Exercises from Linear algebra: the theory of vector spaces and linear transformations

Exercises from Linear algebra: the theory of vector spaces and linear transformations

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Contents

Chapter 1

Systems of linear equations

1.1 Systems of linear equations

Exercises

1. **Geometry of linear systems.** Recall that the set of solutions (x, y) to a single linear equation in 2 variables constitutes a line ℓ in $\mathbb{R}^2$. We denote this $\ell\colon ax + by = c$. Similarly, the set of solutions (x, y, z) to a single linear equation in 3 variables $ax + by + cz = d$ constitutes a plane $\mathcal{P}$ in $\mathbb{R}^3$. We denote this $\mathcal{P}\colon ax + by + cz = d$.

   (a) Fix $m > 1$ and consider a system of m linear equations in the 2 unknowns x and y. What do solutions (x, y) to this *system* of linear equations correspond to geometrically?

   (b) Use your interpretation to give a *geometric* argument that a system of m equations in 2 unknowns will have either (i) 0 solutions, (ii) 1 solution, or (iii) infinitely many solutions.

   (c) Use your geometric interpretation to help produce explicit examples of systems in 2 variables satisfying these three different cases (i)-(iii).

   (d) Now repeat (a)-(b) for systems of linear equations in 3 variables x, y, z.

   Solution. (a) Geometrically, each equation in the system represents a line $\ell_i\colon a_i x + b_i y = c_i$. A solution (x, y) to the i-th equation corresponds to a point on ℓ_i. Thus a solution (x, y) to the system corresponds to a point lying on *all* of the lines: i.e., a point of intersection of the lines.

   (b) First of all to prove the desired "or" statement it suffices to prove that if the number of solutions is greater than 1, then there are infinitely many solutions.

   Now suppose there is more than one solution. Then there are at least two different solutions: $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Take any of the two lines ℓ_i, ℓ_j. By above the intersection of ℓ_i and ℓ_j contains P_1 and P_2. But two *distinct* lines intersect in at most one point. It follows that ℓ_i and ℓ_j must be equal. Since ℓ_i and ℓ_j were arbitrary, it follows *all* of the lines ℓ_i are in fact the same line ℓ. But this means the common intersection of the lines is ℓ, which has infinitely many points. It follows that the system has infinitely many solutions.

   (c) We will get 0 solutions if the system includes two different parallel lines: e.g., $\ell_1\colon x + y = 5$ and $\ell_2\colon x + y = 1$.

We will get exactly one solution when the slopes of each line in the system are distinct.

We will get infinitely many solutions when *all* equations in the system represent the *same line*. This happens when all equations are multiples of one another.

(d) Now each equation in our system defines a plane $\mathcal{P}_i\colon a_ix+b_iy+c_iz = d_i$. A solution (x, y, z) to the system corresponds to a point $P = (x, y, z)$ of intersection of the planes. We recall two facts from Euclidean geometry:

(a) *Fact 1.*

Given two distinct points, there is a unique line containing both of them.

(b) *Fact 2.*

Given any number of distinct planes, they either do not intersect, or intersect in a line.

We proceed as in part (b) above: that is show that if there are two distinct solutions to the system, then there are infinitely many solutions. First, for simplicity, we may assume that the equations $\mathcal{P}_i\colon a_ix+b_iy+c_iz = d_i$ define *distinct* planes; if we have two equations defining the same plane, we can delete one of them and not change the set of solutions to the system.

Now suppose $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are two distinct solutions to the system. Let ℓ be the unique line containing P and Q (Fact 1). I claim that ℓ is precisely the set of solutions to the system. To see this, take any two equations in the system $\mathcal{P}_i\colon a_ix + b_iy + c_iz = d_i$ and $\mathcal{P}_j\colon a_jx + b_jy + c_jz = d_j$. Since the two corresponding planes are distinct, and intersect in at least the points P and Q, they must intersect in a line (Fact 2); since this line contains P and Q, it must be the line ℓ (Fact 1). Thus any two planes in the system intersect in the line ℓ. From this it follows that: (a) a point satisfying the system must lie in ℓ; and (b) all points on ℓ satisfy the system (since we have shown that ℓ lies in all the planes). It follows that ℓ is precisely the set of solutions, and hence that there are infinitely many solutions.

