THE ASTROPHYSICAL JOURNAL, 166:605–619, 1971 June 15

REALIZATION OF ACCURATE CLOSE-BINARY LIGHT CURVES: APPLICATION TO MR CYGNI

ROBERT E. WILSON AND EDWARD J. DEVINNEY
Department of Astronomy, University of South Florida, Tampa
Received 1970 December 21; revised 1971 January 26

ABSTRACT

A general procedure for computing monochromatic light curves of close eclipsing-binary systems is presented, with allowance for rotational and tidal distortion, the reflection effect, limb darkening, and gravity darkening. All basic techniques used to compute light curves are specified. Solution of the inverse problem (finding the elements from observations) is accomplished by differential corrections, and probable errors are obtained for all adjustable parameters. No rectification of any kind is used or needed. Because of the basic flexibility of the scheme, present limitations (e.g., synchronous rotation only, blackbody physics) may be improved upon with reasonable convenience, as time permits. The procedure has been applied to *B* and *V* observations of MR Cyg, and the results of the differential corrections adjustments are given. With the use of available spectroscopic observations and model-atmosphere results by Mihalas, the components are placed in the H-R diagram. Although the primary seems to be on the main sequence, the secondary is found to be above the main sequence—an observation which suggests that the secondary may still be in the gravitational-contraction phase.

I. INTRODUCTION

Nearly all light curves of eclipsing systems displaying sensible proximity effects are analyzed by use of the model proposed by Russell. This model assumes that the binary components are similar ellipsoids, similarly situated, and it requires the isophotes to be similar in shape to the boundary of the star. While the model represents only an approximation to a real binary system, it has the useful property of being transformable to a spherical system of cosine limb-darkened stars. This rectified system may then be analyzed by the methods developed by Russell and others for spherical stars. These "spherical" parameters are easily related to those of the Russell model. However, systematic error is propagated into the values so determined because the model does not represent the true shapes of the stars, postulates an unrealistic distribution of light over the components, and treats the problem of mutual irradiation in an indirect way. These peculiarities render it of dubious value for determining such astrophysically interesting parameters as limb- and gravity-darkening coefficients for binaries exhibiting even modest proximity effects. An example of the difficulties which can arise is that occasionally the analysis yields a negative luminosity for one of the components. While the Russell model serves a limited purpose well, it is not suited to improvements which would make it physically more realistic.

Improved models have been studied by Kopal (1959), who investigated the light changes of stars subject to tidal and rotational distortion, with limb and gravity darkening. His treatment of the reflection effect differs only slightly from Russell's. In principle, Kopal's outline of ideas could be used to calculate light curves, but so far this has not been done, undoubtedly due to the intractability of the formulae. However, introduction of numerical procedures at certain key points overcomes these difficulties and allows one to proceed to a more improved model than the one suggested by Kopal. In fact, the method given below can accommodate advanced refinements of the theories of stellar structure and atmospheres, as required.

Lucy (1968) made the first attempt at direct calculation of light curves, using a code limited to "overcontact" systems describable by a single value of the potential. Only bolometric light curves were computed, and mutual irradiation was neglected. Hill and Hutchings (1970) concentrated on this latter aspect, using a direct computational method which treated the primary as spherical, and tried to determine the effective temperatures of the semidetached system of Algol. Their paper is open to specific criticism on several grounds, but we postpone this to avoid unduly lengthening this communication. The distortion of both components is properly treated in the approach we describe here, which also includes the photometric effects of proximity. Figure 1 displays representative examples of results of the light-curve program.

