MONASH University

MONASH BUSINESS SCHOOL

Forecasting: principles and practice

Rob J Hyndman

2 ARIMA models

Outline

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \ldots, y_{t+s})$ does not depend on t.

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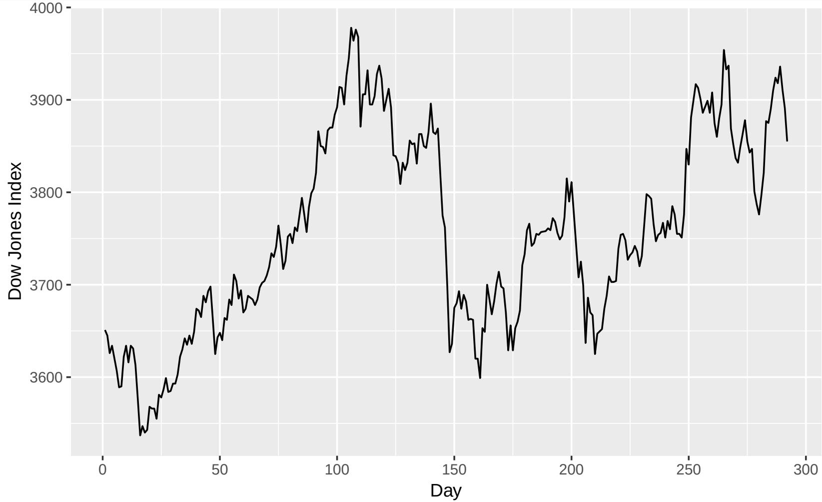
A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

- Transformations (e.g., logs) can help to **stabilize the variance**.
- Differences can help to **stabilize the mean**.

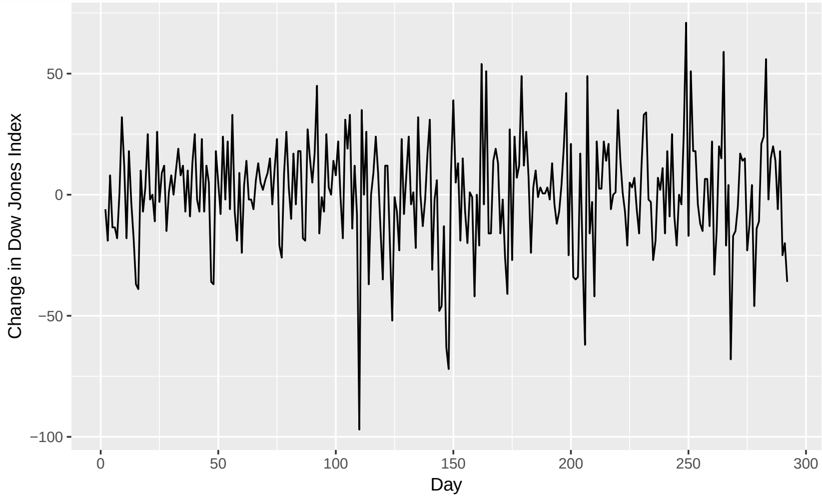
Stationary?

```
dj %>% autoplot() +
  ylab("Dow Jones Index") + xlab("Day")
```



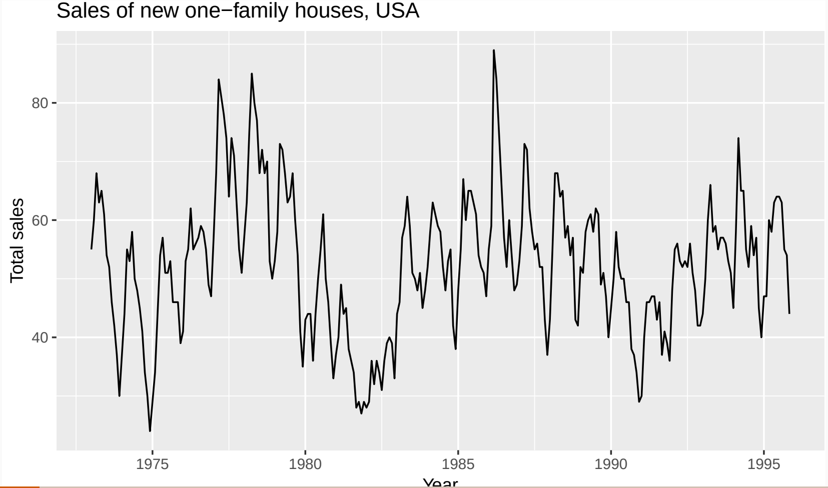
Stationary?

```
dj %>% diff() %>% autoplot() +
  ylab("Change in Dow Jones Index") + xlab("Day")
```



