



# Forecasting: principles and practice

Rob J Hyndman

2.3 Stationarity and differencing

# Outline

- 1 Stationarity
- 2 Differencing
- 3 Unit root tests
- 4 Lab session 15
- 5 Backshift notation

# Stationarity

## Definition

If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

# Stationarity

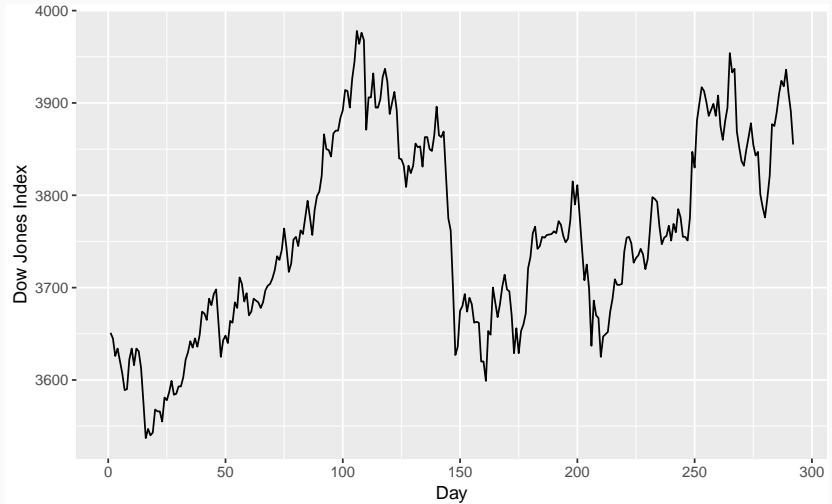
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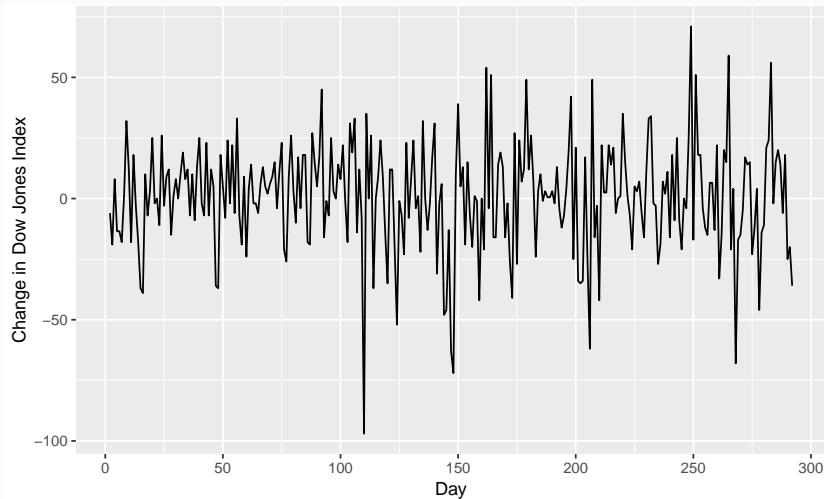
**A stationary series is:**

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

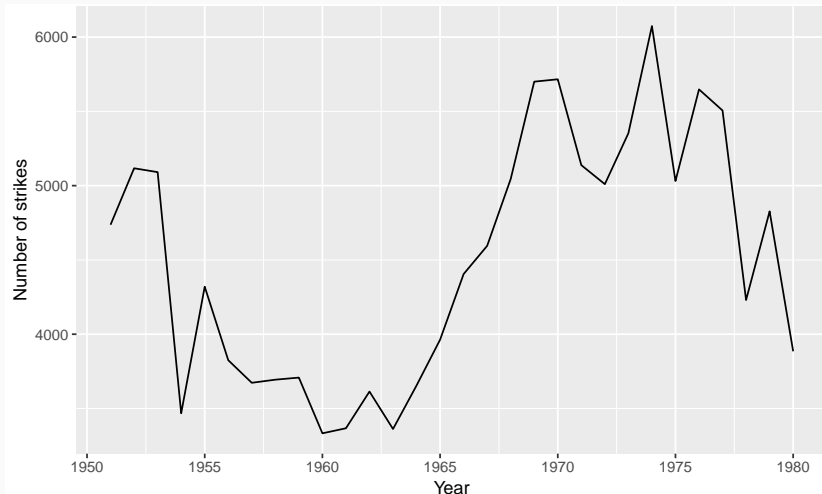
# Stationary?