2. Row operations preserve solutions. We made the claim that each of our three row operations

(a) scalar multiplication ($e_i \mapsto c \cdot e_i$ for $c \neq 0$),

(b) swap ($e_i \leftrightarrow e_j$),

(c) addition ($e_i \mapsto e_i + c \cdot e_j$ for some c)

do not change the set of solutions of a linear system. To prove this claim, let L be a general linear system

$$\begin{array}{rcccccccc} e_1: & a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ e_2: & a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & \vdots & \\ e_m: & a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}.$$

Now consider each type of row operation separately, write down the new system L' you get by applying this row operation, and prove that an n-tuple $s = (s_1, s_2, \ldots, s_n)$ is a solution to the original system L if and only if it is a solution to the new system L'.

Solution. Let L be the original system with equations $e_1, e_2, \ldots, e_m$. For each specified row operation, we will call the resulting new system L' and its equations $e'_1, e'_2, \ldots, e'_m$.

Row swap. In this case systems L and L' have exactly the same equations, just written in a different order. Thus the n-tuple s satisfies L if and only if it satisfies each of the equations e_i, if and only if it satisfies each of the equations e'_i, since these are the same equations! It follows that s is a solution of L if and only if it is a solution to L'.

Scalar multiplication. In this case $e_j = e'_j$ for all $j \neq i$, and $e'_i = c \cdot e_i$ for some $c \neq 0$. Since only the i-th equation has changed, it suffices to show that s is a solution to e_i if and only if s is a solution to $c \cdot e_i$. Let's prove each direction of this if and only if separately.

If s satisfies e_1, then $a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$. Multiplying both sides by c we see that

$$ca_{i1}s_1 + ca_{i2}s_2 + \cdots + ca_{in}s_n = cb_i,$$

and hence that s is also a solution of $c\, e_i = e'_i$.

For the other direction, if s satisfies $ce_i = e'_i$, then

$$ca_{ii}s_1 + ca_{i2}s_2 + \cdots ca_{in}s_n = cb_i.$$

Now, since $c \neq 0$, we can multiply both sides of this equation by $1/c$ to see that

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

and hence that s is a solution to e_i.

Row addition. The only equation of L' that differs from L is

$$e'_i = e_i + ce_j.$$

Writing this equation out in terms of coefficients gives us

$$e'_i : a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c(a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n) = b_i + cb_j.$$

Now if s satisfies L, then it satisfies e_i and e_j, in which case evaluating the RHS of the e'_i above at s yields

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n + c(a_{j1}s_1 + a_{j2}s_2 + \cdots + a_{jn}s_n) = b_i + cb_j$$

showing that s satisfies e'_i. Now suppose $s = (s_1, s_2, \ldots, s_n)$ satisfies L'. Since s satisfies $e'_j = e_j$, we have

$$a_{j1}s_1 + a_{j2}s_2 + \cdots + a_{jn}s_n = b_j. \tag{$\star$}$$

Since s satisfies e'_i, we have

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n + c(a_{j1}s_1 + a_{j2}s_2 + \cdots + a_{jn}s_n) = b_i + cb_j$$

Substituting ($\star$) into the equation above we get

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n + c(b_j) = b_i + cb_j,$$

and hence

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i.$$

This shows that s satisfies e_i. It follows that s satisfies L.

1.2 Gaussian elimination

Exercises

Explain why each of the following matrices fails to be in row echelon form.

1. $A = \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. The first nonzero term in the second row is not a one.

2. $A = \begin{bmatrix} 0 & 1 & 2 & -3 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

For each of the given linear systems, find an equivalent system in row echelon form. Use augmented matrices and follow the Gaussian elimination algorithm to the letter.

