CLASSIFYING TOPOS[1]

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The basic facts about the classifying topos of a stack of groupoids were first stated in [3] and are exposed in detail in [4] Ch. VIII. This construction is useful in cohomology theory and has been introduced independently by D. Mumford to study moduli of elliptic curves [7]. Algebraic stacks of groupoids are used in algebraic geometry df. [1]. Here a simpler and more general approach allows us to treat the case of a stack whose fibers are not supposed to be groupoids. As a by-product we get the existence of fibered products in the bicategory of topos. This result was first announced by M. Hakim several years ago but was never published. I suspect that any written prof would have to deal with rather subtle technical difficulties about finite limits which are overcome here by the results of §1.

Is $\mathscr{S}$ is a site we use the word[2] *stack* for the french champ [4] and prestack for prechamp (a prestack is merely a fibered category over the underlying category of the site) and *split stack* for champ scindé. Up to equivalence a split stack can be viewed as a sheaf of categories over $\mathscr{S}$ (or a category-object of the corresponding topos) satisfying some extra condition namely the patching of objects. As usual we choose and fix a universe $\mathfrak{U}$. For clarity it should be recalled that a $\mathfrak{U}$-topos is a special case of $\mathfrak{U}$-site [5] and that any category can be viewed as a site such that any presheaf is a sheaf and prestack is a stack.

[1]Toposes , algebraic geometry and logic , Lecture Notes in Maths., vol.274 , Springer , 1972.

[2]"work" in the original.

1. Left exact stacks

A category is left exact if it admits finite limits. A functor $f : A \longrightarrow B$ between left exact categories A and B is left exact if it preserves finite limits. A site is said to be left exact if the underlying category is so. A stack C over a site $\mathscr{S}$ is said to be left exact if its fibers are left exact and if for any map $f : T \longrightarrow S$ in $\mathscr{S}$ the inverse image functor induced by f between the fibers of C is left exact.

Lemma **(1.1)**. — *A stack C over a left exact site $\mathscr{S}$ is left exact if and only if the underlying category and the structural functor $p : C \longrightarrow \mathscr{S}$ are left exact.*

The proof rests on the fact that a commutative square of C whose projection is cartesian in $\mathscr{S}$ is cartesian as soon as two opposite sides are $\mathscr{S}$-cartesian.

Lemma **(1.2)**. — *A morphism $m : A \longrightarrow B$ between two left exact stacks is left exact if and only if for any $S \in |\mathscr{S}|$*[3] *the functor $m_S : A_S \longrightarrow B_S$ induced by m between the fibers at S is left exact.*

Proposition **(1.3)**. — *Let $f : \mathscr{S}' \longrightarrow \mathscr{S}$ be a morphism between two sites (e.g. two topos). Then the direct image (resp. inverse image) of a left exact stack and of a left exact morphism of stacks over $\mathscr{S}'$ (resp. $\mathscr{S}$) is left exact.*

1.3.1. The direct image of a stack being nothing but pull-back along the underlying functor $f^* : \mathscr{S} \longrightarrow \mathscr{S}'$ of f, preserves the fibers, hence the left exactness. To treat the case of the inverse image by f of a stack over $\mathscr{S}$ we will use the following characterisation[4] of left-exactness.

1.3.2. First let I be a finite category. For any stack F over $\mathscr{S}$ let F^I be the prestack whose fiber at $S \in |\mathscr{S}|$ is the category of functors from I to the fiber F_S. One checks easily that this is a stack provided with a morphism of stacks (constant diagrams)

$$cF : F \longrightarrow F^I \tag{1}$$

Furthermore *F is left exact if and only if for any finite category I cF admits a right adjoint in the bicategory of stacks*. The if part is obvious since such an adjoint

[3] The set of objects of a category C is denoted by $|C|$

[4] "caracterisation" in the original.

λ induces an adjoint to each functor cF_S, $S \in |\mathscr{S}|$, induced by cF on the fibers at S and since λ is cartesian. The only if part is no more difficult than (1.2). Since the property of having a right adjoint is preserved by morphisms of bicategories and since the inverse image of stacks is such a morphism [4] p.88, it remains to show the following.

