Multilevel Multilingual

Multilevel Models in Stata, R and Julia

Andrew Grogan-Kaylor

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# 1. Multilevel Multilingual

“This curious world which we inhabit is more wonderful than it is convenient…” (Thoreau, 1975)

“Mathematics is my secret. My secret weakness. I feel like a stubborn, helpless fool in the middle of a problem. Trapped and crazed. Also, thrilled.” (Schanen, 2021)

## 1.1 Introduction

Below, I describe the use of [Stata](https://www.stata.com/) (StataCorp, 2023), [R](https://www.r-project.org/) (Bates et al., 2015; R Core Team, 2023), and [Julia](https://www.julialang.org/) (Bates, 2024; Bezanson et al., 2017) to estimate multilevel models and to visualize data.

All of these software packages can estimate multilevel models and can visualize relationships in the data. However, there are substantial differences between the different packages: Stata is proprietary *for cost* software, which is very well documented and very intuitive. While it costs money to purchase Stata, the price is often very reasonable for academic and educational use. R is free open source software which is less intuitive, but there are many excellent resources for learning R. There is often a cost associated with purchasing books and other materials for learning R, which sometimes feels like it offsets the fact that R is free. Julia is newer open source software, and ostensibly much faster than either Stata or R, which may be an important advantage when running multilevel models with very large data sets. At this point in time, both Stata and R feel much more *stable* than Julia which is still evolving software.

While any of these software packages can be used for learning and estimating multilevel models, I will offer my own opinion–based upon 15 years of teaching multilevel models at the doctoral level–that Stata offers the quickest pathway for learning the basic and advanced uses of multilevel models. I also believe the intuitive nature of Stata syntax contributes to accurate and replicable work in this area.

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| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 1.1: Software for Multilevel Modeling   | Software | Cost | Ease of Use | | --- | --- | --- | | Stata | some cost | learning curve, but very intuitive for both multilevel modeling and graphing. | | R | free | learning curve: intuitive for multilevel modeling; but steeper learning curve for graphing (ggplot). | | Julia | free | steep learning curve in general: steep learning curve for multilevel modeling; and very steep learning curve for graphing. Graphics libraries are very much under development and in flux. | |

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|  | **Results Will Vary Somewhat**  Estimating multilevel models is a complex endeavor. The software details of how this is accomplished are beyond the purview of this book. Suffice it to say that across different software packages there will be differences in estimation routines, resulting in some numerical differences in the results provided by different software packages. Substantively speaking, however, results should agree across software. |

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|  | **Multi-Line Commands**  Sometimes I have written commands out over multiple lines. I have done this for especially long commands, but have also sometimes done this simply for the sake of clarity. The different software packages have different approaches to multi-line commands.   1. By default, *Stata* ends a command at the end of a line. If you are going to write a multi-line command you should use the /// line continuation characters. 2. *R* is the software that most naturally can be written using multiple lines, as R commands are usually clearly encased in parentheses (()) or continued with + signs. 3. Like *Stata*, *Julia* expects commands to end at the end of a line. If you are going to write a mult-line command, all commands except for the last line should end in a character that clearly indicates continuation, like a + sign. An alternative is to encase the entire Julia command in an outer set of parentheses (()). |

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|  | **Running Statistical Packages in Quarto**  I used Quarto (Allaire et al., 2024) (<https://quarto.org/>) to create this Appendix. Quarto is a programming and publishing environment that can run multiple programming languages, including Stata, R and Julia, and that can write to multiple output formats including HTML, PDF, and MS Word. To run Stata, I used the Statamarkdown library (Hemken, 2023) in R to connect Stata to Quarto. Quarto has a built in connection to R, and runs R without issue. To run Julia, I used the JuliaCall library (Li, 2019) in R to connect Quarto to Julia.  Of course, each of these programs can be run by itself, if you have them installed on your computer. |

## 1.2 The Data

|  |  |
| --- | --- |
|  | **Datasets**  The examples use the simulated\_multilevel\_data.dta and simulated\_multilevel\_longitudinal\_data.dta files.  Here is a [direct link](https://github.com/agrogan1/multilevel-multilingual/raw/main/simulated_multilevel_data.dta) to download the cross-sectional data.  Here is a [direct link](https://github.com/agrogan1/multilevel-multilingual/raw/main/simulated_multilevel_longitudinal_data.dta) to download the longitudinal data. |

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 1.2: Sample of Simulated Multilevel Data  Table continues below   | country | HDI | family | id | identity | intervention | physical\_punishment | | --- | --- | --- | --- | --- | --- | --- | | 1 | 69 | 1 | 1.1 | 1 | 0 | 3 | | 1 | 69 | 2 | 1.2 | 1 | 1 | 2 | | 1 | 69 | 3 | 1.3 | 0 | 1 | 3 | | 1 | 69 | 4 | 1.4 | 1 | 0 | 0 | | 1 | 69 | 5 | 1.5 | 1 | 0 | 4 | | 1 | 69 | 6 | 1.6 | 0 | 1 | 5 |      | warmth | outcome | | --- | --- | | 3 | 57.47 | | 1 | 50.1 | | 2 | 52.92 | | 5 | 60.17 | | 4 | 55.05 | | 3 | 49.81 | |

## 1.3 An Introduction To Equations and Syntax

To explain statistical syntax for each software, I consider the general case of a multilevel model with dependent variable y, independent variables x and z, clustering variable group, and a random slope for x. *i* is the index for the person, while *j* is the index for the group.

### Stata

In Stata mixed, the syntax for a multilevel model of the form described in [Equation 1.1](#eq-MLMsimple) is:

mixed y x z || group: x

### R

In R lme4, the general syntax for a multilevel model of the form described in [Equation 1.1](#eq-MLMsimple) is:

library(lme4)  
  
lmer(y ~ x + z + (1 + x || group), data = ...)