4.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & = & x_3 & + & x_4 & + & 3 \\ 3x_1 & + & 6x_2 & = & 2x_3 & - & 4x_4 & + & 8 \\ -x_1 & + & 2x_3 & = & 2x_2 & - & x_4 & - & 1 \end{array}$$

Solution. First bring the system into standard form:

$$L\colon \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & + & 4x_4 & = & 8. \\ -x_1 & - & 2x_2 & + & 2x_3 & + & x_4 & = & -1 \end{array}$$

Then perform Gaussian elimination on the associated augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 3 & 6 & -2 & 4 & 8 \\ -1 & -2 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 0 & 0 & 1 & 7 & -1 \\ -1 & -2 & 2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{r_3 + r_1} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 0 & 0 & 1 & 7 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 0 & 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & -7 & 3 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{7}r_3} \begin{bmatrix} 1 & 2 & -1 & -1 & 3 \\ 0 & 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 1 & -\frac{3}{7} \end{bmatrix}.$$

The corresponding equivalent system is

$$L'\colon \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & 3 \\ & & & & x_3 & + & 7x_4 & = & -1. \\ & & & & & & x_4 & = & -\frac{3}{7} \end{array}$$

5.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 1 \\ -2x_1 & - & 2x_2 & + & 2x_3 & - & 2x_4 & = & -2 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & 3 \end{array}$$

6.

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ -2x_1 & + & 5x_2 & + & 2x_3 & = & 1 \\ 8x_1 & + & x_2 & + & 4x_3 & = & -1 \end{array}$$

7.

$$\begin{array}{rcrcrcr} & & -2b & + & 3c & = & 1 \\ 3a & + & 6b & - & 3c & = & -2 \\ 6a & + & 6b & + & 3c & = & 5 \end{array}$$

8.

$$\begin{array}{rcrcrcrcrcr} & & & & T_3 & + & T_4 & + & T_5 & = & 0 \\ -T_1 & - & T_2 & + & 2T_3 & - & 3T_4 & + & T_5 & = & 0 \\ T_1 & + & T_2 & - & 2T_3 & & & - & T_5 & = & 0 \\ 2T_1 & + & 2T_2 & - & T_3 & & & + & T_5 & = & 0 \end{array}$$

1.3 Solving linear systems

Exercises

Solve the following systems of equations.

- When row reducing follow Gaussian elimination to the letter.
- Make sure to indicate how variables are sorted into free and dependent variables.
- Express your answer in both the parametric equation format and set notation format.

1.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & = & x_3 & + & x_4 & + & 3 \\ 3x_1 & + & 6x_2 & = & 2x_3 & - & 4x_4 & + & 8 \\ -x_1 & + & 2x_3 & = & 2x_2 & - & x_4 & - & 1 \end{array}$$

Solution. We saw in Exercise 1.2.4 that the system is equivalent to a system L' with augmented matrix

$$\left[\begin{array}{rrrr|r} \boxed{1} & 2 & -1 & -1 & 3 \\ 0 & 0 & \boxed{1} & 7 & -1 \\ 0 & 0 & 0 & \boxed{1} & -\frac{3}{7} \end{array}\right].$$

The row echelon matrix tells us that $x_2 = t$ is the only free variable of L'. Back substitution then yields the parametric equation description:

$$\begin{aligned} x_1 &= \frac{32}{7} - 2t \\ x_2 &= t \\ x_3 &= 2 \\ x_4 &= -\frac{3}{7}. \end{aligned}$$

Thus the set of solutions is

$$\left\{\left(\frac{32}{7} - 2t, t, 2, -\frac{3}{7}\right) : t \in \mathbb{R}\right\}.$$

2.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 1 \\ -2x_1 & - & 2x_2 & + & 2x_3 & - & 2x_4 & = & -2 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & 3 \end{array}$$

Solution.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ -2 & -2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{r_2+2r_1} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{r_3-r_1} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_2} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The row echelon matrix tells us that $x_2 = s$ and $x_4 = t$ are the free variables. Back substitution then yields the parametric equation description:

$$\begin{aligned} x_1 &= 2 - s - \frac{3t}{2} \\ x_2 &= s \\ x_3 &= 1 - \frac{t}{2} \\ x_4 &= t, \end{aligned}$$

Alternatively, the set of solutions is

$$S = \left\{\left(2 - s - \frac{3t}{2}, s, 1 - \frac{t}{2}, t\right) : s, t \in \mathbb{R}\right\}.$$

3.