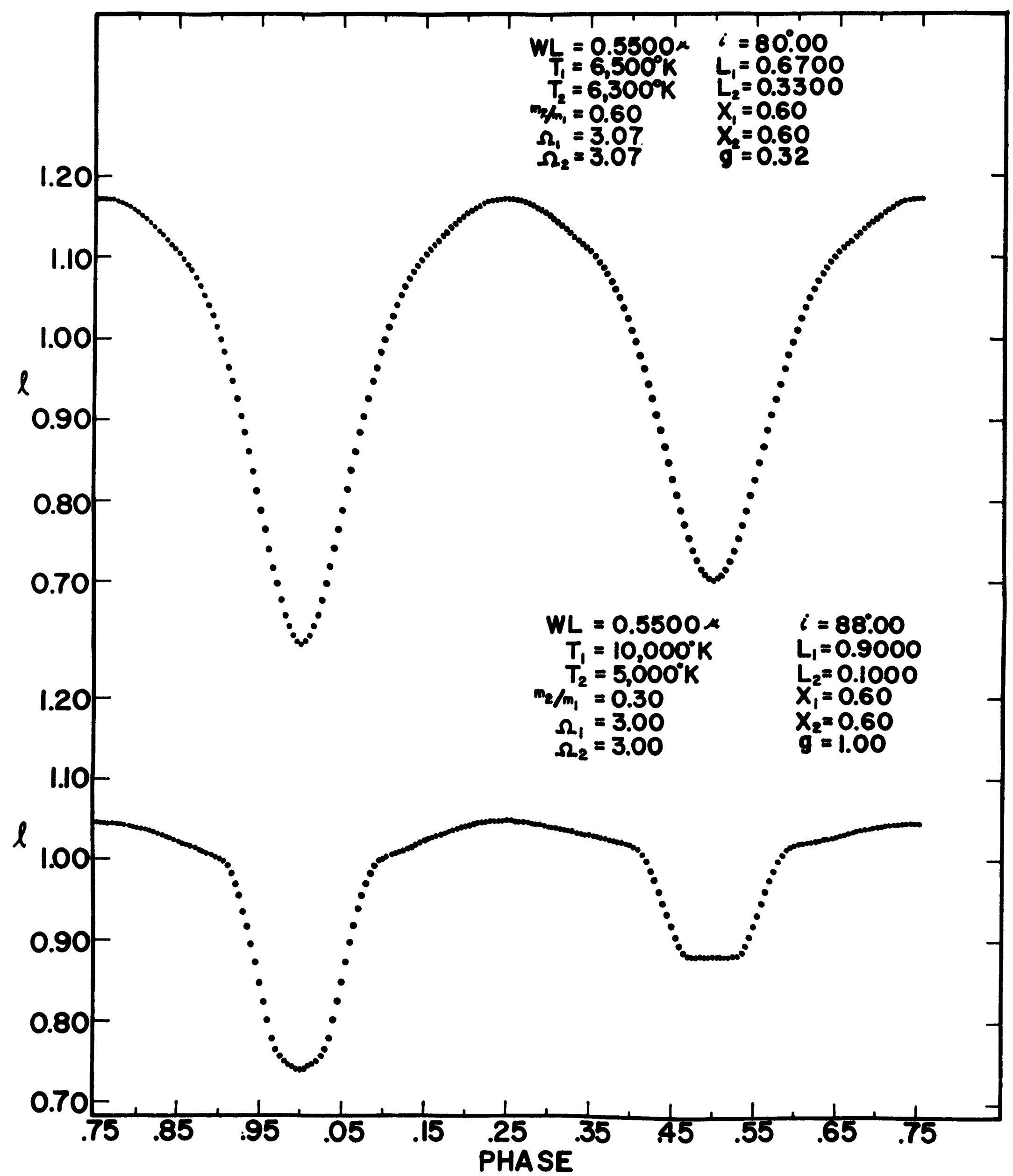

FIG. 1.—Results of the light-curve program. WL means wavelength, and the other listed parameters are defined in the text. The system showing greater distortion is virtually a contact binary. Drafting noise is responsible for any lack of smoothness. Noise due to the program would be unnoticed even on quite a larger scale.

The ideal method of comparing the model with observations of a binary system would be that which yields the most probable values of the model parameters. This set minimizes the sum of the squares of the observed-minus-computed light values. We show below how it is possible to effect this minimization for our directly calculated light curves. In order for a proper adjustment to be obtained, the calculated light curves must have quite small systematic and random errors compared with the probable error of observation. Of course, if the computational errors are only smaller than the errors of good observations, say by a factor of 2, it may be worthwhile to make a trial-and-error adjustment of the parameters, and perhaps to estimate their allowable range of variation, for a subjectively satisfactory fit to the data. Only a rigorous adjustment procedure, however, is capable of properly accounting for correlations among various parameters of the model. In contrast, the estimating technique does not reliably account for correlations, and its use will always result in error estimates which are too small. We shall see that the introduction of more realistic models does not preclude obtaining true least-squares solutions of observed light curves.

II. ADOPTED MODEL

Insofar as the tidal distortion is concerned, the model we adopt here is identical with the classical Roche model for close binaries in synchronous rotation. That is, equipotentials in the system are computed on the assumption of complete central condensation for both components. Obvious future refinements would include allowance for a polytropic mass distribution and for nonsynchronous rotation, neither of which should be extraordinarily difficult since the general computational scheme is very flexible. Even without these modifications, however, the present model is clearly a substantial improvement on a model of similar triaxial ellipsoids. In fact, not only are the figures of the stars better represented, but the general approach permits a more direct treatment of the reflection effect and a much more satisfactory treatment of gravity darkening than we find in the Russell model.

A system will be described by the inclination i, monochromatic L_1 and L_2 luminosities of the components, limb-darkening coefficients x_1 and x_2, gravity-darkening exponents g_1 and g_2, temperatures T_1 and T_2, mass ratio $q = m_2/m_1$, and surface potentials Ω_1 and Ω_2. If we set $g_1 = g_2$, as in the Russell model, we note that we have saved two parameters since we must count the Fourier coefficients of the variation in the maxima (A_0, A_1, A_2) and the spectral types of both components as Russell model parameters.[1] T_1 (or possibly T_2) would only rarely be adjustable since in almost all cases we would know the spectral type of at least one component. Normally, therefore, we shall have only ten adjustable parameters, and, if we have a spectroscopically determined mass ratio, there will be only nine. Notice that by including both luminosities instead of the luminosity ratio as free parameters, we eliminate the necessity for having a special normalization of the light curve. In effect, then, the second luminosity replaces the A_0 Fourier coefficient as a free parameter. Of course, only the ratio of luminosities has physical significance, so we give as final elements the values of L_1 and L_2 normalized such that their sum equals unity.

III. ADJUSTMENT PROCEDURE

One often hears the comment that it is the inverse problem of computing the parameters from the observations rather than computing light curves from given parameters which really bars the way to progress toward improved models of eclipsing systems. It appears to have escaped general notice that there exists a rather simple—indeed, almost trivial—solution to this problem. One merely need apply the well-known method of

[1] The minimum set of Russell model parameters is thirteen. One possible set is i, L_1/L_2, A_0, A_1, A_2, x_1, x_2, N, any three among a_1, b_1, a_2, b_2 and the two spectral types. For detailed definitions see Russell and Merrill (1952).

differential corrections (cf. Wyse 1939; Irwin 1947), using the expression for the total differential of the light values,

$$\Delta l = \frac{\partial l}{\partial i}\Delta i + \frac{\partial l}{\partial x_1}\Delta x_1 + \frac{\partial l}{\partial x_2}\Delta x_2 + \frac{\partial l}{\partial L_1}\Delta L_1 + \ldots,$$

as the equation of condition for a linear least-squares analysis. Of course, one needs a good starting approximation to the solution, but this has not proved unduly difficult in our experience to date. The result is a set of corrections Δi, Δx_1, etc., to the starting elements which, provided the corrections are suitably small and the model basically correct, yields the most probable values for the elements and their probable errors. Why this solution to an apparently insuperable problem was not appreciated earlier is not readily apparent, unless it was due to a failure to realize that the various partial-derivative coefficients $\partial l/\partial i$, $\partial l/\partial x_1$, etc., need not be expressible in analytic form since they are easily evaluated by finite differences. To be sure, we are not completely satisfied with the accuracy with which we now compute these derivatives, but the accuracy is adequate for most practical purposes and we expect it to be improved considerably in the near future. Our differential-corrections program is written so that any subset of the elements can be held fixed while the others are adjusted. One can iterate if a question arises concerning the closeness of the initial solution, so that a true least-squares fit should be achieved. We realize, of course, that the problem of their possibly being in a "local minimum" (which may not be the deepest minimum) of the variance in the nine- or ten-dimensional parameter space may prevent reaching the best solution. It is for this reason that differential corrections, or any other least-squares procedure, should never be viewed as a replacement for graphical or nomograph techniques, but should properly be used as a supplement to them for the purpose of reaching a final solution which is impersonal and which makes use of all the information inherent in the observations.

