Dear Murre,

I am very sorry I did not succeed to convey the intuitive idea behind the general nonsense of my notes. It seems to me that the basic example in order to understand the idea is example 1, where you can take Z to be a standard Kummer covering for definiteness, $Z = Z_a^n$, and S normal. Intuitively, when you look at coverings (normal, say) S' of S whose ramification type is not worse than the one of Z over S, you mean that the normalized inverse image Z' of S' over Z is étale. ~~But by transport de~~ From the birational point of view, assuming S' connected and therefore corresponding to a field extension K' of the field of functions K of S, this means simply that K' is isomorphic to a subextension -K- of an extension L' of the function field L of Z, unramified with respect to the model Z; when S (and whence Z) is strictly local, hence L'=L, this means simply that K' is isomorphic to a subextension of L, a very reasonable way indeed of interpreting the intuitive fact that S'/S has no worse ramification than Z/S. (NB We assume of course now, for simplicity, that Z is connected, and moreover that Z is the normalisation of S in some finite Galois extension L of K). Note that anyhow, we want that the notion in question should be of local nature for the étale topology, thus we can reduce always to the s rictly local case. Now by transport de structure the group G operating on Z must operate on S' (in a way compatible with operations on Z), and it is immediate that we recover S' from the G-covering Z' of Z as being simply S'=Z'/G . Thus ~~we re~~ the category of coverings of S we are interested in is just equivalent to the category of étale coverings of (Z,G). This latter category on the other hand is of a very standard type; the presence of G does not prevent it to share all the properties of catégories of étale coverings

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as developped in Exp VI, and under a mild restriction of a connectedness assumption, we can say that the category in question is ~~classified~~ completely described by a certain pro-finite group, the so-called fundamental group of (Z,G), which in fact is an extension of G by the usual fundamental group of Z. On the other hand, for these latter considerations, all normality assumptions are completely superfluous. Their main use was to insure that the functor Z' Z'/G from étale coverings of (Z,G) to coverings of S is fully faithful; this latter fact, on the other hand, reduces at once to the case when Z is a principal covering of S, with group G (because this condition will be satisfied when restricting to a non empty open set U of S, and on the other hand the restriction functors ~~fr~~ to U resp. Z/U are fully faithful, because of normality); but in this case descent theory tells us that we get in fact an equivalence of the category of étale coverings of S, and the catégory of étale coverings of (Z,G) !

In most applications however, instead of having just one model for ramification (one Z^n_a say), we have an infinity of such (all Z^n_a say, fixed a and variable n), and the condition on coverings of S we are interested in is that the covering should (at least locally) have ramification no worse than one of the Z_i . This situation is described in example two, On the other hand, sometimes the ramification models are not defined globally (for instance in the Kummer case, when we give ~~ideals which ex~~ divisors which are not principal), only locally, in a way that among the set of models given over various open sets (or more generally, objects of the étale site) there are certain compatibility conditions, roughly that the restriction of any model over U to U V has ramification no worse than the restriction of some model given over V. Such an example is

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dealt with in example 3, which has the disadvantage of beigg somewhat involved and artificial in its technical details, but has the merit of allowing explicit construction of the functor "geometric reàlization". Otherwise, the set up outlined before the examples seems more natural, and giving a closer grasp of the really essential features of what "ramification data" should really mean.(By the way, one shoul rather say "category of ramification data").

You are of course right that in the definition of a Galois covering when I assume that the soverings of the various $\underline{s}(M)$ are Galois étale coverings with Galois group G, that beforhand an action of G on X was given, which means that G acts on each of the coverings X(M) of $\underline{s}(M)$ on a way compatible with the base changes $\underline{s}(M')$ $\underline{s}(M)$ stemming from arrows M' M. In the various examples 1 to 3, this means that the action of G commutes to étale localization on S (i.e. base changes S' S' in the étale site of S) <u>and</u> to the action of the groups of automorphisms given on the various Z's .

Now let's come back to the case S strictly local; then (for covering defined over the whole of S) the general situation of example 3 reduces to the one of example 2, and replacing the Z_i's by suitable connected components, we may assume them connected if we want. For a given Z_i, the étale cowerings of $(Z_i,G_i)_x$ are trivial as coverings of Z_i alone (i.e. forgetting the action of G_i) bexcause Z_i is strictly local too. Therefore the catégory of étale coverings of (Z_i,G_i) is just equivalent to the category of finite sets E on which G_i opérates; to such an E is associated the consatnt covering E_{Z_i} of Z_i, but G_i operating on it via it's operation on E; the geometric relaization over S of this is nothing else but $Z_1 \overset{G}{\times} E$, the covering associated to the G-covering

When S is again arbitrary, Z_i of S and the action of G_i on E. This shows in a rather concrete way what it means, for a covering S' of S, to be associated to some M: it means that locally (for étale topology) it is describable by a representation of an inertial group on a finite set, in the obvious way; and the extra structure on one S' involved by deducing it from some S, can be described by saying that we give such local representations of S', in such a way that they match together i.e. satisfy some rather evident compatibility condition (This description makes a sense whenever the functor "geometric realization" is faithful - not necessarily fully faithful). The analogous description holds for Galois coverings with Galois group H, namely in the strictly local case, for a fixed Z_i, the ~~y are classified by system~~ category of these is equivalent to the category of finite sets E on which G_i operates on the left, H on the right, G_i and H commuting, in such a way that E is a principal homogeneous space (or, as we say, a torsor) under H; up to isomorphism, they are classified by $Hom(G_i,H)/H$, group homomorphisms mod. interior automorphism. Using ~~th~~ this description, one easily proves the criterion I indicated for the "geometric realization -functor" to be fully faithful on the category of Galois coverings of type $\underline{R}$; indeed, as usual the question is a local one for the étale topology, which allows one to ~~restr~~ reduce to the stricly local case.

Your interpretation of the Kummer case in the final formulation of example 3 is indeed the one I had in mind. Also, when I wrote n' n, I meant of course the order relation of divisibility (it may be convevient to introduce this order relation explicitly, for simplicity of notations).

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I realize that all the indications I have given you so far are extremely sketshy, and as a consequence that I am charging you with a considerable amount of work to put some sense and order into all that. Thus it is I, not you, who should apologize for causing a lot of trouble! I look forward with great plaäsure meeting you in Bures. As I am having some russian andchinese lessons on fridays, I will probably drop by on June 2.

With best regards

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