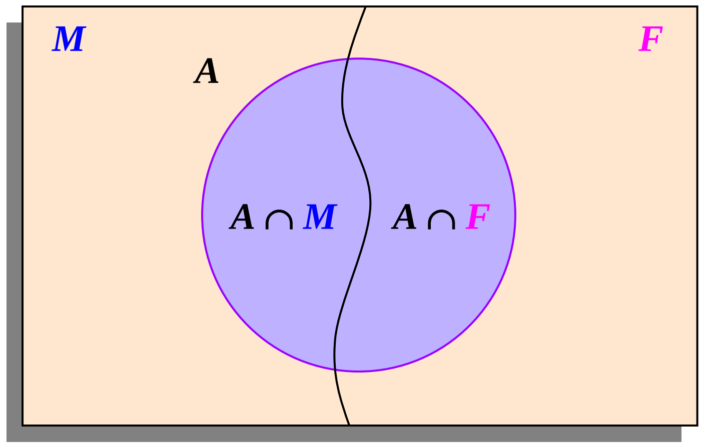
3.3 Bayes' Formula

Suppose that, for a certain population of individuals, we are interested in comparing sleep disorders – in particular, the occurrence of event A = "Apnea" – between $\boldsymbol{M}$ = Males and $\boldsymbol{F}$ = Females.

$\boldsymbol{S}$ = Adults under 50

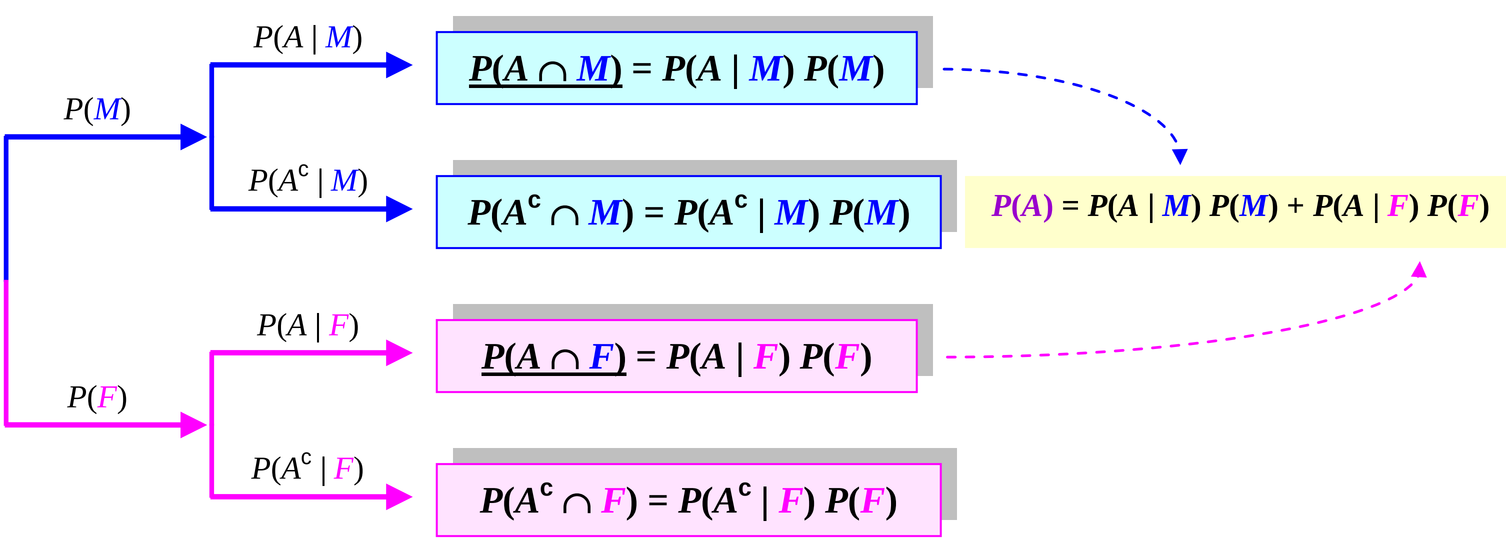


Also assume that we know the following information:

prior probabilities $\begin{cases} \mathbf{P(M) = 0.4} & P(A \mid M) = 0.8 \text{ (80\% of males have apnea)} \\ \mathbf{P(F) = 0.6} & P(A \mid F) = 0.3 \text{ (30\% of females have apnea)} \end{cases}$

Given here are the *conditional* probabilities of having apnea within each respective gender, but these are not necessarily the probabilities of interest. We actually wish to calculate the probability of each gender, *given A*. That is, the ***posterior probabilities*** $\mathbf{P(M \mid A)}$ and $\mathbf{P(F \mid A)}$.

