MONASH University

MONASH BUSINESS SCHOOL

Forecasting: principles and practice

Rob J Hyndman

3.3 Hierarchical forecasting

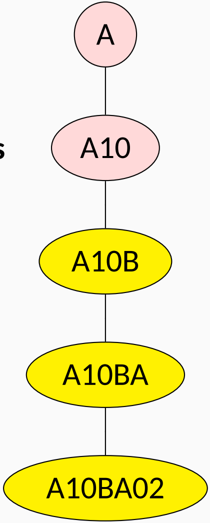
Outline

Forecasting the PBS

ATC drug classification

A	Alimentary tract and metabolism
B	Blood and blood forming organs
C	Cardiovascular system
D	Dermatologicals
G	Genito-urinary system and sex hormones
H	Systemic hormonal preparations, excluding sex hormones and insulins
J	Anti-infectives for systemic use
L	Antineoplastic and immunomodulating agents
M	Musculo-skeletal system
N	Nervous system
P	Antiparasitic products, insecticides and repellents
R	Respiratory system
S	Sensory organs
V	Various

ATC drug classification



14 classes — Alimentary tract and metabolism

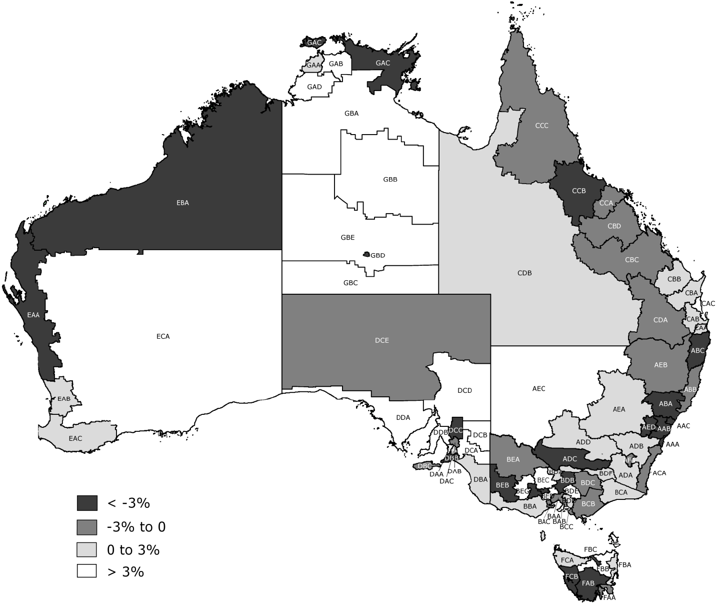
84 classes — Drugs used in diabetes

Blood glucose lowering drugs

Biguanides

Metformin

Australian tourism



Australian tourism

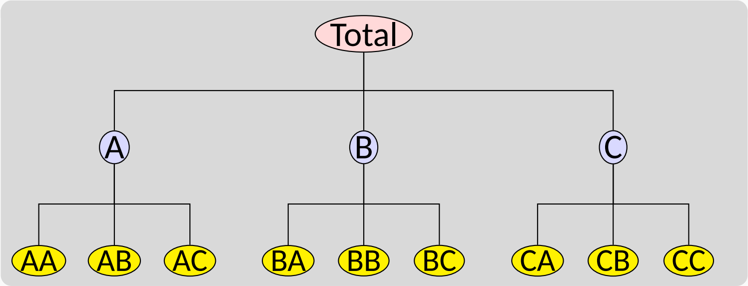
- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series

Spectacle sales

- Monthly UK sales data from 2000 – 2014
- Provided by a large spectacle manufacturer
- Split by brand (26), gender (3), price range (6), materials (4), and stores (600)
- About 1 million bottom-level series

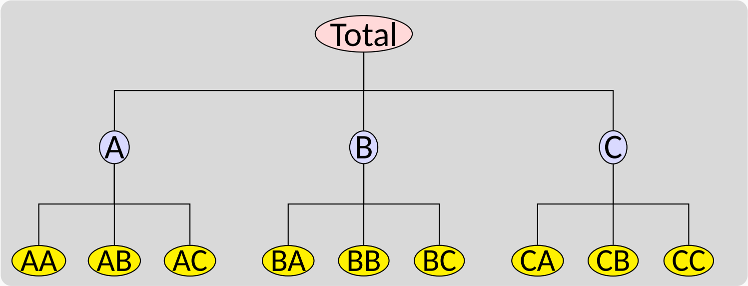
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.

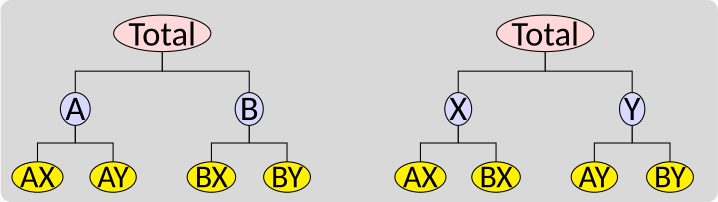


Examples

- Pharmaceutical sales
- Tourism demand by state and region

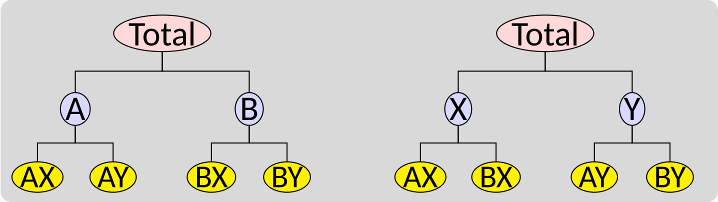
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Examples

- Spectacle sales by brand, gender, stores, etc.
- Tourism by state and purpose of travel

The problem

1. How to forecast time series at all nodes such that the forecasts add up in the same way as the original data?
2. Can we exploit relationships between the series to improve the forecasts?

The problem

1. How to forecast time series at all nodes such that the forecasts add up in the same way as the original data?
2. Can we exploit relationships between the series to improve the forecasts?

The solution

1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm.
   (e.g., `ets`, `auto.arima`, ...)
2. Reconcile the resulting forecasts so they add up correctly using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).
3. This is available in the **hts** package in R.

Outline

hts package for R

hts: Hierarchical and Grouped Time Series

Methods for analysing and forecasting hierarchical and grouped time series

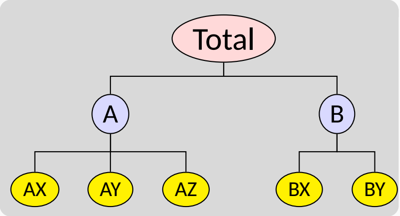
Version:	5.1.5
Depends:	R ($\geq$ 3.2.0), forecast ($\geq$ 8.1)
Imports:	SparseM, Matrix, matrixcalc, parallel, utils, methods, graphics, gr
LinkingTo:	Rcpp ($\geq$ 0.11.0), RcppEigen
Suggests:	testthat, knitr, rmarkdown
Published:	2018-03-26
Author:	Rob J Hyndman, Alan Lee, Earo Wang, Shanika Wickramasuriya
Maintainer:	Rob J Hyndman <Earo.Wang at gmail.com>
BugReports:	https://github.com/earowang/hts/issues
License:	GPL ($\geq$ 2)
URL:	http://pkg.earo.me/hts

Example using R

```r
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
```

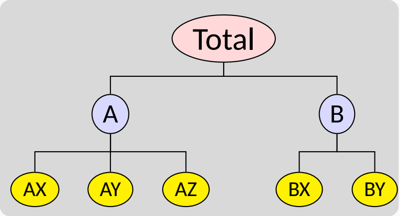
Example using R

```r
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
```



Example using R

```r
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
```




```r
# Forecast 10-step-ahead using WLS combination method
# ETS used for each series by default
fc <- forecast(y, h=10)
```

forecast.gts() function

Usage

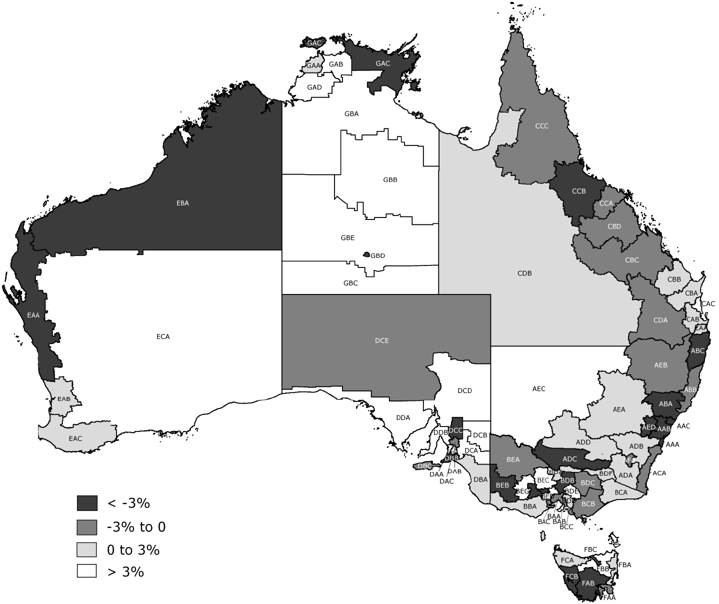
```
forecast(object, h,
  method = c("comb", "bu", "mo", "tdgsf", "tdgsa", "tdfp"),
  fmethod = c("ets", "rw", "arima"),
  weights = c("wls", "ols", "mint", "nseries"),
  covariance = c("shr","sam"),
  positive = TRUE,
  parallel = FALSE, num.cores = 2, ...)
```