IV. OUTLINE OF PROCEDURES FOR COMPUTING LIGHT CURVES

Since realization of accurate binary-star models has previously been prevented more by lack of a suitable scheme for effecting the computations than by lack of understanding of the physics of the problem, we shall describe the logical organization of our program for computing light curves.

In order to maintain maximum flexibility and adaptability, the computations are performed by replaceable subroutines which, in turn, have been designed to be altered with minimum difficulty. The basic simplicity which underlies the entire procedure, however, and which makes possible ready assimilation of future refinements, is that all pertinent physical quantities are computed locally on the tidally distorted components. The flux seen by the observer is then found by summing the flux in his direction contributed by a large number of discrete surface elements spaced approximately uniformly over the components, excluding those which lie over the horizons and those in eclipse.

We typically generate the polar coordinates r, θ, ϕ, of several thousand to several tens of thousands of points over the surface equipotentials defined by the equation (Kopal 1959, p. 127)

$$\Omega = r^{-1} + q[(1 - 2\lambda r + r^2)^{-1/2} - \lambda r] + \tfrac{1}{2}(q + 1)r^2(1 - \nu^2)\,, \tag{1}$$

where $q = m_2/m_1$, λ, ν are direction cosines, and Ω is a linear function of the true potential Ψ. Use of the Ω-function allows a great simplification compared with using the actual potentials because only the mass ratio, rather than the individual masses, appears. The origin of coordinates is taken at the mass center of the component eclipsed at primary minimum. Equation (1) is solved for $r(\theta, \phi, q, \Omega)$ by the Newton-Raphson method, starting at the pole of each component and using each computed radius as the initial approximation to the next radius. The convergence is such that only one or two iterations are required to reach the specified accuracy of 0.00001 in r. While iterating the radii for the component covered at secondary eclipse, the origin of coordinates is temporarily placed at its center. We next compute numbers proportional to the rectangular components of the gradient of potential at each surface point. For this purpose we can use $\partial\Omega/\partial x$, $\partial\Omega/\partial y$, and $\partial\Omega/\partial z$ since these are proportional to the corresponding Ψ-derivatives. These calculations are rather simple since the Ω-derivatives are no more complicated than the Ω-function itself and no iteration is involved. Since the present model has two planes of symmetry, it is only necessary actually to compute and store the radii and potential gradient components for one-fourth of each component. We also avoid actually storing the θ, ϕ coordinates of the surface points because these can be identified by the storage locations assigned to the radii. Such economy is essential if the program is to be run on any but the largest machines.

It has been traditional to equate the luminosity ratio of the components to the flux ratio seen from some particular line of sight. These ratios are equal for the Russell model (similar ellipsoids), but for real binaries the latter is often significantly dependent on phase. We take the luminosity ratio to be the ratio of the fluxes for components 1 and 2 integrated over their surfaces in the absence of the reflection effect. In the bolometric case this is, of course, equal to the ratio of their rates of energy generation. To compute the flux from each surface element, it is necessary to work with a local intensity, so we have a subroutine which computes the mean normal emergent intensity required to yield the 4π steradian luminosity of a component when the resulting local fluxes are suitably integrated over the surface of the star.

The most difficult part of the problem is that of summing the flux in the observer's direction while correctly accounting for horizon and eclipse effects as well as gravity darkening, limb darkening, the reflection effect, and all obliquity factors without being able to use the simplified geometry afforded by sphere or ellipsoid models. Our subroutine LIGHT does this for an arbitrarily specified phase angle, so that numerous calls to LIGHT are made in computing a light curve. In order to have available a function describing the projected limb of the eclipsing component when summing the light of the eclipsed component, the star closer to the observer is always handled first. The cosine of γ is first computed for each surface element, where γ is the angle between the local surface normal and the line of sight. The surface normals are defined by the stored components of the potential gradient, with proper mirror imaging for those which have negative y- or z-components. Those points with negative $\cos\gamma$ lie on the side away from the observer and do not contribute to the summed flux. The first point beyond the horizon on each row of points is treated in a special way. Its projected coordinates in the plane of the sky are stored for later use in representing the boundary of the star. An analytic function for the boundary is then obtained by interpolating among the points for equal spacing and computing the Fourier coefficients up to 5θ through the usual relations

$$a_n = \frac{\Delta\phi}{\pi}\sum_{k=1}^{m} f_k \cos n\phi_k \,, \tag{2}$$

$$b_n = \frac{\Delta\phi}{\pi}\sum_{k=1}^{m} f_k \sin n\phi_k \,, \tag{3}$$

where there are m interpolated boundary points and f_k, ϕ_k are the polar coordinates of the kth point in the plane of the sky. For each point visible to the observer we then compute the monochromatic gravity darkening G, monochromatic reflection effect R, and the limb darkening D. The flux in the observer's direction is then given by

$$\Delta F_\lambda = r^2 \sin\theta \cos\gamma \; GDRI \; \Delta\theta\Delta\phi/\cos\beta \,. \tag{4}$$

Here I is the mean normal emergent intensity required to produce the 4π steradian luminosity of the star, as mentioned above, and need be computed only once for each component; β is the angle between the surface normal and the radius from the center; and $\cos\beta$ may be readily computed since we have the components of both these vectors. The fluxes are summed along each row before adding the total for previous rows, in order to ensure adequate resolution in the sums without using double-precision arithmetic.