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ -2x_1 & + & 5x_2 & + & 2x_3 & = & 1 \\ 8x_1 & + & x_2 & + & 4x_3 & = & -1 \end{array}$$

Solution. The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix},$$

which row reduces first to

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and then further to

$$\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system has solution set

$$S = \left\{ \left(-\frac{1}{7} - \frac{3}{7}r, \frac{1}{7} - \frac{4}{7}r, r \right) : r \in \mathbb{R} \right\}.$$

4.

$$\begin{array}{rcrcrcr} & & -2b & + & 3c & = & 1 \\ 3a & + & 6b & - & 3c & = & -2 \\ 6a & + & 6b & + & 3c & = & 5 \end{array}$$

Solution. Take the corresponding augmented matrix and perform row reduction:

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{r_3 - 2r_1} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}r_2} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$\xrightarrow{r_3 + 6r_2} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}r_1} \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & -2/3 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}r_3} \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & -2/3 \\ 0 & \boxed{1} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

Since there is a leading one in the last column, we conclude that the original system is inconsistent.

5.

$$\begin{array}{rcrcrcrcrcr} & & & & T_3 & + & T_4 & + & T_5 & = & 0 \\ -T_1 & - & T_2 & + & 2T_3 & - & 3T_4 & + & T_5 & = & 0 \\ T_1 & + & T_2 & - & 2T_3 & & & - & T_5 & = & 0 \\ 2T_1 & + & 2T_2 & - & T_3 & & & + & T_5 & = & 0 \end{array}$$

Solution.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-r_2 \leftrightarrow r_1} \begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{r_4-2r_1}\begin{bmatrix}1&1&-2&3&-1&0\\0&0&1&1&1&0\\0&0&0&-3&0&0\\0&0&3&-6&3&0\end{bmatrix}$$
$$\xrightarrow{r_1-2r_2}\begin{bmatrix}1&1&0&5&1&0\\0&0&1&1&1&0\\0&0&0&-3&0&0\\0&0&3&-6&3&0\end{bmatrix}$$
$$\xrightarrow{r_4-3r_2}\begin{bmatrix}1&1&0&5&1&0\\0&0&1&1&1&0\\0&0&0&-3&0&0\\0&0&0&-9&0&0\end{bmatrix}$$
$$\xrightarrow{-\frac{1}{3}r_3}\begin{bmatrix}1&1&0&5&1&0\\0&0&1&1&1&0\\0&0&0&1&0&0\\0&0&0&-9&0&0\end{bmatrix}$$
$$\xrightarrow{r_4+9r_3}\begin{bmatrix}1&1&0&5&1&0\\0&0&1&1&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}$$
$$\xrightarrow{r_2-r_3}\begin{bmatrix}1&1&0&5&1&0\\0&0&1&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}$$
$$\xrightarrow{r_1-5r_3}\begin{bmatrix}1&1&0&0&1&0\\0&0&1&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Now solve. We set the free variables $x_2 = r$ and $x_5 = s$ and substitute:

$$\begin{aligned}T_1 &= -r-s\\T_2 &= r\\T_3 &= -s\\T_4 &= 0\\T_5 &= s\end{aligned}$$

6. For each system below determine all values of a for which the system below has (a) no solutions, (b) a unique solution, and (c) infinitely many solutions.

(a)

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& 2\\ 2x &-& 2y &+& 3z &=& 1\\ x &+& 2y &-& (a^2-3)z &=& a\end{array}$$

(b)

$$\begin{array}{rcrcrcr} x &+& 2y &-& 3z &=& 4\\ 3x &-& y &+& 5z &=& 2\\ 4x &+& y &+& (a^2-14)z &=& a+2\end{array}$$

Solution.

(a) Take the corresponding augmented matrix and row reduce:

$$\begin{bmatrix}1 & 2 & 1 & 2\\ 2 & -2 & 3 & 1\\ 1 & 2 & 3-a^2 & a\end{bmatrix} \xrightarrow{r_1-r_3} \begin{bmatrix}1 & 2 & 1 & 2\\ 2 & -2 & 3 & 1\\ 0 & 0 & a^2-2 & 2-a\end{bmatrix}$$
$$\xrightarrow{2r_1-r_2} \begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 6 & -1 & 3\\ 0 & 0 & a^2-2 & 2-a\end{bmatrix}$$
$$\xrightarrow{\frac{1}{6}r_2} \begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 1 & -\frac{1}{6} & \frac{1}{2}\\ 0 & 0 & a^2-2 & 2-a\end{bmatrix}$$