Arguments

`object`	Hierarchical time series object of class `gts`.
`h`	Forecast horizon
`method`	Method for distributing forecasts within the hierarchy.
`fmethod`	Forecasting method to use
`weights`	Weights used for "optimal combination" method.
`covariance`	Shrinkage estimator or sample estimator for GLS covariance.
`positive`	If TRUE, forecasts are forced to be strictly positive
`parallel`	If TRUE, allow parallel processing
`num.cores`	If parallel = TRUE, specify how many cores to be used

Outline

Australian tourism

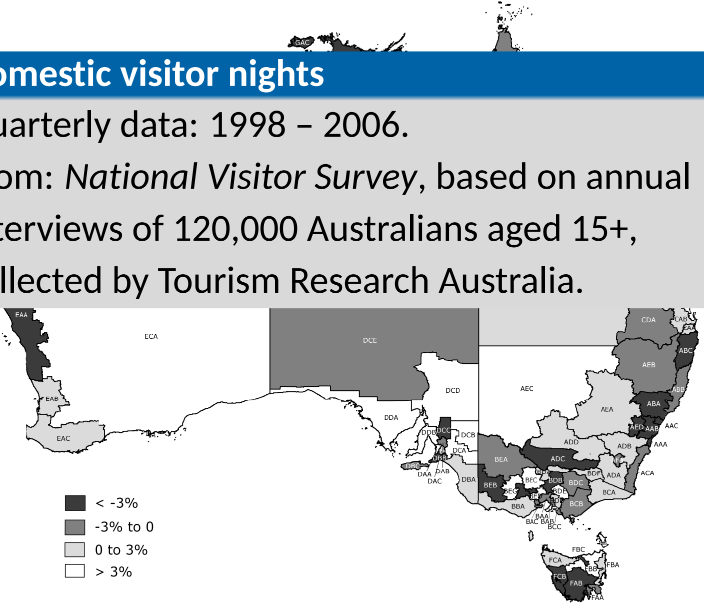


Australian tourism

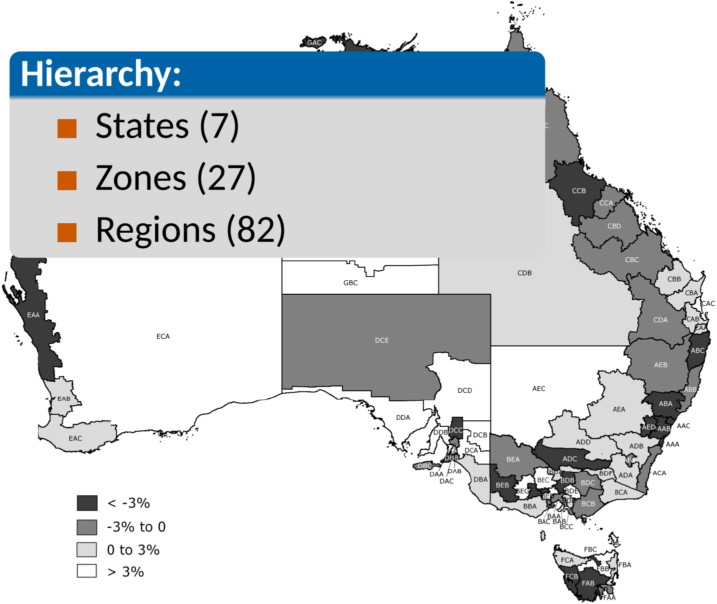
Domestic visitor nights

Quarterly data: 1998 – 2006.

From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.



Australian tourism



Hierarchy:

- States (7)
- Zones (27)
- Regions (82)

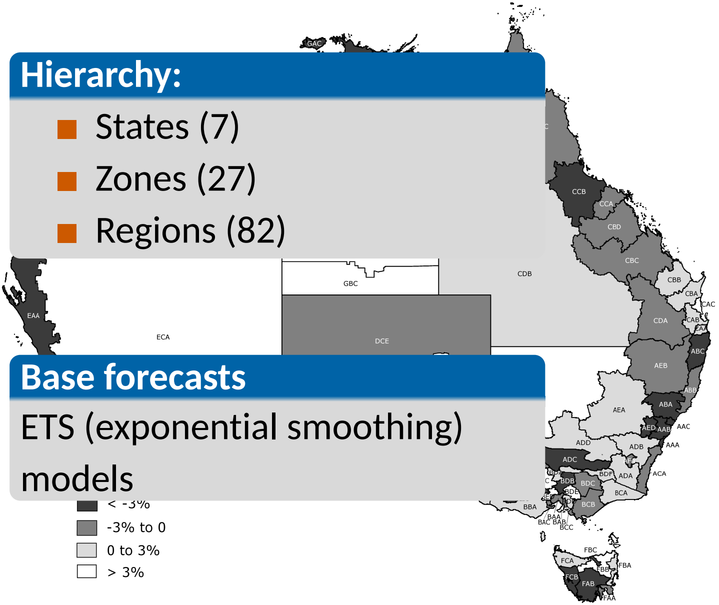
Australian tourism

Hierarchy:

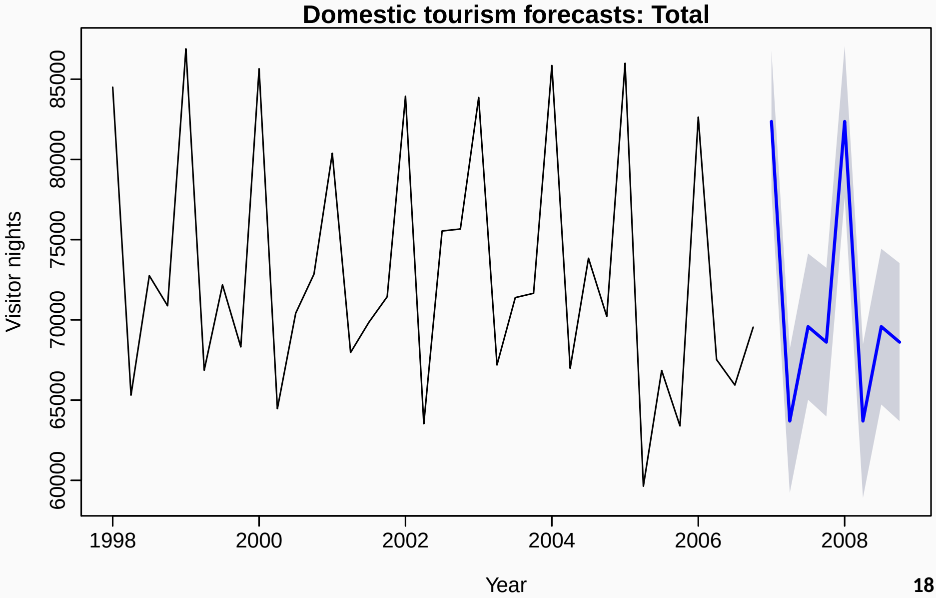
- States (7)
- Zones (27)
- Regions (82)