For the component away from the observer the computations are identical except that we need not save the boundary points but must check each point to determine if it is within the projected boundary of the nearer component. Taking as origin the center of the eclipsing component, we simply compare the projected radius to the point with the radius given by the Fourier boundary function found earlier. We also apply a correction for fractional areas eclipsed, which is quite important since the primary limitation on accuracy is the boundary quantization. We also apply a "fast test" to detect cases in which it is obvious that no eclipse is taking place. This makes it possible to bypass many program steps and compute much faster for points in the maxima. This test compares $\sin^2\theta$ (here θ is the orbital phase angle) with $\sin^2\theta_{\max}$, where $\theta_{\max}$ is the angle of external tangency for the case of two spherical components, each of which is 2 percent larger than the back radius (i.e., the radius along the line of centers but opposite to the direction of the other component) of the corresponding distorted component.

To treat gravity darkening we first find the ratio of the local bolometric flux to that at the pole,

$$\frac{F_{\text{local}}}{F_{\text{pole}}} = (\nabla\Omega)^g\,, \tag{5}$$

where $\nabla\Omega$ has been normalized to the pole. We shall refer to the exponent as g, instead of β, to avoid confusion with the β in equation (4). For von Zeipel (1924) gravity darkening, $g = 1$, but it should have smaller values for stars with convective envelopes (Lucy 1967). The local temperature may then be computed from Stefan's law

$$T_{\text{local}} = T_{\text{pole}}\left(\frac{F_{\text{local}}}{F_{\text{pole}}}\right)^{0.25} \qquad 6)$$

Planck's law then gives the ratio G of the local monochromatic normal emergent intensity to that at the pole.

The geometry of the reflection effect we now use contains some simplifications. Although the distortion of the irradiated component is fully accounted for, the irradiating component is treated as a point source, and the irradiated region is limited to those points within view of the point source. Based on our experience to date in fitting observed light curves, we believe that these simplifications introduce only very minute errors. We consider our geometrical treatment to be far better than that in the Russell model, where light is, in effect, added to the back hemispheres—a procedure which complicates the interpretation in several ways. Naturally we apply the reflection for both components so that the total (not just differential) reflection appears in our computed light curves. The local effect is found by considering the energy balance between the local bolometric energy flux of the affected component and the bolometric energy incident from the other component, taking due account of the tilt of the local surface element with respect to the source of irradiation, and of distance effects. This requires a prior computation of the ratio of bolometric luminosities, which is found from the ratio of monochromatic luminosities (independent variables) and the difference of their bolometric corrections (Harris 1963). A new temperature is then computed from the local temperature previously found for the gravity effect, by Stefan's law

$$T_{\text{new}} = T_{\text{old}}[(L_1/L_2)_{\text{bolo}}\ (\text{geometrical factors})]^{0.25}\,.$$

With this new temperature we can compute the ratio R of the local monochromatic surface brightness to that which would exist in the absence of the reflection effect, and this is the R which appears in equation (4).

At present we use only the linear cosine limb-darkening law,

$$J = J_0(1 - x + x \cos \gamma) . \tag{7}$$

It would be a trivial modification to use a more accurate law such as the one recommended by Klinglesmith and Sobieski (1970), but equation (7) has the advantage of containing only one adjustable parameter. Since only a very few eclipsing systems have been reliably analyzed for the linear coefficient, there appears to be no cause for haste in seeking second-order terms. In addition, Grygar (1963) has shown that it would be very difficult to find these terms with sufficient accuracy to distinguish among various atmosphere models, except perhaps for very early-type stars.

Our program can deal with the case of "overcontact" or dumbbell-shaped binaries, such as considered by Lucy (1968). For those radii from the center of each component which pass through the connecting neck, a "flag" (value = -1) is inserted in the appropriate memory location. When a radius of -1 is encountered in summing the brightness, the light of the point is omitted from the summation. Quantitative comparison of these light curves with those of W UMa stars will be discussed in a future paper. We have plans for reducing the running time of the program, and it seems that a factor of 10 or more may be achieved by the use of various tricks. It should therefore be practical to correct differentially all parameters without the use of normal points within the next year.

Many tests were made to ensure the correct functioning of the program. Initially we made checks on the radii and gradient components to be sure that they varied as expected. The consistency of the system geometry was verified in several ways. For example, both components reach the inner Lagrangian point for precisely the critical value of the potential. All subprograms were thoroughly tested separately as well as together. Among the specific tests of the light-curve program were the following. The light curve for a well-separated, limb-darkened pair with negligible proximity effects was checked against a light curve for the same system computed by an independent spherical model program which, in turn, had been checked against the Merrill (1950) tables. With each component represented by 4636 surface points, agreement was of the order of one part in 1000 in the worst case. We have ideas for improving this accuracy without increasing the number of surface points, but it seems adequate for the present. This worst-case configuration (annular eclipse) does not occur for MR Cyg, the system we study here. The largest errors in the theoretical light curves of MR Cyg are less than 0.0002 fractional light units. A case for 0° inclination showed no light variation, as expected. Also, the symmetry about 0° and 180° phase angle is virtually perfect. A very interesting test was for a system having zero limb darkening but full von Zeipel gravity darkening. A central eclipse for such a system should show an apparent limb brightening in the annular phases because the disk of the larger star is relatively dim at the center since this is the region of maximum gravity darkening at phase angle 0°. The computed light curve showed, as expected, an annular phase in which the curvature was concave downward. Our least-squares subprogram, used in the differential-corrections program, was tested on a problem with known answers. It did find the correct parameters and probable errors. As a final point we should mention that increasing the number of surface elements by a large factor introduces no systematic changes in the light curves, but merely improves their precision.

V. MR CYGNI PARAMETER ADJUSTMENTS

Considerable thought went into our selection of the first binary to be studied in terms of this model. The system MR Cyg, as observed by Hall and Hardie (1969), has nearly all of the characteristics one would prefer in such a test sample. The observations are accurate and well distributed in phase, and the light variation repeats well and shows no obvious complications. Of course, these are filtered observations—a point which excluded some otherwise acceptable examples. The proximity effects are large enough so that we may have significant departures from the Russell model, and the spectral types (middle to late B) are in the range of relatively well-understood model stellar atmospheres. The B spectral types are important also because the envelopes should be fully radiative, so that the simple theory of gravity darkening (von Zeipel 1924) should apply for these stars if it applies for any. Finally, double-lined radial-velocity observations exist (subject to a qualification to follow), which potentially provide important information for placing the components in the H-R diagram. The system is less than ideal in one respect, namely, that the eclipses are probably not complete, although nearly so. The reference made by Hall and Hardie to a complete solution evidently applies only to their nomograph (Merrill 1953) solution, because their subsequent adjustments led to partial eclipses both in their final solution and in their separate B and V solutions. Only in U do their elements correspond to complete eclipses—a point not explicitly mentioned by them. Our results definitely indicate partial, though geometrically deep, eclipses.