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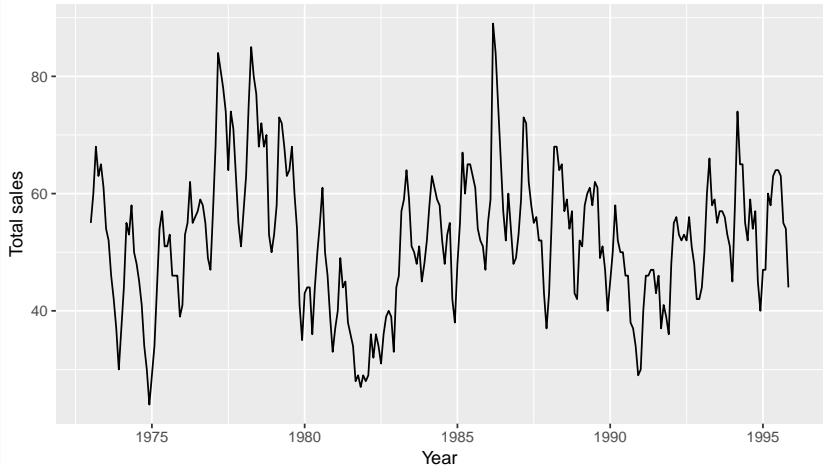


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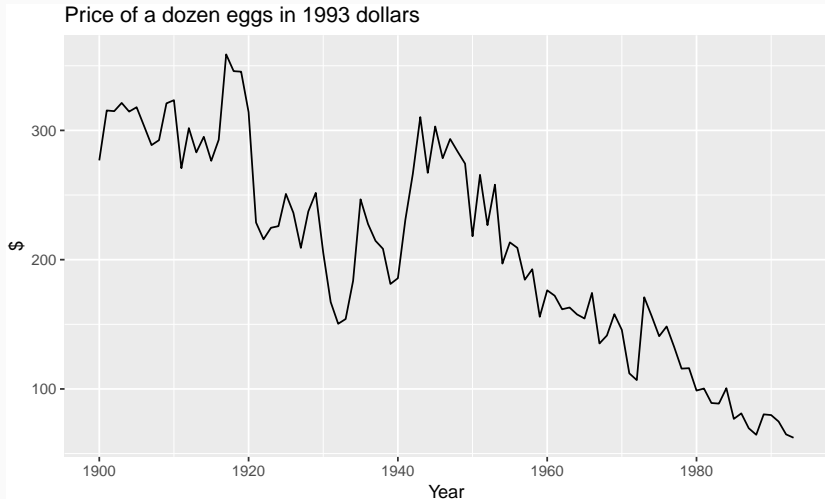
# Stationary?

