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par

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`https://agrothendieck.github.io/`

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# SUR LA COMPLÉTION DU DUAL D'UN ESPACE VECTORIEL LOCALEMENT CONVEXE[1]

Note[2] de M. Alexandre Grothendieck, présentée par M. Élie Cartan.

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Soient $E$ un espace vectoriel localement convexe, $S$ un ensemble de parties bornées, convexes, symétriques et fermées de $E$, $E'_S$ (resp. $\widehat{E}'_S$), l'espace des formes linéaires sur $E$ continues (resp. dont les restrictions aux éléments de $S$ sont continues), munis de la topologie de la convergence uniforme sur les éléments de $S$. Soit $E_0$ le sous-espace engendré par $\cup S$

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Proposition **1**. — *Si les éléments de S sont précompacts, pour que $E'_S$ soit complet, il faut (et il suffit) que la restriction à $E_0$ de tout $u \in \widehat{E}'_S$ soit continue.*

On se ramène en effet facilement à la

Proposition **2**. — *Si les éléments de S sont précompacts, $E'_S$ est dense dans $\widehat{E}'_S$.*

[] Comme le dual de $E$ est le même pour la topologie donnée et sa topologie faible, et que les ensembles bornés sont faiblement précompacts, ces propositions seront d'application assez générale. Signalons la

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[2]Séance du 6 février 1950.

C. R., 1950, i^er^ *Semestre*. (T. 230 N° 7.).

Corollaires. —

1° *Si $E_0 = E$ et si les éléments de $S$ sont précompacts, le complété de $E'_S$ s'identifie à $\widehat{E}'_S$.*

2° *Si $E'_S$ est complet, il en est de même de $E'_T$ pour $T \supset S$ ($S$, $T$ ensembles de parties bornées de $E$, sans plus).*

*En utilisant le fait que $E$ s'identifie à un dual topologique de son dual faible, on obtient de plus :*

3° *Le complété de $E$ s'identifie à l'espace des formes linéaires sur son dual dont les restrictions aux parties équicontinues sont faiblement continues, muni de la topologie de la convergence uniforme sur ces parties. (On retrouve en particulier le complété pour la topologie faible.)*

4° *Si $E$ est complet, toute topologie localement convexe plus fine qui a même dual, et plus généralement qui puisse se définir par une famille de seminormes* semi-continues *(soit : par un système fondamental de voisinages convexes fermés) est encore complète. [En particulier la topologie forte de Mackey associée*[3] *et la topologie induite par le bidual fort*[4] *sont encore complétes.]*

5° *Si $E$ est complet, toute forme linéaire sur son dual dont les restrictions aux parties équicontinues sont faiblement continues est faiblement continue. On retrouve ainsi un fait connu pour les espaces de Fréchet et leurs limites inductives*[4].

[3] Cf. G. W. MACKEY, *Transactions of the Amer. Math. Soc.*, 57, 1945, p. 155-207 et 59, 1946, p. 530-537

[4] Cf. J. DIEUDONNÉ, et L. SCHWARTZ, *La dualité dans les espaces* $(F)$ *et* $(LF)$ (à paraitre aux *Annales de Grenoble*, 1950).

# QUELQUES RÉSULTATS RELATIFS À LA DUALITÉ DANS LES ESPACES $(F)$[5]

Note[6] de M. Alexandre Grothendieck, présentée par M. Arnaud Denjoy.

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Soient $E$ un espace $(F)$, $E'$ son dual fort, $E''$ son bidual fort [?][1]. Les résultats suivants répondent partiellement à certaines questions posées dans [?].

Proposition **1**. — *Toute partie bornée dénombrable de $E''$ est contenue dans l'adhérence faible d'une partie bornée de $E$.*

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[6]Séance du 30 octobre 1950

[1]J. DIEUDONNÉ et L. SCHWARTZ, La dualité dasn les espaces $(F)$ et $(LF)$ (à paraître aux *Annales de Grenoble*, 1950). La terminologie est celle de cet article.