### Julia

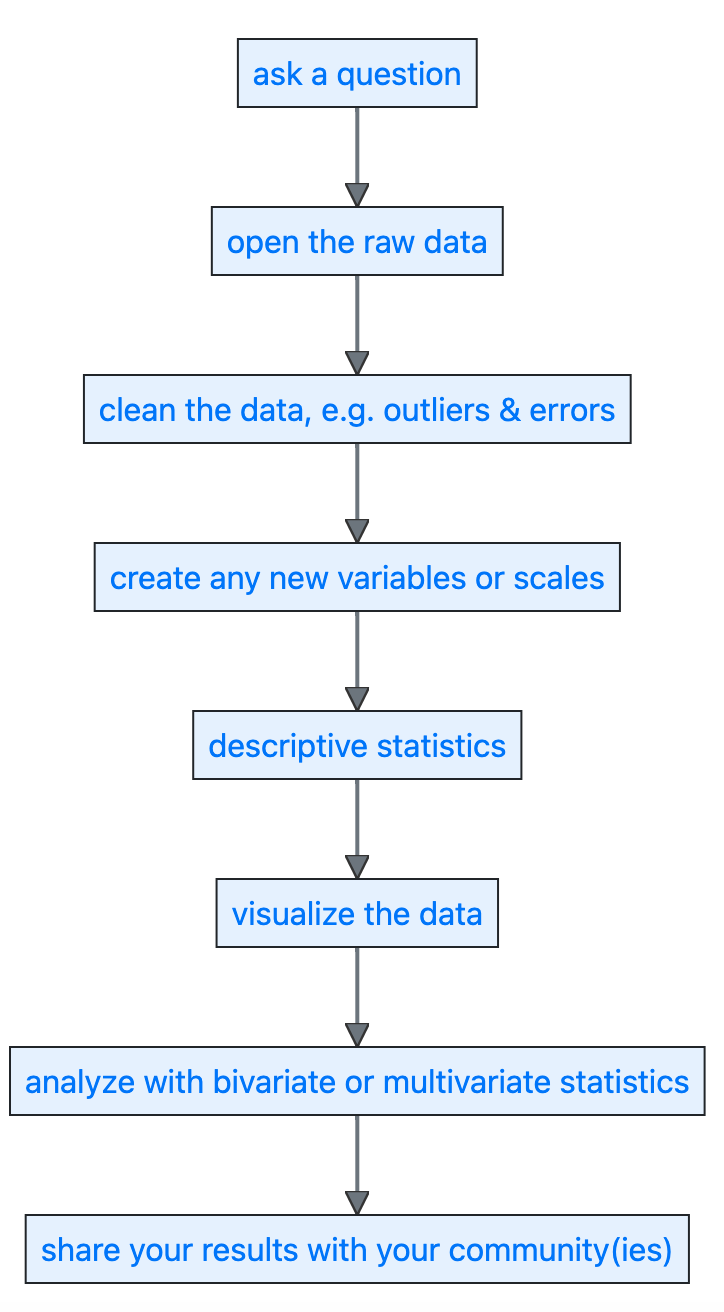
In Julia MixedModels, the general syntax for a multilevel model of the form described in [Equation 1.1](#eq-MLMsimple) is:

using MixedModels  
  
fit(MixedModel, @formula(y ~ x + z + (1 + x | group)), data)

# 2. Statistical Workflows

## 2.1 Statistical Software Is Best Run Using a Script

Many statistical workflows–whatever the statistical package being used–follow the same conceptual pattern.



A Common Statistical Workflow

Increasingly, we want to think about workflows that are

* **documentable**, **transparent**, and **auditable**: We have a record of what we did if we want to double check our work, clarify a result, or develop a new project with a similar process. We, or others, can find the inevitable errors in our work, **and correct them**.
* **replicable**: Others can replicate our findings with the same or new data.
* **scalable**: We are developing a process that can be as easily used with *thousands* or *millions* of rows of data as it can with *ten* rows of data. We are developing a process that can be easily repeated if we are *constantly getting new or updated data*, e.g. getting new data every week, or every month.

## 2.2 Scripts

For most statistical workflows, we will often want to write a script or code. Data analysis scripts can be stored in a Quarto document (Allaire et al., 2024) as they are in this Appendix, or every statistical package has its own unique format for storing scripts as a text file: in Stata, scripts are stored in .do files; in R, scripts are stored in .R files, and in Julia, scripts are stored in .jl files.

## 2.3 Script Flow

A good practice when writing a script, is to have a script that begins with the raw data, moves through any necessary re-coding or cleaning of the data, generates descriptive statistics, generates the appropriate multivariate results, and then generates any necessary visualizations.

## 2.4 Storing Statistical Data

It is usually best to store quantitative data in a statistical format such as R (.Rdata), or Stata (.dta), or even a text format such as .csv. Spreadsheets are likely to be a bad tool for storing quantitative data.

## 2.5 It Is Possible To Use Multiple Statistical Packages

While this Appendix focuses on the use of each individual statistical package on its own, it is certainly possible to use multiple statistical packages as part of the same workflow. For example, one might employ Stata to carry out data management tasks, and then possibly use R to run a multilevel model with a more complicated multilevel structure, such as a cross-classified model, or Julia to more quickly run a model with a large data.

## 2.6 Good Statistical Workflows Require Safe Workspaces

It is also *very important* to be aware that good complex workflows are *highly iterative* and *highly collaborative*. Good complex workflows require a *safe workspace* in which team members feel free to admit their own errors, and help with others’ mistakes in a non-judgmental fashion. Such a *safe environment* is necessary to build an environment where the *overall error rate* is low.

## 2.7 Good Statistical Workflows Require Patience And Can Be Psychologically Demanding

Developing a good documented and auditable workflow that is implemented in code requires a lot of patience, and often, **many iterations**. Working through these many iterations can be psychologically demanding. It is important to remember that careful attention to getting the details right early in the research process, while sometimes tiring and frustrating, will pay large dividends later on when the research is reviewed, presented, published and read.

# 3. Descriptive Statistics

## 3.1 Descriptive Statistics

### Stata

use simulated\_multilevel\_data.dta // use data

We use summarize for *continuous* variables, and tabulate for *categorical* variables.

summarize outcome warmth physical\_punishment HDI  
  
tabulate identity  
  
tabulate intervention

Variable | Obs Mean Std. dev. Min Max  
-------------+---------------------------------------------------------  
 outcome | 3,000 52.43327 6.530996 29.60798 74.83553  
 warmth | 3,000 3.521667 1.888399 0 7  
physical\_p~t | 3,000 2.478667 1.360942 0 5  
 HDI | 3,000 64.76667 17.24562 33 87  
  
  
hypothetica |  
 l identity |  
 group |  
 variable | Freq. Percent Cum.  
------------+-----------------------------------  
 0 | 1,507 50.23 50.23  
 1 | 1,493 49.77 100.00  
------------+-----------------------------------  
 Total | 3,000 100.00  
  
  
 recieved |  
interventio |  
 n | Freq. Percent Cum.  
------------+-----------------------------------  
 0 | 1,547 51.57 51.57  
 1 | 1,453 48.43 100.00  
------------+-----------------------------------  
 Total | 3,000 100.00

