



Forecasting: principles and practice

Rob J Hyndman

2 ARIMA models

Outline

- 1 Stationarity and differencing**
- 2 Backshift notation
- 3 Autoregressive models
- 4 Moving Average models
- 5 Non-seasonal ARIMA models
- 6 Seasonal ARIMA models
- 7 Lab Session 3

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s , the distribution of (y_t, \dots, y_{t+s}) does not depend on t .

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A **stationary series** is:

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- constant variance
- no patterns predictable in the long-term

Stationarity

Definition

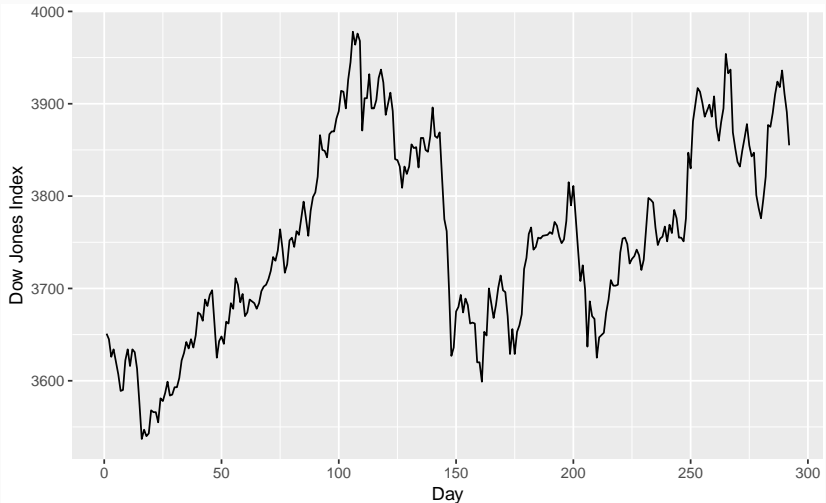
If $\{y_t\}$ is a stationary time series, then for all s , the distribution of (y_t, \dots, y_{t+s}) does not depend on t .

A **stationary series** is:

- roughly horizontal
 - constant variance
 - no patterns predictable in the long-term
-
- Transformations (e.g., logs) can help to **stabilize the variance**.
 - Differences can help to **stabilize the mean**.

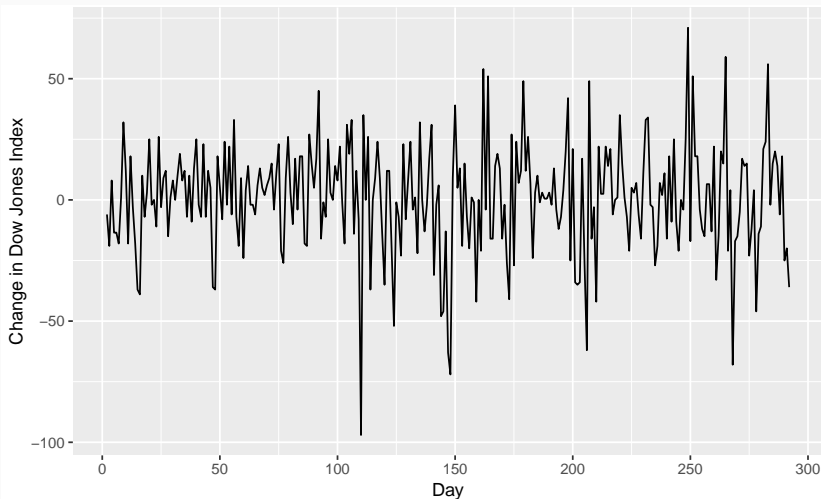
Stationary?

```
dj %>% autoplot() +  
  ylab("Dow Jones Index") + xlab("Day")
```



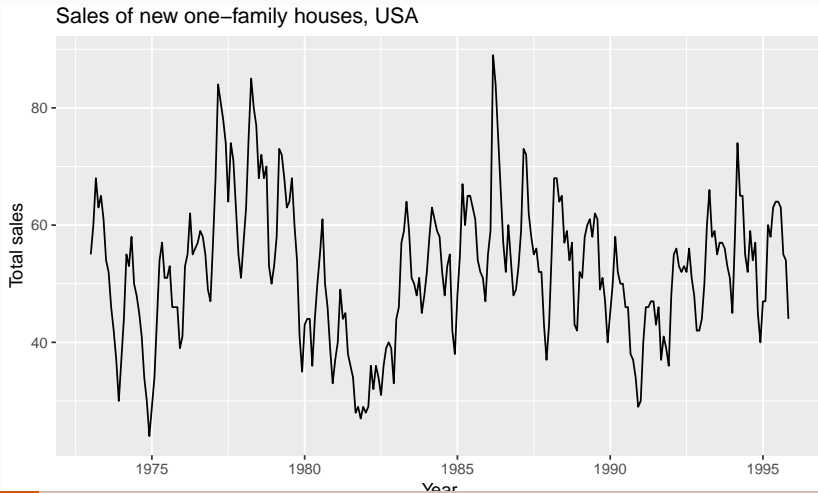
Stationary?

```
dj %>% diff() %>% autoplot() +  
  ylab("Change in Dow Jones Index") + xlab("Day")
```