# CRITÈRES GÉNÉRAUX DE COMPACITÉ DANS LES ESPACES VECTORIELS LOCALEMENT CONVEXES. PATHOLOGIES DES ESPACES $(LF)$[2]

Note[3] de M. Alexandre Grothendieck, présentée par M. Arnaud Denjoy.

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*La première partie de cette Note*

**1. —**

[3]Séance du 30 octobre 1950

Soit $C$ un espace de Banach

# SOME ASPECTS OF HOMOLOGICAL ALGEBRA[1]

[1] Translation by M. L. Barr and M. Barr

**1**. Dans un travail récent [?], Mattuck et Tate déduisent l'inégalité fondamentale de A. Weil qui établit l'hypothèse de Riemann pour les corps de fonctions [?] comme conséquence facile du théorème de Riemann-Roch pour les surfaces. En essayant de comprendre la portée exacte du leur méthode, je suis tombé sur l'énoncé suivant, connu en fait depuis 1937 [?] [?] [?] (comme me l'a signalé J. P. Serre), mais apparemment peu connu et utilisé:

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**2**. Nous allons déduire sur $X$, nous désignerons par $l(D)$ la dimension de l'espace vectoriel des fonctions $f$ sur $X$ telles que $(f) \geq -D$ donc $l(D)$ ne dépend que de la classe de $D$. Rappelons *l'inégalité de Riemann-Roch*

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**3**. Ce qui précède n'utilisait pas à proprement parler la méthode de Mattuck-Tate (si ce n'est en utilisant l'inegalité de Riemann-Roch sur les surfaces). Nous allons indiquer maintenant comment la méthode de ces auteurs, convenablement généralisée, donne d'autres inégalités que celle de A. Weil. Nous nous appuierons sur le

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**Remarques**. Le corollaire 1 devient faux si on ne fait pas l'hypothèse que $K/2$ est encore une classe de diviseurs. En effet, toutes les hypothèses sauf cette dernière sont vérifiées si $X$ est un surface non singulière *rationnelle*. Or, à partir d'une telle surface, on construit facilement une surface birationnellement équivalente par éclatements successifs, dont l'index $\tau$ soit $< 0$ (contrairement à (3.7 ter)).

En effet, on vérifie aisément que lorsqu'on fait éclater un point dans une surface non singulière projective, l'index diminue d'une unité. (Cette remarque, ainsi que l'interprétation de l'inégalité (3.7) a l'aide de l'index, m'a été signalée par J. P Serre).

La disparité des énoncés qu'on déduit du théorème (3.2) est due au fait qu'il n'est pas relatif à un élément arbitraire de l'espace vectoriel $E$ de Néron-Séveri introduit plus haut, mais à un élément du "lattice" provenant des diviseurs sur $X$. On notera d'ailleurs que dans le cas particulier où $X$ est le produit des deux courbes $C$ et $C'$, le théorème 3.2 ne contient rien de plus que l'inégalité de A. Weil.

# THE COHOMOLOGY THEORY OF ABSTRACT ALGEBRAIC VARIETIES

It is less than four years since cohomological methods (i.e. methods of Homological Algebra) were introduced into Algebraic geometry in Serre's fundamental paper [?], and it seems already certain that they are to overflow this part of mathematics in the coming years, from the foundations up to the most advanced parts. All we can do here is to sketch briefly some of the ideas and results. None of these have been published in their final form, but most of them originated in or were suggested by Serre's paper.