### R

library(haven) # read data in Stata format  
  
df <- read\_dta("simulated\_multilevel\_data.dta")

R’s descriptive statistics functions rely heavily on whether a variable is a *numeric* variable, or a *factor* variable. Below, I convert two variables to factors (factor) before using summary[[1]](#footnote-54) to generate descriptive statistics.

df$country <- factor(df$country)  
  
df$identity <- factor(df$identity)  
  
df$intervention <- factor(df$intervention)  
  
summary(df)

country HDI family id identity  
 1 : 100 Min. :33.00 Min. : 1.00 Length:3000 0:1507   
 2 : 100 1st Qu.:53.00 1st Qu.: 25.75 Class :character 1:1493   
 3 : 100 Median :70.00 Median : 50.50 Mode :character   
 4 : 100 Mean :64.77 Mean : 50.50   
 5 : 100 3rd Qu.:81.00 3rd Qu.: 75.25   
 6 : 100 Max. :87.00 Max. :100.00   
 (Other):2400   
 intervention physical\_punishment warmth outcome   
 0:1547 Min. :0.000 Min. :0.000 Min. :29.61   
 1:1453 1st Qu.:2.000 1st Qu.:2.000 1st Qu.:48.02   
 Median :2.000 Median :4.000 Median :52.45   
 Mean :2.479 Mean :3.522 Mean :52.43   
 3rd Qu.:3.000 3rd Qu.:5.000 3rd Qu.:56.86   
 Max. :5.000 Max. :7.000 Max. :74.84

### Julia

using Tables, MixedModels, MixedModelsExtras, StatFiles, DataFrames, CategoricalArrays, DataFramesMeta  
  
df = DataFrame(load("simulated\_multilevel\_data.dta"))

Similarly to R, Julia relies on the idea of *variable type*. I use transform to convert the appropriate variables to *categorical* variables.

@transform!(df, :country = categorical(:country))  
  
@transform!(df, :identity = categorical(:identity))  
  
@transform!(df, :intervention = categorical(:intervention))

describe(df) # descriptive statistics

9×7 DataFrame  
 Row │ variable mean min median max nmissing eltype ⋯  
 │ Symbol Union… Any Union… Any Int64 Union ⋯  
─────┼──────────────────────────────────────────────────────────────────────────  
 1 │ country 1.0 30.0 0 Union{ ⋯  
 2 │ HDI 64.7667 33.0 70.0 87.0 0 Union{  
 3 │ family 50.5 1.0 50.5 100.0 0 Union{  
 4 │ id 1.1 9.99 0 Union{  
 5 │ identity 0.0 1.0 0 Union{ ⋯  
 6 │ intervention 0.0 1.0 0 Union{  
 7 │ physical\_punishment 2.47867 0.0 2.0 5.0 0 Union{  
 8 │ warmth 3.52167 0.0 4.0 7.0 0 Union{  
 9 │ outcome 52.4333 29.608 52.449 74.8355 0 Union{ ⋯  
 1 column omitted

## 3.2 Interpretation

Examining descriptive statistics is an important first step in any analysis. It is important to examine your descriptive statistics first, before skipping ahead to more sophisticated analyses, such as multilevel models.

In examining the descriptive statistics for this data, we get a sense of the data.

* outcome has a mean of approximately 52 and ranges from approximately 30 to 75.
* warmth and physical punishment are both variables that represent the number of times that parents use each of these forms of discipline in a week. The average of the former is about 3.5, while the average of the latter is about 2.5.
* HDI, the Human Development Index has an average of about 65, and a wide range.
* identity is a categorical variable for a hypothetical identity group, and has values of 0 and 1.
* intervention is also a categorical variable, and has values of 0 and 1.

# 4. Unconditional Model

An *unconditional* multilevel model is a model with no independent variables. One should always run an unconditional model as the first step of a multilevel model in order to get a sense of the way that variation is apportioned in the model across the different levels.

## 4.1 The Equation

The Intraclass Correlation Coefficient (ICC) is given by:

In a two level multilevel model, the ICC provides a measure of the proportion of variation attributable to Level 2.

## 4.2 Run Models

### Stata

use simulated\_multilevel\_data.dta // use data

mixed outcome || country: // unconditional model

Performing EM optimization ...  
  
Performing gradient-based optimization:   
Iteration 0: Log likelihood = -9802.8371   
Iteration 1: Log likelihood = -9802.8371   
  
Computing standard errors ...  
  
Mixed-effects ML regression Number of obs = 3,000  
Group variable: country Number of groups = 30  
 Obs per group:  
 min = 100  
 avg = 100.0  
 max = 100  
 Wald chi2(0) = .  
Log likelihood = -9802.8371 Prob > chi2 = .  
  
------------------------------------------------------------------------------  
 outcome | Coefficient Std. err. z P>|z| [95% conf. interval]  
-------------+----------------------------------------------------------------  
 \_cons | 52.43327 .3451217 151.93 0.000 51.75685 53.1097  
------------------------------------------------------------------------------  
  
------------------------------------------------------------------------------  
 Random-effects parameters | Estimate Std. err. [95% conf. interval]  
-----------------------------+------------------------------------------------  
country: Identity |  
 var(\_cons) | 3.178658 .9226737 1.799552 5.614658  
-----------------------------+------------------------------------------------  
 var(Residual) | 39.46106 1.024013 37.50421 41.52  
------------------------------------------------------------------------------  
LR test vs. linear model: chibar2(01) = 166.31 Prob >= chibar2 = 0.0000

estat icc // ICC

Intraclass correlation  
  
------------------------------------------------------------------------------  
 Level | ICC Std. err. [95% conf. interval]  
-----------------------------+------------------------------------------------  
 country | .0745469 .0201254 .0434963 .1248696  
------------------------------------------------------------------------------

### R

library(haven)  
  
df <- read\_dta("simulated\_multilevel\_data.dta")

library(lme4) # estimate multilevel models  
  
fit0 <- lmer(outcome ~ (1 | country),  
 data = df) # unconditional model  
  
summary(fit0)

Linear mixed model fit by REML ['lmerMod']  
Formula: outcome ~ (1 | country)  
 Data: df  
  
REML criterion at convergence: 19605.9  
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-3.3844 -0.6655 -0.0086 0.6725 3.6626   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 country (Intercept) 3.302 1.817   
 Residual 39.461 6.282   
Number of obs: 3000, groups: country, 30  
  