We first made a fairly large number (prehaps 15–20) of trial-and-error adjustments in B and V, using the elements given by Hall and Hardie as an initial guide. It is almost certain that fewer trials would be needed for a system with complete eclipses, and it may be that experience will reduce their number in future work. Not all parameters were adjusted, since it was necessary to complete the project under certain constraints of time and computer availability. In these adjustments we tried to stay close to the theoretically expected darkening coefficients for stars of the estimated spectral types (cf. Grygar 1965). The effective temperature of the hot component was fixed at 18000° K (cf. Harris 1963), and the gravity-darkening exponent was fixed at 1.00, corresponding to von Zeipel darkening. It was at all times clear that a gravity exponent significantly less than 1.00 would degrade the quality of the fit by decreasing the photometric ellipticity. If any thing, the ellipticity effect in the computed light curves was a trifle too small. The agreement between the computed and observed ellipticity effects is certainly satisfactory at present, but one definitely would not want to decrease the gravity exponent. One could not simply increase the sizes of the components in compensation because the eclipse durations would then be too great. This may be the most substantial demonstration of the existence of gravity darkening to date, for the light curves cannot be satisfied at all with $g \simeq 0$. A comparison (Rucinski 1969) of theoretical and observed gravity-darkening effects through the Fourier coefficients in the maxima of four systems showed apparently satisfactory agreement. Unfortunately, this study neglected the reflection effect, which is almost certainly important.

Rather than actually adjust the mass ratio, which would have been difficult by trial and error, we tried two particular values. We first tried $m_2/m_1 = 0.56$, as suggested by Hall and Hardie, and were able to find a marginally acceptable fit to the observations; but there was a decided improvement when we changed to the value 0.83, which follows from the double-lined radial-velocity measures made by J. A. Pearce at Victoria, the elements of which were published by Harper *et al.* (1935). Hall and Hardie were skeptical of the validity of this determination because they thought it unlikely that the lines of the secondary could be measured satisfactorily due to its low fractional luminosity, and also because the absolute masses would not match the assumed spectral types. Although these appear to be valid objections, Hall and Hardie did not actually examine Pearce's observations, which have now been kindly supplied to us by Dr. Pearce (the measured velocities) and by Dr. Alan Batten (selected spectrograms). Figure 2 shows the double-lined radial-velocity curve of MR Cyg, which does seem to establish a rather good value for the mass ratio. On the other hand, we must admit that we cannot be sure we see the lines of the secondary component on any of the plates supplied by Batten, and Dr.

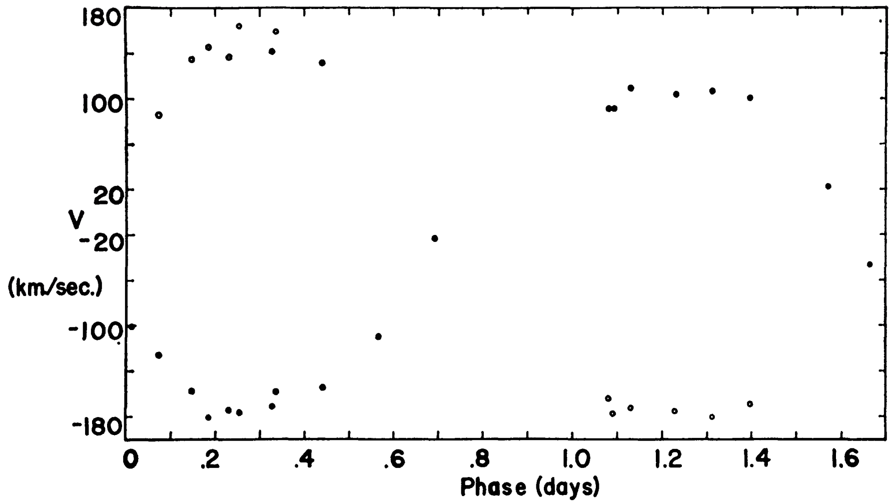

FIG. 2.—Previously unpublished radial-velocity curve of MR Cyg observed by J. A. Pearce in 1932. *Filled circles*, component 1; *open circles*, component 2.

Batten tells us that he is not sure he sees them either! Evidently the experience necessary to measure these lines is very considerable, and we must consider the question of the mass ratio of MR Cyg an open question at present. However, we are not ready to dismiss the spectroscopic mass ratio only on the grounds that it dictates low absolute masses, since it is conceivable that the evolutionary history of such an unusual binary (two B stars nearly in contact) may have produced a pair of quite exceptional objects.

The parameters thus adjusted were then subjected to our differential-corrections analysis. For the differential corrections only we used a fine surface grid of 18456 points on each component. It was necessary to make normal points of five observations each to keep computing time at a reasonable level, and even so the program ran 75 minutes for each iteration on the IBM 360-91 computer at Goddard Space Flight Center. We adjusted seven parameters (i, x_1, x_2, L_1, L_2, Ω_1, and Ω_2), while keeping m_2/m_1, T_2, and g fixed. The program is capable of adjusting these latter parameters, but we felt it prudent not to adjust them in view of the already long running time. The observations were weighted on the assumption that scintillation noise (which we take to include variable transparency) is dominant; that is, the weights were inversely proportional to the square of the light level. The B and V results are given in Table 1. Probable errors for the radii may be computed from those in Ω and the derivative

$$\frac{dr}{d\Omega} = \left\{(q+1)(1-\nu^2)r - \frac{1}{r^2} - \left[\frac{q(r-\lambda)}{(1-2\lambda r + r^2)^{3/2}} - qr\right]\right\}^{-1},$$

where λ is zero for the polar and side radii, $+1$ for the point radius, and -1 for the back radius. The direction cosine ν is $+1$ in the polar case and zero for the other three cases. Since the corrections from the trial-and-error parameters were satisfactorily small, we give only the final results. It was quite obvious that the differential-corrections program was working properly because we made the check of computing the direct light curves before and after the adjustments, and in both colors the differential corrections visibly improved the fit beyond the best previously obtained. The final fit to the V observations is shown by the theoretical light curve plotted in Figure 3. The mean residual from the final solution in B or V for a single normal is about 0.006 mag, a value representative of the eclipses as well as the maxima.