Let us first give an outline of the main topics of cohomological investigation in Algebraic geometry, as they appear at present. The need of a theory of cohomology for 'abstract' algebraic varieties was first emphasized by Weil, in order to be able to give a precise meaning to his celebrated conjectures in Diophantine geometry [?]. Therefore the initial aim was to find the *'Weil cohomology' of an algebraic variety*, which should have as coefficients something 'at least as good' as a field of *characteristic 0*, and have such formal properties (e.g. duality, Künneth formula) as to yield the analogue of Lefschetz's 'fixed-point formula'. Serre's general idea has been that the usual 'Zariski topology' of a variety (in which the closed sets are the algebraic subset) is a suitable one for applying methods of Algebraic Topology. His first approach was hoped to yield at least the right Betti numbers of a variety, it being evident from the start that it could not be considered as the Weil cohomology itself, as the coefficient field for cohomology was the ground field of a variety, and therefore not in general of characteristic $0$. In fact, even the hope of getting the 'true' *Betti numbers* has failed, and so have other attempts of Serre's [?] to get Weil's cohomology by taking the cohomology of the variety with values, not in the sheaf of local rings themselves, but in the sheaves of Witt-vectors constructed on the latter. He gets in this way modules over the ring $W(k)$ of infinite Witt vectors on the ground field $k$, and $W(k)$ is a ring of characteristic $0$ even if $k$ is of characteristic $p \neq 0$. Unfortunately, modules thus obtained over $W(k)$ may be infinitely generated, even when the variety $V$ is an abelian variety [?]. Although interesting relations must certainly exist between these cohomology groups and the 'true ones', it seems certain now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the *connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other* (as explained quite unsystematically in Serre's tentative Mexico paper [?]), and by Serre's idea that a 'reasonable' algebraic principal fiber space with structure group $G$, defined on a variety $V$, if it is not locally trivial, should become locally trivial on some covering of $V$ *unramified* over a given point of $V$. This has been the starting point of a definition of the Weil cohomology (involving both 'spatial' and Galois cohomology), which seems to be the right one, and which gives clear suggestions how Weil's conjectures may be attacked by the machinery of Homological algebra. As I have not begun theses investigations seriously as yet,and as moreover this theory has a quite distinct flavor from the one of the theory of algebraic coherent sheaves which we shall now be concerned with, we shall not dwell any longer on Weil's cohomology. Let us merely remark that the definition alluded to has already been the starting-point of a theory of cohomological dimension of fields, developed recently by Tate [?].

The second main topic for cohomological methods is the *cohomology theory of algebraic coherent sheaves*, as initiated by Serre. Although inadequate for Weil's purposes, it is at present yielding a wealth of new methods and new notions, and gives the key even for results which were not commonly thought to be concerned with sheaves, still less with cohomology, such as Zariski's theorem on 'holomorphic functions' and his 'main theorem' - which can be stated now in a more satisfactory way, as we shall see, and proved by the same uniform elementary methods. The main parts of the theory, at present, can be listed as follows:

(a) General finiteness and asymptotic behaviour theorems.

(b) Duality theorems, including (respectively identical with) a cohomological theory of residues.

(c) Riemann-Roch theorem, including the theory of Chern classes for algebraic coherent sheaves.

(d) Some special results, concerning mainly abelian varieties.

The third main topic consists in the *application of the cohomological methods to local algebra*. Initiated by Koszul and Cartan-Eilenberg in connection with Hilbert's 'theorem of syzygies', the systematic use of these methods is mainly due again to Serre. The results are the *characterization* of regular local rings as those whose global cohomological dimension is finite, the clarification of *Cohen-Macaulay's equidimensionality theorem* by means of the notion of *cohomological codimension* [**?**], and specially the possibility of giving (for the first time as it seems) a *theory of intersections*, really satisfactory by its algebraic simplicity and its generality. Serre's result just quoted, that regular local rings are the the only ones of finite global cohomological dimension, accounts for the fact that only for such local rings does a satisfactory theory of intersections exist. I cannot give any details here on these subjects, nor on various results I have obtained by means of a *local duality theory*, which seems to be the tool which is to replace differential forms in the case of unequal characteristics, and gives, in the general context of commutative algebra, a clarification of the notion of residue, which as yet was not at all well understood. The motivation of this latter work has been the attempt to get a global theory of duality in cohomology for algebraic varieties admitting arbitrary singularities, in order to be able to develop intersection formulae for cycles with arbitrary singularities, in a non-singular algebraic variety, formulas which contain also a 'Lefschetz formula mod $p$' [**?**]. In fact, once a proper local formalism is obtained, the global statements become almost trivial. As a general fact, it appears that, to a great extent, the 'local' results already contain a global one; more precisely, global results on varieties of dimension $n$ can frequently be deduced from corresponding local ones for rings of Krull dimension $n+1$.