Sales of new one-family houses, USA





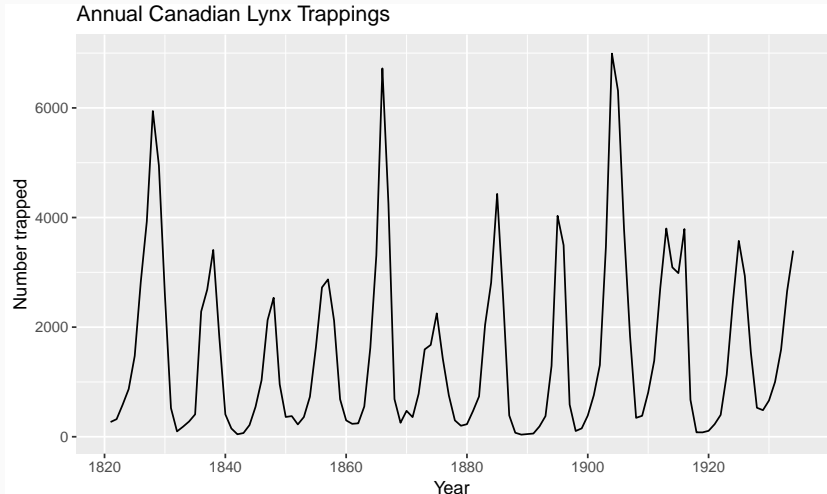
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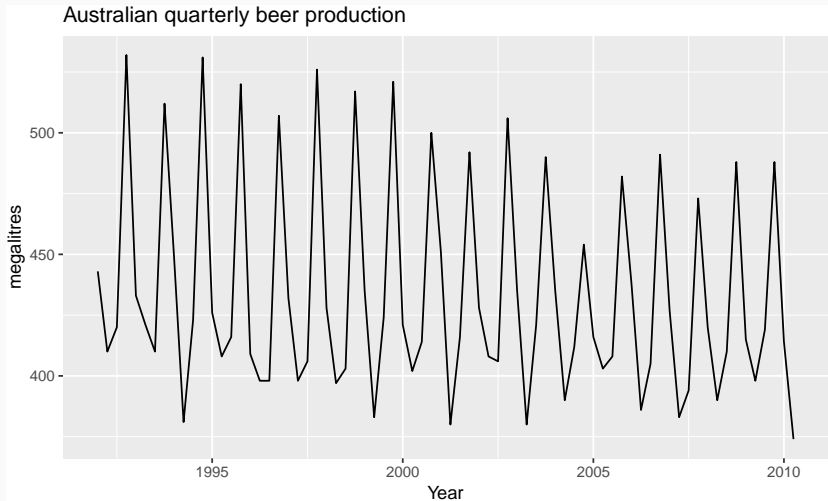
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Transformations help to **stabilize the variance**.

For ARIMA modelling, we also need to **stabilize the mean**.

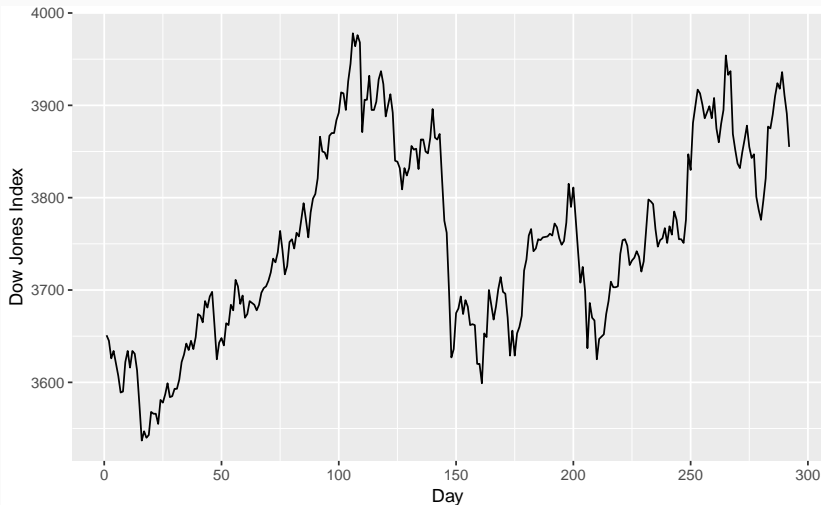
# Non-stationarity in the mean

## Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of  $r_1$  is often large and positive.