Lemma **(1.3.3)**. — *One has a natural equivalence $e : f^*(F^I) \longrightarrow f^*(F)^I$ such that $ef(cF) = cf^*(F)$.*

According to [4] p.88, the inverse image of a stack F is given by the formula

$$f^*(F) = Af^{-1}(LF) \tag{1}$$

where LF is the free split stack associated to F [4] p.39, where f^{-1} denotes the inverse image of LF as category-object of the topos $\widetilde{\mathscr{S}}$ and where A stands for "associated stack". Since there is a natural equivalence $f \longrightarrow LF$ and L is a morphism of bicategories we get a natural equivalence of split stack $L(F^I) to (LF)^I$.

Since the functor "inverse image of sheaves of sets" is left exact one gets a natural isomorphism $f^{-1}((LF)^I) \xrightarrow{\sim} (f^{-1}(LF))^I$ and it remains to find, for any prestack G over $\mathscr{S}'$ a natural equivalence $A(G^I) \longrightarrow (AG)^I$. One has a commutative square

$$\begin{tikzcd} G \arrow[r, "a"] \arrow[d, "cG"'] & AG \arrow[d, "cAG"] \\ G^I \arrow[r, "a^I"'] & (AG)^I \end{tikzcd}$$

where a is the structural map of AG. According to [4] p.77 it suffices to show that a^I is "bicouvrant" [4] p.72, which is an easy consequence of the fact that a has this property. Q.E.D..

Corollary **(1.4)**. — *Let F and F' be left exact stacks on $\mathscr{S}$ and $\mathscr{S}'$, $m : F \longrightarrow f_*(F')$ be a morphism of stacks and $m' : f^*(F) \longrightarrow F'$ the morphism associated to m by the universal property of the inverse image. Then m is left exact if and only if m' is.*

This is a formal consequence of (1.3).

2. Classifying topos of a stack

Proposition **(2.1)**. — *Let $\mathscr{S}$ be a left exact $\mathfrak{U}$-site and C a prestack over $\mathscr{S}$ whose fibers are equivalent to categories which belong to $\mathfrak{U}$ (C is said to be* small*). Let us denote by J the coarsest topology on C such that the projection $p : C \longrightarrow \mathscr{S}$ is a comorphism [5] III 3.1, and by $C - \mathscr{S}$ the category of sheaves on C for J with values in $\mathfrak{U}$.*

(1) J is defined by the pretopology whose covering families are those $(m_i : C_i \longrightarrow C)$, $i \in I \in \mathfrak{U}$, such that each m_i is $\mathscr{S}$-cartesian and such that $p(m_i)$, $i \in I$, is a covering family.

(2) $C - \mathscr{S}$ is a $\mathfrak{U}$-topos and the morphism $\pi : C - \mathscr{S} \longrightarrow \mathscr{S}$ defined by p is essential (i.e. π^* has a left adjoint $\pi_!$). If C is left exact then $\pi_!$ is left exact.

(3) If $\mathscr{S}$ is a $\mathfrak{U}$-topos and C is a stack, then the Yoneda functor $\varepsilon : C \longrightarrow C - \mathscr{S}$ is full and faithful and the composite $C \xrightarrow{\varepsilon} C - \mathscr{S} \xrightarrow{\pi_!} \mathscr{S}$ is equal to p.

Proof. (1) is an easy consequence of the definition of a comorphism and of the observation made in the proof of (1.1). Let S_a, $a \in A \in \mathfrak{U}$, be a family of generators of $\mathscr{S}$ and G_a, $a \in A$, be a subset of $|C_{S_a}|$ which both belongs to $\mathfrak{U}$ and contains an element of each isomorphism class of objects of the fiber C_{S_a}. The union of the G_a is a generator of the site (C,J), hence this one is a $\mathfrak{U}$-site and $C - \mathscr{S}$ is a $\mathfrak{U}$-topos. Using (1) one sees easily that for any sheaf F on $\mathscr{S}$, Fp is a sheaf on C hence $\pi^*(F) = Fp$, hence π^* has a left adjoint hence π is essential. The last assertion of (2) follows from the fact that when C is left exact, p is the underlying functor of a morphism of sites $\mathscr{S} \longrightarrow C$. The fist assertion of (3) follows readily from (1) and the patching condition for morphisms in C. For any $S \in |\mathscr{S}|$, and any $c \in |C_S|$ one has

$$\mathrm{Hom}(\pi_!\varepsilon(c), S) = \mathrm{Hom}(\varepsilon(c), \pi^*(S)) = \pi^*(S)(c) = \mathrm{Hom}(p(c), S)$$

by adjunction, Yoneda and the formula $\pi^*F = Fp$, and this concludes the proof.

2.2. Under the assumptions of (2.1), $C - \mathscr{S}$ is called the *classifying topos of the (pre)stack* C. Note that a morphism of stacks $m : C \longrightarrow C'$ is a comorphism of sites and induces a morphism of topos $m-\mathscr{S} : C-\mathscr{S} \longrightarrow C'-\mathscr{S}$. If m is an equivalence, then so is $m-\mathscr{S}$. If C is a split stack one can define a split stack C^V whose fibers are the opposites of the fibers of C. Note that the underlying category of C^V *is not* the opposite C° of C. Let us consider the category

$$\mathrm{B}_C(\mathscr{S}) = \mathrm{St}_{\mathscr{S}}(C^V, \mathrm{SH}(\mathscr{S})) \tag{1}$$

of morphisms of stacks $F : C^V \longrightarrow \mathrm{SH}(\mathscr{S})$, where $\mathrm{SH}(\mathscr{S})$ is the split stack whose fiber at $S \in |\mathscr{S}|$ is the category of sheaves on $\mathscr{S}/S$ (equivalent to $\mathscr{S}/S$ since $\mathscr{S}$ is a topos). One has a natural functor

$$\tau^* : \mathscr{S} \longrightarrow \mathrm{B}_C(\mathscr{S}), \quad \tau^*(S)(c) = \varepsilon(S \times p(c)), \tag{2}$$

where ε is the Yoneda functor of $\mathscr{S}/S$.

Proposition **(2.3)**. — *If $\mathscr{S}$ is a $\mathfrak{U}$-topos and C a split stack one has an equivalence of categories*

$$b : \mathrm{B}_C(\mathscr{S}) \longrightarrow C-\mathscr{S}, \quad b(F)(c) = F(c)(p(c)) \tag{1}$$

and an isomorphism of functors $b\tau^* \xrightarrow{\sim} \pi^*$.

2.3.1. Note that this proposition proves that $\mathrm{B}_C(\mathscr{S})$ is a $\mathfrak{U}$-topos equivalent to $C-\mathscr{S}$ even when C is not split since one can replace C by an equivalent split stack. Furthermore, by the universal property of the associated stack, $\mathrm{B}_C(\mathscr{S})$ is equivalent to $\mathrm{B}_{C'}(\mathscr{S})$ when C is the stack associated to some prestack C'.

Furthermore, Lawvere and Tierney have introduced for any category-object E of the topos $\mathscr{S}$, the topos of objects of $\mathscr{S}$ provided with operations of E. One can prove that this topos is equivalent to $\mathrm{B}_C(\mathscr{S})$ where C is the split prestack defined by E hence also equivalent to $C'-\mathscr{S}$, where C' is the stack generated by C. Thus we have three constructions of the classifying topos.

2.3.2. For any split stack D, any map $f : T \longrightarrow S$ in $\mathscr{S}$ and any $s \in |D_S|$ we denote by s^f the inverse image of s by f and by $s_f : s^f \longrightarrow s$ the cartesian map given by the splitting. To define b completely one must define for any $m : c \longrightarrow c'$ in C an application $b(F)(m) : b(F)(c') \longrightarrow b(F)(c)$. Let $f = p(m)$, $f : S' \longrightarrow S$. Since C is split there is a canonical factorisation $c' \xrightarrow{m'} c^f \xrightarrow{c_f} c$. Since F

is cartesian one has a canonical isomorphism $i : F(c^f) \longrightarrow F(c)^f$ which for the values at S' (or rather $\mathrm{Id}_{S'}$) of these sheaves induces a bijection $j : F(c^f)(S') \longrightarrow F(c)(f)$ and we take for $b(F)(m)$ the composite

$$F(c)(S) \xrightarrow{f(c)(\dot{f})} F(c)(f) \xrightarrow{j^{-1}} F(c^f)(S') \xrightarrow{f(m')(S')} F(c')(S')$$

where $\dot{f} : f \longrightarrow \mathrm{Id}_S$ is the terminal map in $\mathscr{S}/S$. It is easily checked that $b(F)$ is a functor, recalling that the underlying category of C^V is not the underlying category of C°. The sheaf axiom for $b(F)$ is verified by using (2.1 (1)): for a given family $(c_i \longrightarrow c)$ it is a consequence of the fact that $F(c)$ is a sheaf and F a cartesian functor. The functoriality with respect to F is obvious. To prove that b is an equivalence one constructs explicitly a functor

$$a : \mathrm{C} - \mathscr{S} \longrightarrow \mathrm{B}_{\mathrm{C}}(\mathscr{S}), \quad a(G)(c)(f) = G(c^f),$$

where $a \in |F|$ and $f : T \longrightarrow p(c)$ is a map in $\mathscr{S}$.

Proposition **(2.4)**. — *Let $f : \mathscr{S}' \longrightarrow \mathscr{S}$ be a morphism of $\mathfrak{U}$-topos and let C be a left exact stack over $\mathscr{S}$. One has an equivalence of categories*

(1) $\mathrm{Top}_{\mathscr{S}}(\mathscr{S}, \mathrm{C} - \mathscr{S}) \longrightarrow \mathrm{Stex}_{\mathscr{S}}(\mathrm{C}, f_* \mathrm{SH}(\mathscr{S}'))^\circ$, *where the domain is the category of morphisms of $\mathscr{S}$-topos $n : \mathscr{S}' \longrightarrow \mathscr{S}$, where $f_* \mathrm{SH}(\mathscr{S}')$ is the direct image by f of the stack of sheaves over $\mathscr{S}$ (its fiber at $S \in |\mathscr{S}|$ is the category of sheaves over $S'/f^*(S)$) and where the codomain is the opposite of the category of left exact morphisms of stacks $\mathrm{C} \longrightarrow f_* \mathrm{SH}(\mathscr{S}')$.*

Since C is left exact and $\varepsilon : \mathrm{C} \longrightarrow \mathrm{C} - \mathscr{S}$ full and faithful, a morphism of topos $n : \mathscr{S}' \longrightarrow \mathrm{C} - \mathscr{S}$ is nothing but a left exact functor $n^{-1} : \mathrm{C} \longrightarrow \mathscr{S}'$, $n^{-1} = n^*\varepsilon$. Furthermore, since C is left exact there exists a cartesian section p^{-1} of C whose value at $S \in |\mathscr{S}|$ is the terminal object of the fiber C_S and p^{-1} of C is a morphism of sites defining $\pi : \mathrm{C} - \mathscr{S} \longrightarrow \mathscr{S}$ since $\pi^* F = F p$ for any sheaf F on $\mathscr{S}$. Hence an isomorphism of morphisms of topos $i : \pi \xrightarrow{\sim} f$ is nothing but an isomorphism $i^{-1} : n^{-1} p^{-1} \xrightarrow{\sim} f^*$. In other words the category $\mathrm{Top}_{\mathscr{S}}(\mathscr{S}', \mathrm{C} - \mathscr{S})^\circ$ is equivalent to the category M of pairs $(n^{-1} : \mathrm{C} \longrightarrow \mathscr{S}', i^{-1} : n^{-1} p^{-1} \xrightarrow{\sim} f)$ where n^{-1} is continuous and left exact. Let $\mathrm{Arr}(\mathscr{S}')$ be the category whose objects are arrows of $\mathscr{S}'$ and let $b : \mathrm{Arr}(\mathscr{S}') \longrightarrow \mathscr{S}'$, $b(X \longrightarrow Y) = Y$. Since every object $c \in |\mathrm{C}|$ determines *a terminal map $c \longrightarrow p^{-1}(p(c))$, a pair (n^{-1}, i^{-1}) can be viewed as a functor n' : $C \longrightarrow \mathrm{Arr}(\mathscr{S}')$ such that $bn' = f^* p$ and which is left exact (the continuity condition disappears by (2.1 (1))). Since b makes a stack over $\mathscr{S}'$ out of the category $\mathrm{Arr}(\mathscr{S}')$, by the very definition of the direct image of a stack, n' is nothing but a functor n'' : $C \longrightarrow f_* \mathrm{Arr}(\mathscr{S}')$ and, since n' is left exact, n'' is $\mathscr{S}$-cartesian and left exact, hence an object of $\mathrm{Stex}_{\mathscr{S}}(C, \mathrm{Arr}(\mathscr{S}'))$. The conclusion follows since $\mathrm{Arr}(\mathscr{S}')$ is equivalent to $\mathrm{SH}(\mathscr{S}')$.*

According to the proof, the morphism of topos $n : \mathscr{S}' \longrightarrow C - \mathscr{S}$ which corresponds to a left exact morphism of stacks $n'' : C \longrightarrow f_* \mathrm{Arr}(\mathscr{S}')$ is characterized up to unique isomorphism by the equality $n^* \varepsilon = dqn''$

$$C \xrightarrow{n''} f_* \mathrm{Arr}(\mathscr{S}') \xrightarrow{q} \mathrm{Arr}(\mathscr{S}') \xrightarrow{d} \mathscr{S}', \tag{2}$$

where q is the first projection of $f_* \mathrm{Arr}(\mathscr{S}') = \mathrm{Arr}(\mathscr{S}') \times_{\mathscr{S}'} \mathscr{S}$, d the "domain functor" and ε the Yoneda functor.

Corollary **(2.5)**. — *If C is left exact one has an equivalence*[5]

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{S}', C - \mathscr{S}) \longrightarrow \mathrm{Stex}_{\mathscr{S}'}(f^*(C), \mathrm{SH}(S'))^\circ. \tag{1}$$

This follows immediately from (2.4), (1.4) and the universal property of the inverse image $f^*(C)$ of C. This gives the *universal property* of $C - \mathscr{S}$ in the bicategory of $\mathscr{S}$-topos.

Corollary **(2.6)**. — *Let $C' = f^*(C)$. One has a commutative square of morphisms of topos*

$$\begin{tikzcd}
C - \mathscr{S} \arrow[d] & C' - \mathscr{S}' \arrow[l, "C-f"'] \arrow[d] \\
\mathscr{S} & \mathscr{S}' \arrow[l, "f"]
\end{tikzcd}$$

which is bicartesian.

[5] $\mathrm{Stex}_{\mathscr{S}}(,)$ stands for "category of left exact morphisms of stacks

This means that for any morphism of topos $g : \mathscr{S}'' \longrightarrow \mathscr{S}'$ the functor given by composition with $C - f$

$$\text{(2)} \qquad \mathrm{Top}_{\mathscr{S}'}(\mathscr{S}'', C' - \mathscr{S}') \longrightarrow \mathrm{Top}_{\mathscr{S}}(\mathscr{S}'', C - \mathscr{S})$$

is an equivalence. By the very definition of C' [4] p.87, one has a commutative square

$$\begin{tikzcd} C \arrow[r, "\varphi^{-1}"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\ \mathscr{S} \arrow[r, "f^*"'] & \mathscr{S}' \end{tikzcd}$$

where φ^{-1} is cartesian. Furthermore φ^{-1} is left exact by (1.3). By (1.4) and the universal property of $C' = f^*(C)$, for any $g : \mathscr{S}'' \longrightarrow \mathscr{S}'$, the functor

$$\mathrm{Stex}_{S'}(C', g_* \,\mathrm{SH}(\mathscr{S}'')) \longrightarrow \mathrm{Stex}_S(C, f_* g_* \,\mathrm{SH}(\mathscr{S}'')), \quad u \longrightarrow u\varphi^{-1},$$

is an equivalence. By (2.4) the proof is now an exercise about universal properties in bicategories.

3. Generating stack of a $\mathfrak{U}$-topos

The question of defining a relative notion of generators has been raised by Lawvere and Tierney. We propose here an answer in the language of $\mathfrak{U}$-topos. It is clear that Prop. (3.3) is still valid when working in their framework and that (3.2) is not.

Definition **(3.1)**. — *Let $f : \mathscr{X} \longrightarrow \mathscr{S}$ be a morphism of $\mathfrak{U}$-topos. A generating stack of f is a full substack C of $F = f_*(\mathrm{Arr}(\mathscr{X}))$ which is small (2.1) and such that, for any $S \in |\mathscr{S}|$ and any $x \in |F_S|$, there exists a covering family $(S_i \longrightarrow S)$, $i \in I$, in $\mathscr{S}$ and for each $i \in I$ a covering family $(c_{i,j} \longrightarrow x_i)$ in the fiber $F_S = \mathscr{X}/f^*(S)$, with $c_{i,j} \in |C|$, where x_i is the inverse image of x by $S_i \longrightarrow S$. A generating stack C is said to be left exact if C and the inclusion functor $i : C \longrightarrow F$ are left exact.*

Let us recall that the category of arrows of $\mathscr{X}$ provided with the codomain functor $\mathrm{Arr}(\mathscr{X}) \longrightarrow \mathscr{X}$ is a stack. Hence its direct image F is a stack whose fiber at $S \in |\mathscr{S}|$ is the topos $\mathscr{X}/f^*(S)$ and the inverse image functor $F_u : F_S \longrightarrow F_{S'}$ associated to a map $u : S \hookrightarrow S$ in $\mathscr{S}$ is nothing but pull-back along $f^*(u) : f^*(S') \longrightarrow f^*(S)$.

Hence F is a left exact stack and the condition that a full substack C of F is left exact is that each fiber C_S is stable by finite limits in the fiber F_S.

Proposition **(3.2)**. — *Any $\mathscr{S}$-topos admits a left exact generating stack.*

Let us choose a generator S_i, $i \in I \in \mathfrak{U}$, of $\mathscr{S}$ and for each $i \in I$ a full subcategory C_i of F_{S_i} stable by finite limits, generating F_{s_i} and equivalent to a category which belongs to $\mathfrak{U}$. Let us define C as the full subcategory of F whose objects of projection $S \in |\mathscr{S}|$ are those $x \in |F_S|$ such that there exists a covering family $(c_a : S_a \longrightarrow S)$, such that each S_a is one of the S_i and the inverse image of x by c_a is isomorphic to an object of C_i. This condition being local on $\mathscr{S}$, it is clear that C is a full substack of F and eve a left exact one since F is left exact. Furthermore C is small since for each $S \in |\mathscr{S}|$ the set of classes of equivalent covering families $(S_a \longrightarrow S)$ as above belongs to $\mathfrak{U}$. Eventually C is a generating stack since any $S \in |\mathscr{S}|$ can be covered by the S_i.

Proposition **(3.3)**. — *Let $\mathscr{S}$ be a $\mathfrak{U}$-topos and C a generating stack of an $\mathscr{S}$-topos $f : \mathscr{X} \longrightarrow \mathscr{S}$. Then $\mathrm{C} - \mathscr{S}$ is an $\mathscr{S}$-topos and there exists an $\mathscr{S}$-morphism of topos $n : \mathscr{X} \longrightarrow \mathrm{C} - \mathscr{S}$ such that $n_* : \mathscr{X} \longrightarrow \mathrm{C} - \mathscr{S}$ is full and faithful (in other words $\mathscr{X}$ is a subtopos of $\mathrm{C} - \mathscr{S}$).*

3.3.1. We note first that since C is small, $C - \mathscr{S}$ is a $\mathfrak{U}$-topos. Furthermore there exists a left exact generating stack C' of $\mathscr{X}$ containing C and such that each object of C' is a finite limit of objects of C. Hence the inclusion $C \longrightarrow C'$ induces an equivalence between the $\mathscr{S}$-topos $\mathbb{C} - \mathscr{S}$ and $C' - \mathscr{S}$ and this fact allows us to assume that C *is left exact*. Since the inclusion $i : C \longrightarrow F$, $F = f_* \operatorname{Arr}(\mathscr{X})$, is left exact one has an $\mathscr{S}$-morphism $n : \mathscr{X} \longrightarrow C - \mathscr{S}$, (2.4), whose inverse image functor $n^* : C - \mathscr{S} \longrightarrow \mathscr{X}$ is such that its composition with the Yoneda functor $\varepsilon : C \longrightarrow C - \mathscr{S}$ is equal to the composite of

$$(1) \qquad C \xrightarrow{i} F \xrightarrow{q} \operatorname{Arr}(\mathscr{X}) \xrightarrow{d} X, \quad (2.4(2)).$$