To do this, we first need to reconstruct $P(A)$ itself from the given information.



So, *given A*…

$$\mathbf{P(M \mid A)} = \frac{P(M \cap A)}{P(A)} = \frac{P(A \mid M)\, P(M)}{P(A \mid M)\, P(M) + P(A \mid F)\, P(F)}$$

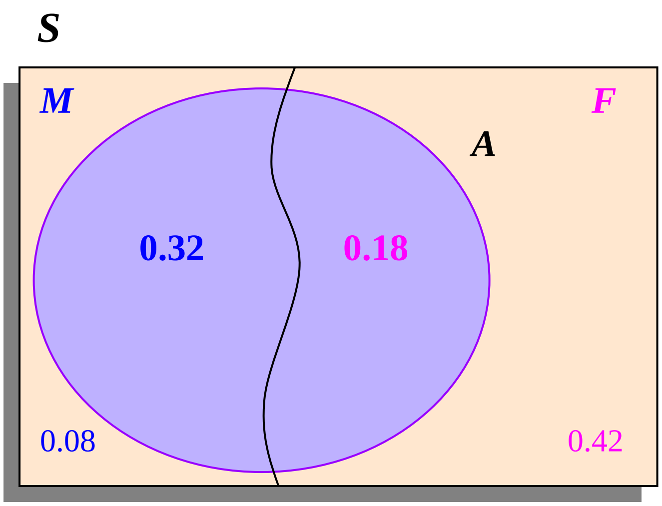
$$= \frac{(0.8)(0.4)}{(0.8)(0.4) + (0.3)(0.6)} = \frac{0.32}{0.50} = \mathbf{0.64}$$

and

$$\mathbf{P(F \mid A)} = \frac{P(F \cap A)}{P(A)} = \frac{P(A \mid F)\, P(F)}{P(A \mid M)\, P(M) + P(A \mid F)\, P(F)}$$

$$= \frac{(0.3)(0.6)}{(0.8)(0.4) + (0.3)(0.6)} = \frac{0.18}{0.50} = \mathbf{0.36}$$

posterior probabilities



Thus, the additional information that a randomly selected individual has apnea (an event with probability 50% – why?) increases the likelihood of being male from a ***prior*** probability of 40% to a ***posterior*** probability of 64%, and likewise, decreases the likelihood of being female from a ***prior*** probability of 60% to a ***posterior*** probability of 36%. That is, knowledge of event A can alter a ***prior probability*** $P(B)$ to a ***posterior probability*** $P(B \mid A)$, of some other event B.

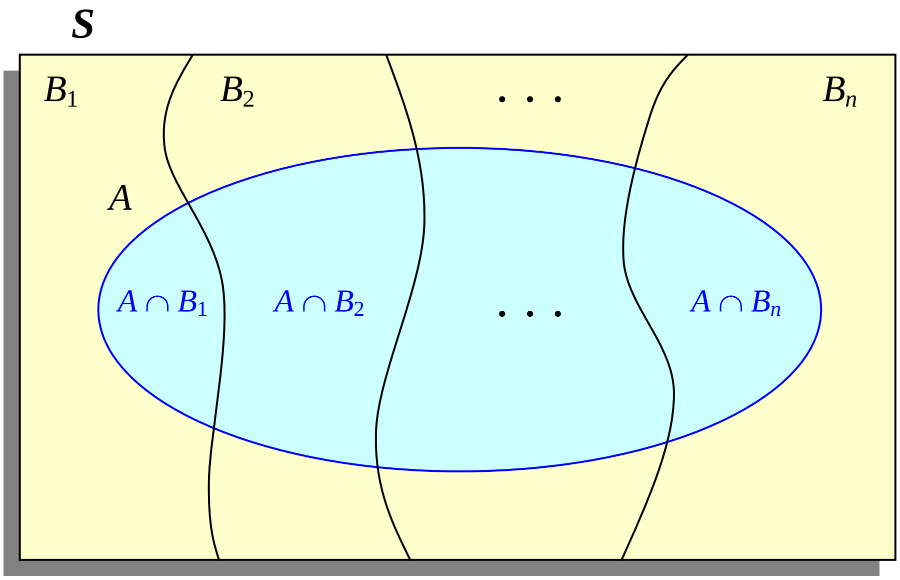
Exercise: Calculate and interpret the ***posterior*** probabilities $\mathbf{P(M \mid A^c)}$ and $\mathbf{P(F \mid A^c)}$ as above, using the ***prior*** probabilities (and conditional probabilities) given.