Fixed effects:  
 Estimate Std. Error t value  
(Intercept) 52.433 0.351 149.4

library(performance)  
  
performance::icc(fit0) # ICC

# Intraclass Correlation Coefficient  
  
 Adjusted ICC: 0.077  
 Unadjusted ICC: 0.077

### Julia

using Tables, MixedModels, MixedModelsExtras,   
StatFiles, DataFrames, CategoricalArrays, DataFramesMeta  
  
df = DataFrame(load("simulated\_multilevel\_data.dta"))

@transform!(df, :country = categorical(:country))

m0 = fit(MixedModel,   
 @formula(outcome ~ (1 | country)), df) # unconditional model

Linear mixed model fit by maximum likelihood  
 outcome ~ 1 + (1 | country)  
 logLik -2 logLik AIC AICc BIC   
 -9802.8371 19605.6742 19611.6742 19611.6822 19629.6933  
  
Variance components:  
 Column Variance Std.Dev.  
country (Intercept) 3.17863 1.78287  
Residual 39.46106 6.28180  
 Number of obs: 3000; levels of grouping factors: 30  
  
 Fixed-effects parameters:  
──────────────────────────────────────────────────  
 Coef. Std. Error z Pr(>|z|)  
──────────────────────────────────────────────────  
(Intercept) 52.4333 0.345121 151.93 <1e-99  
──────────────────────────────────────────────────

icc(m0) # ICC

0.07454637475695493

## 4.3 Interpretation

In each case, the software finds that nearly 8% of the variation in the outcome is explainable by the clustering of the observations in each country.

# 5. Cross Sectional Multilevel Models

## 5.1 The Equation

Recall the general model of [Equation 1.1](#eq-MLMsimple), and the syntax outlined in [Section 1.3](#sec-syntax). Below in [Equation 5.1](#eq-MLMsubstantive), we consider a more substantive example.

## 5.2 Correlated and Uncorrelated Random Effects

Consider the covariance matrix of random effects (e.g.  and ). In [Equation 5.2](#eq-varcovar) the covariances of the random effects are constrained to be zero.

As discussed in the Chapter on multilevel models with cross-sectional data, however, one can consider a multilevel model in which the random effects are correlated, as is the case in [Equation 5.3](#eq-varcovaruns).

Procedures for estimating models with uncorrelated and correlated random effects are detailed below (Bates et al., 2015; Bates, 2024; StataCorp, 2023).

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 5.1: Correlated and Uncorrelated Random Effects   | Software | Uncorrelated Random Effects | Correlated Random Effects | | --- | --- | --- | | Stata | default | add option: , cov(uns) | | R | separate random effects from grouping variable with || | separate random effects from grouping variable with | | | Julia | separate terms for each random effect e.g. (1 | group) + (0 + x | group) | separate random effects from grouping variable with |. | |

All models in the examples below are run with *uncorrelated* random effects, but could just as easily be run with *correlated* random effects.

## 5.3 Run Models

|  |  |
| --- | --- |
|  | **Continuous and Categorical Variables**  Statistically–as noted in the main text–it is important to be clear on whether independent variables in one’s model are continuous or categorical. *Continuous* variables can be entered straightforwardly into statistical syntax. *Categorical* variables, on the other hand usually require specific attention in statistical software. In Stata, categorical variables are indicated in a statistical model by prefixing them with an i.. In R, categorical variables are distinguished by making them into factors e.g. x <- factor(x). In Julia, categorical variables are created by using the @transform syntax detailed below. |

### Stata

#### 5.3.0.1 Get The Data

use simulated\_multilevel\_data.dta

#### 5.3.0.2 Run The Model

mixed outcome warmth physical\_punishment i.identity i.intervention HDI || country: warmth

Performing EM optimization ...  
  
Performing gradient-based optimization:   
Iteration 0: Log likelihood = -9626.6279   
Iteration 1: Log likelihood = -9626.607   
Iteration 2: Log likelihood = -9626.607   
  
Computing standard errors ...  
  
Mixed-effects ML regression Number of obs = 3,000  
Group variable: country Number of groups = 30  
 Obs per group:  
 min = 100  
 avg = 100.0  
 max = 100  
 Wald chi2(5) = 334.14  
Log likelihood = -9626.607 Prob > chi2 = 0.0000  
  
-------------------------------------------------------------------------------------  
 outcome | Coefficient Std. err. z P>|z| [95% conf. interval]  
--------------------+----------------------------------------------------------------  
 warmth | .8345368 .0637213 13.10 0.000 .7096453 .9594282  
physical\_punishment | -.9916657 .0797906 -12.43 0.000 -1.148052 -.8352791  
 1.identity | -.3004767 .2170295 -1.38 0.166 -.7258466 .1248933  
 1.intervention | .6396427 .2174519 2.94 0.003 .2134448 1.065841  
 HDI | -.003228 .0199257 -0.16 0.871 -.0422817 .0358256  
 \_cons | 51.99991 1.371257 37.92 0.000 49.3123 54.68753  
-------------------------------------------------------------------------------------  
  
------------------------------------------------------------------------------  
 Random-effects parameters | Estimate Std. err. [95% conf. interval]  
-----------------------------+------------------------------------------------  
country: Independent |  
 var(warmth) | .0227504 .0257784 .0024689 .2096436  
 var(\_cons) | 2.963975 .9737647 1.556777 5.643163  
-----------------------------+------------------------------------------------  
 var(Residual) | 34.97499 .9097109 33.23668 36.80422  
------------------------------------------------------------------------------  
LR test vs. linear model: chi2(2) = 205.74 Prob > chi2 = 0.0000  
  
Note: LR test is conservative and provided only for reference.