Base forecasts

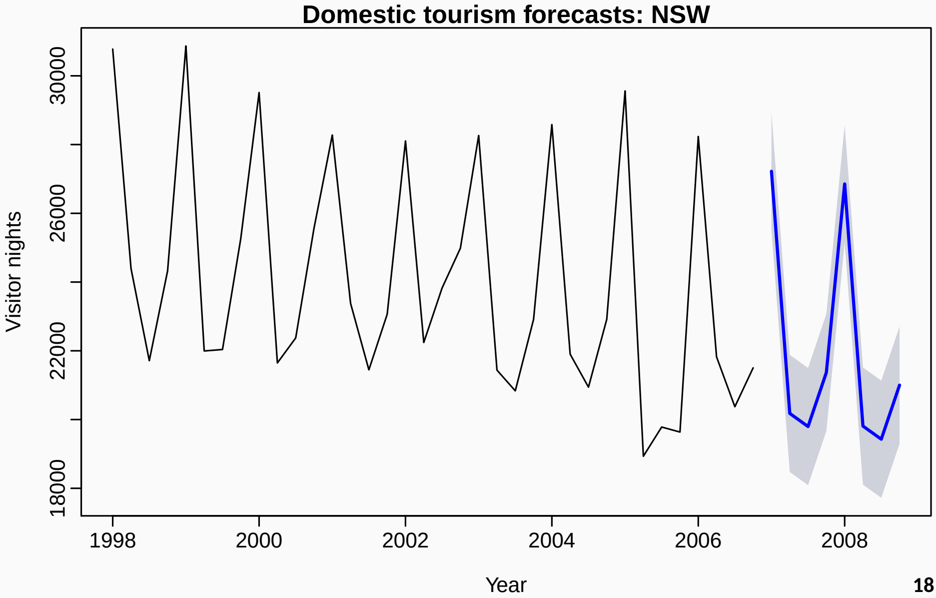
ETS (exponential smoothing) models



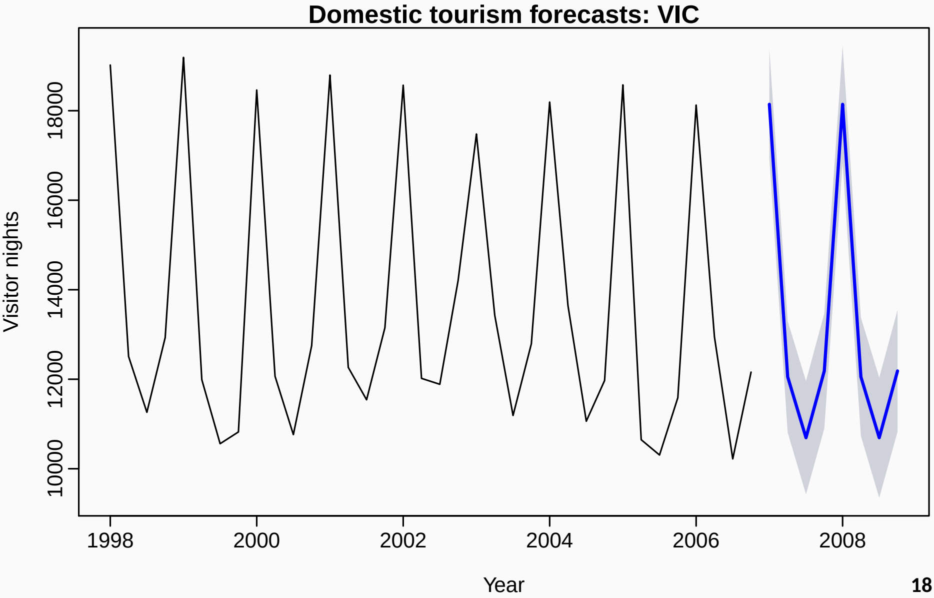
Base forecasts



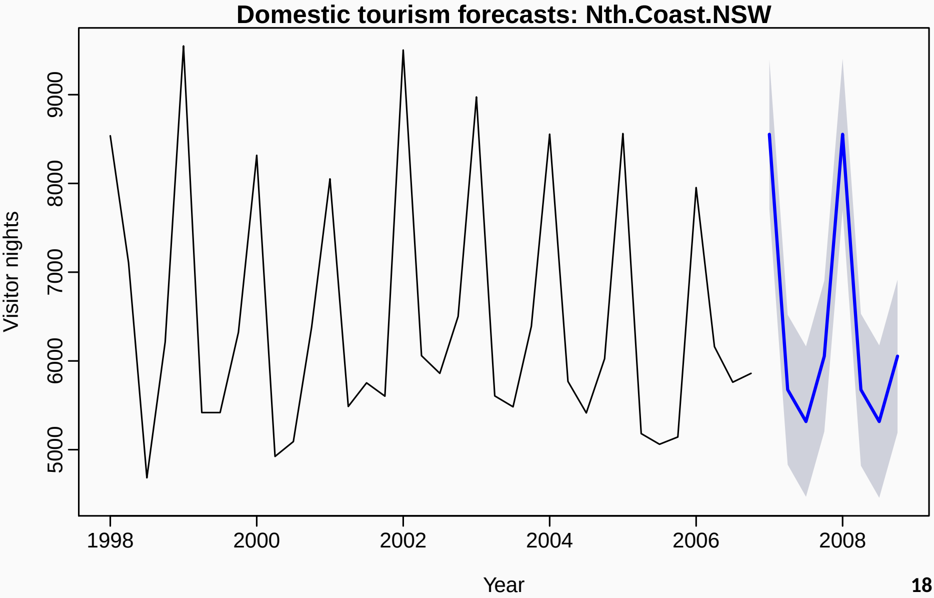
Base forecasts



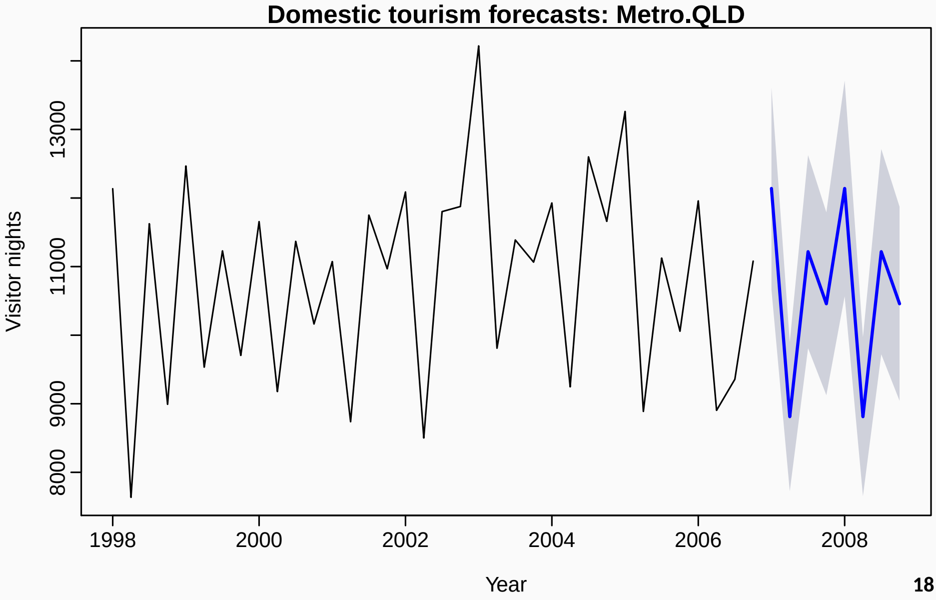
Base forecasts



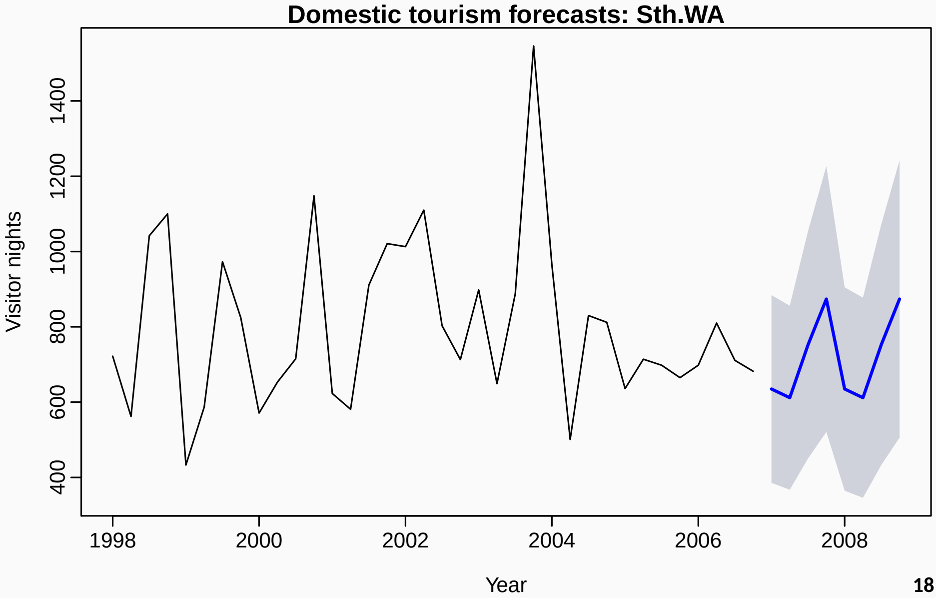
Base forecasts



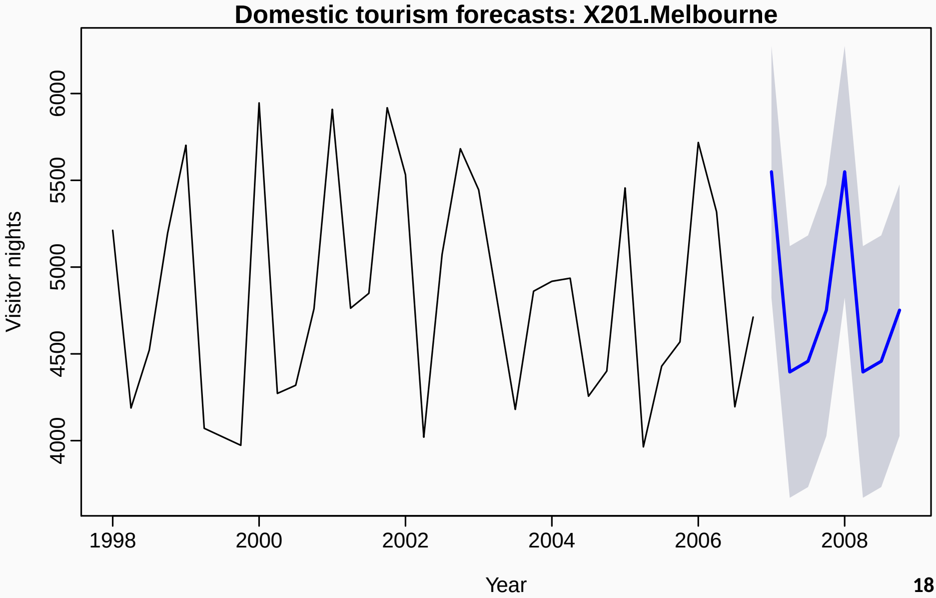
Base forecasts



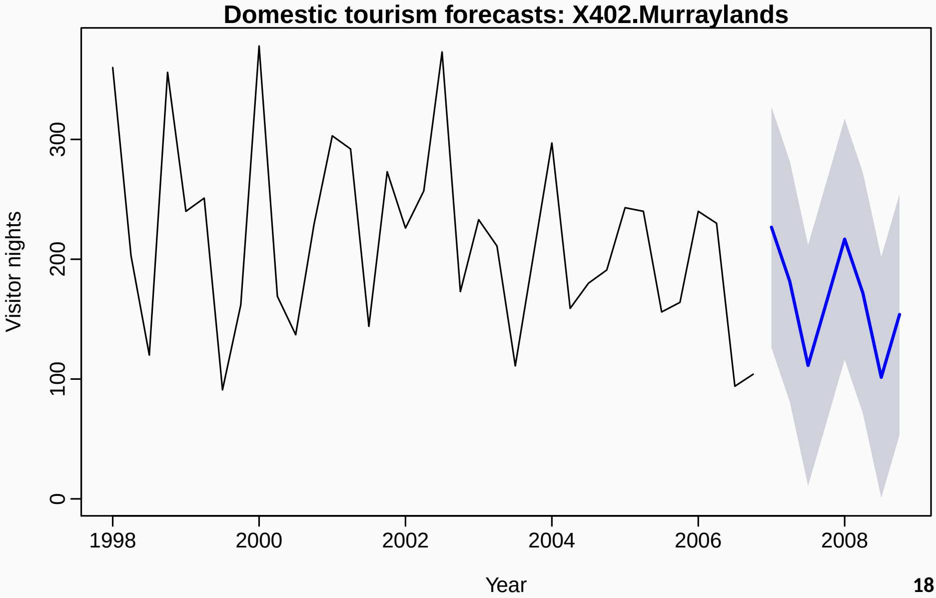
Base forecasts



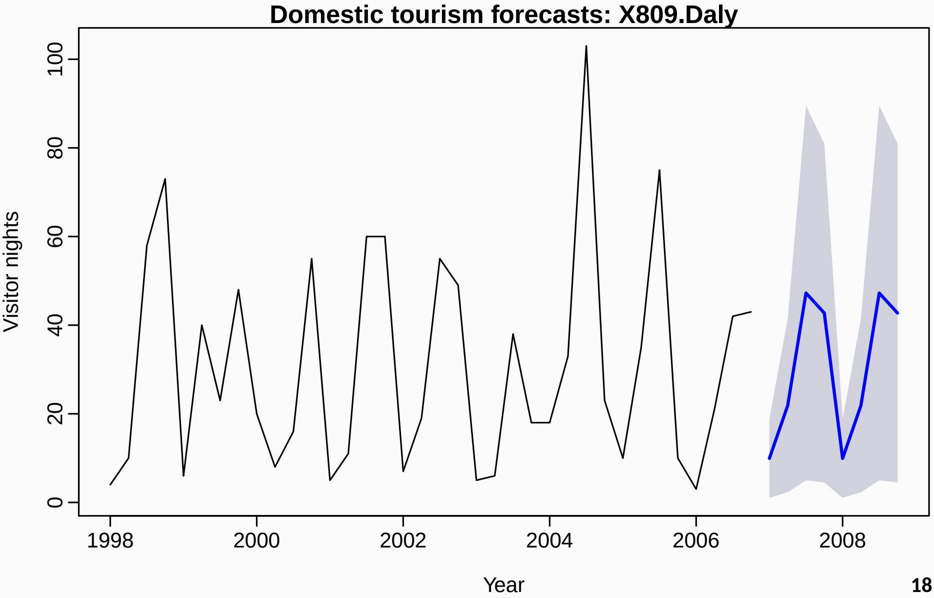
Base forecasts



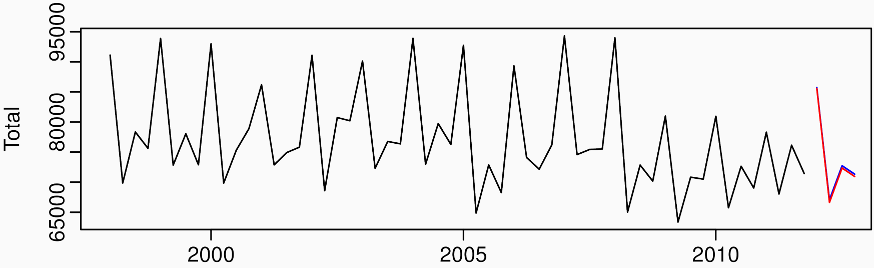
Base forecasts



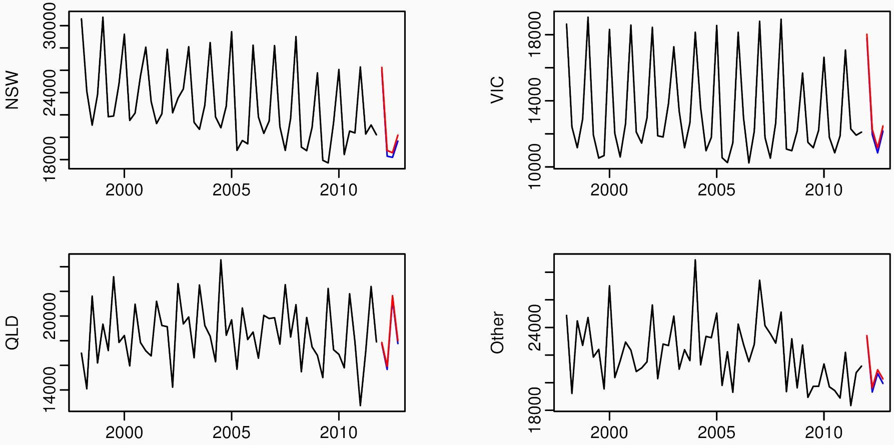
