

ALGEBRAIC CYCLES AND THE WEIL CONJECTURES ‡

S. L. KLEIMAN

INTRODUCTION

With the development of the ℓ -adic étale cohomology, Weil's conjectures about the zeta function have been reduced to formal consequences of certain basic conjectures in the theory of algebraic cycles. It is our purpose to review this formalism. We agree to work over a fixed algebraically closed field k , to let variety mean integral algebraic k -scheme and to assume all varieties smooth, closed subschemes of given projective spaces. While it is often important to keep track of the twisting of cohomology by roots of unity because of its deep arithmetic significance, here it is more natural to choose a (non-canonical) isomorphism $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ and to work formally with the resulting "Weil cohomology" $X \mapsto H^*(X)$, which has coefficients in a field of characteristic zero, satisfies Poincaré duality and the Künneth formula and receives a non-trivial functorial ring homomorphism from the algebraic cycles modulo rational equivalence.

The conjectures spring from two sources: Lefschetz theory [6] and Hodge theory [12]. The strong Lefschetz theorem asserts: Let X be an n -dimensional variety, y the cohomology class of a hyperplane section and L the operator defined by $La = a \cdot y$; then the map $L^{n-i} : H^i(X) \rightarrow H^{2n-i}(X)$ is an isomorphism for $0 \leq i \leq n$. For complex varieties, the theorem may be proved using Hodge theory. For arbitrary varieties, the theorem is not yet established; however, it is expected that Lefschetz' original method will yield a proof.

The weak Lefschetz theorem asserts that if $f : Y \rightarrow X$ is the inclusion morphism of a smooth hyperplane section, then the induced map $f^* : H^i(X) \rightarrow H^i(Y)$ is an isomorphism for $i \leq n-2$ and an injection for $i = n-1$. For complex varieties, the theorem results from the exact sequence $H_c^i(X-Y) \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H_c^{i+1}(X-Y)$ where $H_c^*(X-Y)$ signifies cohomology with compact supports. Since $H_c^i(X-Y) \simeq H_{2n-i}^{lf}(X-Y)$, the weak Lefschetz theorem is therefore equivalent to the Lefschetz affine theorem: $H^j(X-Y) = 0$ for $j \geq n+1$. The affine theorem may be proved by computing with the complex of algebraic differential forms; namely, $H^j(X-Y) = H^j(\Gamma(\Omega_{X-Y/\mathbb{C}}^*))$. For arbitrary varieties, a similar proof may be given using the exact sequence of local cohomology and the Artin-Grothendieck Lefschetz affine theorem [1].

The main conjecture of Lefschetz type, denoted $B(X)$, deals with the cohomology operator Λ defined as zero on the primitive pieces $P^i(X) = \{a \in H^i(X) \mid L^{n-i+1}a = 0\}$ where $i \leq n$ and elsewhere as the natural inverse of L . The conjecture $B(X)$ asserts that Λ is induced by an algebraic cycle with rational coefficients on $X \times X$. (Note that, like $*$ introduced later, Λ is a modification by scalars on the primitive pieces of the classical operator.) The property $B(X)$ is independent of the polarization L and is remarkably stable: under product, smooth hyperplane section and specialization (with possible change of characteristic). Moreover, $B(X)$ is satisfied by projective space and the other usual rational varieties having cellular decompositions, by curves, by surfaces and by abelian varieties;

‡ This research has been partially supported by the National Science Foundation under Grant NSF GP-5177.

in fact, for these varieties Λ is induced by a cycle which does not depend on the choice of Weil cohomology.

Property $B(X)$ implies another property $C(X)$, which asserts that an algebraic cohomology class $a \in H^*(X \times X)$ has Künneth components $a_{pq} \in H^p(X) \otimes H^q(X)$ which are (rationally) algebraic, or equivalently put, the diagonal has algebraic Künneth components $\pi^i \in H^{2n-i}(X) \otimes H^i(X)$. (Hodge proved $C(X)$ holds for a surface using Hodge theory and noted it would hold in general if the Hodge conjecture is proved ($k = \mathbb{C}$.) In turn, $C(X)$ implies that an endomorphism of $H^i(X)$ induced by an algebraic cycle has integer coefficients in its characteristic polynomial. As a consequence, the zeta function of a smooth complete intersection in projective space has the same cohomological decomposition for all theories and the polynomials which occur have integer coefficients. (As Dwork and Washnitzer point out, this assertion is also an easy consequence of the weak Lefschetz theorem.)

