4.1.67.

Dear Coates,

I want to add a few more comments to the talk on algebraic cycles and to what I told you on the phone.

I think the best will be to state the index conjecture right after the statement of the main results of Hodge theory, adding that this conjecture will take its whole significance only when coupled with "conj.A " in the next paragraph. This will give more freedom in the next paragraph to express some extra relationships between various conjectures, such as A + index implies B .

In car. zero, state some known extra features: index theorem holds, the ~~conjectures~~ propositions A_χ to D_χ are independant of χ (because of the existence of Betti cohomology, so that these ~~conjectures~~ propositions are equivalent to the corresponding one's for rational cohomology), ~~and~~ A and C are independant of the choosen polarization ξ (for A because it is equivalent with B, for C because it can be expressed in terms of A, C(X) = (A(XxX) + A(YxY) + ..) Thus the ~~conjectures~~ conditions without ambiguity can be called A(X) to D(X), without ~~index~~ subscript χ and without indication of polarization. Say too that it is known that C(X) is of finite dimension over $\underline{Q}$, (so that A can also be expressed in terms of an equality of dimensions of C^i and C^{n-i}, which again proves it is independant of ξ), bu that this is not known in car $p > 0$. Contrarily to what I hastily stated in my talk (influenced from my recollections of the car 0 case) it is not clear to me if in car $p > 0$ the ~~conditions~~ ~~A(X~~ $A_\chi(X, \xi)$ and $C_\chi(X, \xi)$ are independant of the polarization ξ ; if you do not find some proof of this independance, then the possible dependance should be pointed out, as well as the fact that à priori we do not have a proof that A to D are independant of χ . Of cour-

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se, if the index theorem is proved for X, then $A_\chi(X, \xi) = B_\chi(X)$ is again independant of the polarization, and analoguous remark for $C_\chi(X, \xi)$.

When speaking about condition $C_\chi(X, \xi)$, emphasize at once its stability properties by products (the proof I suggested works indeed) specialization (with possible change of characteristics), hyperplane or more generally linear sections. Give an extra proposition for the relations with the property A , via a formal proposition as follows:

<u>Proposition</u> Condititions équivalentes sur X (varieté <u>polarisée</u>):

(i) $C_\chi(X)$

(ii) $C_\chi(Y)$ et $A_\chi(X\text{x}X)^{\circ}$,

(ii bis) $C_\chi(Y)$ et $A_\chi(X\text{x}X)^{\circ}$, où l'exposant $^{\circ}$ signifie qu'on se borne à exprimer la condition A pour l'homomorphisme en dimension critique $H^{2n-2} \longrightarrow H^{2n+2}$.

(iii) $C_\chi(Y)$ et $A_\chi(X\text{x}Y)$.

(iii bis) $C_\chi(Y)$ et $A_\chi(X\text{x}Y)^{\circ}$, où l'exposant $^{\circ}$ signifie qu'on se borne à exprimer la condition A en dimension critique $H^{(2n-1)-1} \longrightarrow H^{(2n-1)+1}$.

(iv) $C_\chi(Y)$, et pour tout $i \leqslant n-1$, l'homomorphisme naturel $H^i(Y) \to H^i(X)$ inverse à gauche de $\phi^i: H^i(X) \to H^i(Y)$ (induit par $\Lambda_X \phi_*$) est induit par une classe de correspondance algébrique (induisant ce qu'elle veut sur les autres $H^j(Y)$).

(iv bis) $C_\chi(Y)$, et pour $j \geqslant n+1$, l'homomorphisme naturel $H^j(X) \to H^{j-2}(Y)$ inverse à droite de $\phi_{j-2} : H^{j-2}(Y) \to H^j(X)$ (induit par $\phi^* \Lambda_X$) est induit par une classe de correspondance algébrique (induisant ce qu'elle veut sur les autres $H^i(X)$).

<u>Corollaire</u> Ces conditions équivalent aussi à

(v) $A_\chi(X\text{x}X)^{\circ} + A_\chi(Y\text{x}Y)^{\circ} + A_\chi(Z\text{x}Z)^{\circ} +$, où $X \supset Y \supset Z$ est une suite décorissante de sections hyperplanes. [illegible] $i-j$.

(vi) $A_\chi(X\times Y)^{o} + A_\chi(Y\times Z)^{o} + \dots$, avec les mêmes notations .

Of course, the products and hyperplane sections are endowed with the polarizations stemming from the polarization on X. The conditions (v) and (vi) have the slight interest that they allow to express the conjecture $A(k)=C(k)$ in terms of $A(T)^{o}$ for every T of even (resp. odd dimension), where the upper ^{o} means that it is sufficient to look at what happens in critical dimensions.