# Example: Dow-Jones index

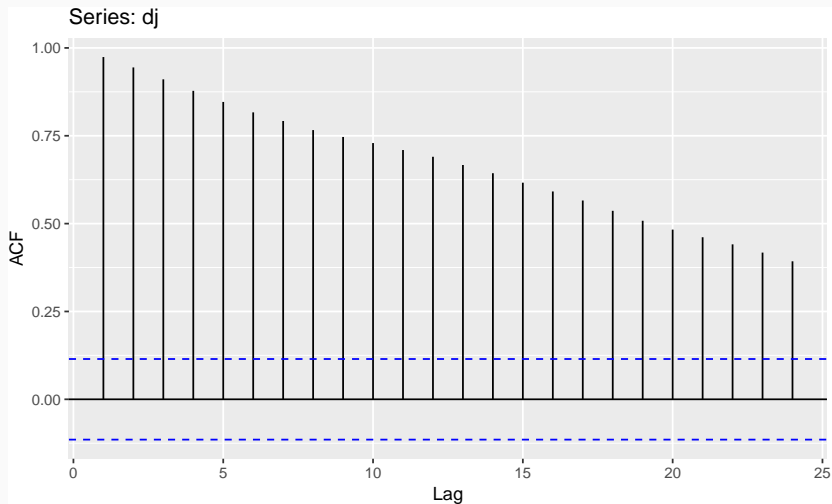
```
autoplot(dj) + ylab("Dow Jones Index") + xlab("Day")
```





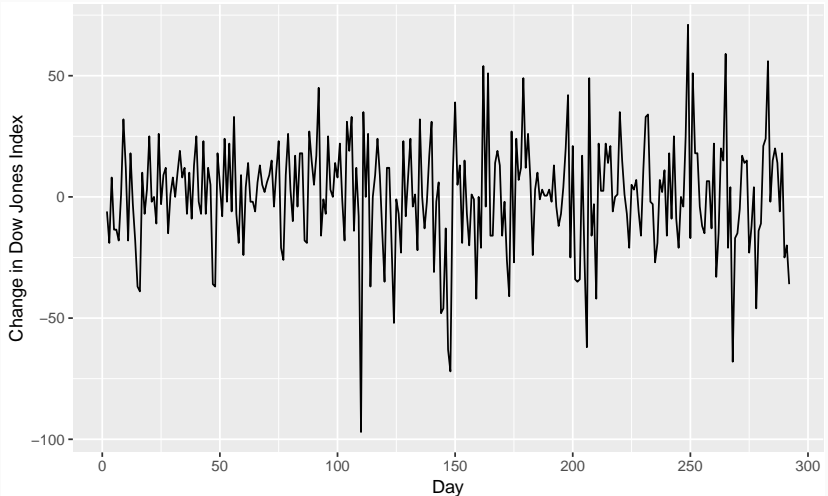
# Example: Dow-Jones index

```
ggAcf(dj)
```



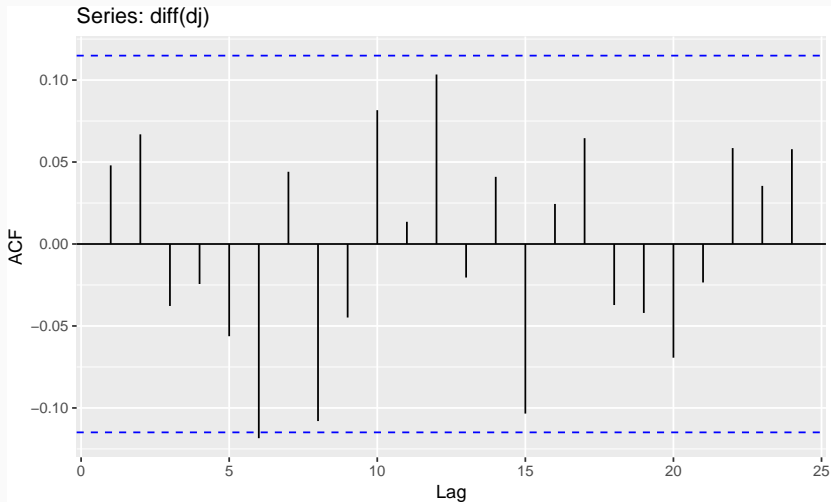
# Example: Dow-Jones index

```
autoplot(diff(dj)) +  
  ylab("Change in Dow Jones Index") + xlab("Day")
```