The Hodge index conjecture $I(X, L)$ asserts that, for $2p \leq n$, the quadratic form $a, b \mapsto (-1)^p \langle L^{n-2p} a, b \rangle$ is positive definite on the \mathbb{Q} -space of algebraic, primitive $2p$ -classes. If $H^*(X)$ is endowed with the non-singular pairing defined as $(-1)^{i(i+1)/2} \langle L^{n-i-2j} a, b \rangle$ on $L^j P^i(X)$, then $B(X)$ implies that the transpose u' of an algebraic endomorphism is again algebraic, and $I(X \times X, L_X \otimes 1 + 1 \otimes L_X)$ implies that $\text{Tr}(u'ou) > 0$ if $u \neq 0$ (compare, Serre [11]). Two celebrated consequences ensue: the semisimplicity of the ring of algebraic endomorphisms; the "Riemann hypothesis" of the absolute value of the zeta function's zeros and poles.

Property $B(X)$ has a weaker form $A(X, L)$ asserting that L^{n-2p} induces an isomorphism of the \mathbb{Q} -space $AP(X)$ of algebraic classes in $H^{2p}(X)$ onto $A^{n-p}(X)$. On the other hand, if Y (resp. Z, \dots) is a smooth hyperplane section of X (resp. Y, \dots) and if $A(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$, $A(Y \times Z, L_Y \otimes 1 + 1 \otimes L_Z), \dots$ hold, then $B(X)$ holds. Moreover, in the presence of the strong Lefschetz theorem and the index conjecture, $A(X, L)$ is equivalent to conjecture $D(X)$ asserting the equality of homological equivalence with numerical equivalence.

In characteristic zero, the classical cohomology theories present new features: the Lefschetz theorems and the index conjecture hold, $\dim_{\mathbb{Q}} A^*(X) < \infty$ and every algebraic endomorphism has a characteristic polynomial with rational coefficients. It follows that $A(X, L)$ holds if and only if $\dim AP(X) = \dim A^{n-p}(X)$ for all $p \leq n$ and that $B(X)$ holds if and only if there exists some algebraic isomorphism $\nu^i: H^{2n-i}(X) \simeq H^i(X)$ for $i \leq n$. Further, $H^*(X, \mathbb{C})$ decomposes into pieces $H^{p,q}(X) = H^q(X, \Omega^p)$; the Hodge conjecture asserts $AP(X) = H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$. This conjecture, proved by Lefschetz for $p=1$, implies $A(X, L)$; hence, $A(X, L)$ holds when $\dim X \leq 4$. In characteristic $p > 0$, there can be no analogous decomposition, related to differentials or not. Otherwise, as Weil points out, the rank of the group of divisorial correspondence classes of a supersingular elliptic curve E would be bounded by $2[\dim H^{0,1}(E)]^2 = 2$.

The two Lefschetz theorems being assumed, the conjecture that for all X over k , $B(X)$ and $I(X, L)$ hold, is often referred to as "the standard conjectures": It is equivalent to the conjecture that for all X over k , $A(X)$ and $I(X, L)$ hold, and to that for all X over k , $D(X)$ and $I(X, L)$ hold. Moreover, if one cohomology theory satisfies these conjectures, then any other does if and only if it satisfies $D(X)$. For classical, ℓ -adic (and perhaps crystalline) cohomologies, the standard conjectures are probably more accessible than the Hodge conjecture (or Tate's variant) and the conjectured equality of τ -, homological and numerical equivalences \dagger . Furthermore, they imply the Weil conjectures and they are basic to Grothendieck's theories of motives and intermediate Picard varieties. At present,

\dagger Since these notes were written, P. Griffiths has announced a counter example to this last conjecture (cf. a paper of his, to appear in Pub. Math. IHES).

known are only the equality of τ -equivalence with numerical equivalence for divisors, proved by Matsusaka, and the index conjecture for surfaces, proved by Hodge-Segre-Bronowski-Grothendieck.