### R

#### 5.3.0.3 Get The Data

library(haven)  
  
df <- read\_dta("simulated\_multilevel\_data.dta")

#### 5.3.0.4 Change Some Variables To Categorical

df$identity <- factor(df$identity)  
  
df$intervention <- factor(df$intervention)

#### 5.3.0.5 Run The Model

|  |  |
| --- | --- |
|  | lme4 does not directly provide p values in results, because of some disagreement over exactly how these p values should be calculated. Therefore, in this Appendix, I also call library lmerTest to provide p values for lme4 results. |

|  |  |
| --- | --- |
|  | R prefers to use scientific notation when possible. I find that the use of scientific notation can be confusing in reading results. I turn off scientific notation by setting a penalty for its use: options(scipen = 999). |

library(lme4)   
  
library(lmerTest)  
  
options(scipen = 999)   
  
fit1 <- lmer(outcome ~ warmth + physical\_punishment +   
 identity + intervention + HDI +  
 (1 + warmth || country),  
 data = df)  
  
summary(fit1)

Linear mixed model fit by REML. t-tests use Satterthwaite's method [  
lmerModLmerTest]  
Formula: outcome ~ warmth + physical\_punishment + identity + intervention +   
 HDI + (1 + warmth || country)  
 Data: df  
  
REML criterion at convergence: 19268.8  
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-3.9774 -0.6563 0.0186 0.6645 3.6730   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 country (Intercept) 3.19120 1.786   
 country.1 warmth 0.02464 0.157   
 Residual 35.01779 5.918   
Number of obs: 3000, groups: country, 30  
  
Fixed effects:  
 Estimate Std. Error df t value  
(Intercept) 52.011324 1.414976 30.293141 36.758  
warmth 0.834562 0.064250 41.896457 12.989  
physical\_punishment -0.991893 0.079845 2968.012381 -12.423  
identity1 -0.300354 0.217179 2970.108153 -1.383  
intervention1 0.639060 0.217603 2971.186718 2.937  
HDI -0.003394 0.020598 27.592814 -0.165  
 Pr(>|t|)   
(Intercept) < 0.0000000000000002 \*\*\*  
warmth 0.000000000000000277 \*\*\*  
physical\_punishment < 0.0000000000000002 \*\*\*  
identity1 0.16678   
intervention1 0.00334 \*\*   
HDI 0.87030   
---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1  
  
Correlation of Fixed Effects:  
 (Intr) warmth physc\_ idntt1 intrv1  
warmth -0.124   
physcl\_pnsh -0.149 -0.003   
identity1 -0.072 -0.012 -0.003   
interventn1 -0.082 0.034 0.022 -0.018   
HDI -0.943 -0.006 0.009 -0.001 0.000

### Julia

#### 5.3.0.6 Get The Data

using Tables, MixedModels, StatFiles, DataFrames, CategoricalArrays, DataFramesMeta  
  
df = DataFrame(load("simulated\_multilevel\_data.dta"))

#### 5.3.0.7 Change Some Variables To Categorical

@transform!(df, :country = categorical(:country))  
  
@transform!(df, :identity = categorical(:identity))  
  
@transform!(df, :intervention = categorical(:intervention))

#### 5.3.0.8 Run The Model

m1 = fit(MixedModel, @formula(outcome ~ warmth + physical\_punishment +   
 identity + intervention + HDI +  
 (1 | country) +  
 (0 + warmth | country)), df)

Linear mixed model fit by maximum likelihood  
 outcome ~ 1 + warmth + physical\_punishment + identity + intervention + HDI + (1 | country) + (0 + warmth | country)  
 logLik -2 logLik AIC AICc BIC   
 -9626.6070 19253.2140 19271.2140 19271.2742 19325.2713  
  
Variance components:  
 Column Variance Std.Dev. Corr.  
country (Intercept) 2.963849 1.721583  
 warmth 0.022756 0.150852 .   
Residual 34.974984 5.913965  
 Number of obs: 3000; levels of grouping factors: 30  
  
 Fixed-effects parameters:  
─────────────────────────────────────────────────────────────  
 Coef. Std. Error z Pr(>|z|)  
─────────────────────────────────────────────────────────────  
(Intercept) 51.9999 1.37124 37.92 <1e-99  
warmth 0.834537 0.0637228 13.10 <1e-38  
physical\_punishment -0.991665 0.0797906 -12.43 <1e-34  
identity: 1.0 -0.300475 0.217029 -1.38 0.1662  
intervention: 1.0 0.639641 0.217452 2.94 0.0033  
HDI -0.0032286 0.0199255 -0.16 0.8713  
─────────────────────────────────────────────────────────────

## 5.4 Interpretation

Models suggest that parental warmth is associated with increases in the beneficial outcome, while physical punishment is associated with decreases in the beneficial outcome. The intervention is associated with increases in the outcome. There is insufficient evidence that either identity group or the Human Development Index are associated with the outcome.

# 6. Longitudinal Multilevel Models

## 6.1 The Data

The data employed in these examples are a longitudinal extension of the data described in [Section 1.2](#sec-data).

## 6.2 The Equation

## 6.3 Growth Trajectories

Remember, following Section 6.4, that in longitudinal multilevel models, the variable for *time* assumes an important role as we are often thinking of a *growth trajectory* over time.

As discussed in Section 6.4, think about a model where *identity* is a (1/0) variable for membership in one of two groups:

Then, each identity group has its own intercept and time trajectory:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 6.1: Slope and Intercept for Each Group   | Group | Intercept | Slope (Time Trajectory) | | --- | --- | --- | | 0 |  |  | | 1 |  |  | |

|  |  |
| --- | --- |
|  | **Main Effects and Interactions**  Thus, again following Section 6.4, in longitudinal multilevel models, *main effects* modify the *intercept* of the time trajectory, while *interactions with time*, modify the *slope* of the time trajectory. Below, we run models with *main effects only*, then models with *main effects, and interactions with time*. |

## 6.4 Run Models

|  |  |
| --- | --- |
|  | Remember that we are estimating a model in which time points are nested inside families, who are in turn nested inside countries. For each software package, it is accordingly important to specify the way in which different levels of the data are nested. Pay careful attention to the syntax examples below with regard to id and country |

### Stata

#### 6.4.0.1 Get The Data

use simulated\_multilevel\_longitudinal\_data.dta

#### 6.4.0.2 Run The Model

##### 6.4.0.2.1 Main Effects Only

mixed outcome t warmth physical\_punishment i.identity i.intervention HDI || country: warmth || id: t

Performing EM optimization ...  
  
Performing gradient-based optimization:   
Iteration 0: Log likelihood = -28523.49   
Iteration 1: Log likelihood = -28499.987   
Iteration 2: Log likelihood = -28499.739   
Iteration 3: Log likelihood = -28499.604   
Iteration 4: Log likelihood = -28499.603   
  
Computing standard errors ...  
  