We will therefore

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## Introduction

These notes are a rough summary of five talks given at I.H.E.S in November and December 1966. The purpose of these talks was to outline a possible definition of a $p$-adic cohomology theory, via a generalization of the De Rham cohomology which was suggested by work of Monsky-Washnitzer [?] and Manin [?].

The contents of the notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out[1].

## 1. De Rham cohomology

**1.1. Differentiable Manifolds.** Let $X$ be a differentiable manifold, and $\underline{\Omega}^{\bullet}_{X/\mathbf{C}}$ the complex of sheaves of differential forms on $X$, whose coefficients are complex valued differentiable functions on $X$.

Theorem **1.1.** (De Rham) — *There is a canonical isomorphism*

$$\mathrm{H}^*(X,\mathbf{C}) \xrightarrow{\sim} \mathrm{H}^*(\Gamma(X,\underline{\Omega}^{\bullet}_{X/\mathbf{C}})),$$

*where* $\mathrm{H}^*(X,\mathbf{C})$ *is the canonical cohomology of* $X$ *with complex coefficients.*

[1] For a more detailed exposition and progress in this direction, we refer to the work of P. Berthelot, to be developed presumably in SGA 8.

To prove this, one observes that, by Poincaré's lemma, the complex $\underline{\Omega}^{\bullet}_{X/\mathrm{C}}$ is a *resolution* of the constant sheaf $\underline{C}$ on $X$, and that the sheaves $\underline{\Omega}^{j}_{X/\mathrm{C}}$ are *fine* for $J \geq 0$, so that $\mathrm{H}^{i}(X, \underline{\Omega}^{j}_{X/\mathrm{C}}) = 0$ for $i > 0$ and $j \geq 0$, whence the assertion.

An analogous result holds for the complex of sheaves of differential forms on $X$, whose coefficients are real valued differentiable functions on $X$.

**1.2.**

**1.3.**

**1.4.**

**1.5.**

**1.6. Criticism of the $\ell$-adic cohomology**. If $X$ is a scheme of finite type over an algebraically closed field $k$, and $\ell$ is any prime number *distinct*[2] from the characteristic of $k$, the $\ell$-adic cohomology of $X$ is defined to be

**1.7.**

**1.8. Proposals for a $p$-adic Cohomology**. We only mention two proposals, namely Monsky and Washnitzer's method via special affine liftings (which we discuss in n° 2), and the method using the fppf (faithfully flat and finite presentation) topology.

By analogy with the $\ell$-adic cohomology, the essential idea of the fppf topology was to consider the cohomology of $X/k$, with respect to the fppf topology, with coefficient groups in the category $C^{\nu}$ of finite schemes of $\mathbf{Z}/p^{\nu}\mathbf{Z}$-modules. Examples of such schemes of modules are

## 2. The cohomology of Monsky and Wishnitzer

### 2.1. Approach via liftings.

Suppose $X_0$ is a scheme on a perfect field $k$

## 3. Connections on the De Rham cohomology

For the definition of a *connection* and a *stratification* on a sheaf, see Appendix I of these notes.

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[2]the $\ell$-adic cohomology is still defined for $\ell$ equal to the characteristic of $k$, but it no longer has too many reasonable properties.

## 4. The infinitesimal topos and stratifying topos

We now turn to the definition of a more general category of coefficients for the De Rham cohomology. To this end we introduce two ringed topos, the *infinitesimal topos* and the *stratifying topos*.

We shall see later that in fact these two topos work well only in characteristic 0

## 5. Cěch calculations

We now consider the cohomology of the infinitesimal topos and the stratifying topos[3]

## 6. Comparison of the Infinitesimal and De Rham Cohomologies

**6.1. The basic idea.** Let $X$ be a scheme above $S$, and $F$ a quasi-coherent Module on $X$ fortified with a stratification relative to $S$.