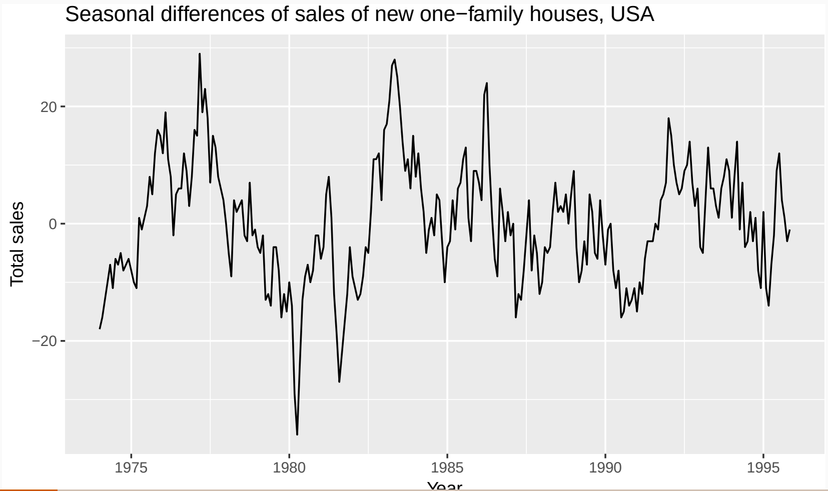
Stationary?

```r
hsales %>% autoplot() +
  xlab("Year") + ylab("Total sales") +
  ggtitle("Sales of new one-family houses, USA")
```



Stationary?

```r
hsales %>% diff(lag=12) %>% autoplot() +
  xlab("Year") + ylab("Total sales") +
  ggtitle("Seasonal differences of sales of new one-family h
```



Stationary?

```
hsales %>% diff(lag=12) %>% diff(lag=1) %>% autoplot() +
  xlab("Year") + ylab("Total sales") +
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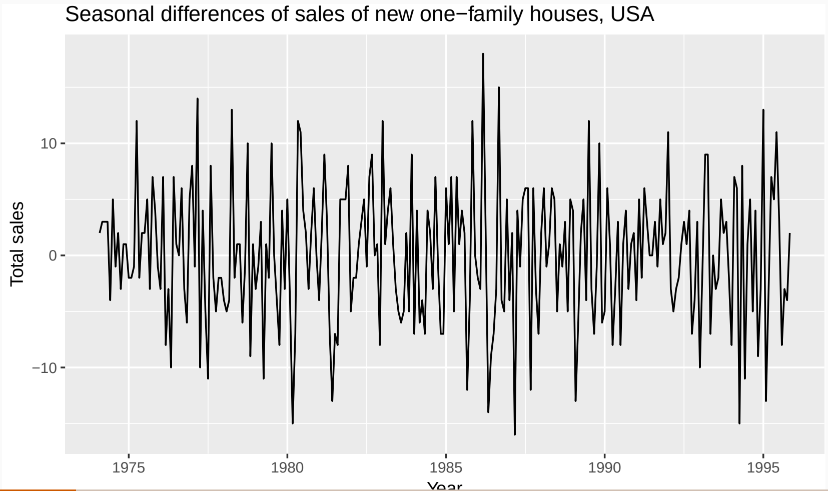
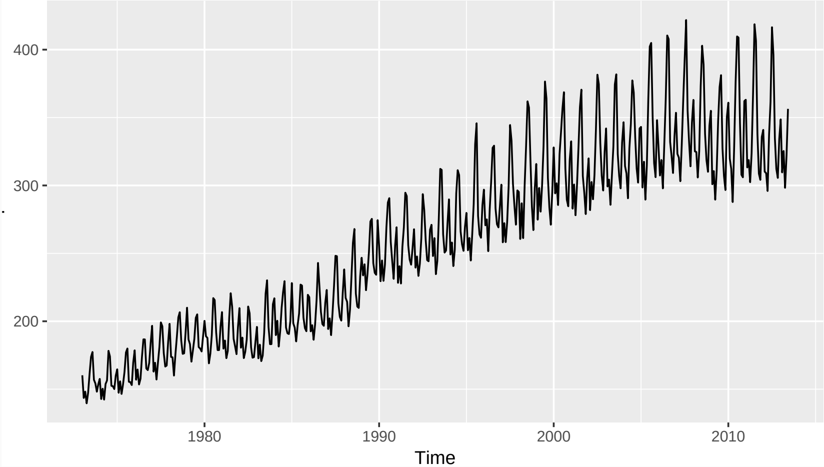


Electricity production