Mixed-effects ML regression Number of obs = 9,000  
  
 Grouping information  
 -------------------------------------------------------------  
 | No. of Observations per group  
 Group variable | groups Minimum Average Maximum  
 ----------------+--------------------------------------------  
 country | 30 300 300.0 300  
 id | 3,000 3 3.0 3  
 -------------------------------------------------------------  
  
 Wald chi2(6) = 1096.15  
Log likelihood = -28499.603 Prob > chi2 = 0.0000  
  
-------------------------------------------------------------------------------------  
 outcome | Coefficient Std. err. z P>|z| [95% conf. interval]  
--------------------+----------------------------------------------------------------  
 t | .943864 .0658716 14.33 0.000 .814758 1.07297  
 warmth | .9134959 .0423732 21.56 0.000 .830446 .9965459  
physical\_punishment | -1.007897 .0497622 -20.25 0.000 -1.105429 -.9103647  
 1.identity | -.1276926 .1515835 -0.84 0.400 -.4247908 .1694057  
 1.intervention | .8589966 .1519095 5.65 0.000 .5612596 1.156734  
 HDI | -.0005657 .0196437 -0.03 0.977 -.0390666 .0379352  
 \_cons | 50.46724 1.338318 37.71 0.000 47.84418 53.09029  
-------------------------------------------------------------------------------------  
  
------------------------------------------------------------------------------  
 Random-effects parameters | Estimate Std. err. [95% conf. interval]  
-----------------------------+------------------------------------------------  
country: Independent |  
 var(warmth) | .0107586 .0127845 .0010478 .1104703  
 var(\_cons) | 3.167085 .9146761 1.798154 5.578181  
-----------------------------+------------------------------------------------  
id: Independent |  
 var(t) | 3.58e-09 7.06e-07 3.5e-177 3.7e+159  
 var(\_cons) | 8.387275 .4724188 7.510631 9.366242  
-----------------------------+------------------------------------------------  
 var(Residual) | 26.02733 .4753701 25.11211 26.97592  
------------------------------------------------------------------------------  
LR test vs. linear model: chi2(4) = 1247.03 Prob > chi2 = 0.0000  
  
Note: LR test is conservative and provided only for reference.

##### 6.4.0.2.2 Interactions With Time

mixed outcome c.t##(c.warmth c.physical\_punishment i.identity i.intervention c.HDI) || country: warmth || id: t

Performing EM optimization ...  
  
Performing gradient-based optimization:   
Iteration 0: Log likelihood = -28522.21   
Iteration 1: Log likelihood = -28498.677   
Iteration 2: Log likelihood = -28498.468   
Iteration 3: Log likelihood = -28498.31   
Iteration 4: Log likelihood = -28498.309   
  
Computing standard errors ...  
  
Mixed-effects ML regression Number of obs = 9,000  
  
 Grouping information  
 -------------------------------------------------------------  
 | No. of Observations per group  
 Group variable | groups Minimum Average Maximum  
 ----------------+--------------------------------------------  
 country | 30 300 300.0 300  
 id | 3,000 3 3.0 3  
 -------------------------------------------------------------  
  
 Wald chi2(11) = 1100.25  
Log likelihood = -28498.309 Prob > chi2 = 0.0000  
  
---------------------------------------------------------------------------------------  
 outcome | Coefficient Std. err. z P>|z| [95% conf. interval]  
----------------------+----------------------------------------------------------------  
 t | .7582075 .326177 2.32 0.020 .1189123 1.397503  
 warmth | .8170757 .082662 9.88 0.000 .6550611 .9790903  
 physical\_punishment | -1.009031 .1112932 -9.07 0.000 -1.227162 -.7909007  
 1.identity | -.2387167 .3039964 -0.79 0.432 -.8345387 .3571053  
 1.intervention | .6607606 .3044503 2.17 0.030 .064049 1.257472  
 HDI | .0013614 .0210842 0.06 0.949 -.0399628 .0426856  
 |  
 c.t#c.warmth | .0483637 .0356074 1.36 0.174 -.0214255 .1181529  
 |  
 c.t#|  
c.physical\_punishment | .0005421 .0494354 0.01 0.991 -.0963496 .0974338  
 |  
 identity#c.t |  
 1 | .0554389 .1317444 0.42 0.674 -.2027754 .3136532  
 |  
 intervention#c.t |  
 1 | .0992811 .131925 0.75 0.452 -.1592872 .3578493  
 |  
 c.t#c.HDI | -.0009551 .0038216 -0.25 0.803 -.0084453 .0065352  
 |  
 \_cons | 50.83632 1.483548 34.27 0.000 47.92862 53.74402  
---------------------------------------------------------------------------------------  
  
------------------------------------------------------------------------------  
 Random-effects parameters | Estimate Std. err. [95% conf. interval]  
-----------------------------+------------------------------------------------  
country: Independent |  
 var(warmth) | .0106014 .0127458 .0010046 .1118779  
 var(\_cons) | 3.170088 .9153355 1.80009 5.582753  
-----------------------------+------------------------------------------------  
id: Independent |  
 var(t) | 9.47e-10 2.07e-07 1.5e-195 6.0e+176  
 var(\_cons) | 8.39189 .4724106 7.515234 9.370809  
-----------------------------+------------------------------------------------  
 var(Residual) | 26.01583 .4751602 25.10101 26.964  
------------------------------------------------------------------------------  
LR test vs. linear model: chi2(4) = 1247.84 Prob > chi2 = 0.0000  
  
Note: LR test is conservative and provided only for reference.

### R

#### 6.4.0.3 Get The Data

library(haven)  
  
dfL <- read\_dta("simulated\_multilevel\_longitudinal\_data.dta")

#### 6.4.0.4 Change Some Variables To Categorical

dfL$identity <- factor(dfL$identity)  
  
dfL$intervention <- factor(dfL$intervention)

#### 6.4.0.5 Run The Model

|  |  |
| --- | --- |
|  | lme4 does not directly provide p values in results, because of some disagreement over exactly how these p values should be calculated. Therefore, in this Appendix, I also call library lmerTest to provide p values for lme4 results. |

|  |  |
| --- | --- |
|  | R prefers to use scientific notation when possible. I find that the use of scientific notation can be confusing in reading results. I turn off scientific notation by setting a penalty for its use: options(scipen = 999). |

##### 6.4.0.5.1 Main Effects Only

library(lme4)   
  
library(lmerTest)  
  
options(scipen = 999)   
  
fit2A <- lmer(outcome ~ t + warmth + physical\_punishment +   
 identity + intervention + HDI +  
 (1 | country/id),  
 data = dfL)  
  
summary(fit2A)

Linear mixed model fit by REML. t-tests use Satterthwaite's method [  
lmerModLmerTest]  
Formula:   
outcome ~ t + warmth + physical\_punishment + identity + intervention +   
 HDI + (1 | country/id)  
 Data: dfL  
  
