (it is important that $\beta < 2$). Then the number of the bodies enclosed inside radius r and the halo mass depend on the radius as:

$$N(r) = N_0\left(\frac{r}{r_b}\right)^{3-\beta} \quad M(r) = \mu N_0\left(\frac{r}{r_b}\right)^{3-\beta} \tag{4}$$

where μ is the test body mass, r_b is the radius where the central profile $\rho \sim r^{-\beta}$ breaks; $r_b = r_s$ in the case of NFW profile. The gravitational potential is

$$\phi(r) = G\frac{\mu N_0}{(2-\beta)r_b}\left(\frac{r}{r_b}\right)^{2-\beta} = \frac{\Phi^2}{(2-\beta)}\left(\frac{r}{r_b}\right)^{2-\beta} \tag{5}$$

Assuming $\phi(0) = 0$ and designating $\Phi = \sqrt{\frac{G\mu N_0}{r_b}}$. We follow the simplifying method applied by Binney and Tremaine [17] to obtain Eq. (2): instead of exact distribution functions, we will use characteristic, averaged over time values of particle radius r, velocity v and momentum $p = \mu v$. In the framework of this approach the task becomes one-dimensional: the system state is totally defined by a single function $N(r)$ depending on the only coordinate (r, for instance). Of course, the halo center is virialised. The average kinetic energy of a particle is $\epsilon_k = p^2/(2\mu)$, the average potential one $\epsilon_p = \mu\phi(r)$. Potential (5) is a power-law of index $(2-\beta)$: according to the virial theorem (see Appendix A), ϵ_k and ϵ_p of a particle being in finite motion in this field are bound

$$\epsilon_k = \frac{2-\beta}{2}\epsilon_p \tag{6}$$

So

$$\frac{p^2}{2\mu} = \frac{2-\beta}{2}\frac{\mu\Phi^2}{(2-\beta)}\left(\frac{r}{r_b}\right)^{2-\beta}$$

$$p = \mu\Phi\left(\frac{r}{r_b}\right)^{1-\frac{\beta}{2}} \qquad r = r_b\left(\frac{p}{\mu\Phi}\right)^{\frac{2}{2-\beta}} \tag{7}$$

The last equation defines the one-to-one relation between the characteristic values of r and p, and we may use p instead of r. In particular, $v = p/\mu$. Considering all particles contained within radius r, their momentum distribution can be calculated as

$$N(p) = N_0\left(\frac{p}{\mu\Phi}\right)^{\frac{6-2\beta}{2-\beta}} \tag{8}$$

$$n(p) \equiv \frac{dN(p)}{dp} = \frac{(6-2\beta)}{(2-\beta)}\frac{N_0}{\mu\Phi}\left(\frac{p}{\mu\Phi}\right)^{\frac{4-\beta}{2-\beta}} \tag{9}$$

2.2. The effect of collisions on the momentum distribution

The collisions lead to a sort of diffusion of the test bodies in the phase space. Since the close encounters are excluded by the potential smoothing, the evolution of distribution function $n(p)$ can be described by the Fokker–Planck equation:

$$\frac{\partial n(p)}{\partial t} = \frac{\partial}{\partial p}\left\{\tilde{A}n(p) + \frac{\partial}{\partial p}[Bn(p)]\right\} \tag{10}$$

where

$$\tilde{A} = \frac{\langle\Delta p\rangle}{\delta t} = \mu\frac{\langle\Delta v\rangle}{\delta t} \qquad B = \frac{\langle\Delta p^2\rangle}{2\delta t} = \frac{\mu^2}{2}\frac{\langle\Delta v^2\rangle}{\delta t} \tag{11}$$

Substituting here Eqs. (2), we obtain $\tilde{A} = 0$ and

$$B = \frac{4p^3\ln\Lambda}{r\mu N} = \frac{4\mu^2\Phi^3\ln\Lambda}{r_b N_0}\left(\frac{p}{\mu\Phi}\right)^{-\frac{2+\beta}{2-\beta}} \tag{12}$$