Speaking without cohomology and working with only the \mathbb{Q} -algebra of cycles modulo τ - (resp. numerical) equivalence, we may formulate analogues of the standard conjectures. For example, the Lefschetz affine "theorem" asserts that, if Y is a hyperplane section of X , then, on the affine variety $X-Y$, every cycle of dimension $\leq n/2$ can be deformed by τ - (resp. numerical) equivalence and pushed to infinity. Assuming these conjectures and speaking only of cycles π^i , Λ , etc. on $X \times X$ characterized by certain simple properties, we may state a Lefschetz fixed-point formula and give a proof, non-cohomological in appearance, of all the Weil conjectures. Of course, a cohomology theory satisfying the Lefschetz theorems satisfies the standard conjectures if and only if homological equivalence is equal to numerical equivalence and the numerical equivalence index conjecture holds. Moreover, the τ -equivalence conjectures imply that τ -equivalence is equal to numerical equivalence, thence to homological equivalence; so, they imply the standard conjectures hold for any cohomology theory satisfying the Lefschetz theorems.

This study is essentially due to Grothendieck, with three noteworthy exceptions: First, Lieberman proved the conjectures of Lefschetz type for abelian varieties (2A11, 2A13, 3.10: in fact, our entire work should be compared with Lieberman's articles [7, 8]). Second, Lubkin suggested the potential elimination of the denominators in the characteristic polynomial of an integrally algebraic endomorphism (2.6). Third, Bombieri independently observed that the Weil conjectures are formal consequences of the standard conjectures. I would also express my debt to the many others who contributed to these notes.

1. FORMALISM OF CYCLES

1.1. Rings of cycle classes [2, 9]

An algebraic cycle on a variety X is defined as a finite formal sum

$$Z = \sum m_\alpha Z_\alpha,$$

where the coefficients m_α are integers and the Z_α are closed, integral subschemes of X . The set of all cycles forms a group

$$C^*(X) = \oplus \mathbb{C}P(X)$$

graded by codimension.

Two closed, integral subschemes Z, W of X are said to intersect properly if every component Y_α of $Z \cap W$ has codimension equal to the sum of the codimensions of Z and W . When Z and W intersect properly, their intersection product is the cycle defined by

$$Z.W = \sum i(Z.W, Y_\alpha; X) Y_\alpha,$$

where the coefficients are the intersection multiplicities. This product, extended as far as possible over $C^*(X)$, is commutative and associative whenever defined.

A set $\{Z_t\}$ of cycles on X indexed by the closed points of a smooth, connected, quasi-projective scheme T is called an algebraic family if there exists a cycle Z on $X \times T$ such that each Z_t is the intersection-theoretic fiber over $t \in T$. Two cycles on X are said to be algebraically equivalent if there exists an algebraic family containing them both. It is easy to see that algebraic equivalence is compatible with subtraction, so it is an equivalence relation.

More generally, an equivalence relation is obtained whenever the parameter scheme T

is suitably restricted. Rational equivalence, for example, is defined to be the relation obtained by always taking T to be the projective line.

Lemma. Given two cycles Z, W on a variety X , there exists a cycle Y rationally equivalent to W such that the intersection cycle $Z.Y$ is defined.

Lemma. Let Z, W, Y be three cycles on a variety X . Suppose that Y is rationally (resp. algebraically) equivalent to W and that $Z.Y$ and $Z.W$ are defined. Then $Z.Y$ is rationally (resp. algebraically) equivalent to $Z.W$.

It follows from these lemmas that the group of cycle classes modulo rational (resp. algebraic) equivalence forms a graded ring $C^*_{\text{rat}}(X)$ (resp. $C^*_{\text{alg}}(X)$) under intersection product. The ring $C^*_{\text{rat}}(X)$ is often called the Chow ring.

