July 18, 1962

My dear Murre,

I recently had some thought on finiteness conditions for Picard preschemes, and substantially improved on the results stated in the last section of my last Bourbaki talk. The main result stated there for a simple projective morphism with connected geometric fibers (namely that the pieces $\underline{\text{pic}}^{P}_{X/S}$ are of finite type over S) has been extended by Mumford to the case where instead of f simple we assume only f flat with integral geometric fibers, (at least if these are normal). Using his result (the proof of which is quite simple and beautiful), I could get rid of the normality assumption, and even (as in theorem 4.1. of my talk) restrict to the consideration of the two first non tivial coefficients of the Hilbert polynomials. The key results for the reduction are the following (the proofs being very technical, and rather different for (i) and (ii), except that (ii) uses (i) to reduce to the normal case; moreover (ii) uses Mumford's result and the equivalence criteria as developped in my last Seminar):

(i) Let X, Y be proper over S neetherian, let $f : X \to Y$ be a <u>surjective</u> S - morphism, assume for simplicity of the statement that the Picard preschemes exist, then $f : \underline{\text{Pic}}_{Y/S} \to \underline{\text{Pic}}_{X/S}$ is of finite type (and in fact affine if S is the spectrum of a field), i. e. a subset M of $\underline{\text{Pic}}_{Y/S}$ is quasi-compact iff its image in $\underline{\text{Pic}}_{X/S}$ is.

(ii) The same conclusion holds for a canonical immersion $X \to Y$, if Y/S is projective with fibers all components of which are of dimension $\geqslant 3$, and if X is the sub-scheme of zeros of a section over Y of an invertible sheaf $\underline{L}$ ample with respect to S.

A connected result is that for any X/S proper, and integer $n \neq 0$, the n. th power homomorphism in the Picard prescheme is of finite type.

I tell you about this, namely (i), because of the method of proof, involving of course considerations of non flat descent. The fact that I do not have any good effectivity criterion does not hamper, by just recalling what the effectivity of a given descent datum means. Now it turns out that by a slightly more careful analysis of the situation, one can prove the following theorem, of a type very close to the one you have proved recently, and to some you still want to prove as I understand it.

.../...

72

__Theorem__ Let S be an integral noetherian scheme, X and X' proper over S and $f: X' X$ a surjective S-morphism, look at the corresponding homomorphisme for the Picard functors $f^{\cdot}: \underline{\underline{\mathrm{Pic}}}_{X/S} \to \underline{\underline{\mathrm{Pic}}}_{X'/S}$. Assume: a) the existence problems A and B defined below for X/S has always a solution (this is certainly true when X/S is projective). b) the morphism $f_s : X'_s \to X_s$ induced on the generic fiber is a morphism of descent, i.e. $\underline{O}_{X_s} \to f(\underline{O}_{X'_s}) \rightrightarrows h(\underline{O}_{X''_s})$ is exact. Then, provided we replace S by a suitable non empty open set, the homomorphism $f^{\cdot}$ is representable by a quasi-affine morphism, more specifically in the factorisation of $f^{\cdot}$ via the functor representing suitable descent data, $f^{\cdot} = vu$ with u affine and v a monomorphism (as you well know), v is in fact representable by finite ~~direct~~ sums of immersions.

__Corollary__ Without assuming b), but instead in a) allowing X/S to be replaced by suitable other schemes X_1 finite over X, the same conclusion holds, namely $f^{\cdot}$ is representable by quasi-affine morphisms.

This follows from the theorem, using a suitable factorisation of f. For instance, using Chow's lemma and the Main existence theorem in my first talk on Picard schemes, one gets:

__Corollary 2__ Assume X/S proper satisfies the condition a') for every X' finite over X, there exists a non empty open subset S_1 of S such that problem A for X' S_1 has allways solution ~~for problems A and B~~ (this condition is satisfied if X/S is projective). Then provided we replace S by a suitable S_1 non empty and open, $\underline{\mathrm{Pic}}_{X/S}$ exists, is separated, its connected components are of finite type over S.

__N.B.__ The proof does not give any evidence towards the fact that in the theorem, one could replace "quasi-affine" by "affine". This is true however over a field, bacause a quasi-affine algebraic group is affine ! It would be interesting to have a couterexample, say, over a ring of dimension 1 such as k[t], X and X' projective and simple over S and $X' \to X$ birational, or alternatively, X and X' projective and normal over S, and $f: X' \to X$ finite. A counterexample in the latter case would of course provide a counterrexample to the effectivity problem for a finite morphism raised in my first talk on descent ....

"Problem A" is the following: given X/S and Module F on X, to represent the functor on the category of S-preschemes taking any S'/S into a one-element or into the empty set, occording as to whether F' on X'/S' is flat with respect to S' or not, where $X' = X \times_S S'$, $F' = F \times_S S'$.

.../...

73

Given X/S, we say that "Problem A for X/S has always a solution " if for every coherent F' on some X'/S', the previous functor on (Sch)/S' is representable by a S'-scheme of finite type. The main step in my proof of existence of Hilbert schemes shows that this condition is satisfied when X/S is projective; In the proof, essential use is made of the Hilbert polynomial, in fact we get a solution as a disjoint sum of subschemes of S corresponding to various Hilbert polynomials. Still I would expect that the functor is representable as soon as X/S is proper. In view of the application we have in mind here, it would be sufficient (for any integral S) to find in S a non empty open set S_1 such Problem A has always a solution for $X_1 = X \times_S S_1$ over S_1. To prove this weeker existence result, it is well possible that a reduction to the projective case is possible, using Chow's lemma and some induction on the relative dimension perhaps. I also would expect that a proof wil be easier when working over a complete noetherian local ring, hence the case of a general noetherian local ring by flat descent. And it is well possible that, putting together two such partial results, a proof of the existence in general could be obtained. (I met with such difficulties allready time ago in a very analoguous non projective existence problem, which beside I did not solve so far !). This problem A has been met also by Hartshorne (A Harvard Student), but I doubt he will work seriously on it. Thus I now wrote you in the hope you may be interested to have a try on this problem. As a general fact, our knowledge of non projective existence theorems is exceedingly poor, and I hope this will change eventually.

Sincerely yours.

A. GROTHENDIECK