The diffusion flux of the particles is [25]

$$s = -\tilde{A}n(p) - \frac{\partial}{\partial p}[Bn(p)] \tag{13}$$

Since our task is one-dimensional, the number of particles crossing the surface $p = const$ (or, what is the same in view of one-to-one relation (7), $r = const$) in a unit time is $dN(r)/dt = -s$. Substituting Eq. (12) for B and Eq. (9) for $n(p)$ into (13), we obtain

$$-s = \frac{16(3-\beta)(1-\beta)}{(2-\beta)^2}\frac{\Phi\ln\Lambda}{r_b}\left(\frac{p}{\mu\Phi}\right)^{-\frac{\beta}{2-\beta}} \tag{14}$$

In order to estimate the timescale τ_{col} influence of collisional diffusion on the density profile, we should consider the ratio of the number of particles diffused through radius r in simulation time t_0 to the particle number $N(r)$ inside this radius, i.e. $\tau_{col} = N(r)\Big/\left|\frac{dN(r)}{dt}\right|$. Substituting here $\tau_r(r)$ instead of $N(r)$ in accordance with (3), we obtain from (14)

$$\tau_{col} = \frac{(2-\beta)^2}{2(3-\beta)|1-\beta|}\tau_r \tag{15}$$

It is remarkable that the typical evolution time in units of τ_r depends only on β, and not on radius.

Obviously, $\tau_{col} \gg \tau_r$ for $\beta \simeq 1$, implying that the system evolves very slowly on account of collisions, despite being collisional, if the profile is approximately $\rho \propto r^{-1}$. This result suggests that stability of a r^{-1} profile does not guarantee the absence of collisional effects. In fact, collisions might provide a stable r^{-1} density profile. Moreover, profile $\rho \propto r^{-1}$ is a sort of attractor: we may see from (14) that the diffusion flux changes its sign at $\beta = 1$. If the profile is shallower than $\rho \propto r^{-1}$, the flux is pointed towards the center and tends to sharpen it, and vice versa, the flux shallows a profile with $\beta > 1$. Fig. 1 illustrates the characteristic time necessary for the central part of the system to converge towards the $\rho \propto r^{-1}$ profile. As we can see, if β significantly differs from 1, the profile evolves in a time shorter than τ_r. On the contrary, profile $\rho \propto r^{-1}$ may survive for a much longer time interval ($t_0 \gg \tau_r$).

3. Discussion

Thus we have come to the main conclusion of this paper: the very fact of the simulation results converging to some 'stationary' (or quasi-stationary) solution does not guarantee that the results have converged to the physical distribution. Numerical effects may also create some stable pseudosolutions.

3.1. The immanence of a NFW-like attractor solution

Of course, our model was rather crude: we considered β as a constant: so we can only ascertain the characteristic time, in which the profile will significantly change, and the direction of evolution, being unable to follow the evolution in time. Moreover, we used 'characteristic', averaged values of the particle velocities v and radii r, instead of real distributions; our analysis certainly fails if $r \lesssim \varrho$. This simplified approach allowed us to follow exactly the collision estimation method utilized by Power et al. [18] and to make the calculations short and obvious.

However, the presence of a NFW-like attractor solution ($\rho \propto r^{-\beta}$, where $\beta \simeq 1$) in the halo center seems to be an immanent result of the particle collisions, independent on the simplifications we made. Indeed, Spitzer [26] and Quinlan [27] consider a spherical star cluster with the distribution function depending only on energy and time (so the velocity distribution is assumed to be isotropic). The diffusion coefficients of the Fokker–Planck equation are evaluated in the local approximation. Though the equations are much more complex in this case, an attractor solution, very similar to that obtained in this paper, occurs as well [28]. The only difference is that the power index is $\beta = 4/3$ instead of NFW's value $\beta = 1$. However, the discrepancy does not seem very significant. First, the convergency radius r_{conv} typically contains hundreds or thousands of test particles in the case of N-body simulations, and so the statistical noise is rather high and does not allow to reconstruct the density profile and determine β at $r \sim r_{conv}$ very precisely. Second, the model [26,27] is also not precise: it calculates the diffusion coefficients in the local approximation (which is hardly suitable for an N-body halo at $r \sim r_{conv}$ with only a moderate number of particles inside) and disregards the potential smoothing. Finally, there is an effect that shallows the attractor profile, making it even closer to NFW.