For the proof of the proposition, I told you already the equivalence of (i) and (ii), (ii bis). The equivalence of (iv) and (iv bis) is trivial by transposition, they ~~are of course implied by (i) because of the stability of $C_\chi(X)$ and $C_\chi(Y)$~~ imply (i) because $H^{2n-i}(X) \to H^{i}(X)$ is the compositum $H^{2n-i}(X) \to H^{2n-i-2}(Y) \to H^{i}(Y) \to H^{i}(X)$ where the extreme arrows are the ones of (iv bis) and (iv) and the middle one is induced by $\Lambda_Y^{(n-1)-i}$, and they are implied by (iii bis) because of the formula

$$(\Lambda_X\phi_*)L_Y + L_X(\Lambda_X\phi_*) = (\phi_*\Lambda_Y\phi^* + \mathrm{id}_X)\phi_* .$$

On the other hand (iii) $\Longrightarrow$ (iii bis) is trivial, and so is (i) $\Longrightarrow$ (iii) because of the stabilities.

N.B. $(\varphi^*\Lambda_X)L_X + L_Y(\varphi^*\Lambda_X) = \varphi^*(\varphi_*\Lambda_Y\varphi^* + \mathrm{id}_X)$

For the list of the known facts, you can state that :

1) In arbitrary caracteristics, C(X) is known if $\dim X \leqslant 2$, because more generally, it is known that in arbitrary dimension n, $H^{2n-1}(X) \to H^{1}(X)$ is induced by an algebraic correspondance class; also, in arbitrary dimension, it is known that $\pi_0, \pi_{2n}, \pi_1, \pi_{2n-1}$ are algebraic (trivial for the first two, not quite trivial for the two next one's). If $\dim X = 3$, it is not known however, even in car. 0 , if C(X) or only D(X) hold, nor A(X) and B(X) in car. $p>0$; also if for 1-cycles, τ-equiv. = num. equivalence ....

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By the way, the fact that the π_i for a surface are algebraic was pointed out (Tate tells me) by Hodge in Algebraic correspondances between surfaces, Proc. London Math. Soc. Seeies 2, Vol XLIV, 1938, p.226. It is rather striking that this statement should not have struck the algebraic geometers more, and has fallen into oblivion for nearly thirty years !

2) In car.0, $A(X)$ is known for $\dim X \leqslant 4$. But $A(X)^0$ is not known if $\dim X = 5$; the first interesting case would be for a variety $X \times Y$, X of dim 3 and Y a hyperplane section, as this would prove $C(X)$, see above.

Thus the main problems arise already for 1-cycles on threefolds, and partially even in car.0 . Urged by Kleiman's question, I will look again at my old scribbles on that subject (when I pretend to reduce the "strong" form of Lefschetz to the "week" one). As for the suggestion I made on the phone, to try to get any X as birationally equivalent to a non singular X', which is a spcialisation of a non singular X", itself birationally equivalent to a non singular hypersurface - this cannot work as Serre pointed out, because such an X would have to be simply connected ! Thus if one wants to reduce somehow to the case of hypersurfaces, one will have to work also with singular ones, and see how to reformulate for singular varieties the standard conjectures ...

Sincerely your's

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6.1.1966

Dear Coates,

Here a few more comments to my talk on the conjectures. The following proposition shows that the conjecture $C_\chi(X)$ is independant of the choosen polarization, and has also some extra interest, in showing the part playd by the fact that $H^i(X)$ should be "motive-theoretically" isomorphic to its natural dual $H^{2n-i}(X)$ (as usual, I drop the twist for simplicity).

<u>Proposition</u> The condition $C_\chi(X)$ is equivalent also to each of the follwing conditions:

a) $D_\chi(X)$ holds, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \to H^i(X)$ which is algebraic (i.e. induced by an algebraic correspondance class; we do not make any assertion on what it induces in degrees different from 2n-i).

b) For every endomorphism $H^i(X) \to H^i(X)$ which is algebraic, the coefficients of the characteristic polynomial are rational, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \to H^i(X)$ which is algebraic.

Proof. I sketched already how $D_\chi(X)$ implies the fact that for an algebraic endomorphism of $H^i(X)$, the coefficients of the characteristic polynomial are rational numbers, Therefore we know that a) implies b), and of course $C_\chi(X)$ implies a). It remains to prove that b) implies $C_\chi(X)$. Let u: $H^{2n-i}(X) \to H^i(X)$ be the given isomorphism which is algebraic, and v: $H^i(X) \to H^{2n-i}(X)$ ~~an~~ the algebraic isomorphism in the opposite direction, induced by L_X^{n-i}. Then uv = w is an automorphism of $H^i(X)$ which is algebraic, and the Hamilton-Cayley formula $w^b - \sigma_1(w)w^{b-1} + \dots + (-1)^b \sigma_b(w) = 0$ (where the $\sigma_i(w)$ are the coefficients of the car. pol. of w) whos that w^{-1} is a linear combination of the w^i, with coefficients of the type $\pm \sigma_i(w)/\sigma_b(w)$ (NB b = rank H^i). The assumption implies

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that these coefficients are rational, which implies that w^{-1} is algebraic, and so is $w^{-1}u = v^{-1}$, which was to be proved.