Two cycles Z, W on X are said to be τ -equivalent if, for some integer $m \neq 0$, mZ is algebraically equivalent to mW . Clearly, τ -equivalence defines a corresponding ring of cycle classes $C^*_{\tau}(X)$.

The degree map $\langle \rangle : C^*(X) \rightarrow \mathbb{Z}$ is defined as zero on $C^p(X)$ for $p < n = \dim(X)$ and as $\sum \alpha_i$ at $\sum \alpha_i P_i \in C^n(X)$. Two cycles Z, W on X are said to be numerically equivalent if $\langle Z, Y \rangle = \langle W, Y \rangle$ for all cycles Y on X . Clearly, numerical equivalence defines a corresponding ring of cycle classes $C^*_{\text{num}}(X)$ and the canonical pairing

$$C^*_{\text{num}}(X) \times C^*_{\text{num}}(X) \rightarrow \mathbb{Z}$$

is separated.

Let $f: X \rightarrow Y$ be a morphism of varieties and let eq stand for one of the above four equivalence relations. Then f induces a functorial ring homomorphism

$$f^*: C^*_{\text{eq}}(Y) \rightarrow C^*_{\text{eq}}(X)$$

and a functorial group homomorphism

$$f_*: C^p_{\text{eq}}(X) \rightarrow C^{p+r}_{\text{eq}}(Y)$$

for each p where $r = \dim(Y) - \dim(X)$. In fact, f^* comes from the map which takes a closed, integral subscheme W of Y into its intersection-theoretic inverse image $f^{-1}(W)$ whenever $f^{-1}(W)$ is defined. And f_* comes from the map which takes a closed, integral subscheme Z of X into the cycle $d[f(Z)]$ where d is $[k(Z):k(f(Z))]$ if $\dim(Z) = \dim(f(Z))$ and d is 0 if $\dim(Z) > \dim(f(Z))$. The homomorphisms f^* and f_* are related through the projection formula,

$$f_*(f^*W.Z) = W.f_*Z$$

which holds already on the cycle level when all the terms are defined.

An equivalence relation is called adequate if it defines a ring of cycle classes $C^*_{\text{eq}}(X)$ for every variety X and homomorphisms f^*, f_* as above for every morphism $f: X \rightarrow Y$. It can be proved that rational equivalence is the smallest adequate relation and that numerical equivalence, the largest such that $C^*_{\text{eq}}(\text{Point}) = \mathbb{Z}$.

1.2. Weil cohomology

Fix a field K of characteristic zero, to be called the coefficient field. A contravariant functor $X \mapsto H^*(X)$ from varieties to augmented, finite dimensional, graded (by \dim), anti-commutative K -algebras is said to be a Weil cohomology if it satisfies the following three properties:

A. Poincaré duality - Let n be the dimension of X . Then:

- (i) The groups $H^i(X)$ are zero for $i \notin [0, 2n]$.
- (ii) There is given an "orientation" isomorphism $H^{2n}(X) \simeq K$.

(iii) The canonical pairings

$$H^i(X) \times H^{2n-i}(X) \rightarrow H^{2n}(X)$$

are non-singular.

Define a degree map $\langle \cdot \rangle : H^*(X) \rightarrow K$ as zero on $H^i(X)$ for $i < 2n$ and as the orientation isomorphism on $H^{2n}(X)$. Let $H_1(X)$ denote the K -vector space dual to $H^1(X)$. Then Poincaré duality states that the map $a \mapsto \langle \cdot, a \rangle$ induces isomorphisms $H^{2n-i}(X) \simeq H_1(X)$, which will be viewed as identifications.

Let $f: X \rightarrow Y$ be a morphism and $f^* = H^*(f): H^*(Y) \rightarrow H^*(X)$. Then define a K -linear map $f_*: H^*(X) \rightarrow H^*(Y)$ as being the transpose of f^* . Since f^* is a ring homomorphism, it follows that f^* and f_* are related through the projection formula,

$$f_*((f^*a) \cdot b) = a \cdot f_*b$$

which simply expresses that $H_*(X)$ is a left $H^*(X)$ -module functorially.