The row echelon form, and thus the set of solutions, now depends on whether $a^2 - 2 = 0$ or not: equivalently, whether $a = \pm\sqrt{2}$ or not. This gives us two cases:

- *Case:* $a = \pm\sqrt{2}$.
  In this case $2 - a \neq 0$, which means the row echelon matrix will end up having a leading 1 in the last column, resulting in an inconsistent system. There are no solutions in this case.
- *Case:* $a \neq \pm\sqrt{2}$.
  In this case the third column of the row echelon form will have a leading 1, and all variables are leading variables. Thus there is a unique solution in this case, obtained by back substitution.

  Since our two cases above are exhaustive, we see that there is no choice of a that yields infinitely many solutions in this case

(b) The augmented matrix row reduces to

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -7 & 14 & -10\\ 0 & 0 & (a^2-16) & a-4\end{bmatrix}$$

From this it follows that the system has:

a) 0 solutions iff $a^2 - 16 = 0$ and $a - 4 \neq 0$ iff $a = -4$;

(b) exactly one solution iff $a^2 - 16 \neq 0$ iff $a \neq \pm 4$;

(c) infinitely many solutions iff $a^2 - 16 = 0$ and $a - 4 = 0$ iff $a = 4$.

7. Show that a linear system with more unknowns than equations has either 0 solutions or infinitely many solutions.

Solution. Suppose we have a system of m equations in n unknowns $x_1, x_2, \ldots, x_n$. We assume $n > m$. Let A be the augmented matrix associated to the system, and suppose A is reduced to a matrix U in row echelon form.

Since U has m rows, there are *at most* m leading ones in U, which means there are at most m leading variables among the x_i. Since $n > m$, not all the x_i can be leading. Thus the system *must* have a free variable.

What does this mean? Note that the system could still be inconsistent, meaning no solutions. However, the existence of a free variable means if there is a solution, then there are infinitely many, because the parametric equations for the x_i will involve at least one parameter.

We conclude that the system is either inconsistent, or has infinitely many solutions.

8. True or false. If true, provide a proof; if false, provide an explicit counterexample.

(a) Every matrix has a unique row echelon form.

(b) Any homogeneous linear system with more unknowns than equations has infinitely many solutions.

(c) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading ones, then the linear system has the unique solution $s = (0, 0, \ldots, 0)$.

(d) All leading ones in of a matrix in row echelon form must occur in distinct columns.

(e) If the reduced row echelon form of the augmented matrix for a linear system has a zero row, then the system must have infinitely many solutions.

(f) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Solution.

(a) False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then A is already in row echelon form, but can be further reduced to $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is also in row echelon form. Thus A and I are two different row echelon forms of A.

(b) True. First observe that since the system is homogeneous, it is consistent: thus we have either 1 or infinitely many solutions.

Let m be the number of equations, and let n be the number of unknowns. We assume that $n > m$. The corresponding augmented matrix A is $m \times (n+1)$. Suppose it reduces to a matrix U in row echelon form.

Since there are *at most* m leading ones in U (at most one leading one per row), and since $n > m$, it follows that at least one of the first n columns does not contain a leading one. The corresponding variable in the system is free, and we see that the system has infinitely many solutions.

(c) True. Let U be the matrix mentioned. Since the system is homogenous, the as last column of U is a zero column. Since U has n leading ones, and since a row echelon matrix has *at most* one leading one per column (to get the staircase pattern), we see that each of the first n columns must contain a leading one (remember, the last column is a zero column). It follows that the corresponding system has no free variables, and hence that $s = (0, 0, \ldots, 0)$ is the only solution.

(d) True. If one of the columns of the matrix contained two leading ones, say in the ith and jth rows, with $i < j$, then the matrix would fail the third condition of being in row echelon form.

(e) False. Such a system might be inconsistent. For example, consider a system with augmented matrix

$$\left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

(f) False. The inconsistent system $0x_1 + 0x_2 = 1$ has more variables then equations.