3.2. The cusp disappearance at $t \sim 50\tau_r$

As was emphasized in [28], condition $s = 0$ (i.e., the cessation of the Fokker–Planck diffusion) means the dynamical, but not thermodynamical equilibrium of the system. The full Boltzmann equation has the only stationary solution — the isothermal sphere. The Fokker–Planck equation is obtained by expanding the full Boltzmann collision integral into Taylor's series and dropping out all the terms beyond second order [25]. Therefore a stationary solution of the Fokker–Planck equation is not obligatory stationary with respect to the full collision integral. Indeed, the particle velocity dispersion (and so the 'temperature') drops towards the halo center for any density profile softer than $\rho \propto r^{-2}$ (see, for instance, the first part of Eq. (7)). So the NFW-like solutions, like those considered in this paper or in [28], have a nonuniform temperature distribution, and the particle collisions should still redistribute energy, leading to additional transport processes. However, the fact that we consider a stationary solution of the Fokker–Planck equation results in an important consequence: the main terms of the collision integral are self-compensated in this case. So the transport processes are possible, but they are caused by the higher-order terms of the collision integral, which are notably smaller. The attractor solutions should survive for much longer than any other central distribution, i.e. for much longer than τ_r. However, they are finally smeared out by the kinetic terms of higher orders. Since the process is caused by the collisions and by the nonuniform distribution of the system 'temperature', this effect may be considered as an 'additional thermal conductivity'.

We can estimate the lifetime of the NFW-like attractor profiles. According to [28, Eq. (22)], the heat flux $Q = dE/dt$ towards the center in the case of the attractor solution is

$$Q = \frac{25\pi}{768\sqrt{2}} \log\Lambda \frac{\mu\phi^{3/2}(r)}{r} \tag{16}$$

In the case under consideration $\beta = 4/3$. With the help of (6), we can estimate $\phi(r) \simeq \frac{2}{2-\beta}\frac{v^2}{2} = \frac{3}{2}v^2$ and the total kinetic energy of the particles inside radius $r : E_{tot}(r) \simeq N(r)\frac{\mu v^2}{2}$. Substituting these estimations and Eq. (3) for τ_r into (16), we obtain:

$$Q = \frac{25\pi\sqrt{3}}{8192}\frac{E_{tot}(r)}{\tau_r} \tag{17}$$

The ratio $\tau_{th} = E_{tot}/Q$ gives the time scale τ_{th} on which the 'additional heat flux' significantly affects the system. The 'heat' relaxation time is much larger than the collisional one, $\tau_{th} \simeq 60\tau_r$, and this ratio does not depend on the radius. As we can see, the attractor solutions survive much longer then τ_r. However, they are eventually smeared out by the higher order terms of the kinetic equation at tens of relaxation times.

Indeed, Hayashi et al. [24], Klypin et al. [19] report that the NFW-like cusp smears out, when t_0 reaches several tens of τ_r. They considered this effect as the first sign of the influence of the test body collisions. However, our calculations suggest that collisions become important much earlier, at $t_0 \lesssim \tau_r$: it may, in fact, be the collisions that form the cusp itself. The smearing of the cusp at $t_0 \sim 50$ may be a result of the 'additional thermal conductivity': the characteristic time and the sense of the process coincide. The additional heat flux, being also a result of the test body collisions, is much slower than the Fokker–Planck diffusion and becomes important approximately 50 times later.

Thus the property that the particle encounters form a cuspy, NFW-like profile in the center of the halo in a time interval $\sim \tau_r$ seems to be quite general and model-independent. Since this is an attractor solution, the profile will survive for $t_0 \gg \tau_r$. However, considering that the process is driven by the unphysical test-body collisions, it would be an artefact of the simulations. The NFW profile may still be valid. If the halo concentration is small ($c_{vir} \sim 3$), $\rho \propto r^{-1}$ profile occurs at large radii, where certainly $t_0 \ll \tau_r$, and the collisional relaxation has nothing to do with it. Perhaps, the violent relaxation (as we could see, a strong energy relaxation is mandatory to form the cusp) may form the same profile even closer to the center, where $t_0 \geqslant \tau_r$. However, the simulation results require further verification if $t_0 \simeq \tau_r$: the Fokker–Planck diffusion may produce the same profile as well. Since there is no obvious way to distinguish these two processes, the criterion $t_0 \ll \tau_r$ seems to be the only reliable condition of the negligibility of the unphysical numerical effects, while criterion $\tau_r \geqslant 0.6t_0$ [18] significantly overestimates the resolution of N-body simulations in the halo center.

