Some Stuff About Logarithms

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1 Introduction (The Linear Function)

We begin with the linear function:

$$y = ax$$

The coefficient a tells us how much y increases for a 1 unit increase in x.

```
twoway function y = 2 * x, lwidth(thick) ///
title("Linear Function") subtitle("With Coefficient 2") ///
range(-10 10)

graph export linear0.png, replace
```

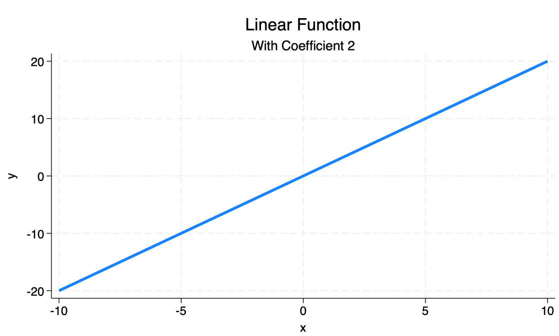

Figure 1: Graph of linear function with coefficient 2

2 The Exponential Function

We then introduce the exponential function:

$$y = \text{base}^{\text{exponent}}$$

The *exponent* tells us how many times to multiply the base by itself to get the result.

For example: $2^3 = 8$ because $2 \times 2 \times 2 = 8$

```
twoway function y = 2^x, lwidth(thick) ///
title("Exponential Function") subtitle("With Base 2") ///
range(-10 10)

graph export exponential0.png, replace
```

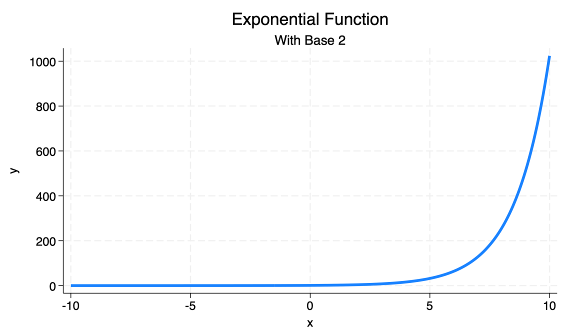

Figure 2: Graph of exponential function with base 2

Applications of Exponential Functions

Exponential functions–and hence their inverses, logarithmic functions described below–have applications in the study of categorical data analysis, radioactive decay, disease spread, and population growth.

3 A Definition of the Logarithm

We then consider the logarithm.

If

$$y = b^x$$

then

$$\log_b(y) = x$$

.

In words: If $\text{number} = \text{base}^{\text{exponent}}$ then $\log_{\text{base}}(\text{number}) = \text{exponent}$.

For example, $2^3 = 8$, therefore $log_2 8 = 3$.

The logarithm answers the question: *What is the power to which we have to raise the number to get the result?*

The logarithm may thus be thought of as *the inverse of the exponential function*.

```
twoway function y = ln(x)/ln(2), lwidth(thick) ///
title("Logarithmic Function") subtitle("Base 2") ///
range(-10 10)

graph export logarithmic0.png, replace
```

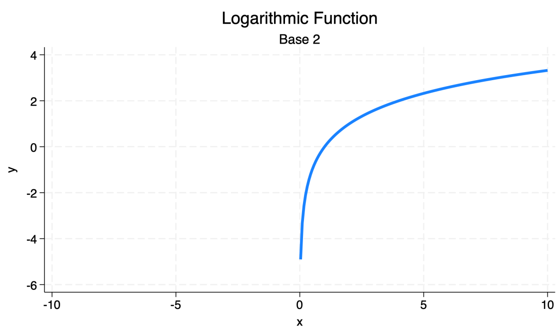

Figure 3: Graph of logarithmic function with base 2

4 A Definition of the Natural Logarithm

For deep mathematical reasons, it is often useful to use logarithms with base e which is often termed the *natural logarithm*, written ln

e is a kind of fundamental mathematical constant, like π, but without the easy geometric definition that π has. (For any ○, $\pi = \frac{\text{circumference}}{\text{diameter}}$.)

e emerges in many contexts, including some aspects of compound growth processes[1]. e is approximately equal to 2.71828....

If

$$y = e^x$$

then

$$\ln(y) = x$$

[1] A common definition of e is

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

.

```
twoway function y = exp(x), lwidth(thick) ///
title("Exponential Function") subtitle("Base e") ///
range(-10 10)

graph export exponential.png, replace
```

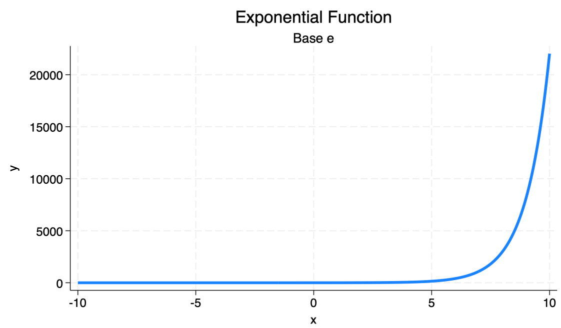

Figure 4: Graph of exponential function with base e

```
twoway function y = ln(x), lwidth(thick) ///
title("Logarithmic Function") subtitle("Base e") ///
range(-10 10)

graph export logarithmic.png, replace
```