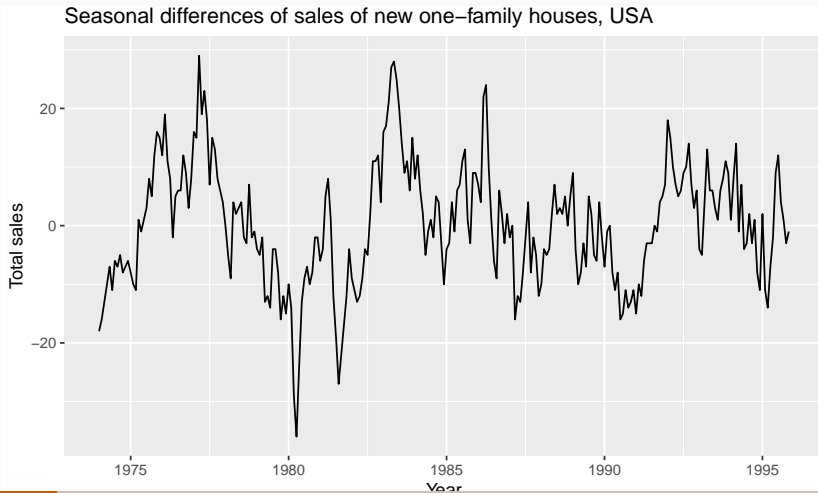
Stationary?

```
hsales %>% autoplot() +  
  xlab("Year") + ylab("Total sales") +  
  ggtitle("Sales of new one-family houses, USA")
```



Stationary?

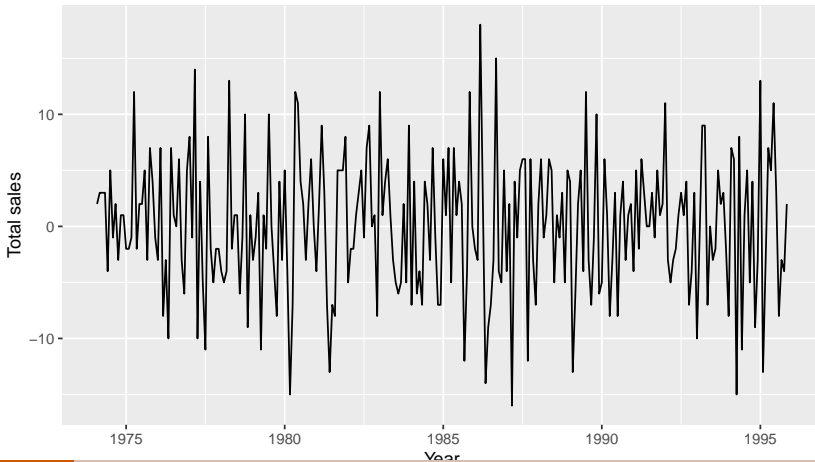
```
hsales %>% diff(lag=12) %>% autoplot() +  
  xlab("Year") + ylab("Total sales") +  
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```



Stationary?

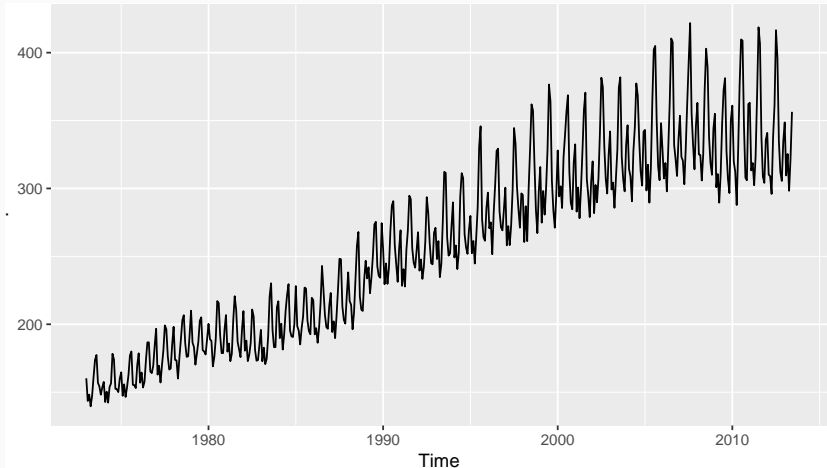
```
hsales %>% diff(lag=12) %>% diff(lag=1) %>% autoplot() +  
  xlab("Year") + ylab("Total sales") +  
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```