B. Künneth formula - Let $p: X \times Y \rightarrow X$ and $q: X \times Y \rightarrow Y$ be the projections. Then the canonical map $a \otimes b \mapsto p^*a \cdot q^*b$ is an isomorphism

$$H^*(X) \otimes_K H^*(Y) \simeq H^*(X \times Y).$$

C. Cycle map - There exist group homomorphisms

$$\gamma_X: \mathbb{C}P(X) \rightarrow H^{2p}(X)$$

satisfying: (i) (functoriality) - If $f: X \rightarrow Y$ is a morphism, then

$$f^*\gamma_Y = \gamma_X f^* \quad \text{and} \quad f_*\gamma_X = \gamma_Y f_*.$$

(ii) (multiplicativity) - $\gamma_{X \times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$.

(iii) (non-triviality) - If P is a point, then

$$\gamma_P: \mathbb{C}^*(P) = \mathbb{Z} \rightarrow H^*(P) = K$$

is the canonical inclusion.

The elements of $H^*(X)$ are often called cohomology classes and the multiplication, cup product. A cohomology class is said to be (integrally) algebraic if it is the image under γ_X of an algebraic cycle. Two algebraic cycles are said to be homologically equivalent if they define the same cohomology class.

1.2.1. Proposition. If two algebraic cycles on X are τ -equivalent, then they are homologically equivalent.

Indeed, since the coefficient field K has characteristic zero, we may assume the cycles are algebraically equivalent. Let $\{Z_t\}$ be an algebraic family; we have to show that the cohomology class $\gamma_X(Z_t)$ is independent of t . Let Z be a cycle in $X \times T$ defining $\{Z_t\}$. Let P be a point, $\alpha_t: P \rightarrow T$ the injection defined by the closed point t of T and $f_t = \text{id}_X \times \alpha_t: X = X \times P \rightarrow X \times T$. Then

$$\gamma_X(Z_t) = \gamma_X f_t^*(Z) = f_t^* \gamma_{X \times T}(Z).$$

However, $f_t^* = \text{id} \otimes \alpha_t^*$ and $\alpha_t^*: H^*(T) \rightarrow H^*(P) = K$ is independent of t , being the unique homomorphism of augmented algebras.

1.2.2. Proposition. Let eq stand for an adequate equivalence relation finer than τ -equivalence. Then the cycle map induces a ring homomorphism

$$\gamma_X: \mathbb{C}_{\text{eq}}^*(X) \rightarrow H^*(X).$$

Indeed, by 1.2.1, γ_X is well-defined. Now let $\Delta: X \rightarrow X \times X$ be the diagonal morphism. Then

$$\gamma_X(Z \cdot W) = \gamma_X \Delta^*(Z \times W) = \Delta^*(\gamma_X(Z) \otimes \gamma_X(W)) = \gamma_X(Z) \cdot \gamma_X(W).$$

1.2.3. Proposition. Homological equivalence is an adequate relation, finer than numerical equivalence.

Indeed, by 1.2.2, $C^*_{\text{hom}}(X) = \gamma_X(C^*_{\tau}(X))$ is a ring; hence, by the functoriality of γ_X , homological equivalence is adequate. It is finer than numerical equivalence because $C^*_{\text{hom}}(\text{Point}) = \mathbb{Z}$.

1.2.4. Proposition. Let $f: X \rightarrow Y$ be a surjective morphism. Then $f^*: H^*(Y) \rightarrow H^*(X)$ is injective.

Indeed, let x be a closed point of the generic fibre of F , Z the closure of x in X and $z = \gamma_X(Z)$. Then $f_* z \neq 0$. Let $a \in H^*(Y)$ and suppose $f^* a = 0$. Then, for any $b \in H^*(Y)$, $0 = f_*(f^* a \cdot f^* b \cdot z) = a \cdot b \cdot f_* z$. Hence, by Poincaré duality, $a = 0$.