```
usmelec %>% autoplot()
```



Electricity production

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usmelec %>% log() %>% autoplot()
```

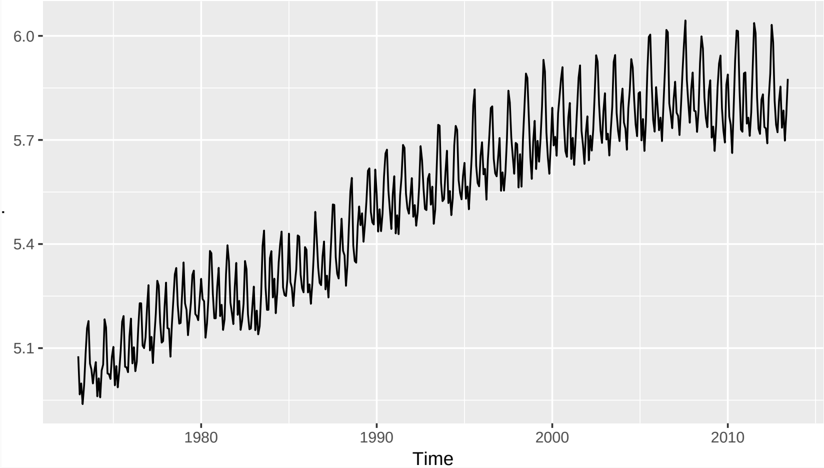

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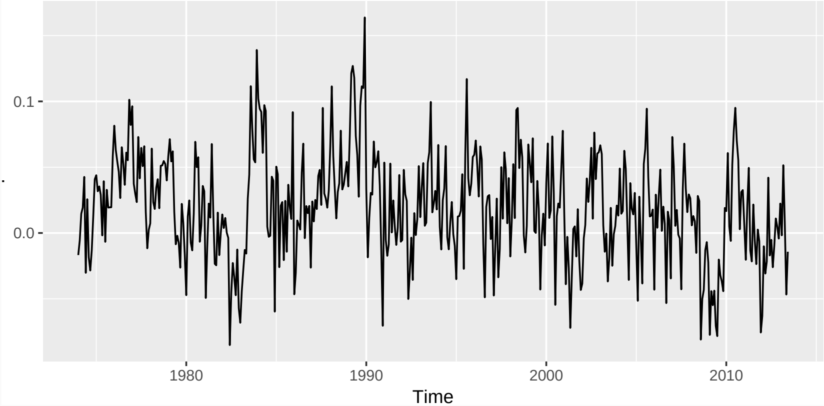

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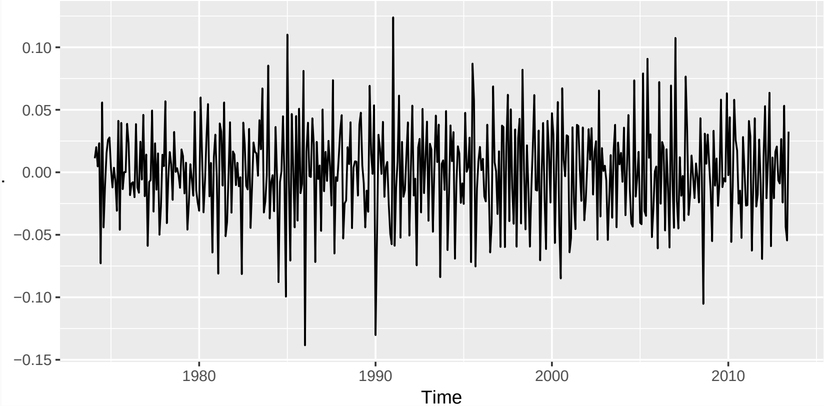

Outline

Backshift notation

Backward shift operator

Shift back one period

$$By_t = y_{t-1}$$

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Backward shift operator

Shift back one period

$$By_t = y_{t-1}$$

Shift back two periods:

$$B(By_t) = B^2y_t = y_{t-2}$$

Shift back 12 periods

$$B^{12}y_t = y_{t-12}$$

Backshift notation

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$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t\ .$$

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$$y''_t = (1 - B)^2 y_t\ .$$
- *d*th-order differences:
$$(1 - B)^d y_t.$$
- Seasonal difference followed by first difference:
$$(1 - B)(1 - B^m)y_t\ .$$