## 7. The crystalline topos and connecting topos

**7.1. Inadequacy of infinitesimal topos.** Let $X_0$ be a scheme above a perfect field $k$ of characteristic $p > 0$. Then, regarding $X_0$ as being above $S = \operatorname{Spec} W(k)$ instead of $k$, the infinitesimal cohomology

$$\mathrm{H}^*((X_0/S)_{\mathrm{inf}}, \underline{O}X_0)$$

is a graded module

## Appendix

Let $X$ be a scheme above the base $S$, and $F$ a Module on $X$. For each positive integer $n$,

[3]For a general discussion of the cohomology of a topos, see (SGA 4 V).

# HODGE'S GENERAL CONJECTURE IS FALSE FOR TRIVIAL REASONS

A. Grothendieck

(Received 27 October 1968)[4]

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§1. — The startling title is somewhat misleading, as everybody will think about a part of the Hodge conjecture which is most generally remembered, namely the part concerned with a criterion for a cohomology class (on a projective smooth connected scheme $X$ over $\mathbf{C}$) to be "algebraic", i.e. to come from an algebraic cycle with rational[5] coefficients. This conjecture is plausible enough, and (as long as it is not disproved) should certainly be regarded as the deepest conjecture in the "analytic" theory of algebraic varieties. However in [6, p. 184], Hodge gave a more general formulation of his conjecture in terms of filtrations of cohomology spaces, and the main aim of my note is to show that for a rather trivial reason, this formulation has to be slightly corrected.

Consider on the complex cohomology

$$\mathrm{H}^i(X^{an}, \mathbf{C}) = \mathrm{H}^i(X^{an}, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$

($X^{an}$ denotes the analytic space associated to the scheme $X$) the "Hodge filtration"

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[4] *Topology* Vol. 8, pp. 299-303. Pergamon Press, 1969. Printed in Great Britain

[5] In fact, Hodge states his conjecture for integral cohomology. That this is too optimistic was proved in [1]

§2. — This makes clear how the Hodge conjecture should be corrected, to eliminate trivial counterexamples: namely the left hand side of $(*)$ should be the largest sub-space of the right hand side, generating a subspace of $\mathrm{H}^i(X^{an}, \mathbf{C})$ which is a sub-Hodge structure, i.e. stable under decomposition into $p$, $q$ types. In other words, an element of $\mathrm{H}^i(X^{an}, \mathbf{C})$ should belong to $\mathrm{Filt}'^p$ if and only if all its bihomogeneous components belong to the **C**-vector space generated by the right hand side of $(*)$.

This formulation may seem a little too cumbersome to inspire confidence. To make it look better, we may remark that it is equivalent to the conjunction of the usual Hodge conjecture

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§3. — It may be of interest to review here the few non trivial instances known to the author where the Hodge conjecture has been checked.

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§4. — In most concrete examples, it seems very hard to *check* the Hodge conjecture, due to the difficulty in explicitly determining the filtration $\mathrm{Filt}'$ of the cohomology, and even in determining simply the part of the cohomology coming from algebraic classes. It may be easier, for the time being, to *test* the Hodge conjectures in various non trivial cases, through various consequences of the Hodge conjectures which should be more amenable to direct verification. I would like to mention here two such consequences, which can be seen in act to be consequences already of the *usual* Hodge conjecture.

First, if $X$ is as before, the dimensions of the graded components of the vector space associated to the arithmetic filtration $\mathrm{Filt}'$ (and indeed this very filtration itself, if we interpret complex cohomology as the de Rham cohomology, which makes a purely algebraic sense) is clearly invariant if we transform $X$ by any automorphism of the field $\mathbf{C}$, or equivalently, if we change the topology of $C$ by such an automorphism. In other words, if we have a smooth projective scheme $X$ over a field $K$ of char 0, then the invariants we get by different embeddings of $K$ into the field $\mathbf{C}$ are the same. Granting the Hodge conjecture, the same should be true if we replace the $\mathrm{Filt}'$ filtration by the filtration described in §2 in terms of the Hodge structure (which is a transcendental description). What if we take for instance for $X$ a "general" abelian variety of given dimension of powers of it, or powers of a "general" curve $C$ of given genus? The case of genus 1 checks by Tate's result recalled in example c) above.