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```
ggAcf(diff(dj))
```



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# Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series:  
$$y'_t = y_t - y_{t-1}.$$
- The differenced series will have only  $T - 1$  values since it is not possible to calculate a difference  $y'_1$  for the first observation.

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Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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- $y_t''$  will have  $T - 2$  values.
- In practice, it is almost never necessary to go beyond second-order differences.



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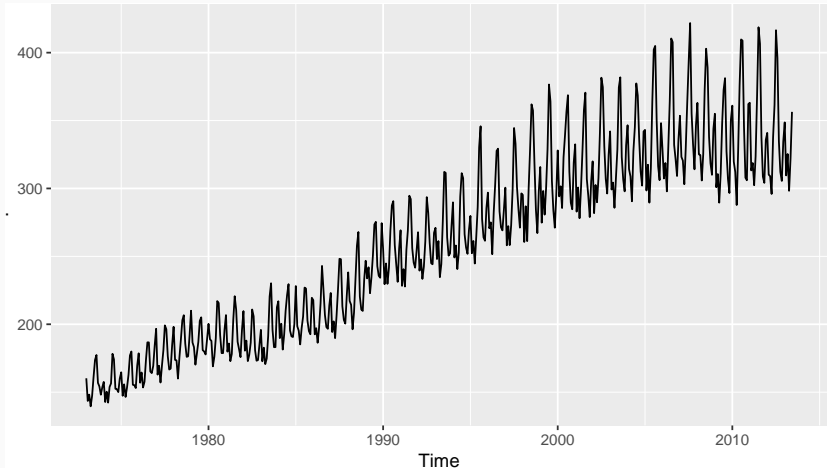
$$y'_t = y_t - y_{t-m}$$

where  $m$  = number of seasons.

- For monthly data  $m = 12$ .
- For quarterly data  $m = 4$ .

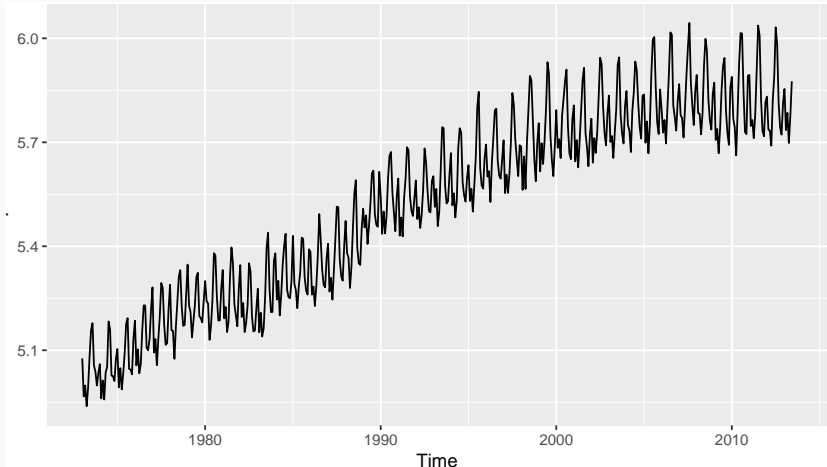
# Electricity production

```
usmelec %>% autoplot()
```



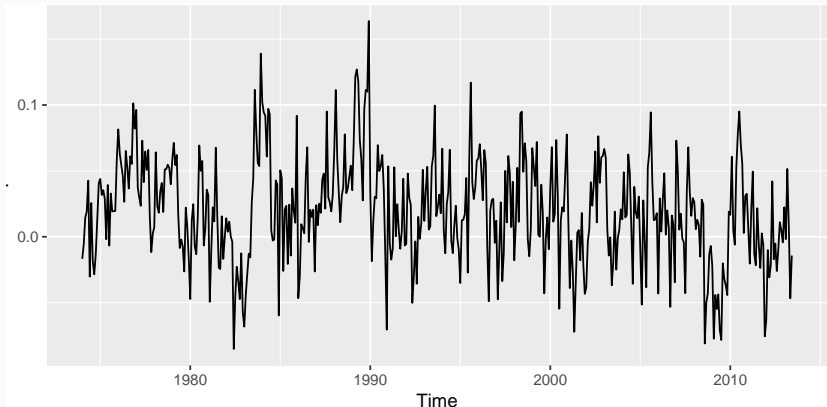
# Electricity production