1.2.5. Example. If the ground field k is the complex numbers, then the classical cohomology theory is a Weil cohomology; if k is an arbitrary algebraically closed field, then the ℓ -adic étale cohomology $X \mapsto H^*_{\text{ét}}(X)$, where $H^*_{\text{ét}}(X) = [\varprojlim_{\nu} H^*_{\text{ét}}(X, \mathbb{Z}/\ell^{\nu} \mathbb{Z})] \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$ and ℓ is prime to $\text{char}(k)$, becomes a Weil cohomology after the (non-canonical) choice of an isomorphism $\mathbb{Z}_{\ell}(1) \simeq \mathbb{Z}_{\ell}$.

1.2.6. Example. For the usual rational varieties having cellular decompositions (e.g., projective spaces, Grassmanns, flag manifolds [3]) and, more generally, for any variety X , where τ -equivalence equals numerical equivalence and whose K -algebra $A^*(X)$ of cycles modulo τ -equivalence satisfies the Künneth formula $A^*(X \times X) = A^*(X) \otimes A^*(X)$, the cohomology ring $H^*(X)$ is equal to $A^*(X)$.

Indeed, by 1.2.2 and 1.2.3, $A^*(X)$ may be considered a subring of $H^*(X)$. Let $p, q: X \times X \rightarrow X$ be the projections and $\Delta = \sum x_i \otimes y_i$ a decomposition of the diagonal such that $x_i, y_i \in A^*(X)$. If $a \in H^*(X)$ is an arbitrary element, then $a = q_*(\Delta \cdot p^* a) = \sum (x_i \cdot a) y_i$ is in $A^*(X)$.

1.3. Correspondences

Let $X \mapsto H^*(X)$ be a Weil cohomology. Then there are canonical K -linear isomorphisms

$$H^*(X \times Y) \simeq H^*(X) \otimes_K H^*(Y) \quad (\text{Künneth formula})$$

$$\simeq H_*(X) \otimes_K H^*(Y) \quad (\text{Poincaré duality})$$

$$\simeq \text{Hom}_K(H^*(X), H^*(Y)).$$

Specifically, an element $u = a \otimes b \in H^*(X \times Y)$ corresponds to the K -linear map $u^*: H^*(X) \rightarrow H^*(Y)$ defined by $u^*(c) = \langle c, a \rangle b$. Again, an arbitrary element $u \in H^*(X \times Y)$ corresponds to the composition

$$u^*: H^*(X) \xrightarrow{p^*} H^*(X \times Y) \xrightarrow{v \mapsto v \cdot u} H^*(X \times Y) \xrightarrow{q^*} H^*(Y),$$

where p, q are the projections. Furthermore, if $n = \dim(X)$, then the elements $u \in H^{2n+d}(X \times Y)$ correspond to the K -linear maps

$$u^*: H^*(X) \rightarrow H^*(Y)$$

which are homogeneous of degree d .

The elements $u \in H^*(X \times Y)$, thought of as the K -linear maps $u = u^* : H^*(X) \rightarrow H^*(Y)$, are called (homological) correspondences. Composition of linear maps defines a composition of correspondences: explicitly, if $u \in H^*(X \times Y)$ and $v \in H^*(Y \times Z)$, then

$$1.3.1. \quad \text{you} = p_*(u \otimes 1_Z \cdot 1_X \otimes v),$$

where $p : X \times Z \rightarrow X \times Z$ is the projection. In particular, the self-correspondences of X form a ring composition; namely,

$$H^*(X \times X) \simeq \text{End}_K(H^*(X)).$$

Analogously, there is a canonical isomorphism

$$H^*(X \times Y) \simeq \text{Hom}_K(H_*(X), H_*(Y))$$

in which $v = b \otimes a \in H^\beta(X) \otimes H^\alpha(Y)$ corresponds to the linear map $v_* : H_*(X) \rightarrow H_*(Y)$ defined by

$$v_*(d) = (-1)^{\alpha\delta} \langle b, d \rangle a = q_*(v \cdot p^* d)$$

for $d \in H_\delta(X) = H^{2n-\delta}(X)$, where p, q are the appropriate projections. Note that, if $v \in \oplus H^{2i}(X \times Y)$, then

$$v_* = v^* : H^*(X) \rightarrow H^*(Y),$$

but not otherwise.