Seasonal differences of sales of new one-family houses, USA



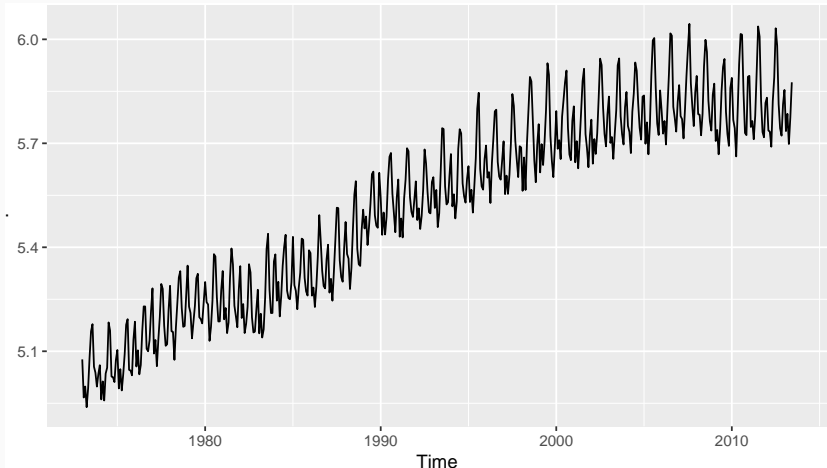
Electricity production

```
usmelec %>% autoplot()
```



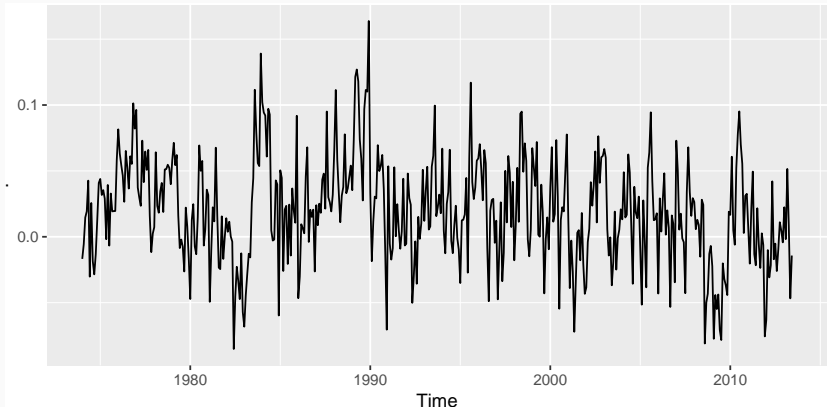
Electricity production

```
usmelec %>% log() %>% autoplot()
```



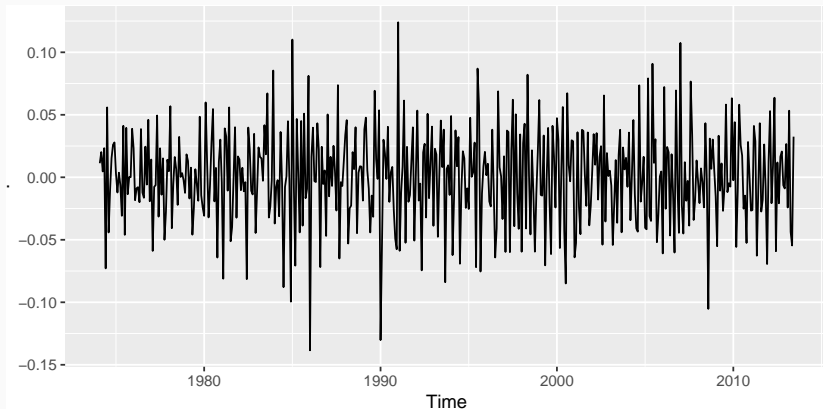
Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%  
autoplot()
```



Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%  
diff(lag=1) %>% autoplot()
```



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Backshift notation

Backward shift operator

Shift back one period

$$By_t = y_{t-1}$$

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$$B(By_t) = B^2y_t = y_{t-2}$$