Secondly, and more coarsely, if we have a projective and smooth morphism $f : X \longrightarrow S$ of algebraic schemes over $\mathbf{C}$, we can for every $s \in S$ consider the complex cohomology of the fiber $X$, as a Hodge structure, and look at the filtration "rational over $\mathbf{Q}$" which it defines (and which conjecturally should be the arithmetic filtration). Hodge's conjecture would imply that the set of points $s \in S^{an}$ where the dimensions of the components of the associated graded space have fixed values has a very special structure: it should be the difference of two countable unions of Zariski-closed subsets of $S$, which in fact should even be definable over a fixed subfield of $\mathbf{C}$, of finite type over the field $\mathbf{Q}$. (A simple application of Baire's theorem, not using Hodge's conjecture, would give us only a considerably weaker structure theorem for the set in question, where Zariski-closed subsets would be reaplced by the images, under the projection of the universal covering $\widetilde{S}$ of $S^{an}$, of analytic subsets of $\widetilde{S}$[6].)

## REFERENCES

1.

[6](Added April 1969) David Lieberman has informed me that he can prove the stronger result obtained by replacing $\widetilde{S}$ by $S^{an}$ itself.

# STANDARD CONJECTURES ON ALGEBRAIC CYCLES

## 1. Introduction

We state two conjectures on algebraic cycles, which arose fro an attempt at understanding the conjectures of Weil on the $\zeta$-functions of algebraic varieties. These are not really new, and they were worked out about three years ago independently by Bombieri and myself.

The first is an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures), and is inspired by and formally analogous to Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field.

The second is a statement of positivity, generalising Weil's well-known positivity theorem in the theory of abelian varieties. It is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero.

WHAT REMAINS TO BE PROVED OF WEIL'S CONJECTURES? Before stating our conjectures, let us recall what remains to be proved in respect of the Weil conjectures, when approached through $\ell$-adic cohomology.

Let $X/\mathbf{F}_q$ be a smooth irreducible projective variety of dimension $n$ over the finite field $\overline{\mathbf{F}}_q$ with $q$ elements, and $\ell$ a prime different from the characteristic. It has then been proved by M. Artin and myself that the $Z$-function of $X$ can be expressed as

$$
\begin{aligned}
Z(t) &= \frac{L'(t)}{L(t)},\\
L(t) &= \frac{L_0(t)L_2(t)\ldots L_{2n}(t)}{L_1(t)L_3(t)\ldots L_{2n-1}(t)},\\
L_i(t) &= \frac{1}{P_i(t)},
\end{aligned}
$$

where $P_i(t) = t^{\dim \mathrm{H}^i(\overline{X})} Q_i(t^{-1})$, $Q_i$ being the characteristic polynomial of the action of the Frobenius endomorphism of $X$ on $\mathrm{H}^i(\overline{X})$ (here $\mathrm{H}^i$ stands for the $i^{\text{th}}$ $\ell$-adic cohomology group and $\overline{X}$ is deduced from $X$ by base extension to the algebraic closure of $\mathbf{F}_q$). But it has not been proved so far that

(a) the $P_i(t)$ have integral coefficients, independent of $\ell (\neq \operatorname{char} \mathbf{F})$;

(b) the eigenvalues of the Forbenius endomorhisms on $\mathrm{H}^i(\overline{X})$, i.e., the reciprocals of the roots of $P_i(t)$, are of absolute value $q^{i/2}$.

Our first conjecture meets question (a). The first and second together would, by an idea essentially due to Serre [**?**], imply (b).

## 2. A weak form of conjecture 1

From now on, we work with varieties over a ground field $k$ which is algebraically closed and of arbitrary characteristic. Then (a) leads to the following question: If $f$ is an endomorphism of a variety $X/k$ and $\ell \neq \operatorname{char} k$, $f$ induces

$$f^i : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^i(x),$$

and each of these $f^i$ has a characteristic polynomial. *Are the coefficients of these polynomials rational integers, and are they independent of $\ell$?* When $X$ is smooth and proper of dimension $n$, the same question is meaningful when $f$ is replaced by any cycle of dimension $n$ in $X \times X$, considered as an algebraic correspondence.