Outline

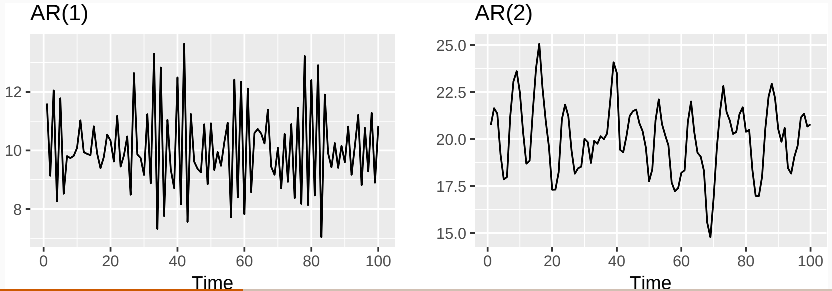
Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

$$(1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + \varepsilon_t$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:

  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1.$
- More complicated conditions hold for $p \geq 3$.

Outline

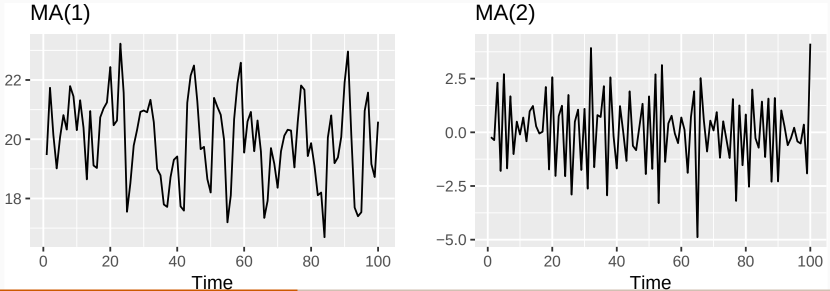
Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}$$

$$y_t = c + (1 + \theta_1 B + \cdots + \theta_1 B^q)\varepsilon_t$$

where ε_t is white noise. This is a multiple regression with ***past errors*** as predictors.



Invertibility

- Invertible models have property that distant past has negligible effect on forecasts. Requires consraints on MA parameters.

General condition for invertibility

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General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

- For $q = 1$: $-1 < \theta_1 < 1$.
- For $q = 2$:

$$-1 < \theta_2 < 1 \qquad \theta_2 + \theta_1 > -1 \qquad \theta_1 - \theta_2 < 1.$$

- More complicated conditions hold for $q \geq 3$.

Outline

ARIMA models

Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t
\end{aligned}
$$

$$\phi_p(B) y_t = \theta_q(B) \varepsilon_t$$

ARIMA models

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$$\phi_p(B) y_t = \theta_q(B) \varepsilon_t$$

- Predictors include both **lagged values of y_t and lagged errors.**
- $\phi_p(B)$ is a pth order polynomial in B
- $\theta_q(B)$ is a qth order polynomial in B

ARIMA models

Autoregressive Moving Average models:

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Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- $(1 - B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR:	p = order of the autoregressive part
I:	d = degree of first differencing involved
MA:	q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)
- MA(q): ARIMA(0,0,q)

Backshift notation for ARIMA

- ARIMA($p, 0, q$) model:

$$
\begin{aligned}
y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t \\
y_t &= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t \\
\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t &= c + (1 + \theta_1 B + \cdots + \theta_q B^q) \varepsilon_t
\end{aligned}
$$

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$$

- ARIMA(1,1,1) model:

$$
\underbrace{(1 - \phi_1 B)}_{\text{AR(1)}} \quad \underbrace{(1 - B) y_t}_{\text{First difference}} \quad = \quad c + \underbrace{(1 + \theta_1 B) \varepsilon_t}_{\text{MA(1)}}
$$

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- ARIMA(1,1,1) model:

$$
\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B) y_t} \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B) \varepsilon_t}
$$

Written out:

$$
y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t
$$

R model

Intercept form

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)^d y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

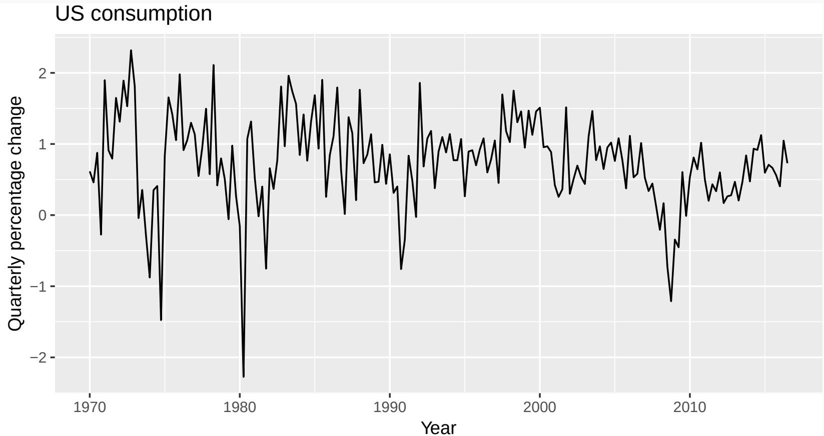
Mean form

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)^d (y_t - \mu t^d / d!) = (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- μ is the mean of $(1 - B)^d y_t$.
- $c = \mu(1 - \phi_1 - \cdots - \phi_p)$.
- R uses mean form.
- Including c equivalent to y_t having dth order polynomial trend.