3.3. How to test the veracity of the simulation results?

Our suggestions could eliminate the 'cusp vs. core' problem, which actually appears mainly on the scale of galaxies and dwarf satellites. First, N-body simulations results come into conflict with the observational data only if soft criteria like $\tau_r \geqslant 0.6t_0$ are true. As we saw, we probably should use a more conservative criterion $t_0 \ll \tau_r$, which increases several times the convergence radius. Then the numerical resolution can be just too low to resolve cores in the centers of the galaxy haloes, that are typically a few percent of the virial radius.

Second, a supposition of the moderate energy relaxation of galactic haloes inevitably leads to the central profile that fits observational data much better than the cuspy profiles suggested by N-body simulations: the profile has a core, an extensive region with $\rho \propto r^{-2}$, and the product of the central density and the core radius is almost independent of the halo mass [15]. On the contrary, N-body simulations suggest a very strong energy relaxation of the system [1,3]. However, we could see that the residual test body collisions might boost the energy exchange in the halo center, leading to disagreement with the observations.

Thus the criteria based on the stability of the central profile, like [18], seem to be questionable. The absence of the collisional relaxation is guaranteed only if $t_0 \ll \tau_r$. A question appears: how can we quantitatively estimate the influence of the collisions and optimize the limit for t_0/τ_r? The most direct way is to consider the evolution of the total particle energy ϵ depending on their average radius and time. Indeed, violent relaxation is effective only during the short interval of halo collapse [16], after which the gravitational field of the halo becomes stationary, and the total energy of each particle remains more or less constant. There are several factors that can yet affect the particle energy distribution of the formed halo.

I. An already formed halo may gravitationally capture more substance from the surroundings [29,30]. The accretion of the additional mass can lead to the 'secondary violent relaxation'.
II. Tidal influence of the nearby halos.
III. The substructures.
IV. A slow evolution of the formed halo, like its concentration growth.
V. Unphysical test body collisions.

The first four items correspond to quite physical phenomena, while the fifth one is purely numerical and should be avoided. As we could see, the collisions appear to be the most important in the halo center, so we should focus our attention on this area. The central region contains few if any subhaloes, as they are typically tidally destroyed. The tidal influence of the nearby halos is minor because of the smallness of the region comparing to the halo size. We can completely avoid the influence of late accretion (item I) and the tidal influences by considering an isolated halo. Moreover, even in the case of more sophisticated simulations containing many haloes, the fraction of matter accreted to the central area after the main relaxation and the presence of subhaloes are easily measurable. Thus we may estimate the influence of factors I–III.

In the end, we may wish to discriminate between the physical process IV and the unphysical process V. In the following we propose an independent test of the absence or presence of significant test-body collisions in numerical simulations, that can be performed in real time as the simulations progress. First of all, we should determine the total energy ϵ of each particle at the moment, when the halo has just been formed, and its gravitational field becomes stationary. A marker of this moment could be the attenuation of initial density waves and caustics, i.e. the absence of rapid oscillations of the density profile and the gravitational potential at the halo center. Then we can extract from the particle distribution over the fractional variation $\Delta\epsilon/\epsilon$ of their total energies as a function of time and the average radius of particles. The effects of processes IV and V on the energy evolution differ drastically. Process IV leads to a slow regular shift of the particle energies. The energies of the bodies with similar orbits evolve alike. On the contrary, process V leads to stochastic variations of the particle energy. The relative energy change $\Delta\epsilon/\epsilon$ should have a near-Gaussian distribution with dispersion σ roughly proportional to $\sqrt{t}$, i.e. $\sigma \simeq \sqrt{t/\tau_r} \propto \sqrt{t/N(r)}$. The energy evolution driven by the unphysical test body collisions is stochastic and can be distinguished from the physical processes like (I–IV). A more sophisticated method to investigate the influence of the collisions is to consider the adiabatic invariants associated with the particles. Indeed, the quantities $\oint p_i dq_i$ (where q_i and p_i are a generalized coordinate and the corresponding generalized momentum of the particle) are conserved, if the gravitational field changes slowly [31]. On the contrary, particle collisions should lead to accidental variations of the adiabatic invariants.