NB In char. 0, ~~itxis~~ the statement simplifies to: $C(X)$ equivalent to the existence of algebraic isomorphisms $H^{2n-i}(X) \to H^{i}(X)$, (as the preliminary condition in b) is then automatically satisfied). Maybe with some extra care this can be proved too in arbitrary characteristics.

<u>Corollary</u> Assume X and X' satisfy condition C_{χ} , and let $u: H^{i}(X) \to H^{i+2j}(X')$ $(j \in \mathbb{Z})$ be an isomorphism which is algebraic. Then u^{-1} is algebraic.

Indeed, the two spaces can be identified "algebraically" (both directions !) to their dual, so taht the transpose of u can be viewed as an isomorphism $u' : H^{i+2j}(X') \to H^{i}(X)$. Thus $u'u$ is an algebraic automorphism w of $H^{i}(X)$, and by the previous argument we see that w^{-1} is algebraic, hence so is $u^{-1} = w^{-1}u'$.

As a consequence, we see that if $x \in H^{i}(X)$ is such that $u(x)$ is algebraic (i being now assumed to be even), then so is x. The same result should hold in fact if u is a monomorphism, the reason being that in this case there should exist a left-inverse which is algebraic; this exists indeed in a case like $H^{n-1}(X) \to H^{n-1}(Y)$ (where we take the left inverse $\Lambda_X \phi_*$). But to get it in general, it seems we need moreover the Hodge index relation. (The complete yoga then being that we have the category of motives which is semi-simple !). Without speeking of motives, and staying down on earth, it would be nice to explain in the notes that $C(X)$ together with the index relation $I(X \times X)$ implies that the ring of correspondance classes for X is semi-simple, and how one deduces from this the existence of left and right inverses as looked for above.

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This could be given in an extra paragraph (which I did not really touch upon in the talk), containing also the deduction of the Weil conjectures from the conjectures C and A.

A last and rather trivial remark is the following. Let's introduce variants $A'_\chi(X)$ and $A''_\chi(X)$ as follows:

$A'_\chi(X)$: if $2i \leqslant n-1$, any element x of $H^i(X)$ whose image in $H^i(Y)$ is algebraic, is algebraic.

$A''_\chi(X)$: if $2i \geqslant n-1$, any algebraic element of $H^{i+2}(X)$ is the image of an algebraic element of $H^i(Y)$.

Let us consider also the specifications $A'_\chi(X)^\circ$ and $A''_\chi(X)^\circ$, where we restrict to the critical dimensions $2i = n-1$ if n odd, $2i = n-2$ if n even. All these conditions are in the nature of "week" Lefschets relations, and they are trivially implied by $A_\chi(X)$ resp. $C_\chi(Y)$ (in the first case, applying ϕ we see that $L_X x$ is algebraic; in the second, we take $y = \Lambda_Y \phi^+(x))$. The remark then is that these pretended "week" variants in fact imply the full Lefschets relations for algebraic cycles, namely:

<u>Proposition</u> $A_\chi(X)$ is equivalent to the conjunction $C_\chi(Y) + A'_\chi(X\times X)^\circ + A''_\chi(X\times X)^\circ$, hence (by induction) also to the conjunction of the conditions $A'_\chi{}^\circ$ and $A''_\chi{}^\circ$ for all of the varieties XxX, YxY, ZxZ .... . Analoguous statement with XxY, YxZ etc instead of XxX, YxY etc.

This comes from the remark that $A_\chi(X)^\circ$ follows from the conjuction of $A'_\chi(X)^\circ$ and $A''_\chi(X)^\circ$, as one sees by decomposing $L_X{}^2 : H^{2m-2}(X) \to H^{2m+2}(X)$ into $H^{2m-2}(X) \xrightarrow{\varphi^*} H^{2m-2}(Y) \xrightarrow{\varphi_*} H^{2m}(X) \xrightarrow{L_X} H^{2m+2}(X)$ if $\dim X = 2m$ is even, and $H^{2m+1-1}(X) \to H^{2m+1+1}(X)$ into $H^{2m}(X) \xrightarrow{\varphi^*} H^{2m}(Y) \xrightarrow{\varphi_*} H^{2m+2}(X)$ if $\dim X = 2m+1$ is odd.

Sincerely your's

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