The reader may notice in Table 1 that the luminosity ratio actually changes in the

TABLE 1

ELEMENTS AND AUXILIARY INFORMATION IN B AND V

Parameter	B	p.e.	V	p.e.
i(degrees)	82.29	±0.30	83.28	±0.61
L_1	0.7698	±0.0087	0.7848	±0.0081
L_2	0.2302	±0.0080	0.2152	±0.0076
x_1	0.75	±0.06	0.62	±0.08
x_2	−0.36	±0.21	−0.11	±0.12
g_1	1.00	...	1.00	...
g_2	1.00	...	1.00	...
$\lambda(\mu)$	0.435	...	0.550	...
T_1(° K)	18000	...	18000	...
T_2(° K)	13500	...	13500	...
m_2/m_1	0.83	...	0.83	...
Ω_1	3.828	±0.017	3.749	±0.017
Ω_2	3.906	±0.030	4.009	±0.035
r_1(pole)	0.329	...	0.337	...
r_1(point)	0.371	...	0.388	...
r_1(side)	0.340	...	0.350	...
r_1(back)	0.356	...	0.369	...
r_2(pole)	0.290	...	0.280	...
r_2(point)	0.322	...	0.307	...
r_2(side)	0.298	...	0.287	...
r_2(back)	0.312	...	0.299	...

wrong sense from yellow to blue. This is entirely due to the relatively poor determinacy of partial-eclipse solutions and to the fact that we have not constrained the geometry of the system to be the same in both colors, contrary to usual practice. Rather, we have treated the two light curves as independent sources of information for the geometric as well as the photometric elements. Consider, for example, the orbital inclination i. Although the yellow and blue values of i differ by more than 1°, this is almost within their overlapped probable errors. The resulting differences in the overall geometry cause substantial changes in the yellow and blue luminosity ratios which, by chance, happen in the sense such that the normal variation of L_1/L_2 with wavelength is reversed. It is quite clear from inspection of the light curves that the true luminosity ratio varies in the correct sense with wavelength, and by about the expected amount. One could realize this result within our solution scheme either by adjusting the geometrical parameters in one color only and leaving them fixed in all others or, preferably, by adjusting observations in all colors simultaneously in one very large differential-corrections solution. In view of our present limitations of computer accessibility, we have chosen not to do this for MR Cyg.

The negative values for x_2 in both colors may cause some concern, but it must be remembered that x_2 is normally considered an extremely weakly determined quantity (Kopal 1959, p. 376; Wilson 1968) for partially eclipsing systems as, in fact, is x_1. The arguments by Kopal and by Wilson were within the context of the Russell model, and, since this actually has more adjustable parameters than the present model, there may yet be some hope for finding darkening coefficients in the case of partial eclipses. Further experience is certainly needed, however, before many definite statements can be made on this question. If there is actually some physical significance to these negative x_2 values, we note that the effect is in the proper direction to have been induced by the irradiation of the primary component (i.e., by diminishing the temperature gradient in the atmosphere of the secondary on its inner-facing hemisphere). Our experience definitely indicates, however, that the gravity darkening must be close to that given by the