1.3.2. Proposition. Let $u \in H^*(X \times Y)$ and let $t_u \in H^*(Y \times X)$ denote its canonical transpose. Then $(t_u)_* : H_*(Y) \rightarrow H_*(X)$ is the K -transpose of $u^* : H^*(X) \rightarrow H^*(Y)$.

Indeed, by linearity it suffices to note that, for $u = a \otimes b \in H^\alpha(X) \otimes H^\beta(Y)$, $c \in H^\gamma(X)$ and $d \in H_\delta(Y)$, we have $t_u = (-1)^{\alpha\beta} b \otimes a$ and $\langle b, d \rangle = 0$ when $\beta \neq \delta$; so, $\langle c, t_{u*}(d) \rangle = (-1)^{\alpha\beta} \langle c, a \rangle \langle b, d \rangle = \langle u^*(c), d \rangle$.

1.3.3. Proposition. If $u \in H^*(X \times Y)$ and $v \in H^*(Z \times W)$, then the tensor product of $u^* : H^*(X) \rightarrow H^*(Y)$ and $v^* : H^*(Z) \rightarrow H^*(W)$ corresponds to the map $H^*(X \times Z) \rightarrow H^*(Y \times W)$ defined by the cycle

$$u \otimes v = p^* u \cdot q^* v \in H^*(X \times Z \times Y \times W),$$

where p, q are the appropriate projections.

Indeed, the assertion follows easily from the definition:

$$u^* \otimes v^*(a \otimes b) = (-1)^{\beta\delta} u^*(a) \otimes v^*(b)$$

for $u \in H^\delta(X \times Y)$, $b \in H^\beta(Z)$.

1.3.4. Proposition. Consider a diagram of correspondences

$$\begin{array}{ccc} H^*(X) & \xrightarrow{u^*} & H^*(Y) \\ \uparrow t_x^* & & \downarrow y^* \\ H^*(W) & \xrightarrow{v^*} & H^*(Z). \end{array}$$

Suppose $x \in \oplus H^{2i}(X \times W)$. If $v = (x \otimes y)^* u$, then $v = \text{you} \circ t_x$.

Indeed, by linearity, we may assume $u = a \otimes b$. Then, for $c \in H^*(W)$, we have

$$v^*(c) = (x^* a \otimes y^* b)^* c = \langle c, x^* a \rangle y^* b = y^*(\langle t_x^* c, a \rangle b) = y^*(u^*(t_x^* c)).$$

1.3.5. Proposition. Let $a \in H^*(X)$. The operator $M_a: H^*(X) \rightarrow H^*(X)$ defined by $M_a(b) = b.a$ corresponds to the cycle $\Delta_*(a)$ where $\Delta: X \rightarrow X \times X$ is the diagonal morphism.

Indeed, if $q: X \times X \rightarrow X$ is the projection onto the second factor, then $(\Delta_*(a))^*b = q_*(b \otimes 1, \Delta_*(a)) = q_* \circ \Delta_*(\Delta^*(b \otimes 1).a) = b.a$.

1.3.6. Proposition. Let X, Y be varieties of dimension n, m . Let $\Delta \in H^*(X \times X)$ be the diagonal class and $\pi^i \in H^{2n-i}(X) \otimes H^i(X)$ the Künneth components of Δ . Let $u \in H^*(X \times X)$, $v \in H^*(X \times Y)$, $w \in H^*(Y \times X)$ be correspondences of degrees $0, d, -d$ respectively and let $\text{Tr}_i(u)$ denote the trace of the map $H^i(X) \rightarrow H^i(X)$ induced by u . Then:

(i) π^i corresponds to the i th-projection operator, $\pi^i: H^*(X) \rightarrow H^i(X)$.