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Backward shift operator

Shift back one period

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Shift back two periods:

$$B(By_t) = B^2y_t = y_{t-2}$$

Shift back 12 periods

$$B^{12}y_t = y_{t-12}$$

Backshift notation

- First differences

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t .$$

Backshift notation

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Backshift notation

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$$y''_t = (1 - B)^2 y_t .$$

- d th-order differences:

$$(1 - B)^d y_t .$$

- Seasonal difference followed by first difference:

$$(1 - B)(1 - B^m)y_t .$$

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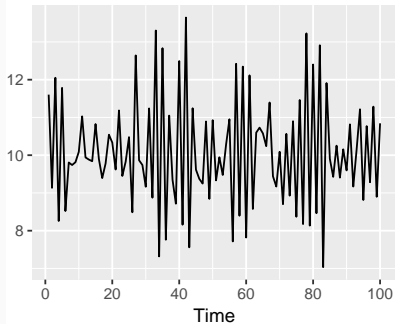
Autoregressive models

Autoregressive (AR) models:

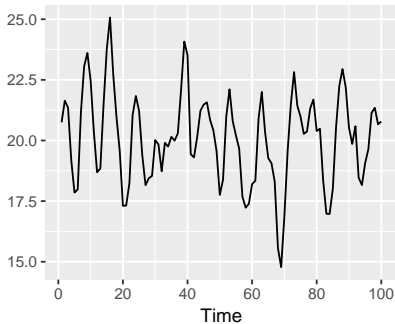
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.

AR(1)



AR(2)



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
 $-1 < \phi_2 < 1 \quad \phi_2 + \phi_1 < 1 \quad \phi_2 - \phi_1 < 1$.
- More complicated conditions hold for $p \geq 3$.

Outline

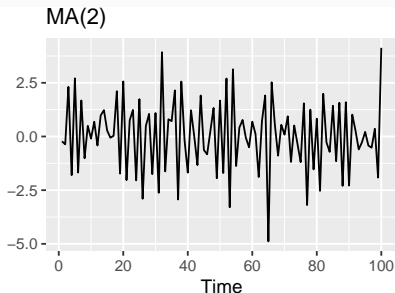
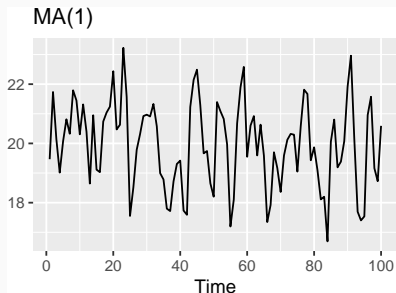
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Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise. This is a multiple regression with **past errors** as predictors. *Don't confuse this with moving average smoothing!*



Invertibility

- Invertible models have property that distant past has negligible effect on forecasts. Requires constraints on MA parameters.

General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$ lie outside the unit circle on the complex plane.

Invertibility

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General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$ lie outside the unit circle on the complex plane.

- For $q = 1$: $-1 < \theta_1 < 1$.
- For $q = 2$:
 $-1 < \theta_2 < 1 \quad \theta_2 + \theta_1 > -1 \quad \theta_1 - \theta_2 < 1$.
- More complicated conditions hold for $q \geq 3$.

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ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t.$$

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
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- Conditions on coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- $(1 - B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR: p = order of the autoregressive part

I: d = degree of first differencing involved

MA: q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA($p,0,0$)
- MA(q): ARIMA(0,0, q)

Backshift notation for ARIMA

■ ARIMA(p, q) model:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

$$\text{or } (1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

■ ARIMA(1,1,1) model:

$$(1 - \phi_1 B) (1 - B) y_t = c + (1 + \theta_1 B) \varepsilon_t$$

↑

AR(1)

↑

First

difference

↑

MA(1)

Backshift notation for ARIMA

■ ARIMA(p, q) model:

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$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{AR}(1) & \text{First} & \text{MA}(1) \\ & \text{difference} & \end{array}$$

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

R model

Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

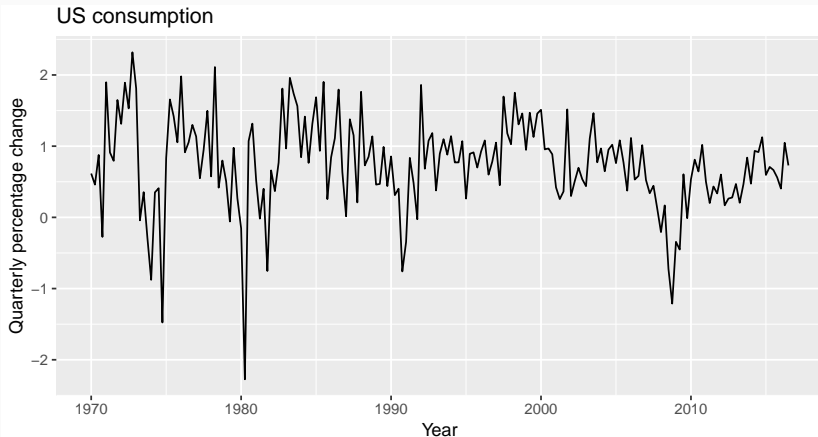
Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- μ is the mean of y'_t .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$.
- R uses mean form.