One can offer several practical ways to test the influence of the unphysical particle collisions in real N-body simulations. Any method has its strong and weak points. We can consider standard cosmological simulations containing many halos and choose a halo of interest. Then we need to find a moment t_{in} when the halo density distribution becomes almost stationary. Even the most massive halos are quite formed at $z = 1$ in the ΛCDM cosmological model. We remind that $z = 1$ corresponds to ~40% of the present Universe age, that is, a halo has a quasi-stationary density profile and gravitation field during the major part of its physical age. Then we need to consider the energy evolution of the test bodies $|\epsilon(t) - \epsilon(t_{in})/\epsilon(t_{in})|$, as this was described above. A slow physical evolution of the halo, such as the concentration growth, leads to a regular energy change similar for all the particles that can be easily estimated by the gravitational potential change. A random walk of the particle energies undoubtedly reveals the influence of the unphysical collisions. A possible difficulty of this method is the fact that halos in cosmological simulations are never spherically symmetric. This significantly complicates the task making it three-dimensional. In particular, one needs to perform accurate calculations of the gravitational potential going beyond the commonly-used spherical approximation.

The second way to estimate the influence of the collisions is a simulating of one of the well-known analytical models of s stationary isolated halo, such as Plummer, Hernquist, or Osipkov distributions [17]. The particle velocity distribution and the density profile are time-independent in this case. There should be no relaxation, since the gravitational field is stationary. Therefore, any evolution of the distribution function (in particular, the cusp formation) would be a clear sign of numerical effects.

Both above-mentioned testing methods are only applicable for a formed halo and unsuitable for the halo collapse consideration. However, test body collisions may cause undesirable numerical effects during the collapse as well. Indeed, in the well-known case of Tolman collapse (the initial perturbation is spherically symmetric and uniform $\rho = const$, the matter has no angular momentum [32]) all the particles reach the center simultaneously, and the halo density becomes infinite. In the instance of real initial perturbation, ρ grows towards the center and particle angular momentum is not exactly zero. However, in the very general case the initial density contrast is low $\delta\rho/\rho \ll 1$. The initial angular momentum is zero [33], though it can be gained later by the tidal interaction. Therefore, the collapse dynamics is similar in the early phase to the Tolman case, that is, any real halo passes through a stage of high compression at the beginning of the collapse. Though the stage is fairly short, the test particle collisions may significantly redistribute the particle energy, since the collision rate is proportional to ρ^2. Thus the collisions of test bodies make an unphysical contribution to the violent relaxation in simulations. Moreover, a halo during the violent relaxation has a complex structure of the density and gravitational field. The ability of simulations to model adequately the fine structure of numerous caustics and other inhomogeneities by a fairly small number of test bodies is questionable, while the efficiency of relaxation strongly depends on this. Thus the problem of the reliability limits of the N-body simulations needs further consideration. The profile stability criterion along is insufficient.

The convergence criteria obtained with the help of a direct and detailed consideration of the energy evolution of the system should be significantly more reliable than those based on the density profile stability. The stationary solutions obtained in this paper or by Evans and Collett [28] serve as a good illustration. The attractor density profiles are quite stable even if $t \gg \tau_r$, though the unphysical collisions of the test bodies form them. Despite of the profile stability, the energies of the particles exhibit accidental variations, revealing the collision influence. Eqs. (2) and (3) show that $\Delta\epsilon/\epsilon \simeq 1$ when $t = \tau_r$, i.e. the random energy fluctuations are fully visible. Thus the insensitivity of the density profile to the simulation parameters is not a sufficient criterion of convergence per se, and a more detailed consideration of the phase evolution of test bodies is necessary to judge the simulation reliability. We have proposed a simple test that can be performed in any simulation code.

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Appendix A. The virial theorem for the case of a single particle moving in a potential field $\phi(\boldsymbol{r})$

Let us denote by x_i and p_i the three components of the particle radius and momentum. Then (we use the Einstein summation convention: when an index variable appears twice in a single term it implies summation of that term over all the values of the index.)

$$\frac{d(p_i x_i)}{dt} = x_i \frac{dp_i}{dt} + p_i \frac{dx_i}{dt} = -m x_i \frac{d\phi}{dx_i} + 2T \tag{A.1}$$

Here we took into account that $\frac{dp_i}{dt} = F_i = -m\frac{d\phi}{dx_i}$ and $p_i \frac{dx_i}{dt} = p_i v_i = 2T$, where T is the particle kinetic energy. We may introduce the time averaging of a quantity $A : \bar{A} \equiv \frac{1}{t}\int_0^t A dt$ and apply it to Eq. (A.1).