In characteristic zero, one sees that this is so by using integral cohomology. If $\operatorname{char} k > 0$, one feels certain that this is so, but this has not been proved so far.

Let us fix for simplicity an isomorphism

$$\ell^{\infty k^* \simeq \mathbf{Q}_\ell/\mathbf{Z}_\ell} \qquad \text{(a heresy!)}.$$

We than have a map

$$: F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathrm{H}_\ell^{2i}(X)$$

which associates to an algebraic cycle its cohomology class. We denote by $C_\ell^i(X)$, and refer to its elements as *algebraic cohomology classes*.

A known result, due to Dwork-Faton, shows that for the integrality question (not to speak of the independence of the characteristic polynomial of $\ell$), it suffices to prove that

$$f_i^N \in \frac{1}{m}\mathbf{Z} \quad \text{for every } N \geq 0,$$

where $m$ is a fixed positive integer[7]. Now, the graph $\Gamma_{f^N}$ in $X \times X$ of $f^N$ defines a cohomology class on $X \times X$, and if the cohomology class $\Delta$ of the diagonal in $X \times X$ is written as

$$\Delta = \sum_0^n \pi_i$$

where $\pi_i$ are the projections of $\Delta$ onto $\mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$ for the canonical decomposition $\mathrm{H}^n(X \times X) \simeq \sum_{i=0}^n \mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$, a known calculation shows that

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\Gamma_{f^N})\pi_i \in \mathrm{H}^{4n}(X \times X) \approx \mathbf{Q}_\ell.$$

*Assume that the $\pi_i$ are algebraic*. Then $\pi_i = \frac{1}{m}(\prod_i)$, where $\prod_i$ is an algebraic cycle, hence

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\prod_i \Gamma_{f^N}) \in \frac{1}{m}\mathbf{Z}$$

and we are through.

WEAK FORM OF CONJECTURE 1. $(C(X))$: The elements $\pi_i^\ell$ are algebraic, (and come from an element of $F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is independent of $\ell$).

N.B.

1. The statement in parenthesis is needed to establish the independence of $P_i$ on $\ell$.

[7] This was pointed out to me by S. Kleimann.

2. If $C(X)$ and $C(Y)$ hold, $C(X \times Y)$ holds, and more generally, the Künneth components of any algebraic cohomology class on $X \times Y$ are algebraic.

## 3. The conjecture 1 (of Lefschetz type)

Let $X$ be smooth and projective, and $\xi \in \mathrm{H}^2(X)$ the class of a hyperplane section. Then we have a homomorphism

$$(*) \qquad \cup \xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X) \quad (i \leq n).$$

It is expected (and has been established by Lefschetz [?], [?] over the complex field by transcendental methods) that this is an isomorphism for all characteristics. For $i = 2j$, we have the commutative square

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Our conjecture is then: $(A(X))$:

*(a) (*) is always an isomorphism (the mild form);*

*(b) if* $i = 2j$. *(*) induces an isomorphism (or equivalently, an epimorphism)* $C^j(X) \longrightarrow C^{n-j}(X)$.

N.B. If $C^j(X)$ is assumed to be finite dimensional, (b) is equivalent to the assertion that $\dim C^{n-j}(X) \leq \dim C^j(X)$ (which in particular implies the equality of these dimensions in view of (a)).

An equivalent formulation of the above conjecture (for all varieties $X$ as above) is the following.