US personal consumption

```
autoplot(uschange[, "Consumption"]) +  
  xlab("Year") + ylab("Quarterly percentage change") +  
  ggtitle("US consumption")
```



US personal consumption

```
(fit <- auto.arima(uschange[, "Consumption"]))
```

```
## Series: uschange[, "Consumption"]  
## ARIMA(2,0,2) with non-zero mean  
##  
## Coefficients:  
##          ar1      ar2      ma1      ma2      mean  
##          1.391   -0.581   -1.180    0.558    0.746  
## s.e.    0.255    0.208    0.238    0.140    0.084  
##  
## sigma^2 estimated as 0.351:  log likelihood=-165.1  
## AIC=342.3   AICc=342.8   BIC=361.7
```


US personal consumption

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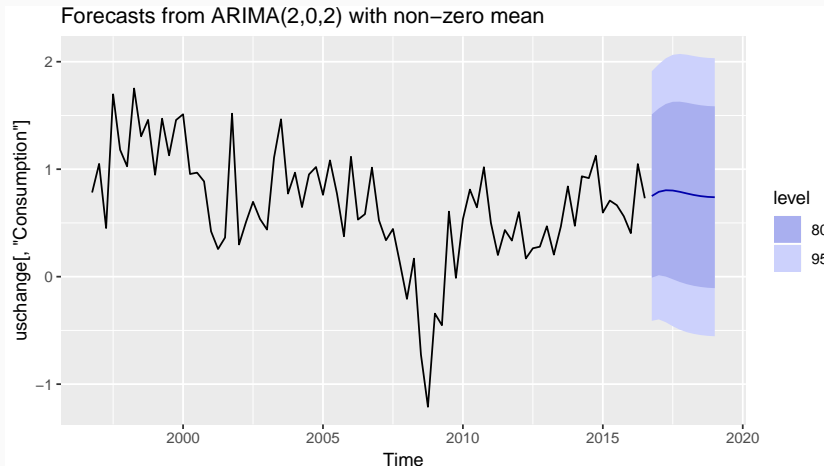
ARIMA(2,0,2) model:

$$y_t = c + 1.391y_{t-1} - 0.581y_{t-2} - 1.180\varepsilon_{t-1} + 0.558\varepsilon_{t-2} + \varepsilon_t,$$

where $c = 0.746 \times (1 - 1.391 + 0.581) = 0.142$ and $\varepsilon_t \sim N(0, 0.351)$.

US personal consumption

```
fit %>% forecast(h=10) %>% autoplot(include=80)
```



Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2 \log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

Information criteria

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

Information criteria

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

Good models are obtained by minimizing the AICc.

How does auto.arima() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via KPSS test and seasonal strength measure.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does `auto.arima()` work?

Step 1: Select values of d and D .

Step 2: Select current model (with smallest AICc) from:

ARIMA(2, d , 2)

ARIMA(0, d , 0)

ARIMA(1, d , 0)

ARIMA(0, d , 1)

How does `auto.arima()` work?

Step 1: Select values of d and D .

Step 2: Select current model (with smallest AICc) from:

ARIMA(2, d , 2)

ARIMA(0, d , 0)

ARIMA(1, d , 0)

ARIMA(0, d , 1)

Step 3: Consider variations of current model:

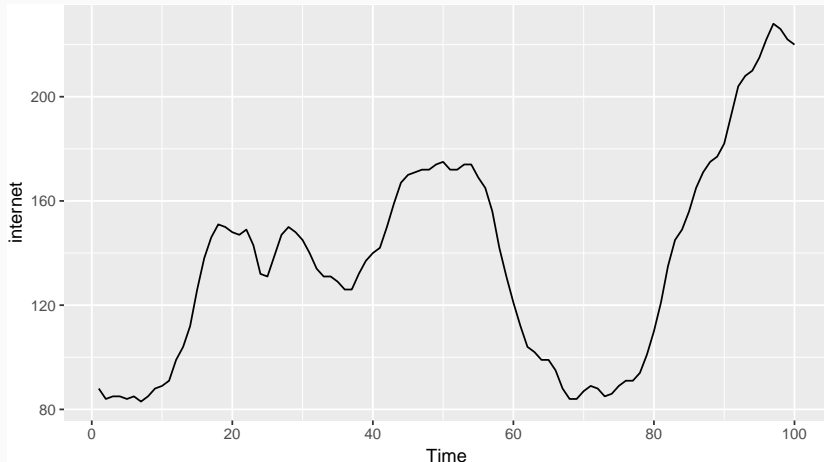
- vary one of p , q , from current model by ± 1 ;
- p , q both vary from current model by ± 1 ;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 3 until no lower AICc can be found.