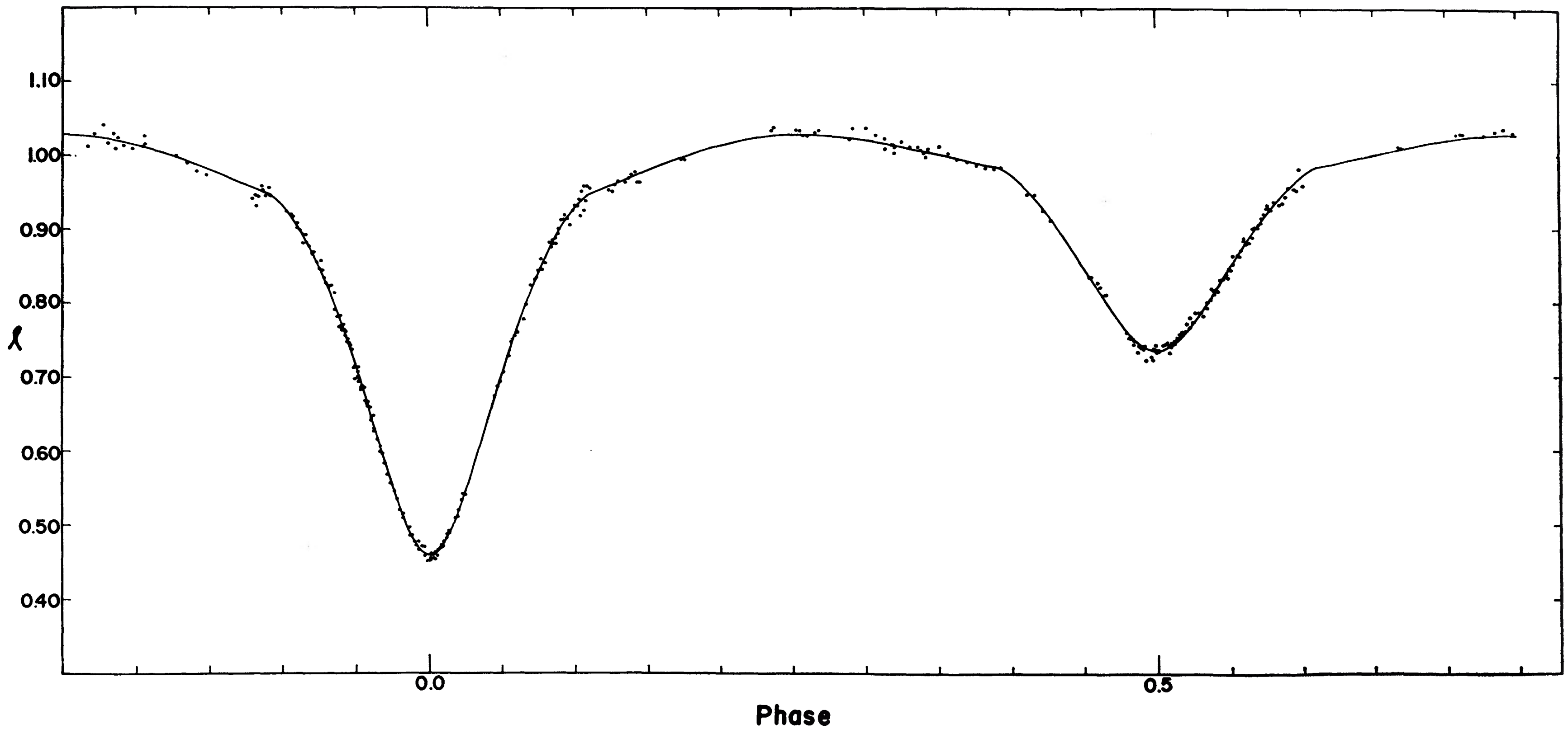

Fig. 3.—Theoretical *V* light curve of MR Cyg graphed among the observations

von Zeipel law. This result suggests that investigations of W UMa stars may lead to substantial checks on the theory of convection in stars, since according to Lucy (1967) convection lowers the gravity darkening substantially. Finally, it should be noted that quite a good fit to the observations was accomplished, even though the number of adjustable parameters was two fewer than in the Russell model.

We consider it extremely important to provide the reader with a means of checking our results. Therefore, our theoretical light curve for the V solution is given in Table 2. This was computed with the V parameters of Table 1, except that the potentials were rounded to two decimal places, the unnormalized luminosities ($L_1 = 9.8416$, $L_2 = 2.6984$) were used, and x_2 was set equal to 0.00 instead of -0.11 (which makes extremely little difference).

VI. INTERPRETATION OF MR CYGNI

The only previous discussion of the basic nature of MR Cyg was by Hall and Hardie, who concluded that both components were on the main sequence and "essentially normal and uncomplicated." We are now in a stronger position to test these conclusions because of our improved analysis, as described above. When placing the components in the H-R diagram, we can find the differences of their coordinates much more reliably than the absolute coordinates of either one, so we begin by establishing ΔM_v and $\Delta(B - V)$. ΔM_v is equal to 1.41 mag and comes directly from the luminosity ratio L_1/L_2 for the V light curve. These are the true 4π steradian luminosities and are determined already freed from the reflection effect. Hall and Hardie tried to infer $\Delta(B - V)$ directly from the color curve and the parameters of their solution. It is well known, however, that $B - V$ is a weak function of temperature for hot stars, so the value of $\Delta(B - V)$ they

TABLE 2

THEORETICAL V LIGHT CURVE

θ	l	θ	l
0.00	0.4577	0.26	1.0273
0.01	0.4750	0.27	1.0265
0.02	0.5196	0.28	1.0251
0.03	0.5783	0.29	1.0229
0.04	0.6422	0.30	1.0201
0.05	0.7055	0.31	1.0168
0.06	0.7650	0.32	1.0132
0.07	0.8183	0.33	1.0092
0.08	0.8642	0.34	1.0050
0.09	0.9014	6.35	1.0008
0.10	0.9295	0.36	0.9965
0.11	0.9474	0.37	0.9924
0.12	0.9557	0.38	0.9883
0.13	0.9639	0.39	0.9842
0.14	0.9721	0.40	0.9725
0.15	0.9801	0.41	0.9547
0.16	0.9878	0.42	0.9324
0.17	0.9951	0.43	0.9066
0.18	1.0019	0.44	0.8781
0.19	1.0080	0.45	0.8480
0.20	1.0134	0.46	0.8174
0.21	1.0179	0.47	0.7878
0.22	1.0216	0.48	0.7615
0.23	1.0244	0.49	0.7424
0.24	1.0262	0.50	0.7352
0.25	1.0272		

were seeking is only about as large as the observational uncertainty. If there existed an observable quantity which varied much more strongly with temperature than does $B - V$, we could find our small $\Delta(B - V)$ more accurately, even though indirectly. Such a quantity is the emergent (monochromatic) flux. That is, for stars in the temperature range we are considering, a change in temperature causes only a small change in color, but a much larger change in the absolute emission. The emergent flux for either component, F_1 or F_2, can be computed simply by dividing the luminosity by the surface area. Since the effective temperature of the primary is reasonably well determined at about 18000° K by its B3 spectral type and the temperature calibration of the MK system (Harris 1963), we can find the temperature of the secondary from our value of F_1/F_2 and the tables of Mihalas (1965), which give emergent flux as a function of wavelength, temperature, log g, and helium abundance for early-type model stellar atmospheres. We find, taking $\log g = 4.5$ and He/H = 0.15, a temperature of about 11000° K for the secondary, corresponding to an intrinsic $B - V$ of about -0.01 (Harris 1963). The disagreement between this temperature and the temperature of Table 1 (which mainly satisfies the reflection effect) is understandable in terms of the incomplete allowance for blanketing in our present model. Hopefully this will be improved in future work.