9. Interpret each matrix below as an augmented matrix of a linear system. Asterisks represent an unspecified real number. For each matrix, determine whether the corresponding system is consistent or inconsistent. If the system is consistent, decide further whether the solution is unique or not. If there is not enough information answer 'inconclusive' and back up your claim by giving an explicit example where the system is consistent, and an explicit example where the system is inconsistent.

(a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$

(d) $\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Solution. (a) The corresponding system is consistent since the row echelon form of the augmented matrix has no leading 1 in the last column. Since the three columns corresponding to the three variables all have leading 1's, there are no free variables. Hence the system has a unique solution.

(b) This system has a unique solution. You can see this either by noting that the "reverse staircase pattern" allows us to do "forwards substitution", solving first for x_1, then for x_2, etc., or else by noting that the 1's along the diagonal (and 0's above them) allow us to row reduce the matrix further to one have exactly three leading 1's in the first three columns.

(c) Inconsistent. Rows 1 and 2 give

$$x_1 = 0 \qquad x_1 = 1$$

(d) Inconclusive. Consider

$$\begin{bmatrix} 1 & a & b & c \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

If $a = b = 0$ and $c = 2$ the system is inconsistent: the matrix row reduces to one with a leading 1 in the last column. If $a = b = 0$ and $c = 1$, the system has infinitely many solutions: the matrix row reduces to one with a leading 1 in the first column only.

10. What condition must a, b, and c satisfy in order for the system below to be consistent? Express your answer as an equation involving a, b, and c.

$$\begin{array}{rcrcrcl} x & + & 3y & + & z & = & a \\ -x & - & 2y & + & z & = & b \\ 3x & + & 7y & - & z & = & c \end{array}$$

Solution. Take the corresponding augmented matrix and row reduce:

$$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix} \xrightarrow{r_1+r_2} \begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 3 & 7 & -1 & c \end{bmatrix}$$
$$\xrightarrow{3r_1-r_3} \begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 2 & 4 & 3a-c \end{bmatrix}$$
$$\xrightarrow{2r_2-r_3} \begin{bmatrix} \boxed{1} & 3 & 1 & a \\ 0 & \boxed{1} & 2 & a+b \\ 0 & 0 & 0 & a-2b-c \end{bmatrix}$$

We see the system is consistent as long as $a-2b-c=0$, which guarantees there is no leading 1 in the last column.

11. Solve the system of equations below for x, y, and z.

$$\begin{aligned} \tfrac{1}{x} + \tfrac{2}{y} - \tfrac{4}{z} &= 1 \\ \tfrac{2}{x} + \tfrac{3}{y} + \tfrac{8}{z} &= 0 \\ -\tfrac{1}{x} + \tfrac{9}{y} + \tfrac{10}{z} &= 5 \end{aligned}$$

Hint. First replace the given *nonlinear* system with a linear one using a change of variable substitution.

Solution. Start by replacing variables. Let $X = \frac{1}{x}$, $Y = \frac{1}{y}$, and $Z = \frac{1}{z}$. Now we can solve the new system as we normally would.

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{bmatrix} \xrightarrow{r_1+r_3} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ 0 & 11 & 6 & 6 \end{bmatrix}$$
$$\xrightarrow{2r_1-r_2} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{bmatrix}$$
$$\xrightarrow{11r_2-r_3} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & -182 & 16 \end{bmatrix}$$

Now solve the system for X, Y, Z:

$$X = -\frac{7}{13}$$
$$Y = \frac{54}{91}$$
$$Z = -\frac{8}{91}$$

Now we solve for the original x, y, and z:

$$x = -\frac{13}{7}$$
$$y = \frac{91}{54}$$
$$z = -\frac{91}{8}$$

12. If A is a matrix with three rows and five columns, then what is the maximum possible number of leading ones in its reduced row echelon form? Justify your answer.

Provide an explicit example of a matrix that attains this maximum number of leading ones.

Solution. The maximum possible number of leading 1's in the reduced row echelon form of a matrix with 3 rows and 5 columns is 3. It is indeed possible to obtain this maximal number, as the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

illustrates.

13. If A is a matrix with three rows and six columns, then what is the maximum possible number of free variables in the general solution of the linear system with augmented matrix A? Justify your answer.