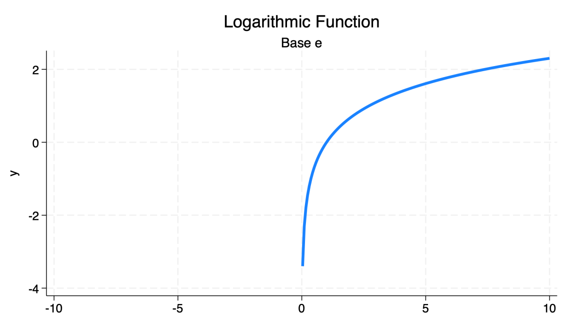

Figure 5: Graph of logarithmic function with base e

5 Exponential Regression

In categorical data analysis–especially later in the course–we are often thinking about some equation like $\ln(y) = \beta x$. This is equivalent to $y = e^{\beta x}$ so many models–particularly later in the course–will have us thinking about *exponential* relationships.

6 Logistic Regression

Early on in this course, we will think about logistic regression. In logistic regression, we start by thinking about the *odds* of our outcome:

$$\frac{p(y)}{1 - p(y)}$$

We will ultimately be working with the *logarithm* of the odds, or the *log odds*:

$$\ln\left(\frac{p(y)}{1 - p(y)}\right) = x$$

To *graph* these *log odds*, we need to solve for $p(y)$:

$$p(y) = \frac{e^x}{1 + e^x}$$

```
twoway function y = exp(x)/(1 + exp(x)), lwidth(thick) ///
title("Logistic Function") ///
range(-10 10)

graph export logistic.png, replace
```

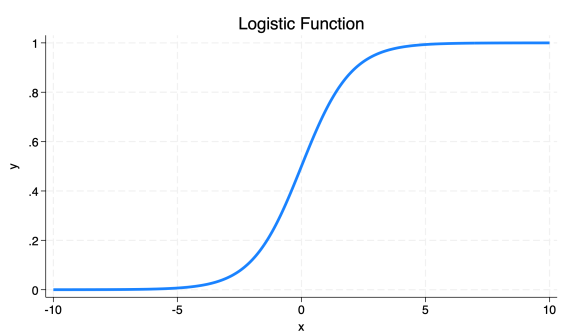

Figure 6: Logistic curve

This function is sometimes called a *sigmoid*, and has the interesting property of mapping the interval $-\infty < x < \infty$ to $0 < y < 1$. (This is the first step in mapping a *continuous* predictor to a *categorical* outcome.)

7 Logarithmic Spiral

An interesting sidenote is that the logarithm forms the basis of the logarithmic spiral. The equation for a logarithmic spiral in polar coordinates is: $r = ae^{b\theta}$, where θ is the angle, r is the radius, and a and b are constants.

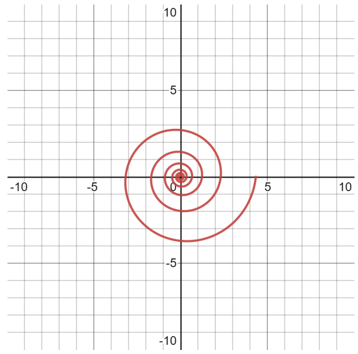

Figure 7: Desmos Graph Logarithmic Spiral

Logarithmic spirals can be found in nature in the *nautilus shell*, and in *sunflowers* and in the *flight of hawks*.

Figure 1: Nautilus Shell, Courtesy Wikipedia

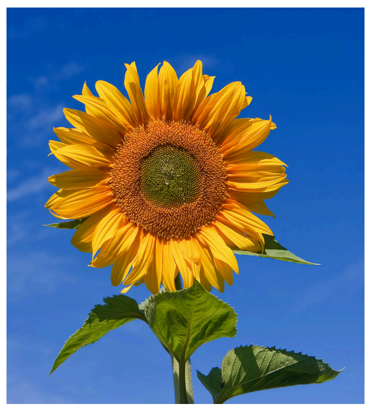

Figure 2: Sunflower, Courtesy Wikipedia

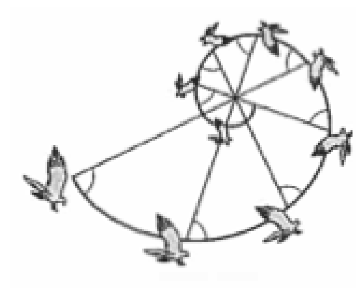

Figure 3: Spiral Flight of Hawk from M. Livio [1]

Logarithmic Spirals

Bibliography

[1] M. Livio, *The golden ratio: the story of phi, the world's most astonishing number*, 1st ed. Broadway Books, 2002.