$$\frac{(p_i x_i)_t - (p_i x_i)_0}{t} = -m\overline{\left(x_i \frac{d\phi}{dx_i}\right)} + 2\overline{T} \tag{A.2}$$

If the particle motion is finite, $(p_i x_i)$ is also finite, while t can be arbitrarily large. So, the left part of the equation approaches zero for $t \to \infty$. Eq. (A.2) may be significantly simplified, if ϕ is a homogeneous function, i.e., $\phi(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^a \phi(x_1, x_2, x_3)$. Then $x_i \frac{d\phi}{dx_i} = a\phi$. Since $m\phi$ is the potential energy of the particle Π, we obtain

$$\frac{a}{2}\,\overline{\Pi} = \overline{T} \tag{A.3}$$

In particular, if $\phi(r) \propto r^{(2-\beta)}$, the potential is a homogeneous function with $a = 2 - \beta$.

References

[1] B. Moore, T. Quinn, F. Governato, J. Stadel, G. Lake, Cold collapse and the core catastrophe, MNRAS 310 (1999) 1147–1152, http://dx.doi.org/10.1046/j.1365-8711.1999.03039.x.
[2] A.F. Neto, L. Gao, P. Bett, S. Cole, J.F. Navarro, C.S. Frenk, S.D.M. White, V. Springel, A. Jenkins, The statistics of Λ CDM halo concentrations, MNRAS 381 (2007) 1450–1462, http://dx.doi.org/10.1111/j.1365-2966.2007.12381.x.
[3] J. Stadel, D. Potter, B. Moore, J. Diemand, P. Madau, M. Zemp, M. Kuhlen, V. Quilis, Quantifying the heart of darkness with GHALO – a multibillion particle simulation of a galactic halo, MNRAS 398 (2009) L21–L25, http://dx.doi.org/10.1111/j.1745-3933.2009.00699.x.
[4] J.F. Navarro, A. Ludlow, V. Springel, J. Wang, M. Vogelsberger, S.D.M. White, A. Jenkins, C.S. Frenk, A. Helmi, The diversity and similarity of simulated cold dark matter haloes, MNRAS 402 (2010) 21–34, http://dx.doi.org/10.1111/j.1365-2966.2009.15878.x.
[5] W.J.G. de Blok, S.S. McGaugh, V.C. Rubin, High-resolution rotation curves of low surface brightness galaxies. II. Mass models, AJ 122 (2001) 2396–2427, http://dx.doi.org/10.1086/323450.
[6] W.J.G. de Blok, A. Bosma, High-resolution rotation curves of low surface brightness galaxies, A&A 385 (2002) 816–846, http://dx.doi.org/10.1051/0004-6361:20020080.
[7] D. Marchesini, E. D'Onghia, G. Chincarini, C. Firmani, P. Conconi, E. Molinari, A. Zacchei, Hα rotation curves: the soft core question, ApJ 575 (2002) 801–813, http://dx.doi.org/10.1086/341475.
[8] G. Gentile, P. Salucci, U. Klein, G.L. Granato, NGC 3741: the dark halo profile from the most extended rotation curve, MNRAS 375 (2007) 199–212, http://dx.doi.org/10.1111/j.1365-2966.2006.11283.x.
[9] A.N. Baushev, Principal properties of the velocity distribution of dark matter particles on the outskirts of the Solar system, MNRAS 417 (2011) L83–L87, http://dx.doi.org/10.1111/j.1745-3933.2011.01125.x.
[10] L. Chemin, W.J.G. de Blok, G.A. Mamon, Improved modeling of the mass distribution of disk galaxies by the Einasto halo model, AJ 142 (109), http://dx.doi.org/10.1088/0004-6256/142/4/109.
[11] S.-H. Oh, W.J.G. de Blok, E. Brinks, F. Walter, R.C. Kennicutt, Jr., Dark and luminous matter in things dwarf galaxies, AJ 141 (193), http://dx.doi.org/10.1088/0004-6256/141/6/193.
[12] F. Governato, A. Zolotov, A. Pontzen, C. Christensen, S.H. Oh, A.M. Brooks, T. Quinn, S. Shen, J. Wadsley, Cuspy no more: how outflows affect the central dark matter and baryon distribution in Λ cold dark matter galaxies, MNRAS 422 (2012) 1231–1240, http://dx.doi.org/10.1111/j.1365-2966.2012.20696.x.