REML criterion at convergence: 57022.7  
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-3.6850 -0.6094 -0.0035 0.6133 3.6792   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 id:country (Intercept) 8.438 2.905   
 country (Intercept) 3.675 1.917   
 Residual 26.036 5.103   
Number of obs: 9000, groups: id:country, 3000; country, 30  
  
Fixed effects:  
 Estimate Std. Error df t value  
(Intercept) 50.3842343 1.4139114 29.8246912 35.635  
t 0.9433806 0.0658755 5998.3764548 14.321  
warmth 0.9140307 0.0379336 4745.3497493 24.096  
physical\_punishment -1.0087537 0.0497972 6483.6771808 -20.257  
identity1 -0.1319548 0.1517350 2968.7828107 -0.870  
intervention1 0.8591494 0.1520510 2971.8111995 5.650  
HDI 0.0007909 0.0207656 28.0001855 0.038  
 Pr(>|t|)   
(Intercept) < 0.0000000000000002 \*\*\*  
t < 0.0000000000000002 \*\*\*  
warmth < 0.0000000000000002 \*\*\*  
physical\_punishment < 0.0000000000000002 \*\*\*  
identity1 0.385   
intervention1 0.0000000175 \*\*\*  
HDI 0.970   
---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1  
  
Correlation of Fixed Effects:  
 (Intr) t warmth physc\_ idntt1 intrv1  
t -0.092   
warmth -0.091 -0.002   
physcl\_pnsh -0.092 -0.007 -0.012   
identity1 -0.051 0.000 -0.013 -0.003   
interventn1 -0.058 0.000 0.039 0.019 -0.018   
HDI -0.951 0.000 -0.004 0.005 0.000 0.002

##### 6.4.0.5.2 Interactions With Time

fit2B <- lmer(outcome ~ t \*(warmth + physical\_punishment +   
 identity + intervention + HDI) +  
 (1 | country/id),  
 data = dfL)  
  
summary(fit2B)

Linear mixed model fit by REML. t-tests use Satterthwaite's method [  
lmerModLmerTest]  
Formula:   
outcome ~ t \* (warmth + physical\_punishment + identity + intervention +   
 HDI) + (1 | country/id)  
 Data: dfL  
  
REML criterion at convergence: 57042.8  
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-3.7118 -0.6092 -0.0024 0.6150 3.6779   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 id:country (Intercept) 8.436 2.905   
 country (Intercept) 3.675 1.917   
 Residual 26.046 5.104   
Number of obs: 9000, groups: id:country, 3000; country, 30  
  
Fixed effects:  
 Estimate Std. Error df t value  
(Intercept) 50.7590272 1.5518360 43.2608620 32.709  
t 0.7552909 0.3263028 6176.7440549 2.315  
warmth 0.8170912 0.0805355 8274.9995422 10.146  
physical\_punishment -1.0097729 0.1113557 8084.6084915 -9.068  
identity1 -0.2446453 0.3041604 8695.8966197 -0.804  
intervention1 0.6604671 0.3046286 8697.0843430 2.168  
HDI 0.0026692 0.0221295 36.1037733 0.121  
t:warmth 0.0486211 0.0356217 6404.8723333 1.365  
t:physical\_punishment 0.0004964 0.0494590 6753.0158441 0.010  
t:identity1 0.0563140 0.1318043 5993.4518022 0.427  
t:intervention1 0.0995037 0.1319917 5994.1433001 0.754  
t:HDI -0.0009379 0.0038233 5993.9090880 -0.245  
 Pr(>|t|)   
(Intercept) <0.0000000000000002 \*\*\*  
t 0.0207 \*   
warmth <0.0000000000000002 \*\*\*  
physical\_punishment <0.0000000000000002 \*\*\*  
identity1 0.4212   
intervention1 0.0302 \*   
HDI 0.9047   
t:warmth 0.1723   
t:physical\_punishment 0.9920   
t:identity1 0.6692   
t:intervention1 0.4510   
t:HDI 0.8062   
---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1  
  
Correlation of Fixed Effects:  
 (Intr) t warmth physc\_ idntt1 intrv1 HDI t:wrmt t:phy\_  
t -0.421   
warmth -0.178 0.331   
physcl\_pnsh -0.190 0.360 -0.005   
identity1 -0.093 0.166 -0.013 -0.002   
interventn1 -0.107 0.192 0.039 0.019 -0.017   
HDI -0.925 0.264 -0.007 0.012 -0.001 0.003   
t:warmth 0.158 -0.377 -0.882 0.001 0.011 -0.035 0.006   
t:physcl\_pn 0.170 -0.402 0.004 -0.894 -0.001 -0.017 -0.010 -0.003   
t:identity1 0.081 -0.192 0.011 0.000 -0.867 0.014 0.001 -0.013 0.002  
t:intrvntn1 0.093 -0.222 -0.035 -0.017 0.014 -0.867 -0.003 0.041 0.019  
t:HDI 0.322 -0.765 0.015 -0.027 0.002 -0.007 -0.346 -0.016 0.029  
 t:dnt1 t:ntr1  
t   
warmth   
physcl\_pnsh   
identity1   
interventn1   
HDI   
t:warmth   
t:physcl\_pn   
t:identity1   
t:intrvntn1 -0.016   
t:HDI -0.002 0.008

### Julia

#### 6.4.0.6 Get The Data

using Tables, MixedModels, StatFiles, DataFrames, CategoricalArrays, DataFramesMeta  
  
dfL = DataFrame(load("simulated\_multilevel\_longitudinal\_data.dta"))

#### 6.4.0.7 Run The Model

##### 6.4.0.7.1 Change Country To Categorical

@transform!(dfL, :country = categorical(:country))  
  
@transform!(dfL, :identity = categorical(:identity))  
  
@transform!(dfL, :intervention = categorical(:intervention))

##### 6.4.0.7.2 Main Effects Only

m2A = fit(MixedModel, @formula(outcome ~ t + warmth +   
 physical\_punishment +   
 identity + intervention +   
 HDI +  
 (1 | country) +   
 (0 + warmth | country) +  
 (1 | id)), dfL)

Linear mixed model fit by maximum likelihood  
 outcome ~ 1 + t + warmth + physical\_punishment + identity + intervention + HDI + (1 | country) + (0 + warmth | country) + (1 | id)  
 logLik -2 logLik AIC AICc BIC   
 -28499.6031 56999.2063 57021.2063 57021.2356 57099.3610  
  