Choosing an ARIMA model

```
autoplot(internet)
```



Choosing an ARIMA model

```
(fit <- auto.arima(internet))
```

```
## Series: internet
## ARIMA(1,1,1)
##
## Coefficients:
##          ar1      ma1
##          0.650   0.526
## s.e.    0.084   0.090
##
## sigma^2 estimated as 10:  log likelihood=-254.2
## AIC=514.3   AICc=514.5   BIC=522.1
```

Choosing an ARIMA model

```
(fit <- auto.arima(internet, stepwise=FALSE,  
  approximation=FALSE))
```

```
## Series: internet
```

```
## ARIMA(3,1,0)
```

```
##
```

```
## Coefficients:
```

```
##          ar1      ar2      ar3
```

```
##          1.151   -0.661   0.341
```

```
## s.e.    0.095    0.135    0.094
```

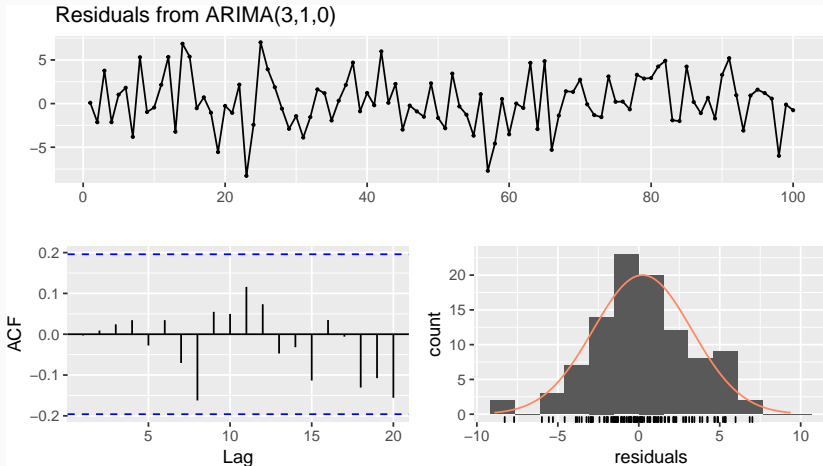
```
##
```

```
## sigma^2 estimated as 9.66:  log likelihood=-252
```

```
## AIC=512    AICc=512.4    BIC=522.4
```

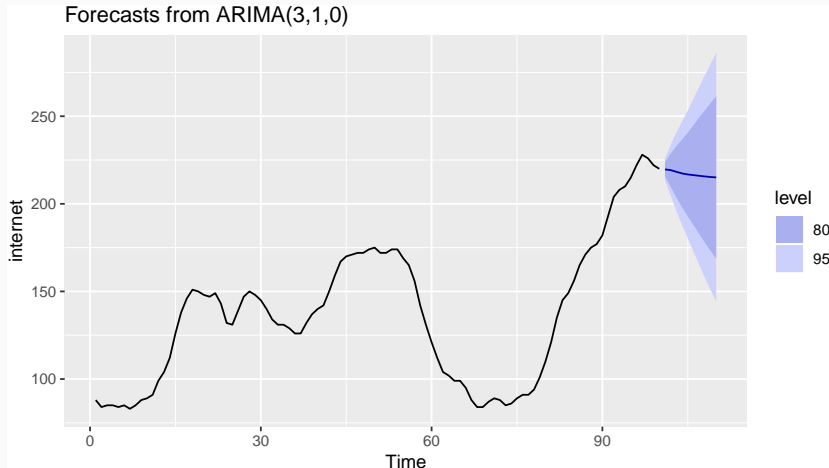
Choosing an ARIMA model

```
checkresiduals(fit, plot=TRUE)
```



Choosing an ARIMA model

```
fit %>% forecast() %>% autoplot()
```



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Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

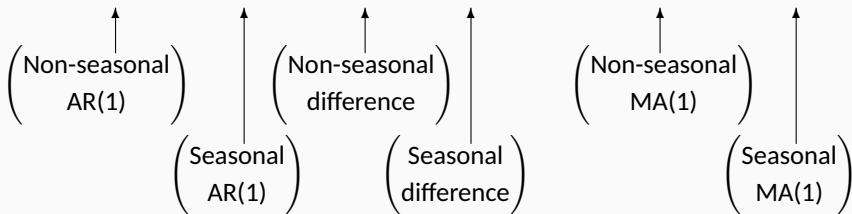
E.g., $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$ model (without constant)