US personal consumption

```r
autoplot(uschange[,"Consumption"]) +
  xlab("Year") + ylab("Quarterly percentage change") +
  ggtitle("US consumption")
```



US personal consumption

```
(fit <- auto.arima(uschange[,"Consumption"]))
```

```
## Series: uschange[, "Consumption"]
## ARIMA(2,0,2) with non-zero mean
##
## Coefficients:
##          ar1     ar2     ma1    ma2   mean
##        1.391  -0.581  -1.180  0.558  0.746
## s.e.   0.255   0.208   0.238  0.140  0.084
##
## sigma^2 estimated as 0.351:  log likelihood=-165.1
## AIC=342.3   AICc=342.8   BIC=361.7
```

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```

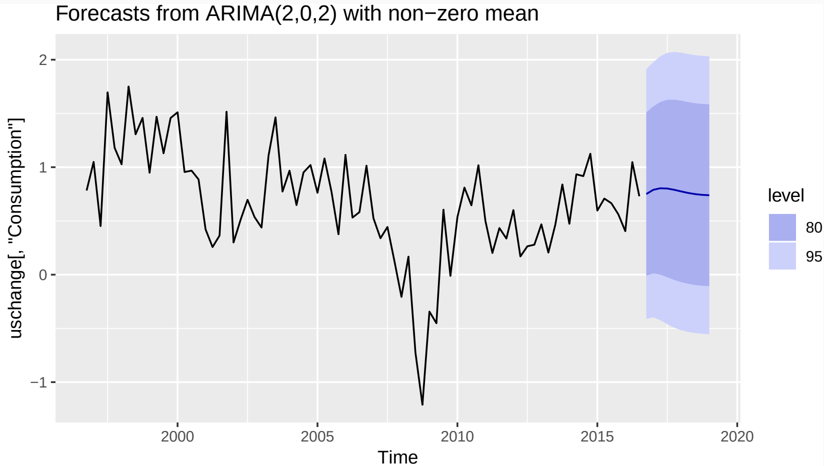
ARIMA(2,0,2) model:

$$y_t = c + 1.391y_{t-1} - 0.581y_{t-2} - 1.180\varepsilon_{t-1} + 0.558\varepsilon_{t-2} + \varepsilon_t,$$

where $c = 0.746 \times (1 - 1.391 + 0.581) = 0.142$ and $\varepsilon_t \sim N(0, 0.351)$.

US personal consumption

```
fit %>% forecast(h=10) %>% autoplot(include=80)
```



Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(L) + 2(p+q+k+1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

Good models are obtained by minimizing the AICc.

How does auto.arima() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via KPSS test and seasonal strength measure.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does auto.arima() work?

Step 1: Select values of d and D.

Step 2: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

How does auto.arima() work?

Step 1: Select values of d and D.

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ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

Step 3: Consider variations of current model:

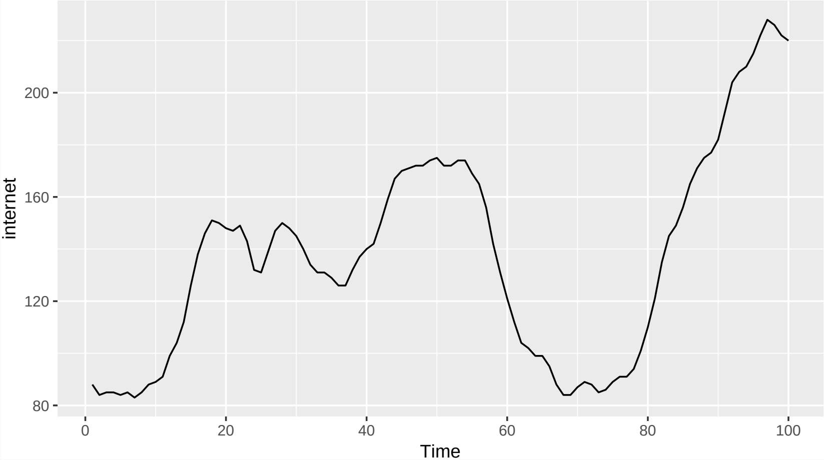
- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 3 until no lower AICc can be found.