Base forecasts



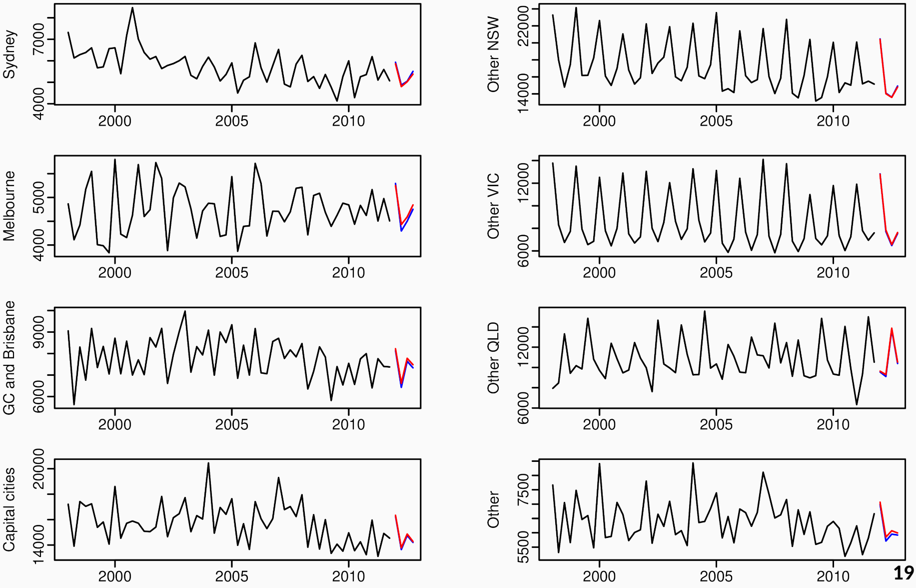
Reconciled forecasts



Reconciled forecasts



Reconciled forecasts

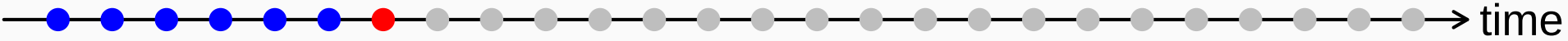


Forecast evaluation

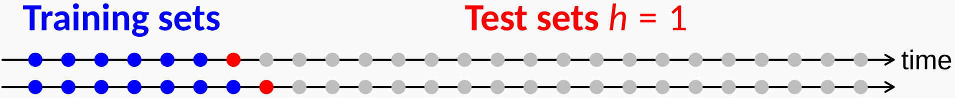
- Select models using all observations;
- Re-estimate models using first 12 observations and generate 1- to 8-step-ahead forecasts;
- Increase sample size one observation at a time, re-estimate models, generate forecasts until the end of the sample;
- In total 24 1-step-ahead, 23 2-steps-ahead, up to 17 8-steps-ahead for forecast evaluation.