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)₄ model (without constant)
$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

Seasonal ARIMA models

E.g., $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$ model (without constant)
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$



Seasonal ARIMA models

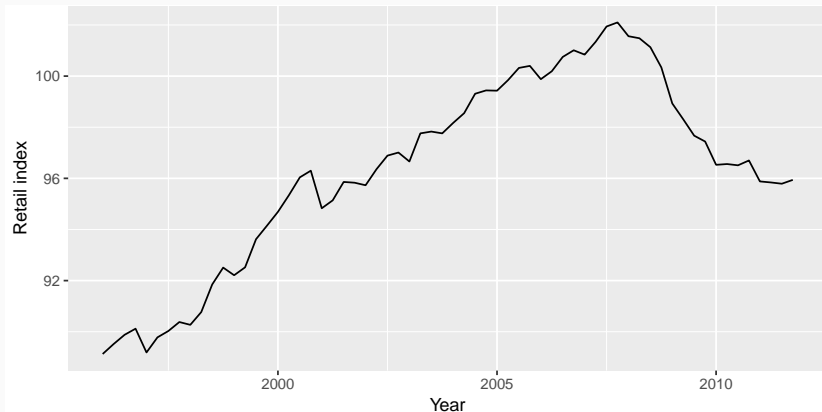
E.g., ARIMA(1, 1, 1)(1, 1, 1)₄ model (without constant)
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

European quarterly retail trade

```
autoplot(euretail) +  
  xlab("Year") + ylab("Retail index")
```



European quarterly retail trade

```
(fit <- auto.arima(euretail))
```

```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1      ma1      ma2      sma1
##          0.736   -0.466   0.216   -0.843
## s.e.    0.224    0.199   0.210    0.188
##
## sigma^2 estimated as 0.159:  log likelihood=-29.62
## AIC=69.24   AICc=70.38   BIC=79.63
```

European quarterly retail trade

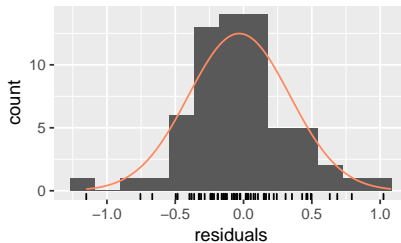
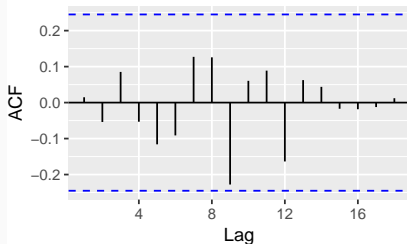
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(fit <- auto.arima(euretail, stepwise=TRUE,  
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```

```
## Series: euretail  
## ARIMA(1,1,2)(0,1,1)[4]  
##  
## Coefficients:  
##          ar1      ma1      ma2      sma1  
##          0.736  -0.466   0.216  -0.843  
## s.e.    0.224    0.199   0.210   0.188  
##  
## sigma^2 estimated as 0.159:  log likelihood=-29.62  
## AIC=69.24   AICc=70.38   BIC=79.63
```

European quarterly retail trade

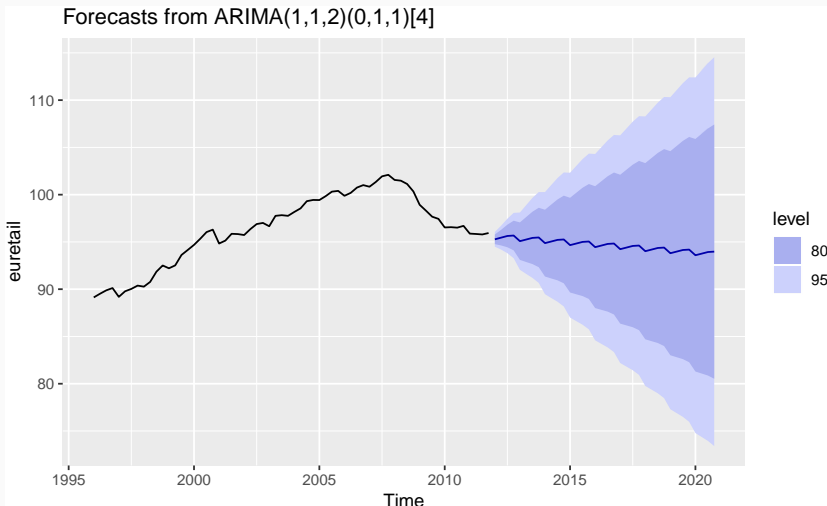
```
checkresiduals(fit, test=FALSE)
```

Residuals from ARIMA(1,1,2)(0,1,1)[4]