$(B(X))$: *The* $\Lambda$*-operation (c.f. [?]) of Hodge theory is algebraic.*

By this, we mean that there is an algebraic cohomology class $\lambda$ in $\mathrm{H}^*(X \times X)$ such that the map $\Lambda : \mathrm{H}^*(X) \longrightarrow \mathrm{H}^*(X)$ is got by lifting a class from $X$ to $X \times X$ by the first projection, cupping with $\lambda$ and taking the image in $\mathrm{H}^*(X)$ by the Gysin homomorphism associated to the second projection

Note that $B(X) \Rightarrow A(X)$, since the algebricity of $\lambda$ implies that of $\lambda^{n-i}$, and $\lambda^{n-i}$ provides an inverse to $\cup \xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X)$. On the other hand, it is easy to show that $A(X \times X) \Rightarrow B(X)$ and this proves the equivalence of conjectures $A$ and $B$.

The conjecture seems to be most amenable in the form of $B$. Note that $B(X)$ is stable for products, hyperplane sections and specialisations. In particular, since it holds for projective spaces, it is also true or smooth varieties which are complete intersections in some projective space. (As a consequence, we deduce for such varieties the wished-for integrality theorem for the Z-function!). It is also verified for Grassmannians, and for abelian varieties (Liebermann [?]).

I have an idea of a possible approach to Conjecture $B$, which relies in turn on certain unsolved geometric questions, and which should be settled in any case.

Finally, we have the implication $B(X) \Rightarrow C(X)$ (first part), since the $\pi_i$ can be expressed as polynomials with coefficients in $\mathbf{Q}$ of $\lambda$ and $L = \cup\xi$. To get the whole of $C(X)$, one should naturally assume further that there is an element of $F(X \times X) \otimes_{\mathbf{Z}} \mathbf{Q}$ which gives $\lambda$ for every $\ell$.

## 4. Conjecture 2 (of Hodge type)

For any $i \leq n$, let $P^i(X)$ be the 'primitive part' of $\mathrm{H}^i(X)$, that is, the kernel of $\cup\xi^{n-i+1} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i+2}(X)$, and put $C^j_{Pr}(X) = P^{2j} \cap C^j(X)$. On $C\widehat{\otimes}_{Pr}(X)$, we have a $\mathbf{Q}$-valued symmetric bilinear form given by

$$(x, y) \longrightarrow (-1)^j K(xy\xi^{n-2j})$$

where $K$ stands for the isomorphism $\mathrm{H}^{2n}(X) \simeq \mathbf{Q}_\ell$. Our conjecture is then that

(Hdg($X$)): *The above form is positive definite.*

One is easily reduced to the case when $\dim X = 2m$ is even, and $j = m$.

REMARKS.

(1) In characteristic zero, this follows readily from Hodge theory [?].

(2) $B(X)$ and Hdg($X \times X$) imply, by certain arguments of Weil and Serre, the following: if $f$ is an endomorphism of $X$ such that $f^*(\xi) = q\xi$ for some $q \in \mathbf{Q}$ (which is necessarily $> 0$), then the eigenvalues of $f_{\mathrm{H}^i(X)}$ are algebraic integers of absolute value $q^{i/2}$. Thus, this implies all of Weil's conjectures.

(3) The conjecture $Hdg(X)$ together with $A(X)(a)$ (the Lefschetz conjecture in cohomology) implies that numerical equivalence of cycles is the same as

cohomological equivalence for any $\ell$-adic cohomology if and only if $A(X)$ holds.

(4) In view of (3), $B(X)$ and $Hdg(X)$ imply that numerical equivalence of cycles coincides wit $\mathbf{Q}_\ell$-equivalence for any $\ell$. Further the natural map

$$Z^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \longrightarrow \mathrm{H}^i_\ell(X)$$

is a monomorphism, and in particular, we have

$$\dim_{\mathbf{Q}} C^i(X) \leq \dim_{\mathbf{Q}_\ell} \mathrm{H}^i_\ell(X).$$

Note that for the deduction of this, we do not make use of the positivity of the form considered in Hdg($X$),but only the fact that it is non-degenerate.

Another consequence of $Hdg(X)$ and $B(X)$ is that the stronger version of $B(X)$, viz. that $\lambda$ comes from an algebraic cycle with rational coefficients *independent of* $\ell$, holds.

## Conclusions

The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties.

Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry.