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The Work of Pierre Deligne

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My purpose here is to convey to you some idea of the scope and the depth of the work for which we are today honoring Pierre Deligne with the Fields Medal. Deligne's work centers around the remarkable relations, first envisioned by Weil, which exist between the cohomological structure of algebraic varieties over the complex numbers, and the diophantine structure of algebraic varieties over finite fields.

I. The Weil conjectures. Let us first consider an algebraic variety Y over a finite field $\boldsymbol{F}_q$. For each integer $n \geqslant 1$ there is a unique field extension $\boldsymbol{F}_{q^n}$ of degree n over $\boldsymbol{F}_q$. We denote by $Y(\boldsymbol{F}_{q^n})$ the (finite) set of points of Y with coordinates in $\boldsymbol{F}_{q^n}$, and by $\# Y(\boldsymbol{F}_{q^n})$ the cardinality of this set. The zeta function of Y over $\boldsymbol{F}_q$ is the formal series defined by

$$Z(Y/\boldsymbol{F}_q, T) = \exp\left(\sum_{n\geqq 1} \frac{T^n}{n} \# Y(\boldsymbol{F}_{q^n})\right).$$

Knowledge of the zeta function is equivalent to knowledge of the numbers $\{\# Y(\boldsymbol{F}_{q^n})\}$.

After the pioneering work of E. Artin, W. K. Schmid, H. Hasse, M. Deuring and A. Weil on the zeta functions of curves and abelian varieties, Weil in 1949 made the following conjectures about the zeta function of a projective non-singular n-dimensional variety Y over a finite field $\mathbf{F}_q$.

(1) The zeta function is a rational function of T, i.e. it lies in $\boldsymbol{Q}(T)$.

(2) There exists a factorization of the zeta function as an alternating product of polynomials $P_0(T), \ldots, P_{2n}(T)$,

$$Z(Y/\boldsymbol{F}_q, T) = \frac{P_1(T)P_3(T)\ldots P_{2n-1}(T)}{P_0(T)P_2(T)\ldots P_{2n}(T)}$$

of the form

$$P_i(T) = \prod_{j=1}^{b_i} (1-\alpha_{i,j}T),$$

such that the map $\alpha \mapsto q^n/\alpha$ carries the $\alpha_{i,j}$ bijectively to the $\alpha_{2n-i,j}$.

(3) The polynomials $P_i(T)$ lie in $\mathbf{Z}[T]$, and their reciprocal roots $\alpha_{i,j}$ are algebraic integers which, together with all their conjugates, satisfy

$$|\alpha_{i,j}| = q^{i/2}.$$

This is the "Riemann Hypothesis" for varieties over finite fields.

(4) If Y is the "reduction mod p" of a projective smooth variety $\mathbf{Y}$ in characteristic zero, then the degree b_i of P_i is the ith Betti number of $\mathbf{Y}$ as complex manifold.

Underlying these conjectures was Weil's belief in the existence of a "cohomology theory", with a coefficient field of characteristic zero, for varieties over finite fields. In this theory, the polynomial $P_i(T)$ would be the "inverse" characteristic polynomial det $(1-TF)$ of the "Frobenius endomorphism" acting on H^i. Conjectures (1) and (2) would then follow from a Lefschetz trace formula for F and its iterates, and from a suitable form of Poincaré duality. Conjecture (4) would follow if the cohomology of the "reduction mod p" Y of a projective smooth variety $\mathbf{Y}$ in characteristic zero were (essentially) equal to the topological cohomology of $\mathbf{Y}$ as complex manifold.

The next years saw the systematic introduction of sheaf-theoretic and cohomological methods into algebraic geometry. By the mid–1960s, M. Artin and A. Grothendieck had developed the étale cohomology theory of arbitrary schemes, along the lines foreseen in Grothendieck's 1958 Edinburgh address. For each prime number l, this gives a cohomology theory, "l-adic cohomology", with coefficients in the field $\mathbf{Q}_l$ of l-adic numbers, which is adequate to give parts (1), (2) and (4) of the Weil conjectures for projective smooth varieties over finite fields of characteristic $p \neq l$. In their theory, the cohomology of the "reduction mod p" Y of a projective smooth $\mathbf{Y}$ in characteristic zero is just the singular cohomology, with coefficients in $\mathbf{Q}_l$, of "$\mathbf{Y}$ as complex manifold". In the case of curves and abelian varieties, these constructions agree with those already given by Weil.

For a given projective smooth $Y/\mathbf{F}_q$, we now have, for *each* $l \neq p$, a factorization of the zeta function as an alternating product of l-adic polynomials $P_{i,l}(T)$. There is, however, no assurance that the $P_{i,l}$ have coefficients in $\mathbf{Q}$ rather than in $\mathbf{Q}_l$, much less that their reciprocal zeros are algebraic integers with the predicted absolute values. Of course, if one could prove directly that the reciprocal zeros of $P_{i,l}$ were algebraic integers which, together with all their conjugates, had the correct absolute value $q^{i/2}$, then the polynomials $P_{i,l}$ could be described intrinsically in terms of the zeros and poles of the zeta function itself, and hence would have rational coefficients independent of l. But how could one even introduce archimedean considerations into the l-adic theory without first knowing the rationality of the cofficients of the $P_{i,l}$?

Let me now try to indicate the brilliant synthesis of ideas involved in Deligne's solution of these problems.

Initially, he tries to prove à priori that the $P_{i,l}$ have rational coefficients independent of l. The idea is to proceed by induction on the dimension of Y. If Y is n-dimensional, then Poincaré duality and the fact that the zeta function is itself rational and independent of l reduce us to treating the polynomials $P_{i,l}$ for $i \leqslant n-1$. Now let Z be a smooth hyperplane section of Y. The "weak" Lefschetz theorem assures us that Y and Z have the same $P_{i,l}$ for $i \leqslant n-2$, and that the $P_{n-1,l}$ for Y divides that for Z. This alone is enough to show inductively that the reciprocal zeroes of the $P_{i,l}$ are algebraic integers.

In order to go further, and show that the $P_{i,l}$ actually have rational coefficients independent of l, the idea is to show that $P_{n-1,l}$ for Y is a generalized "greatest common divisor" of the $P_{n-1,l}$ of all possible smooth hyperplane sections. Unfortunately, this "g.c.d." argument, which itself depends on the full strength of the monodromy theory of Lefschetz pencils, works only when Y satisfies the "hard" Lefschetz theorem (existence of the "primitive decomposition" on its cohomology), otherwise the "g.c.d." will be too big at some stage of the induction. But Deligne will later prove the hard Lefschetz theorem in arbitrary characteristic as a *consequence* of the Weil conjectures. What is to be done?

With characteristic daring, Deligne simply ignores the preliminary problem of establishing independence of l. Fixing one $l \neq p$, he turns to a direct attack on the absolute values of the algebraic integers which occur as the reciprocal roots of the $P_{i,l}$.

Consider a smooth projective even dimensional Y, and a Lefschetz pencil Z_t of hyperplane sections, "fibering" Y over the t-line. Factor the $P_{n-1,l}$ of each Z_t as the product of the "g.c.d." of all of them, and of the "variable" part. Deligne shows à priori that these "variable" parts are each polynomials with rational coefficients whose reciprocal zeroes all satisfy the Riemann Hypothesis

$$|\alpha_{n-1,\,\mathrm{variable}}| = q^{(n-1)/2}.$$

Deligne's proof of this is simply spectacular; no other word will do. He first uses a theorem of Kazdan–Margoulis, according to which the monodromy group of a Lefschetz pencil of odd fibre dimension is "as big as possible", to establish the rationality of the coefficients of the "variable" parts. Then he considers the L-function over the t-line whose Euler factors are the reciprocals of the "variable" parts. This L-function has rational Dirichlet coefficients. Deligne realizes that Rankin's method of estimating Ramanujan's function $\tau(n)$ by "squaring" might be applied in this context to estimate the reciprocal poles of the individual Euler factors (i.e. the reciprocal zeroes of the "variable" parts!). The problem is to control the *poles* of all the L-functions obtained from this one by passing to *even* tensor powers ("squaring"). Deligne gains this control by ingeniously combining Grothendieck's cohomological theory of such L-functions, the Kazdan–Margoulis theorem, and the classical invariant theory of the symplectic group!

Once he has this à priori estimate for the variable parts of the $P_{n-1,t}$ of the hyperplane sections Z_t, a Leray spectral sequence argument shows that in the $P_{n,i}$ for Y itself, all the reciprocal zeroes are algebraic integers which, together with all their conjugates, satisfy the apparently too weak estimate

$$|\alpha_{n,j}| \leqslant q^{(n+1)/2} \quad \text{(instead of } q^{n/2}\text{)}.$$

But this estimate is valid for Y of any *even* dimension n. The actual Riemann hypothesis for any projective smooth variety X follows by applying this estimate to all the even cartesian powers of X.

II. Consequences for number theory. That there are many spectacular consequences for number theory comes as no surprise. Let us indicate a few of them.

(1) Estimation of $\# Y(F_q)$ when Y has a "simple" cohomological structure. For example, if Y is a smooth n-dimensional hypersurface of degree d, we get

$$|\# Y(F_q)-(1+q+\ldots+q^n)| \leqslant \left(\frac{(d-1)^{n+2}+(-1)^{n+2}(d-1)}{d}\right) q^{n/2}.$$

(2) Estimates for exponential sums in several variables, e.g

(a) if f is a polynomial over F_p in n variables of degree d prime to p, whose part of highest degree defines a nonsingular projective hypersurface, then

$$\left|\sum_{x_i\in F_p} \exp\left(\frac{2\pi i}{p} f(x_1, \ldots, x_n)\right)\right| \leqslant (d-1)^n p^{n/2};$$

(b) "multiple Kloosterman sums":

$$\left|\sum_{x_i\in F_p^\times} \exp\left(\frac{2\pi i}{p}\left(x_1+\ldots+x_n+\frac{1}{x_1\ldots x_n}\right)\right)\right| \leqslant (n+1)\cdot p^{n/2}.$$

(3) The Ramanujan–Petersson conjecture. Already in 1968 Deligne had combined techniques of l-adic cohomology and the arithmetic moduli of elliptic curves with earlier ideas of Kuga, Sato, Shimura and Ihara to reduce this conjecture to the Weil conjectures. Thus if $\sum a(n)q^n$ is the q-expansion of a normalized $(a(1)=1)$ cusp form on $\Gamma_1(N)$ of weight $k\geqslant 2$ which is a simultaneous eigenfunction of all Hecke operators, then

$$|a(p)| \leqslant 2p^{(k-1)/2} \quad \text{for all primes} \quad p\nmid N.$$

III. Cohomological consequences; weights. As Grothendieck foresaw in the 1960s with his "yoga of weights", the truth of the Weil conjectures for varieties over finite fields would have important consequences for the cohomological structure of varieties over the complex numbers. The idea is that any reasonable algebro-geometric situation over C is actually defined over a subring of C which, as a ring, is finitely generated over Z. Reducing modulo a maximal ideal $\mathfrak{m}$ of this ring, we find a situation over a finite field, and the corresponding Frobenius endomorphism $F(\mathfrak{m})$ operating on this situation. This Frobenius operates by functoriality on the l-adic cohomology, which is none other than the singular cohomology, with $\mathbf{Q}_l$-coefficients, of our original situation over $\mathbf{C}$. This natural operation of Frobenius imposes a previously unsuspected structure on the cohomology of complex algebraic varieties, the so-called "weight filtration", or filtration by the magnitude of the eigenvalues of Frobenius.