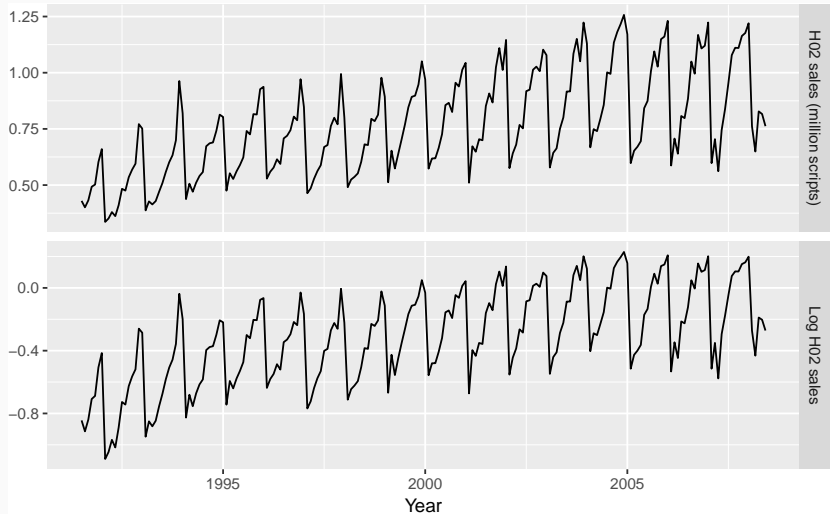


European quarterly retail trade

```
forecast(fit, h=36) %>% autoplot()
```



Corticosteroid drug sales



Corticosteroid drug sales

```
(fit <- auto.arima(h02, lambda=0, max.order=9,  
  stepwise=FALSE, approximation=FALSE))
```

```
## Series: h02
```

```
## ARIMA(4,1,1)(2,1,2)[12]
```

```
## Box Cox transformation: lambda= 0
```

```
##
```

```
## Coefficients:
```

```
##          ar1      ar2      ar3      ar4      ma1      sar1
```

```
##          -0.042  0.210  0.202  -0.227  -0.742  0.621
```

```
## s.e.      0.217  0.181  0.114   0.081   0.207  0.242
```

```
##          sar2      sma1      sma2
```

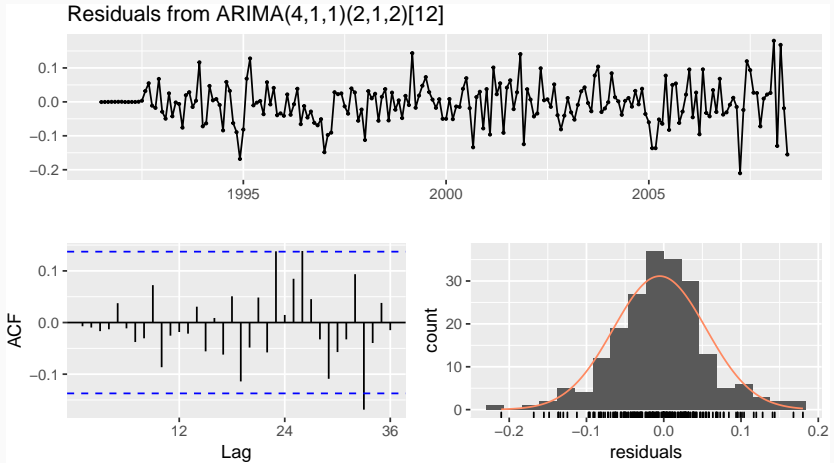
```
##          -0.383  -1.202  0.496
```

```
## s.e.      0.118   0.249  0.214
```

```
##
```

Corticosteroid drug sales

`checkresiduals(fit)`



Understanding ARIMA models

Long-term forecasts

zero	$c = 0, d + D = 0$	
non-zero constant	$c = 0, d + D = 1$	$c \neq 0, d + D = 0$
linear	$c = 0, d + D = 2$	$c \neq 0, d + D = 1$
quadratic	$c = 0, d + D = 3$	$c \neq 0, d + D = 2$

Forecast variance and $d + D$

- The higher the value of $d + D$, the more rapidly the prediction intervals increase in size.
- For $d + D = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
 - the uncertainty in the parameter estimates has not been accounted for.
 - the ARIMA model assumes historical patterns will not change during the forecast period.
 - the ARIMA model assumes uncorrelated future errors

Outline

- 1 Stationarity and differencing
- 2 Backshift notation
- 3 Autoregressive models
- 4 Moving Average models
- 5 Non-seasonal ARIMA models
- 6 Seasonal ARIMA models
- 7 Lab Session 3

Lab Session 3