Forecast evaluation

Training sets **Test sets h = 1**

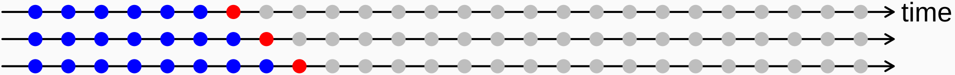


Forecast evaluation

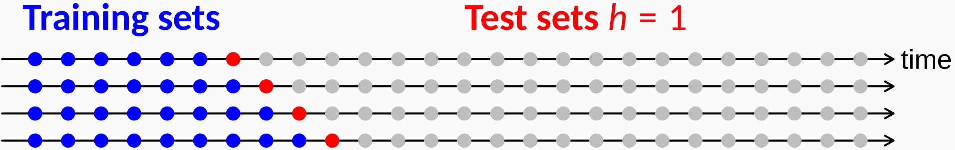


Forecast evaluation

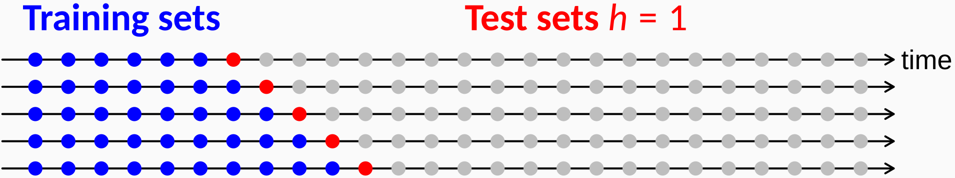
Training sets **Test sets h = 1**



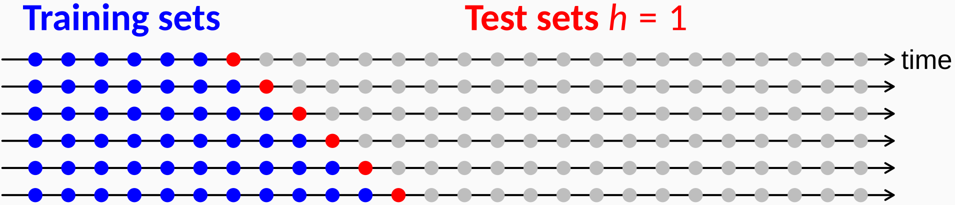
Forecast evaluation



Forecast evaluation

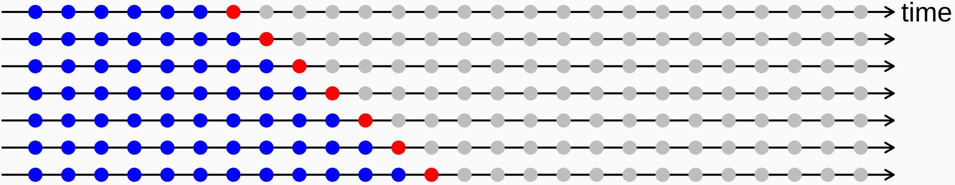


Forecast evaluation



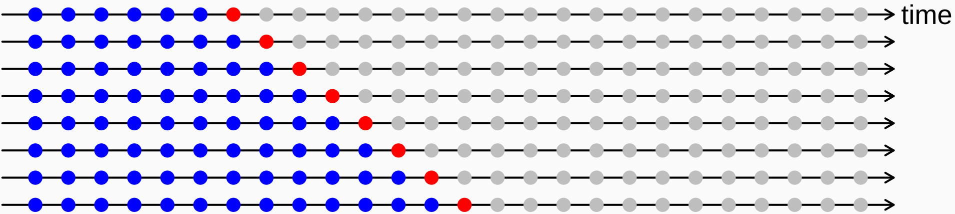
Forecast evaluation

Training sets **Test sets h = 1**

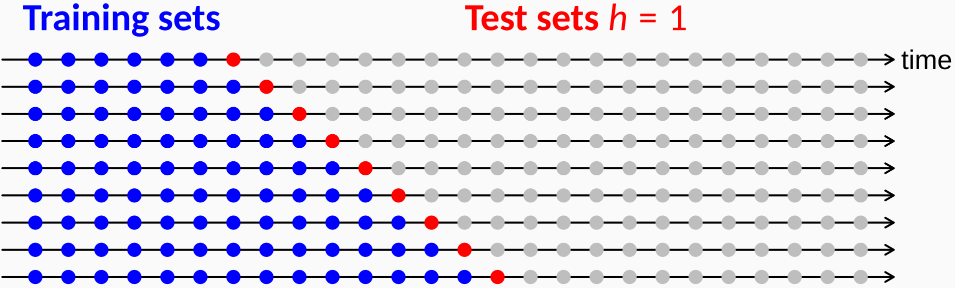


Forecast evaluation

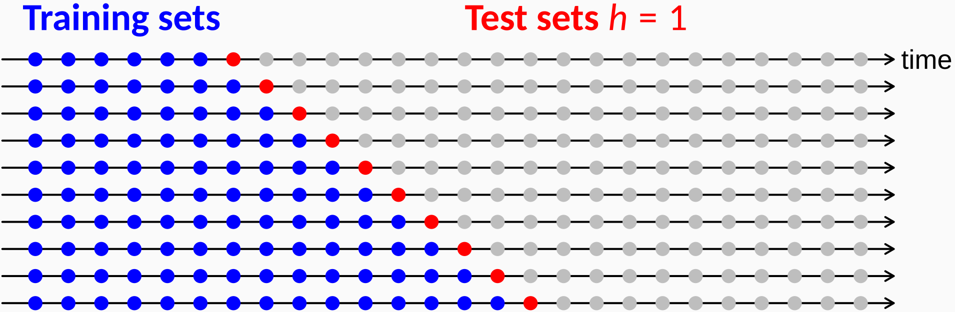
Training sets **Test sets h = 1**



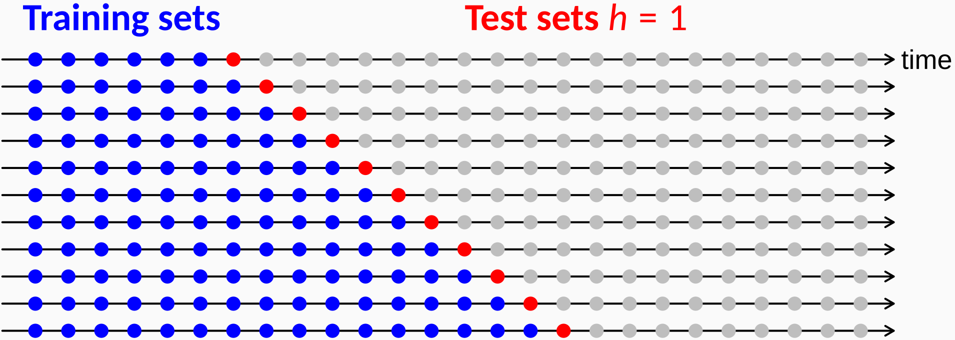
Forecast evaluation



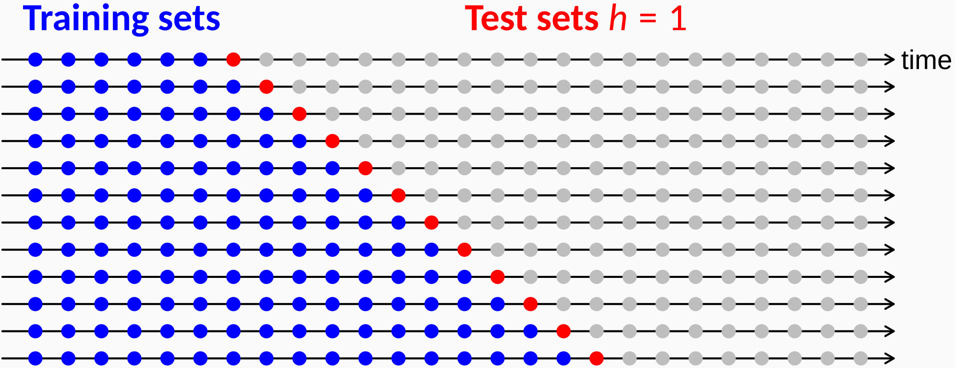
Forecast evaluation



Forecast evaluation

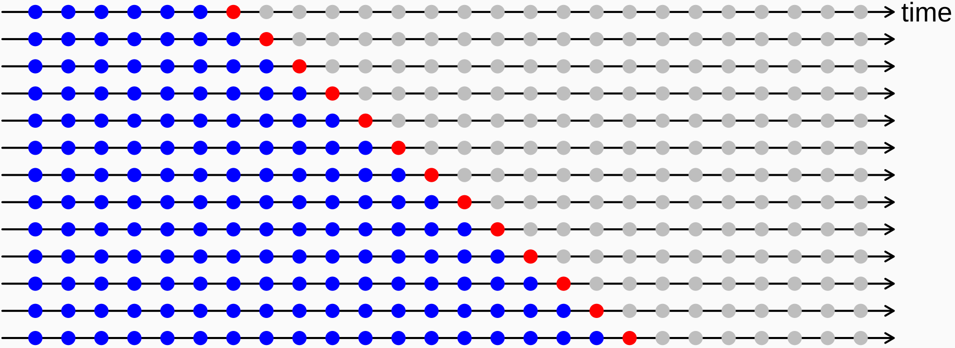


Forecast evaluation

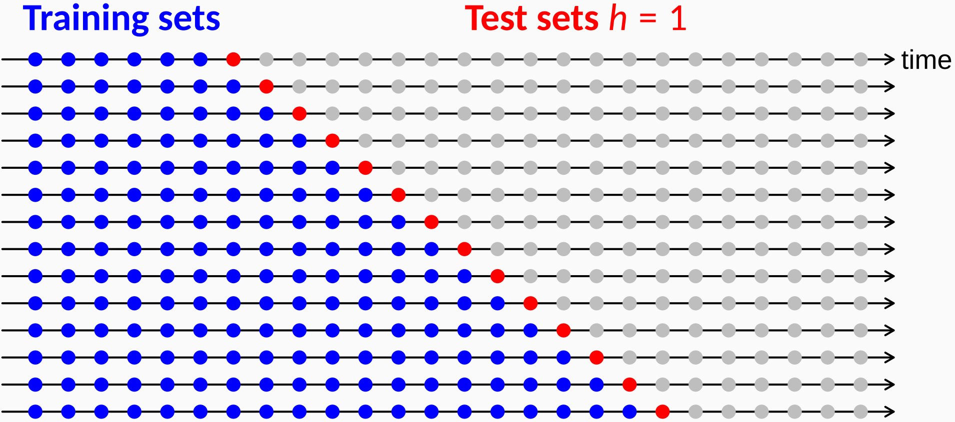


Forecast evaluation

Training sets **Test sets $h = 1$**

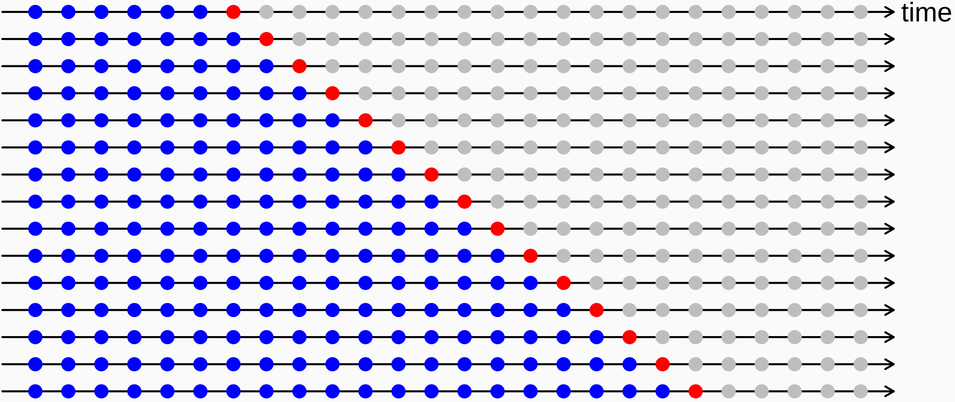


Forecast evaluation

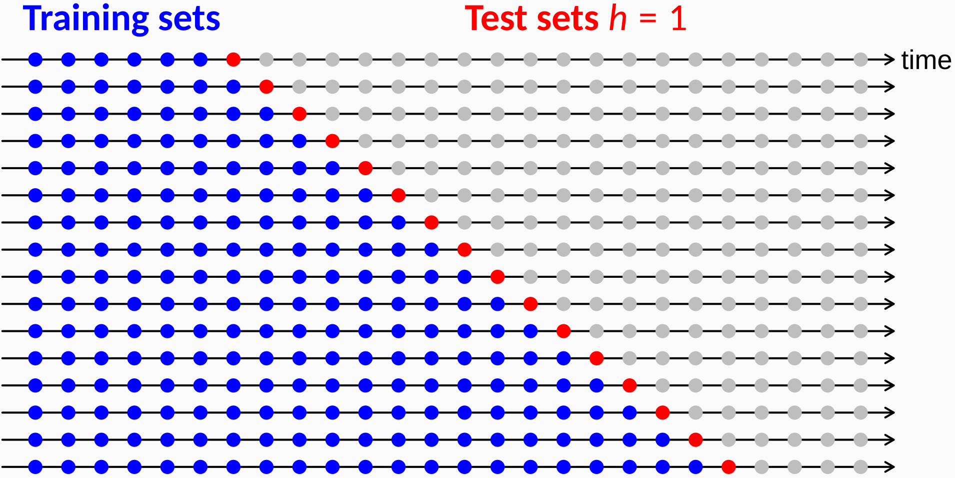


Forecast evaluation

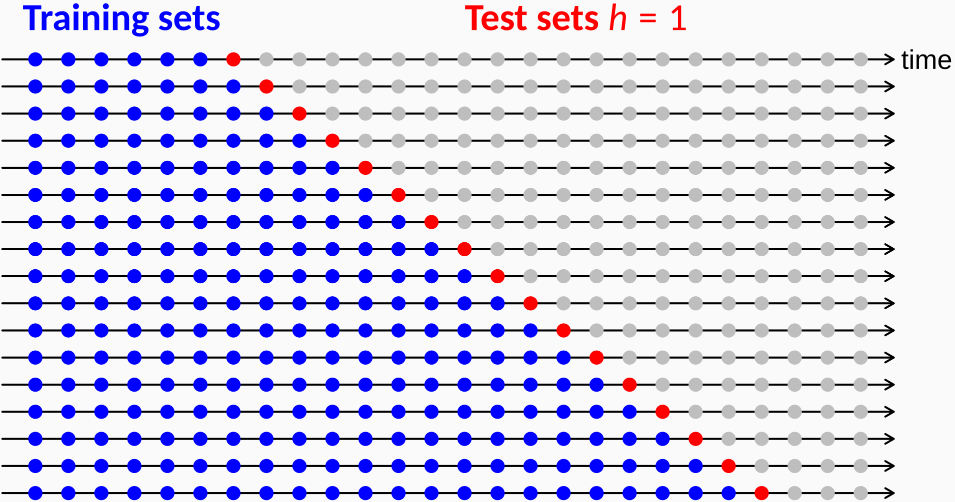
Training sets **Test sets h = 1**



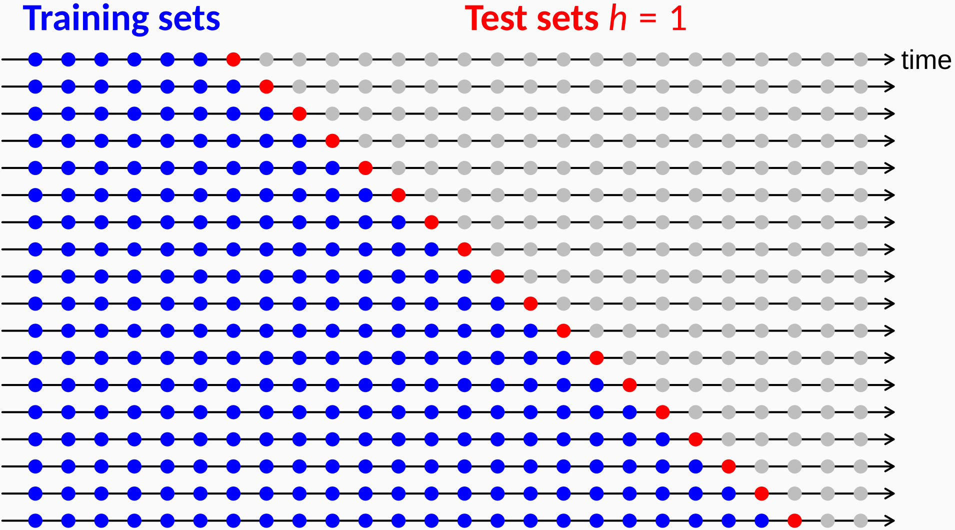
Forecast evaluation



Forecast evaluation

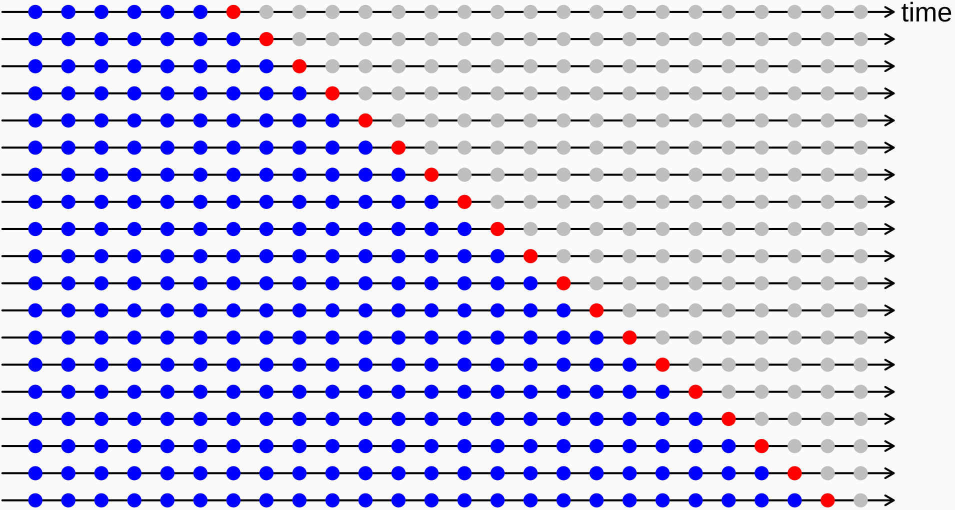


Forecast evaluation

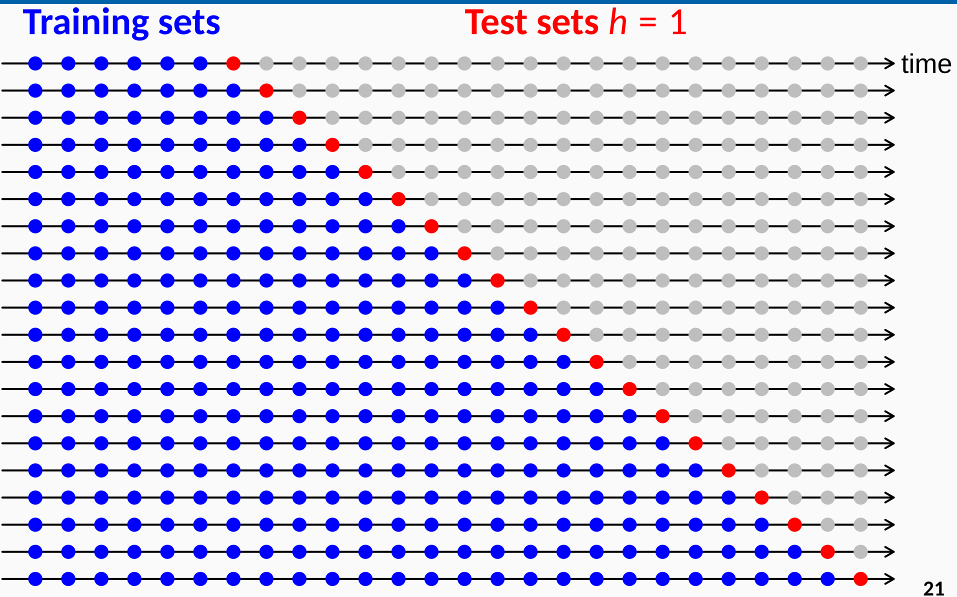


Forecast evaluation

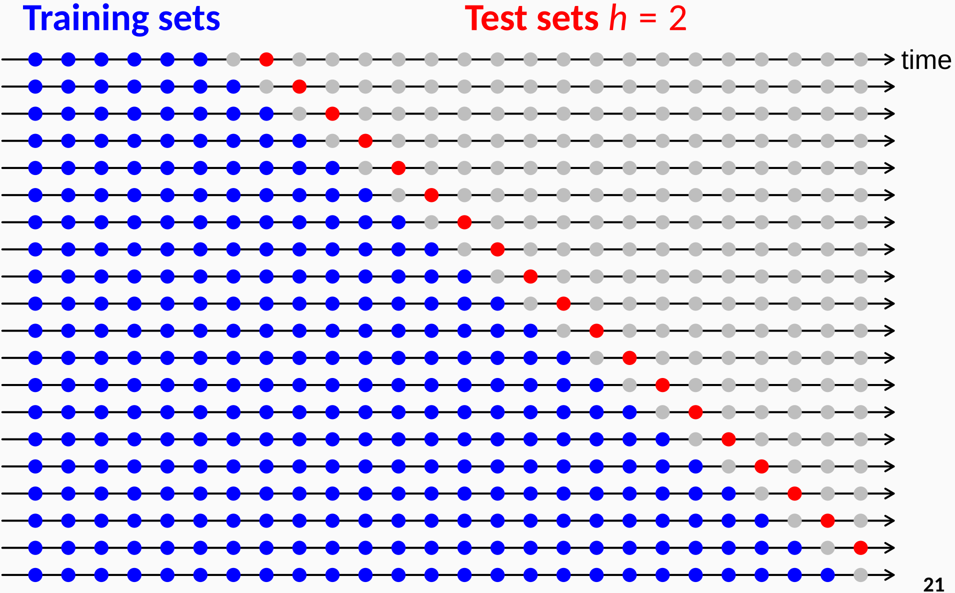
Training sets **Test sets h = 1**