```
usmelec %>% log() %>% autoplot()
```



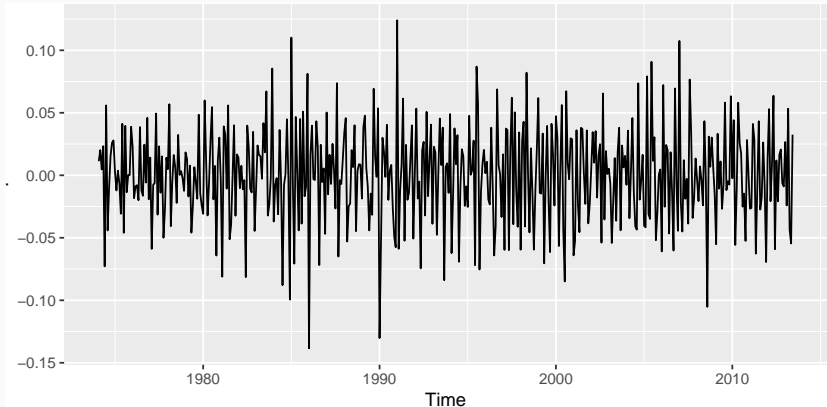
# Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%  
  autoplot()
```



# Electricity production

```
usmelec %>% log() %>% diff(lag=12) %>%  
diff(lag=1) %>% autoplot()
```



# Electricity production

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If  $y'_t = y_t - y_{t-12}$  denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned}y_t^* &= y'_t - y'_{t-1} \\&= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\&= y_t - y_{t-1} - y_{t-12} + y_{t-13} .\end{aligned}$$



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It is important that if differencing is used, the differences are interpretable.

# Interpretation of differencing

- first differences are the change between **one observation and the next**;
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But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

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## Statistical tests to determine the required order of differencing.

- 1 Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
- 2 Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
- 3 Other tests available for seasonal data.

# KPSS test

```
library(urca)
summary(ur.kpss(goog))
```

```
##
## #####
## # KPSS Unit Root Test #
## #####
##
## Test is of type: mu with 7 lags.
##
## Value of test-statistic is: 10.72
##
## Critical value for a significance level of:
##           10pct  5pct 2.5pct  1pct
## critical values 0.347 0.463 0.574 0.739
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```
ndiffs(goog)
```

```
## [1] 1
```

# Automatically selecting differences

STL decomposition:  $y_t = T_t + S_t + R_t$

Seasonal strength  $F_s = \max \left( 0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)} \right)$

If  $F_s > 0.64$ , do one seasonal difference.

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If  $F_s > 0.64$ , do one seasonal difference.

```
usmelec %>% log() %>% nsdiffs()
```

```
## [1] 1
```

```
usmelec %>% log() %>% diff(lag=12) %>% ndiffs()
```

```
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$$B(By_t) = B^2y_t = y_{t-2} .$$

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$$B(By_t) = B^2y_t = y_{t-2} .$$

For monthly data, if we wish to shift attention to “the same month last year,” then  $B^{12}$  is used, and the notation is  $B^{12}y_t = y_{t-12}$ .

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Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t .$$

# Backshift notation

- Second-order difference is denoted  $(1 - B)^2$ .
- *Second-order difference* is not the same as a *second difference*, which would be denoted  $1 - B^2$ ;
- In general, a  $d$ th-order difference can be written as

$$(1 - B)^d y_t.$$

- A seasonal difference followed by a first difference can be written as

$$(1 - B)(1 - B^m)y_t.$$

# Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$



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For monthly data,  $m = 12$  and we obtain the same result as earlier.