Provide an explicit example of a matrix that attains this maximal number of free variables.

Solution. The matrix B corresponds to a linear system of 3 equations in 5 unknowns $x_1, x_2, \ldots, x_5$.

Let U be a row echelon form of B, and let k be the number of leading 1's *among the first five columns* of U. Then the number of parameters in the general solution to the system corresponding to B is $5 - k$. Thus we see, that the number of parameters is at most 5 (when $k = 5$).

This case is indeed possible, as the matrix $B = \underset{3\times 6}{\mathbf{0}}$ illustrates.

14. If A is a matrix with five rows and three columns, then what is the minimum possible number of zero rows in any row echelon form of A?

Provide an explicit example of a matrix that attains this minimal number of zero rows.

Solution. If a row echelon form of A has r zero rows, then all other rows have leading 1's. Thus there are $5 - r$ leading 1's in this case. Since the number of leading 1's is at most 3 (the number of columns), we have $5 - r \leq 3$. It follows that $2 \leq r$, and thus there are at least 2 zero rows in a row echelon form of A. It is indeed possible to achieve this minimum number of zero rows, as the matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

illustrates.

Chapter 2

Matrics, their arithmetic, and their algebra

2.1 Matrices and their arithmetic

Exercises

1. For each part below write down the most general 3×3 matrix $A = [a_{ij}]$ satisfying the given condition (use letter names a, b, c,etc. for entries).

(a) $a_{ij} = a_{ji}$ for all i, j.

(b) $a_{ij} = -a_{ji}$ for all i, j

(c) $a_{ij} = 0$ for $i \neq j$.

Solution.

(a) $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

2. Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Compute the following matrices, or else explain why the given expression is not well defined.

(a) $(2D^T - E)A$

(b) $(4B)C + 2B$

(c) $B^T(CC^T - A^TA)$

Solution. (a)

$$\left(\begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 16 & -4 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 60 & 12 \\ 24 & 8 & 40 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 0 & 4 \end{bmatrix}$$

The matrix sum on the right is not defined. Thus the operation is not defined.

(c)

$$\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix} - \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 40 & 72 \\ 26 & 42 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}.$$

Compute the following using either the row or column method of matrix multiplication. Make sure to show how you are using the relevant method.

(a) the first column of AB;

(b) the second row of BB;

(c) the third column of AA.

Solution.

(a) Using expansion by columns, the first column of AB is given by A times the first column of B. We compute:

$$\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} = 6\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix}$$

(b) The second row of BB is given by the second row of B times B it self. We compute:

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = 0\begin{bmatrix} 6 & -2 & 4 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 & 3 \end{bmatrix} + 3\begin{bmatrix} 7 & 7 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 21 & 22 & 18 \end{bmatrix}$$

(c) The third column of AA is given by A times the third column of A. We compute:

$$A\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

4. Use the row or column method to quickly compute the following product:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution. I'll just describe the row method here.

Note that the rows of A are all identical, and equal to $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix}$. From the row method it follows that each row of AB is given by

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix} B.$$

Thus the rows of AB are all identical, and the row method computes the product above by taking the corresponding alternating sum of the rows of B:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 2 & -1 & 4 \end{bmatrix}.$$

Thus AB is the the 5×4 matrix, all of whose rows are $\begin{bmatrix} 2 & 2 & -1 & 4 \end{bmatrix}$.

5. Each of the 3×3 matrices B_i below performs a specific row operation when multiplying a $3 \times n$ matrix $A = \begin{bmatrix} -\mathbf{r}_1- \\ -\mathbf{r}_2- \\ -\mathbf{r}_3- \end{bmatrix}$ on the left; i.e., the matrix B_iA is the result of performing a certain row operation on the matrix A. Use the row method of matrix multiplication to decide what row operation each B_i performs.

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution. The matrix B_1, when multiplied on the left, replaces the third row of A with $\mathbf{r}_3 - 2\mathbf{r}_2$.

The matrix B_2, when multiplied on the left, replaces the second row of A with $\frac{1}{2}\mathbf{r}_2$.

The matrix B_3, when multiplied on the left, swaps $\mathbf{r}_1$ and $\mathbf{r}_3$.