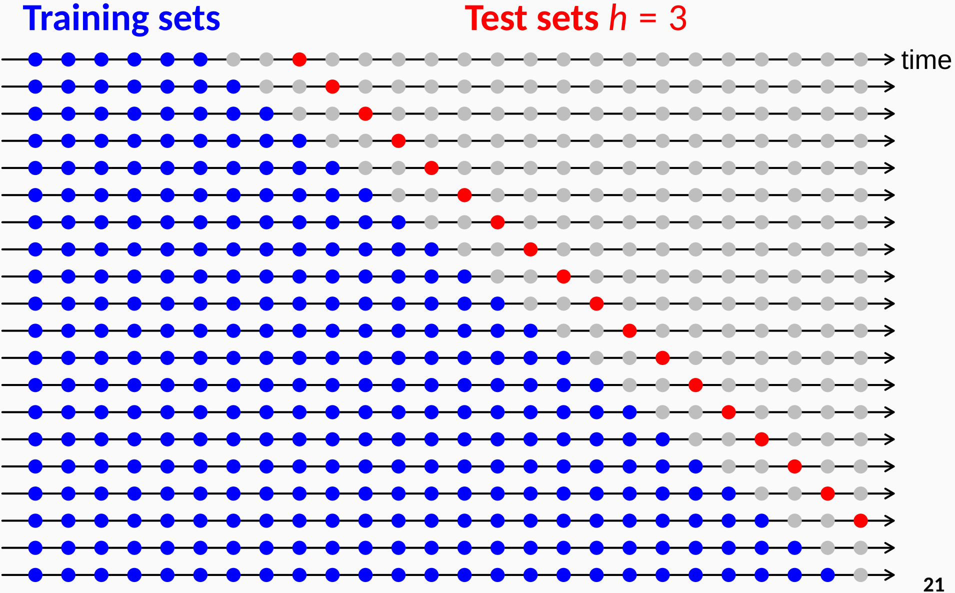
Forecast evaluation



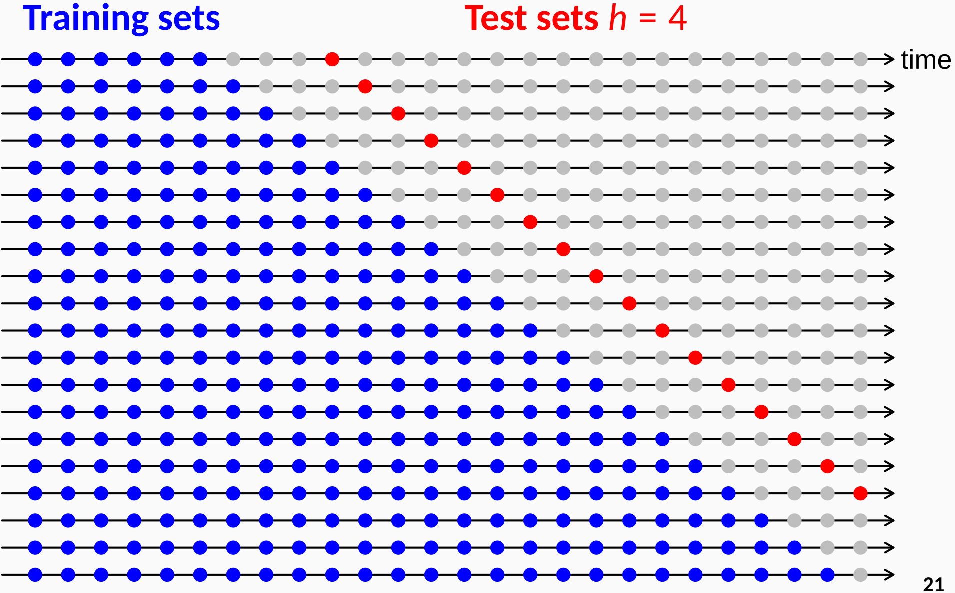
Forecast evaluation



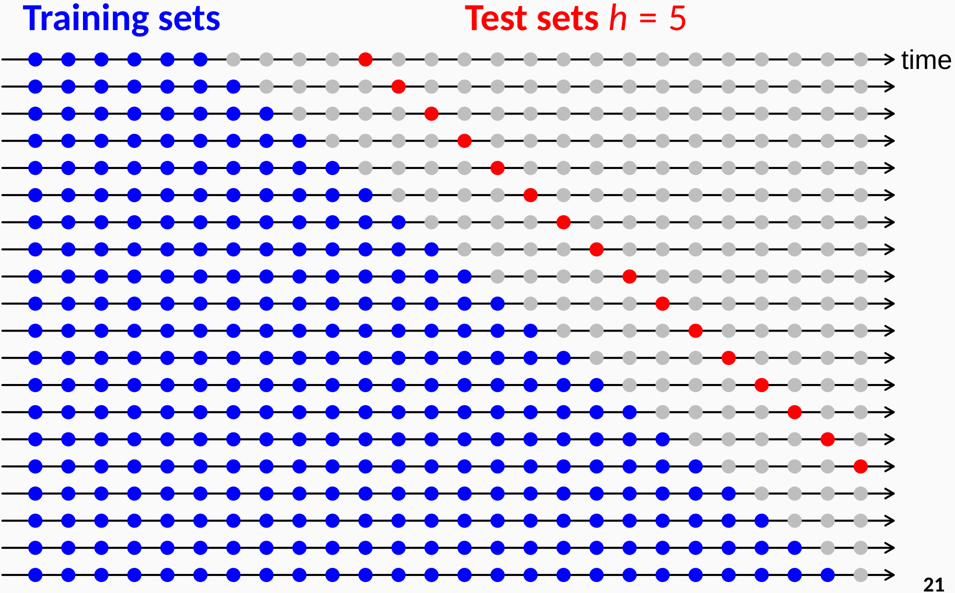
Forecast evaluation



Forecast evaluation

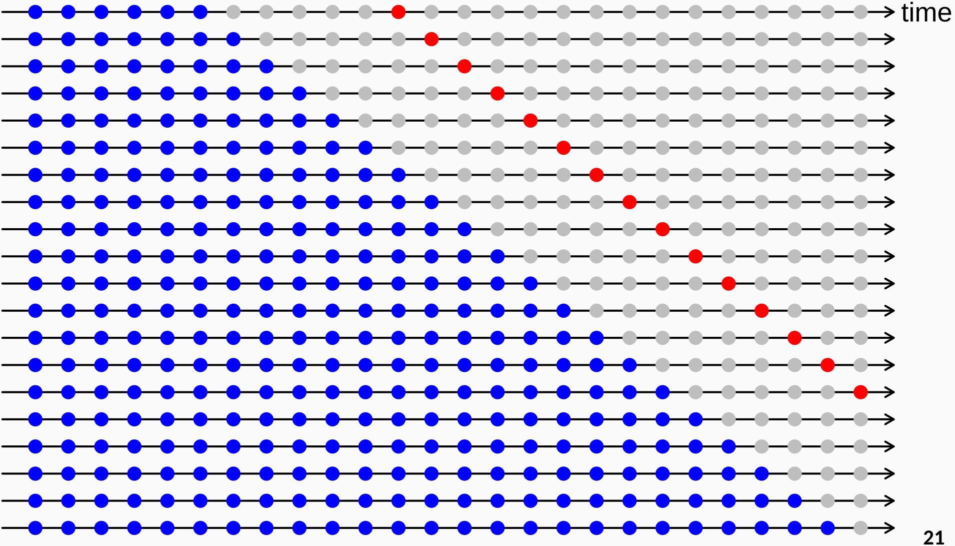


Forecast evaluation



Forecast evaluation

Training sets **Test sets h = 6**

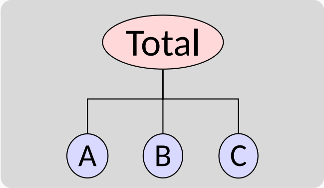


Hierarchy: states, zones, regions

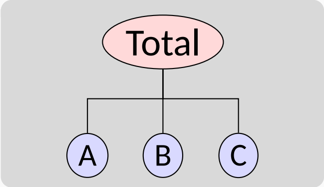
			Forecast horizon				
RMSE	h = 1	h = 2	h = 3	h = 4	h = 5	h = 6	**Ave**
			Australia				
Base	1762.04	1770.29	1766.02	1818.82	1705.35	1721.17	**1757.28**
Bottom	1736.92	1742.69	1722.79	1752.74	1666.73	1687.43	**1718.22**
WLS	1705.21	1715.87	1703.75	1729.56	1627.79	1661.24	**1690.57**
GLS	1704.64	1715.60	1705.31	1729.04	1626.36	1661.64	**1690.43**
			States				
Base	399.77	404.16	401.92	407.26	395.38	401.17	**401.61**
Bottom	404.29	406.95	404.96	409.02	399.80	401.55	**404.43**
WLS	398.84	402.12	400.71	405.03	394.76	398.23	**399.95**
GLS	398.84	402.16	400.86	405.03	394.59	398.22	**399.95**
			Regions				
Base	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
Bottom	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
WLS	93.02	93.32	93.38	93.72	93.39	93.53	**93.39**
GLS	92.98	93.27	93.34	93.66	93.34	93.46	**93.34**

Outline

Hierarchical time series



Hierarchical time series

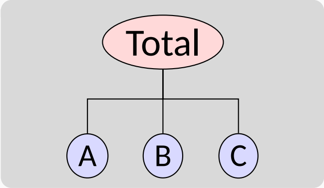


y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