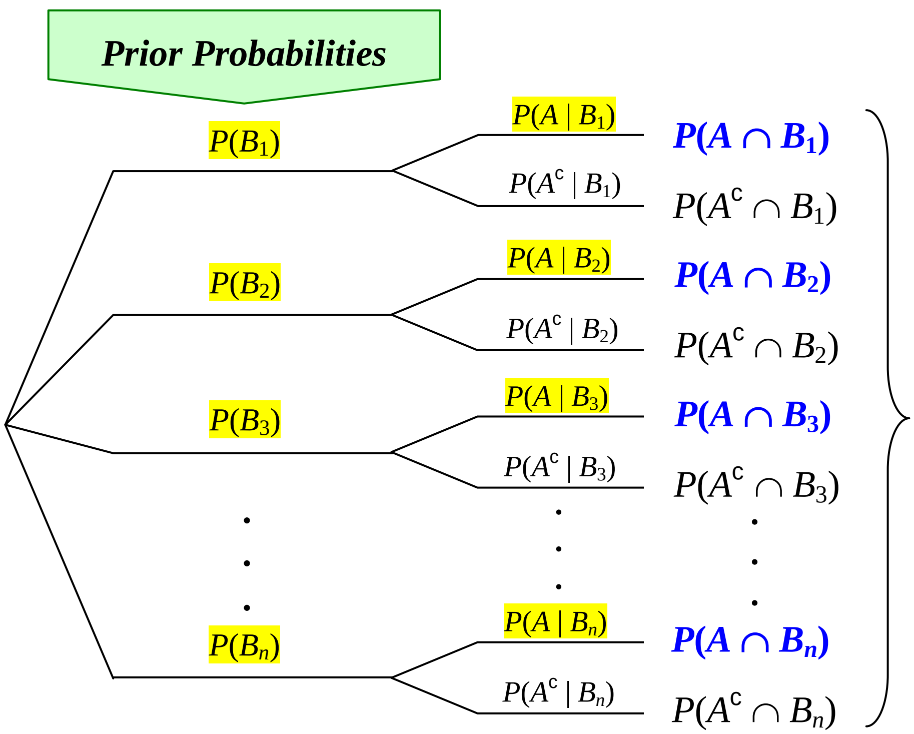
More formally, consider any event A, and two *complementary* events B_1 and B_2, (e.g., $\mathbf{M}$ and $\mathbf{F}$) in a sample space $\mathbf{S}$. How do we express the ***posterior probabilities*** $P(B_1 \mid A)$ and $P(B_2 \mid A)$ in terms of the conditional probabilities $P(A \mid B_1)$ and $P(A \mid B_2)$, and the ***prior probabilities*** $P(B_1)$ and $P(B_2)$?

Bayes' Formula *for* ***posterior probabilities*** $P(B_i \mid A)$ *in terms of* ***prior probabilities*** $P(B_i)$, $i = 1, 2$

$$\mathbf{P(B_i \mid A)} = \frac{P(B_i \cap A)}{P(A)} = \frac{\mathbf{P(A \mid B_i)\, P(B_i)}}{\mathbf{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2)}}$$

In general, consider an event A, and events $B_1, B_2, \ldots, B_n$, *disjoint* and *exhaustive*.





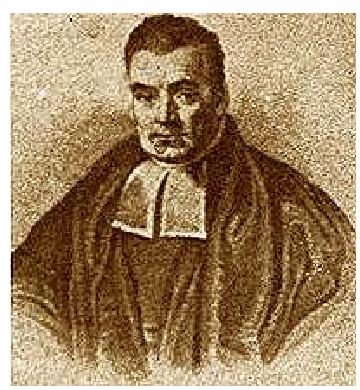
Law of Total Probability

$$P(A) = \sum_{j=1}^{n} \overbrace{P(A \mid B_j)\, P(B_j)}^{P(A \cap B_j)}$$

Bayes' Formula (general version)

For $i = 1, 2, \ldots, n$, the ***posterior probabilities*** are…

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)}.$$



Reverend Thomas Bayes
1702 - 1761