For any $c \in |C|$ and any $X \in |\mathscr{X}|$ one has $n_*(X)(c) = \operatorname{Hom}(\varepsilon(c), n_*(X)) = \operatorname{Hom}(n^*\varepsilon(c), X) = \operatorname{Hom}(dqi(c), X) = \operatorname{Hom}_S(i(c), X \times f^*(S))$ where the last set of morphisms is taken in the fiber $\mathscr{X}/f^*(S)$ of F with $S = p(c)$, and the last equality sign is justified by the definition of F as a fibered product. Hence the formula

$$(2)\quad n_* : \mathscr{X} \longrightarrow \mathrm{C}-\mathscr{S}, \quad n_*(X)(c) = \mathrm{Hom}_S(i(c), X \times f^*(S)), \quad S = p(c).$$

3.3.2. To prove that n_* is full and faithful we will first compose it with the inverse $a : \mathrm{C} - \mathscr{S} \longrightarrow \mathrm{B}_C(\mathscr{S})$ of (2.3 (1))

$$(3)\qquad an_* : \mathscr{X} \longrightarrow \mathrm{B}_C(\mathscr{S}), \quad an_*(X)(c) = \mathscr{H}om_S(i(c), X \times f^*(S)),$$

$$S = p(c), c \in |C|,$$

the above formula being justified by (2.3 (2)), where $\mathscr{H}om_S(u,v)$ stands for the sheaf (over S) of S-morphisms between the objects u and v of the fiber at S of the stack F. Let us prove that (3) is the effect on the fibers at the terminal object of $\mathscr{S}$ of a morphism of stacks

$$(4)\qquad m : F \longrightarrow \mathrm{ST}(C^V, \mathrm{SH}(\mathscr{S})),$$

where $\mathrm{ST}(A,B)$ stands for the (split) *stack* or morphisms of stacks between A and B (internal Hom in the bicategory of stacks [4] p.57, 77), whose fiber at $S \in |\mathscr{S}|$ is the category of morphisms $A/S \longrightarrow B$§ of stacks over $\mathscr{S}/S$. We obtain (4) by composition of

$$(5)\qquad F \xrightarrow{y} \mathrm{ST}(F^V, \mathrm{SH}(\mathscr{S})) \xrightarrow{j} \mathrm{ST}(C^V, \mathrm{SH}(\mathscr{S}))$$

where j is induced by composition with $i : C \longrightarrow F$ and where y is a "relative Yoneda functor" defined by

$$(6)\qquad y(a)(b) = \mathscr{H}om_S(b, a^f)$$

where $f : T \longrightarrow S$ is a map in $\mathscr{S}$ and $a \in |F_S|$, $b \in |F_T|$. One should note that the restriction of y to the terminal fiber of F is also the restriction of the composite $F \xrightarrow{\varepsilon} F - \mathscr{S} \xrightarrow{a} \mathrm{B}_F(\mathscr{S})$, (2.1(3)), (2.3(2)). By localisation it follows that the restriction of y to each fiber is full and faithful hence y is such. On the other hand, since any object of F can be covered for the canonical topology of F by objects of $i(C)$ and since i is full and faithful it is easy to show that j is also full and faithful and the conclusion follows.

Proposition **(3.4)**. — *Fibered products exist in the bicategory of* $\mathfrak{U}$*-topos.*

according to (3.2) and (3.3) any morphism of $\mathfrak{U}$-topos $\mathscr{X} \longrightarrow \mathscr{S}$ can be factored in $\mathscr{X} \xrightarrow{n} C - \mathscr{S} \xrightarrow{\pi} \mathscr{S}$ where n_* is full and faithful and where C is a left exact small stack over $\mathscr{S}$. By (2.6) the pullback of π along any morphism of $\mathfrak{U}$-topos $f : \mathscr{S}' \longrightarrow \mathscr{S}$ exists. On the other hand the pull-back of n along any morphism of $\mathfrak{U}$-topos $y : \mathscr{Y} \longrightarrow C - \mathscr{S}$ exists because $\mathscr{X}$ is a subtopos of $C - \mathscr{S}$ hence is defined by some topology J on $C - \mathscr{S}$ and it suffices to take as a pullback the subtopos of $\mathscr{Y}$ defined by the finest topology J' on $\mathscr{Y}$ such that the inverse image functor $y^* : C - \mathscr{S} \longrightarrow \mathscr{Y}$ is continuous. The conclusion follows by transitivity of pullback in a bicategory.

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