Choosing an ARIMA model

```r
autoplot(internet)
```



Choosing an ARIMA model

```r
(fit <- auto.arima(internet))
```

```
## Series: internet
## ARIMA(1,1,1)
##
## Coefficients:
##          ar1    ma1
##        0.650  0.526
## s.e.   0.084  0.090
##
## sigma^2 estimated as 10:  log likelihood=-254.2
## AIC=514.3   AICc=514.5   BIC=522.1
```

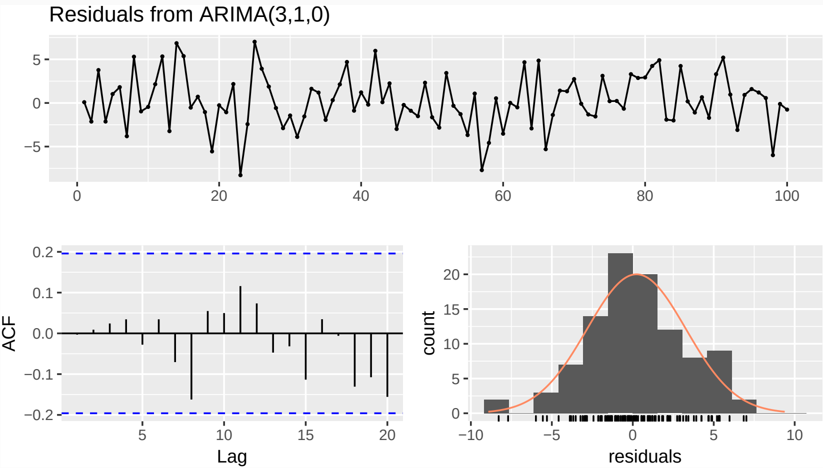
Choosing an ARIMA model

```r
(fit <- auto.arima(internet, stepwise=FALSE,
  approximation=FALSE))
```

```
## Series: internet
## ARIMA(3,1,0)
##
## Coefficients:
##          ar1     ar2    ar3
##        1.151  -0.661  0.341
## s.e.   0.095   0.135  0.094
##
## sigma^2 estimated as 9.66:  log likelihood=-252
## AIC=512   AICc=512.4   BIC=522.4
```

Choosing an ARIMA model

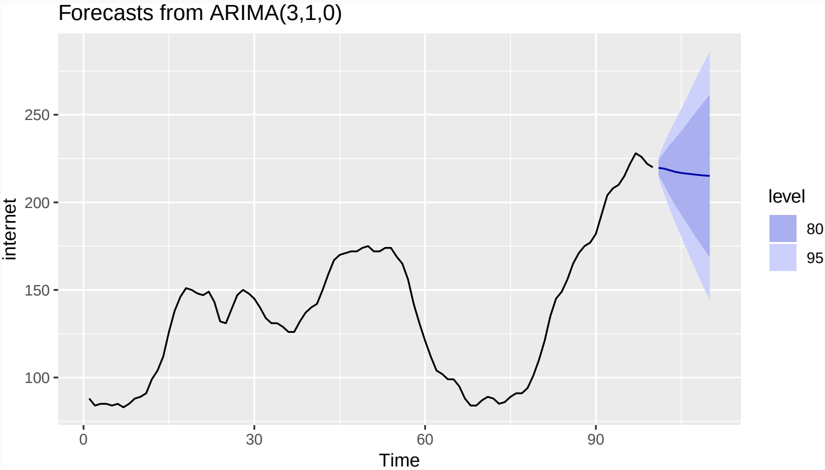
```
checkresiduals(fit, plot=TRUE)
```



#

Choosing an ARIMA model

```
fit %>% forecast() %>% autoplot()
```



Outline

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

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E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

Seasonal ARIMA models

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$\begin{pmatrix}\text{Non-seasonal}\\ \text{AR(1)}\end{pmatrix}$ $\begin{pmatrix}\text{Seasonal}\\ \text{AR(1)}\end{pmatrix}$ $\begin{pmatrix}\text{Non-seasonal}\\ \text{difference}\end{pmatrix}$ $\begin{pmatrix}\text{Seasonal}\\ \text{difference}\end{pmatrix}$ $\begin{pmatrix}\text{Non-seasonal}\\ \text{MA(1)}\end{pmatrix}$ $\begin{pmatrix}\text{Seasonal}\\ \text{MA(1)}\end{pmatrix}$

Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

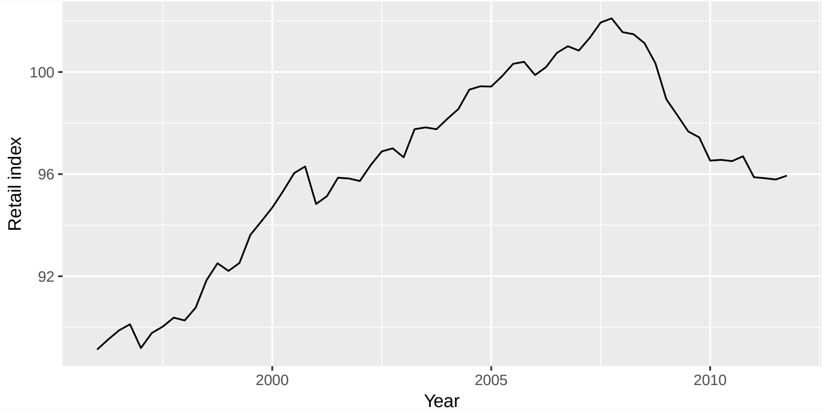
$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned}
y_t &= (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\
&\quad - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\
&\quad - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\
&\quad + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}.
\end{aligned}$$

European quarterly retail trade

```r
autoplot(euretail) +
  xlab("Year") + ylab("Retail index")
```



European quarterly retail trade

```r
(fit <- auto.arima(euretail))
```

```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1      ma1     ma2     sma1
##        0.736  -0.466  0.216  -0.843
## s.e.   0.224   0.199  0.210   0.188
##
## sigma^2 estimated as 0.159:  log likelihood=-29.62
## AIC=69.24   AICc=70.38   BIC=79.63
```

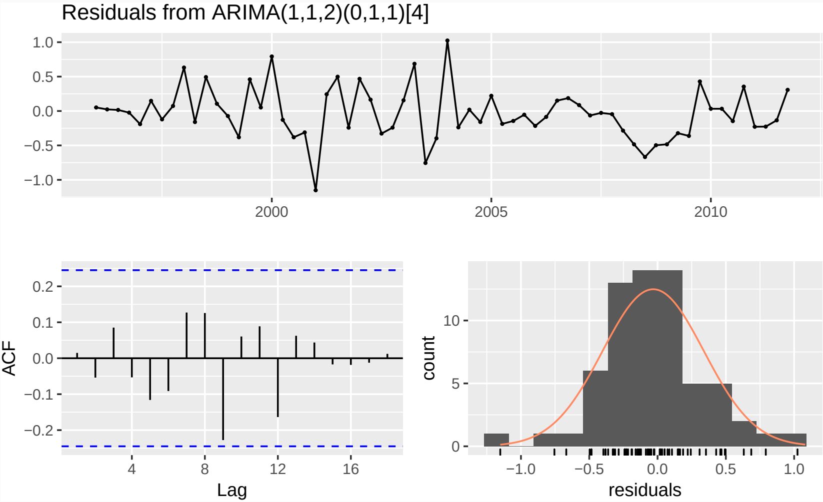
European quarterly retail trade

```r
(fit <- auto.arima(euretail, stepwise=TRUE,
  approximation=FALSE))
```

```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1     ma1    ma2     sma1
##        0.736  -0.466  0.216  -0.843
## s.e.   0.224   0.199  0.210   0.188
##
## sigma^2 estimated as 0.159:  log likelihood=-29.62
## AIC=69.24   AICc=70.38   BIC=79.63
```

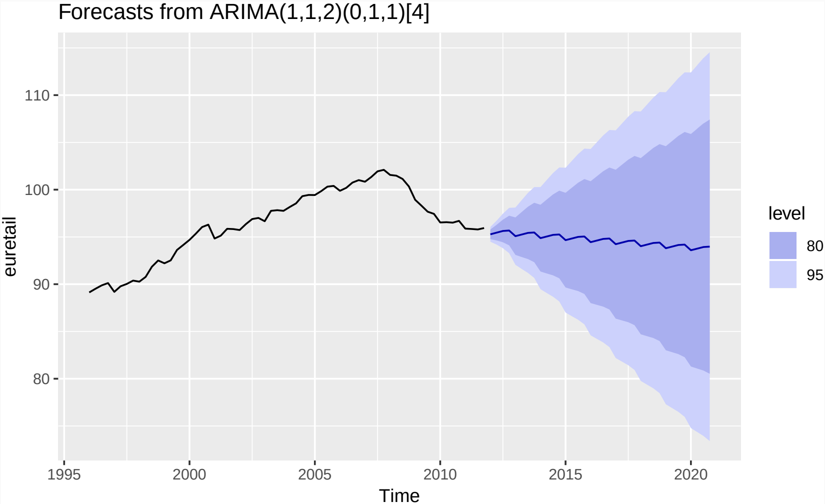
European quarterly retail trade

```r
checkresiduals(fit, test=FALSE)
```