Hierarchical time series



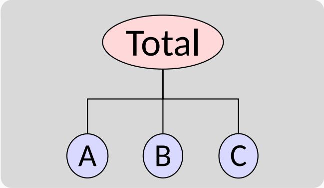
y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

Hierarchical time series



y_t : observed aggregate of all series at time t.

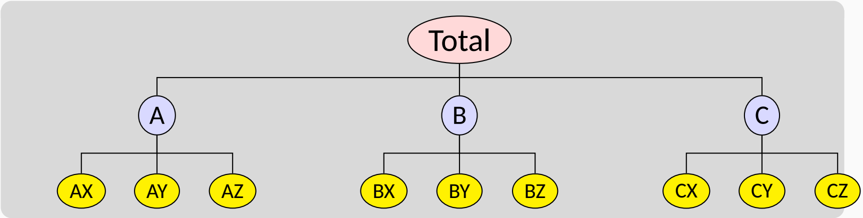
$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

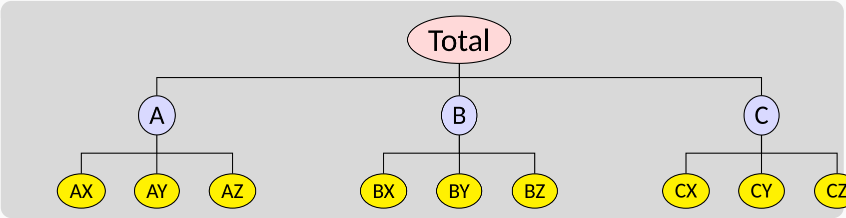
$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Hierarchical time series

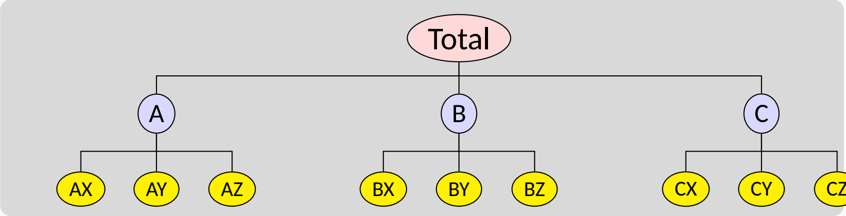


Hierarchical time series



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

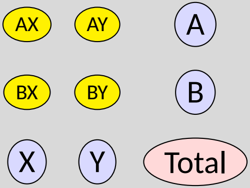
Hierarchical time series



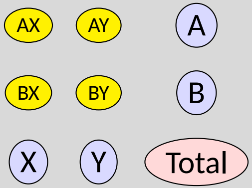
$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Grouped time series

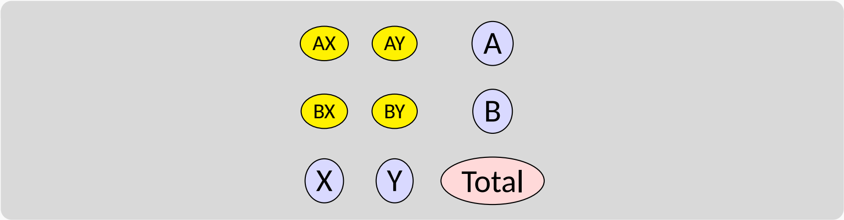


Grouped time series



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Grouped time series



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boxed{\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t}$$

Hierarchical and grouped time series

Every collection of time series with aggregation constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

where

- $\boldsymbol{y}_t$ is a vector of all series at time t
- $\boldsymbol{b}_t$ is a vector of the most disaggregated series at time t
- $\boldsymbol{S}$ is a "summing matrix" containing the aggregation constraints.

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(In general, they will not "add up".)

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(In general, they will not “add up”.)

Reconciled forecasts must be of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{P}$.

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(In general, they will not "add up".)

Reconciled forecasts must be of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{P}$.

- $\boldsymbol{P}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_n(h)$ to get bottom-level forecasts.
- $\boldsymbol{S}$ adds them up

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\boldsymbol{P} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$, where Σ_h is the h-step base forecast error covariance matrix.

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\boldsymbol{P} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$, where Σ_h is the h-step base forecast error covariance matrix.

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}\hat{\boldsymbol{y}}_n(h)$$

Problem: Σ_h hard to estimate, especially for $h > 1$.

Solutions:

- Ignore Σ_h (OLS)
- Assume Σ_h diagonal (WLS) [Default in `hts`]
- Try to estimate Σ_h (GLS)

Features

- Covariates can be included in initial forecasts.
- Adjustments can be made to initial forecasts at any level.
- Very simple and flexible method. Can work with *any* hierarchical or grouped time series.
- Conceptually easy to implement: regression of base forecasts on structure matrix.

Outline

Lab Session 22