[13] E.J. Tollerud, R.L. Beaton, M.C. Geha, J.S. Bullock, P. Guhathakurta, J.S. Kalirai, S.R. Majewski, E.N. Kirby, K.M. Gilbert, B. Yniguez, R.J. Patterson, J.C. Ostheimer, J. Cooke, C.E. Dorman, A. Choudhury, M.C. Cooper, The SPLASH survey: spectroscopy of 15 M31 dwarf spheroidal satellite galaxies, ApJ 752 (45), http://dx.doi.org/10.1088/0004-637X/752/1/45.
[14] A.N. Baushev, Galaxy halo formation in the absence of violent relaxation and a universal density profile of the halo center, ApJ 786 (65), http://dx.doi.org/10.1088/0004-637X/786/1/65.
[15] A.N. Baushev, Relaxation of dark matter halos: how to match observational data?, A&A, Available from: <ArXiv:1309.5162>, in press.
[16] D. Lynden-Bell, Statistical mechanics of violent relaxation in stellar systems, MNRAS 136 (1967) 101.
[17] J. Binney, S. Tremaine, Galactic Dynamics, second ed., Princeton University Press, 2008.
[18] C. Power, J.F. Navarro, A. Jenkins, C.S. Frenk, S.D.M. White, V. Springel, J. Stadel, T. Quinn, The inner structure of ΛCDM haloes – I. A numerical convergence study, MNRAS 338 (2003) 14–34, http://dx.doi.org/10.1046/j.1365-8711.2003.05925.x.
[19] A. Klypin, F. Prada, G. Yepes, S. Hess, S. Gottlober, Halo abundance matching: accuracy and conditions for numerical convergence, ArXiv e-prints.
[20] B. Moore, F. Governato, T. Quinn, J. Stadel, G. Lake, Resolving the structure of cold dark matter halos, ApJL 499 (1998) L5, http://dx.doi.org/10.1086/311333.
[21] S. Ghigna, B. Moore, F. Governato, G. Lake, T. Quinn, J. Stadel, Density profiles and substructure of dark matter halos: converging results at ultra-high numerical resolution, ApJ 544 (2000) 616–628, http://dx.doi.org/10.1086/317221.
[22] A. Klypin, A.V. Kravtsov, J.S. Bullock, J.R. Primack, Resolving the structure of cold dark matter halos, ApJ 554 (2001) 903–915, http://dx.doi.org/10.1086/321400.
[23] J. Diemand, B. Moore, J. Stadel, S. Kazantzidis, Two-body relaxation in cold dark matter simulations, MNRAS 348 (2004) 977–986, http://dx.doi.org/10.1111/j.1365-2966.2004.07424.x.
[24] E. Hayashi, J.F. Navarro, J.E. Taylor, J. Stadel, T. Quinn, The structural evolution of substructure, ApJ 584 (2003) 541–558, http://dx.doi.org/10.1086/345788.
[25] L.D. Landau, E.M. Lifshitz, Statistical physics. Pt.1, Pt.2, 1980.
[26] L. Spitzer, Dynamical evolution of globular clusters, 1987.
[27] G.D. Quinlan, The time-scale for core collapse in spherical star clusters, New Astron. 1 (1996) 255–270. doi: 10.1016/S1384-1076(96)00018-8..
[28] N.W. Evans, J.L. Collett, On the origin of cusps in stellar systems, ApJL 480 (1997) L103, http://dx.doi.org/10.1086/310629.

[29] J. Wang, C.S. Frenk, J.F. Navarro, L. Gao, T. Sawala, The missing massive satellites of the Milky Way, MNRAS 424 (2012) 2715–2721, http://dx.doi.org/10.1111/j.1365-2966.2012.21357.x.
[30] A.N. Baushev, Extragalactic dark matter and direct detection experiments, ApJ 771 (117), http://dx.doi.org/10.1088/0004-637X/771/2/117.
[31] L.D. Landau, E.M. Lifshitz, Mechanics (1969).
[32] L.D. Landau, E.M. Lifshitz, The classical theory of fields, 1975.
[33] D.S. Gorbunov, V.A. Rubakov, Introduction to the early universe theory, Cosmological Perturbations, vol. 2, LKI Publishing House, Moscow, 2010.