We have now fixed the absolute horizontal coordinates of components 1 and 2 at $B - V = -0.19$ and -0.01, and we have a value of 1.41 mag for the difference of their vertical coordinates. If we could establish the absolute visual magnitude of the primary, the positions of both stars could be accurately plotted. Since Pearce's double-lined radial-velocity curve exists, we should be able to find the absolute dimensions of both components and, with absolute flux values from Mihalas's tables, compute M_v. In view of the controversy over the velocity curve of the secondary, however, we have decided to use only the radial velocities of the primary. In this case the dimensions of the stars cannot be determined unless we *assume* a value for the mass ratio. This assumption yields directly the ratio of the absolute orbital radii, a_1/a_2, and, since we know a_1 from the single-lined curve, we have $a = a_1 + a_2$, the separation of centers in kilometers. The radii from our photometric solution are expressed in this unit, so we arrive at the dimensions of each component in kilometers. Figure 4 shows the components of MR Cyg plotted in the (color, absolute magnitude) diagram for various assumed mass ratios, with the zero-age main sequence (Johnson 1963) also shown. One should keep in mind that the only degree of freedom presently allowed, except for those due to ordinary errors in estimating parameters, is a vertical displacement of components 1 and 2 *moved as a pair*. That is, since both differential coordinates are well determined, component 2 must always be placed in the same position relative to component 1.

Unless we accept the extreme upper values for m_1/m_2, the present evidence is that the primary is on the main sequence. Indeed, the uncertainty introduced by the mass-ratio problem is not much larger than the present uncertainty in the position of the standard zero-age main sequence. The secondary, however, clearly is above the main sequence, and there is no adjustment capable of placing it on the zero-age main sequence unless the primary is actually below, which possibility we do not consider. The overluminosity of component 2 amounts to a little over 1 mag, only about one-third of which could be accounted for by using Hall and Hardie's value for the luminosity ratio, so it appears that the secondary really is not a main-sequence star, contrary to their estimates.

The present theories of stellar evolution allow two possible interpretations of the system. First, the secondary may still be in the phase of gravitational contraction. According to Iben (1965), the time required for a 3 $\mathfrak{M}_\odot$ star to reach the observed position of MR Cyg B on the H-R diagram is close to 1.2×10^6 years. Since the main-sequence lifetime of the primary is at least 5×10^6 years, this hypothesis satisfies the observed locations for both stars. Alternatively, it cannot be ruled out that the system has arrived at its present configuration by way of a complex process of mass exchange. Seemingly, this

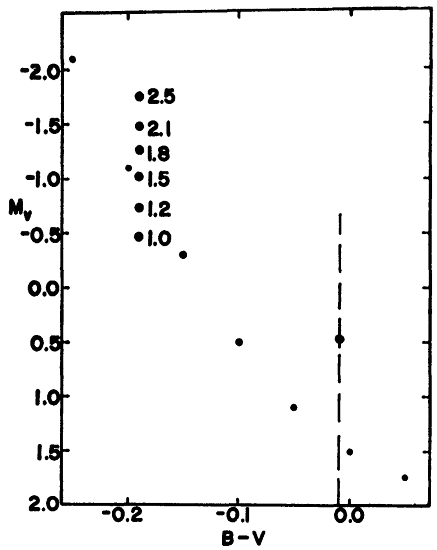

FIG. 4.—Components of MR Cyg plotted in the color-magnitude diagram for six assumed mass ratios. Only one position is actually plotted for the secondary (*large filled circle*). This position is for the primary being exactly on the zero-age main sequence (*small filled circles*), and the dashed line shows the degree of freedom permitted for the secondary by the uncertainty in the mass ratio. See additional remarks in the text.

would result in at least a semidetached system, so that we feel the first alternative is the likelier possibility. With further detailed study, MR Cyg may shed some light on the hypotheses (e.g., fission) of the origin of binary systems.

It is not practical to thank all persons who assisted at various stages of this work, but at the U.S.F. Computer Research Center Fred Brisard, Tom Walters, Gerhard Stoopman, and Dr. William Miller have been particularly helpful. At the University of Pennsylvania, Dr. Scott Shaw arranged for computer time during one crucial 2-week period, and at Goddard Space Flight Center Mr. Lawrence Draper ran programs for us when we could not be physically present. We also thank Drs. Alan Batten and J. A. Pearce for their contributions, which were mentioned in the text, and Dr. Pearce for permission to publish the graph of his radial-velocity observations. Dr. Douglas Hall was consulted on several points concerning his interpretation of the MR Cyg observations. Most of the working graphs and all of the final figures were made by Mr. Lawrence Twigg. Some additional graphs were made by Mr. Geoffrey Forden. Finally we express our gratitude to the National Science Foundation for support through grant GP18181.

REFERENCES

Grygar, J. 1963, *Bull. Astr. Inst. Czechoslovakia*, **14**, 127.
———. 1965, *ibid.*, **16**, 195.
Hall, D. A., and Hardie, R. H. 1969, *Pub. A.S.P.*, **81**, 754.
Harper, W. E., Pearce, J. A., Petrie, R. M., and McKellar, A. 1935, *J.R.A.S. Canada*, **29**, 411.
Harris, D. L. 1963, in *Basic Astronomical Data*, ed. K. Aa. Strand (Chicago: University of Chicago Press), p. 263.
Hill, G., and Hutchings, J. B. 1970, *Ap. J.*, **162**, 265.
Iben, I. 1965, *Ap. J.*, **141**, 993.
Irwin, J. 1947, *Ap. J.*, **106**, 380.
Johnson, H. L. 1963, in *Basic Astronomical Data*, ed. K. Aa. Strand (Chicago: University of Chicago Press), p. 204.
Klinglesmith, D. A., and Sobieski, S. 1970, *A.J.*, **75**, 175.
Kopal, Z. 1959, *Close Binary Systems* (New York: John Wiley & Sons).

Lucy, L. B. 1967, *Zs. f. Ap.*, **65**, 89.
———. 1968, *Ap. J.*, **153**, 877.
Merrill, J. E. 1950, *Contrib. Princeton Obs.*, Vol. **23**.
———. 1953, *ibid.*, Vol. **24**.
Mihalas, D. 1965, *Ap. J. Suppl.*, No. 92, **9**, 321.
Rucinski, S. M. 1969, *Acta Astr.*, **19**, 125 (No. 2).
Russell, H. N., and Merrill, J. E. 1952, *Contrib. Princeton Obs.*, Vol. **26**.
Wilson, R. E. 1968, *A.J.*, **73**, S124.
Wyse, A. 1939, *Lick Obs. Bull.*, **19**, 17 (No. 496).
Zeipel, H. von. 1924, *M.N.R.A.S.*, **84**, 665, 684, 702.