In a remarkable tour de force in the late 1960's and early 1970's, Deligne developed, independently of the Weil conjectures, a complete theory of the weight filtration of complex algebraic varieties, by making systematic use of Hironaka's resolution of singularities, of the notion of differential forms with "logarithmic poles" (i.e. products of dt/t's), and of his own earlier work on cohomological descent. The resulting theory, which Deligne named "mixed Hodge theory", should be seen as a far-reaching generalization of the classical theory of "differentials of the second kind" on algebraic varieties, as well as of "usual" Hodge theory.

Consider, for example, a smooth affine variety U over $\mathbf{C}$. By one of Hironaka's fundamental results, we can find a projective smooth variety X and a collection of smooth divisors D_i in X which cross transversally, such that $U \cong X - \cup D_i$. Deligne shows that the Leray spectral sequence, in rational cohomology, of the inclusion map $U \hookrightarrow X$, degenerates at E_3, and that the filtration it defines on the cohomology of U is independent of the choice of the compactification. This is the desired weight filtration; its smallest filtrant is the image of $H^\bullet(X)$ in $H^\bullet(U)$, i.e. the space of "differentials of the second kind on U".

One of the many applications of this theory is to the global monodromy of families of projective smooth varieties. Given a projective smooth map $X \to S$ of smooth complex varieties, and a point $s \in S$, the fundamental group $\pi_1(S, s)$ acts on each of the cohomology groups $H^i(X_s, \mathbf{C})$ of the fibre X_s. Deligne shows that these representations are all completely reducible, and that each of their isotypical components, especially the space of invariants, is stable under the Hodge decomposition into (p, q)-components.

By means of an extremely ingenious and difficult argument drawing upon Grothendieck's cohomological theory of L-functions and the ideas of the Hadamard–de la Vallée Poussin proof of the prime number theorem, Deligne later established an l-adic analogue of this theorem of complete reducibility for l-adic "local systems" in characteristic p over open subsets of the projective t-line $\mathbf{P}^1$, provided that all of the "fibres" of the local system satisfy the Riemann Hypothesis (with a fixed power of $\sqrt{q}$). Once he had *proven* the Riemann Hypothesis for varieties over finite fields, he could apply this theorem to the local system coming from a Lefschetz pencil on a projective smooth variety over a finite field. The resulting complete reducibility is easily seen to imply the hard Lefschetz theorem. This theorem, previously known only over $\mathbf{C}$, and there by Hodge's theory of harmonic integrals, is thus established, in all characteristics as a consequence of the Weil conjectures.

Other outgrowths of Deligne's work on weights include his theory of differential equations with regular singular points on smooth complex varieties of arbitrary dimension, which yields a new solution of Hilbert's 21st problem, and his affirmative solution of the "local invariant cycle problem" in the local monodromy theory of families of projective smooth varieties.

Still another application of "weights" is to homotopy theory. Deligne, Griffiths, Morgan and Sullivan jointly apply mixed Hodge theory to prove that the homotopy theory ($\otimes Q$) of a projective smooth complex variety is a "formal consequence" of its cohomology.

IV. Other work. We have passed over in silence a considerable body of Deligne's work, which alone would be sufficient to mark him as a truly exceptional mathematician; duality in coherent cohomology, moduli of curves (jointly with Mumford), arithmetic moduli (jointly with Rapoport), the Ramanujan–Petersson conjecture for forms of weight one (jointly with Serre), the Macdonald conjecture (jointly with Lusztig), local "roots numbers", p-adic L-functions (jointly with Ribet), motivic L-functions, "Hodge cycles" on abelian varieties, and much more.

I hope that I have conveyed to you some sense, not only of Deligne's accomplishments, but also of the combination of incredible technical power, brilliant clarity, and sheer mathematical daring which so characterizes his work.

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