Variance components:  
 Column Variance Std.Dev. Corr.  
id (Intercept) 8.387214 2.896069  
country (Intercept) 3.167143 1.779647  
 warmth 0.010762 0.103739 .   
Residual 26.027363 5.101702  
 Number of obs: 9000; levels of grouping factors: 3000, 30  
  
 Fixed-effects parameters:  
───────────────────────────────────────────────────────────────  
 Coef. Std. Error z Pr(>|z|)  
───────────────────────────────────────────────────────────────  
(Intercept) 50.4673 1.33833 37.71 <1e-99  
t 0.943864 0.0658717 14.33 <1e-45  
warmth 0.913496 0.0423744 21.56 <1e-99  
physical\_punishment -1.0079 0.0497622 -20.25 <1e-90  
identity: 1.0 -0.127692 0.151583 -0.84 0.3996  
intervention: 1.0 0.858997 0.151909 5.65 <1e-07  
HDI -0.000566026 0.0196439 -0.03 0.9770  
───────────────────────────────────────────────────────────────

##### 6.4.0.7.3 Interactions With Time

m2B = fit(MixedModel, @formula(outcome ~ t \* (warmth +   
 physical\_punishment +   
 identity + intervention +   
 HDI) +  
 (1 | country) +  
 (0 + warmth | country) +  
 (1 | id)), dfL)

Linear mixed model fit by maximum likelihood  
 outcome ~ 1 + t + warmth + physical\_punishment + identity + intervention + HDI + t & warmth + t & physical\_punishment + t & identity + t & intervention + t & HDI + (1 | country) + (0 + warmth | country) + (1 | id)  
 logLik -2 logLik AIC AICc BIC   
 -28498.3091 56996.6182 57028.6182 57028.6788 57142.2979  
  
Variance components:  
 Column Variance Std.Dev. Corr.  
id (Intercept) 8.391746 2.896851  
country (Intercept) 3.170032 1.780458  
 warmth 0.010609 0.102999 .   
Residual 26.015906 5.100579  
 Number of obs: 9000; levels of grouping factors: 3000, 30  
  
 Fixed-effects parameters:  
──────────────────────────────────────────────────────────────────  
 Coef. Std. Error z Pr(>|z|)  
──────────────────────────────────────────────────────────────────  
(Intercept) 50.8364 1.48355 34.27 <1e-99  
t 0.758209 0.326177 2.32 0.0201  
warmth 0.817076 0.0826636 9.88 <1e-22  
physical\_punishment -1.00903 0.111293 -9.07 <1e-18  
identity: 1.0 -0.238714 0.303996 -0.79 0.4323  
intervention: 1.0 0.660761 0.30445 2.17 0.0300  
HDI 0.00136065 0.0210842 0.06 0.9485  
t & warmth 0.0483635 0.0356074 1.36 0.1744  
t & physical\_punishment 0.000542203 0.0494355 0.01 0.9912  
t & identity: 1.0 0.0554385 0.131745 0.42 0.6739  
t & intervention: 1.0 0.0992809 0.131925 0.75 0.4517  
t & HDI -0.000955067 0.00382162 -0.25 0.8027  
──────────────────────────────────────────────────────────────────

## 6.5 Interpretation

The *main effects only model* suggests that time is associated with increases in the outcome. In the main effects model, main effects other than time, indicate whether a particular variable is associated with higher or lower intercepts of the time trajectory, at the beginning of the study time. Warmth is again associated with increases in the outcome, while physical punishment is associated with decreases in the outcome. Identity is again not associated with the outcome, while the intervention is associated with higher levels of the outcome. The Human Development Index is again not associated with the outcome.

The second model adds interactions with time to the first model. Results are largely similar to the prior model. However, here we not only examine whether main effects other than time are associated with higher or lower time trajectories, but also whether particular variables are associated with differences in the slope of the time trajectory. In this case, we find insufficient evidence that any independent variable is associated with changes in the slope of the time trajectory.

|  |  |
| --- | --- |
|  | **Which Interactions To Test?**  In this example–for the sake of illustration–I test the interaction of *every* independent variable with time. In many cases, it may make sense to test only only one or two interactions of time with particular variables of key interest. Also, after finding, as I did in this model, that none of the interactions of other independent variables with time are significant, I might report the model with interactions, or might report only the results of the model with only main effects. |

It may be illustrative to imagine how we would interpret the results had a particular interaction term been statistically significant. Let us consider one of the interaction terms with the largest coefficient, intervention#time. The interaction of the intervention with time is positive. Had this coefficient been statistically signifcant, it would have indicated that the intervention was associated with more rapid increases in the outcome over time *in addition to* the fact that the intervention is associated with higher initial levels of the outcome.

# References

Allaire, J. J., Teague, C., Scheidegger, C., Xie, Y., & Dervieux, C. (2024). *Quarto* (Version 1.4). <https://doi.org/10.5281/zenodo.5960048>

Bates, D. (2024). *MixedModels.jl Documentation*. <https://juliastats.org/MixedModels.jl/stable/>

Bates, D., Mächler, M., Bolker, B., & Walker, S. (2015). Fitting linear mixed-effects models using lme4. *Journal of Statistical Software*, *67*(1), 1–48. <https://doi.org/10.18637/jss.v067.i01>

Bezanson, J., Edelman, A., Karpinski, S., & Shah, V. B. (2017). Julia: A fresh approach to numerical computing. *SIAM Review*, *59*(1), 65–98. <https://doi.org/10.1137/141000671>

Hemken, D. (2023). *Statamarkdown: ’Stata’ markdown*. <https://CRAN.R-project.org/package=Statamarkdown>

Li, C. (2019). JuliaCall: An R package for seamless integration between R and Julia. *The Journal of Open Source Software*, *4*(35), 1284. <https://doi.org/10.21105/joss.01284>

R Core Team. (2023). *R: A language and environment for statistical computing*. R Foundation for Statistical Computing. <https://www.R-project.org/>

Schanen, J. (2021). *Math person (Strogatz Prize entry)*. National Museum of Mathematics.

StataCorp. (2023). *Stata 18 mixed effects reference manual*. Stata Press.

Thoreau, H. D. (1975). The commercial spirit of modern times [1837]. In J. J. Moldenhauer, E. Moser, & A. C. Kern (Eds.), *Early essays and miscellanies*. Princeton University Press.

1. skimr is an excellent new alternative library for generating descriptive statistics in R. [↑](#footnote-ref-54)