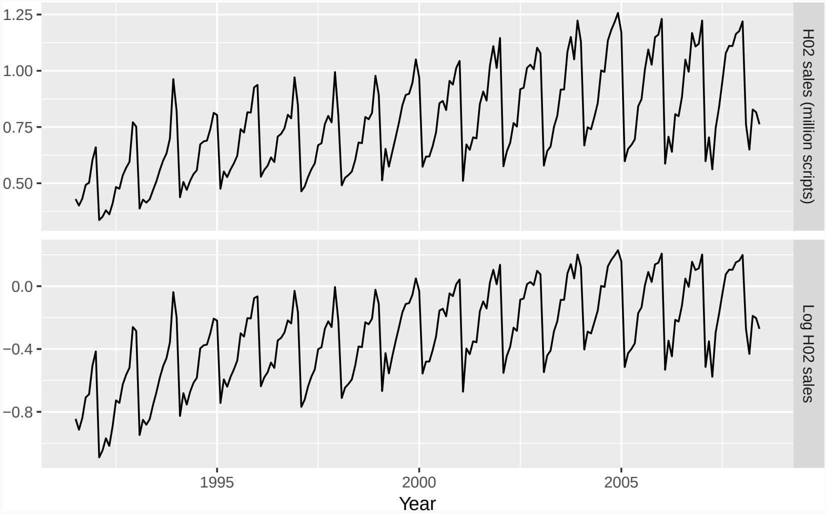


European quarterly retail trade

```r
forecast(fit, h=36) %>% autoplot()
```



Cortecosteroid drug sales



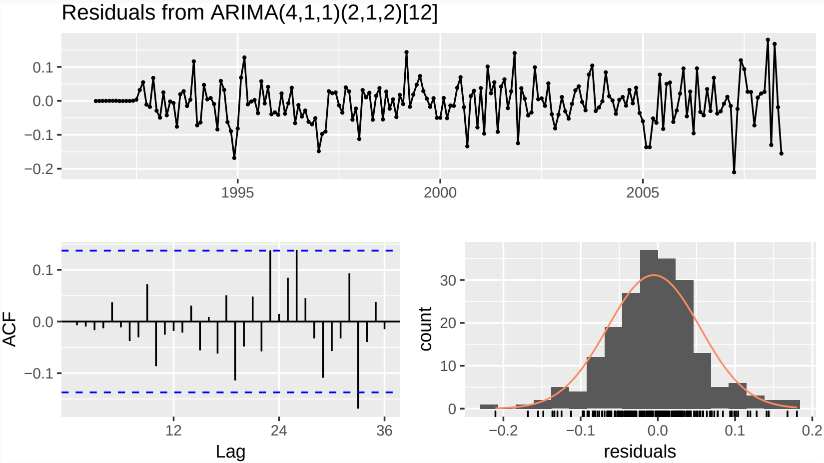
Cortecosteroid drug sales

```r
(fit <- auto.arima(h02, lambda=0, max.order=9,
  stepwise=FALSE, approximation=FALSE))
```

```
## Series: h02
## ARIMA(4,1,1)(2,1,2)[12]
## Box Cox transformation: lambda= 0
##
## Coefficients:
##          ar1    ar2    ar3     ar4     ma1   sar1
##       -0.042  0.210  0.202  -0.227  -0.742  0.621
## s.e.   0.217  0.181  0.114   0.081   0.207  0.242
##         sar2    sma1   sma2
##       -0.383  -1.202  0.496
## s.e.   0.118   0.249  0.214
##
```

Cortecosteroid drug sales

```
checkresiduals(fit)
```



##

Understanding ARIMA models

Long-term forecasts

zero	$c = 0, d + D = 0$	
non-zero constant	$c = 0, d + D = 1$	$c \neq 0, d + D = 0$
linear	$c = 0, d + D = 2$	$c \neq 0, d + D = 1$
quadratic	$c = 0, d + D = 3$	$c \neq 0, d + D = 2$

Forecast variance and $d + D$

- The higher the value of $d + D$, the more rapidly the prediction intervals increase in size.
- For $d + D = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period.
  - the ARIMA model assumes uncorrelated future errors

Outline

Lab Session 3