(ii) a. The trace formula: $\text{Tr}_i(u) = (-1)^i \langle u, \pi^{2n-i} \rangle$.

b. The Lefschetz fixed-point formula:

$$\langle u, \Delta \rangle = \sum_{i=0}^{2n} (-1)^i \text{Tr}_i(u).$$

c.

$$\langle v, tw \rangle = \sum_{i=0}^{2n} (-1)^i \text{Tr}_i(\text{wov}).$$

Indeed, (i) is clear and (ii) a, b are special cases of (ii) c. To prove (ii) c, we may assume $v \in H^{2n-i}(X) \otimes H^i(Y)$, $w \in H^{2m-j}(Y) \otimes H^j(X)$; say, $v = \sum a_\ell \otimes b$, $w = \sum c_\ell \otimes a_\ell$ with $\langle a_\ell, a_m \rangle = \delta_{\ell m}$. Then $\langle v, tw \rangle = \sum \langle b_\ell, c_\ell \rangle$ and $(\text{wov})(a_\ell) = (-1)^i w(b_\ell) = (-1)^i \langle b_\ell, c_\ell \rangle a_\ell + \dots$. Thus, $\text{Tr}_i(\text{wov}) = (-1)^i \sum \langle b_\ell, c_\ell \rangle = (-1)^i \langle v, tw \rangle$.

A correspondence $u \in H^*(X \times Y)$ (resp. a linear map $u: H^i(X) \rightarrow H^i(Y)$) is called (rationally) algebraic if it is (resp. is the restriction of a map $H^*(X) \rightarrow H^*(Y)$ induced by) an element of the \mathbb{Q} -vector space generated by the (integrally) algebraic classes.

1.3.7. Proposition. (i) If $u \in H^*(X \times Y)$, $w \in H^*(Z \times W)$ are algebraic correspondences, then $tu \in H^*(Y \times X)$, $u \otimes w \in H^*(X \times Z \times Y \times W)$, and if $Y = Z$, $wou \in H^*(X \times W)$ are all algebraic correspondences.

(ii) If $a \in H^*(X)$ is algebraic, then the map $M_a: H^*(X) \rightarrow H^*(X)$ defined by $M_a(b) = b.a$ is algebraic.

(iii) Let $f: Y \rightarrow X$ be a morphism and $u \in H^*(X \times Y)$ the class of its graph. Then $f^* = u^*: H^*(X) \rightarrow H^*(Y)$ and $f_* = t_{u*}: H^*(Y) \rightarrow H^*(X)$; hence, f^* and f_* are algebraic.

(iv) Let $f: Y \rightarrow X$ and $g: Z \rightarrow W$ be morphisms. Then

$$(f \times g)^* = f^* \otimes g^*: H^*(X \times W) \rightarrow H^*(Y \times Z)$$

and they are algebraic.

Indeed, (i) and (ii) follow immediately from 1.3.2, 1.3.3, 1.3.1 and 1.3.5. In (iv), $(f \times g)^* = f^* \otimes g^*$ by the Künneth formula and they are algebraic by (iii). In (iii),

$$u = \gamma_{X \times Y}(\text{id} \times f)^* \Delta = (\text{id} \otimes f^*) \gamma_{X \times X} \Delta,$$

where Δ is the diagonal; hence, $f^* = u^*$ by 1.3.4 and $f_* = t_{u*}$ by 1.3.2.

1.3.8. In view of 1.3.7, the algebraic self-correspondence of X form a \mathbb{Q} -algebra, denoted $\mathcal{A}^*(X)$. It is stable under transposition and contains the operators M_a , f^* and f_* for any algebraic element $a \in H^*(X)$ and any morphism $f: X \rightarrow X$.

1.4. Lefschetz theory

Fix a Weil cohomology $X \mapsto H^*(X)$. For each variety X , let Y be a hyperplane section and $y = \gamma_X(Y) \in H^2(X)$. Define an operator of degree 2 (or polarization)

$$L = L_X: H^*(X) \rightarrow H^*(X) \quad \text{by} \quad L(a) = a.y.$$

In the propositions below, assume X satisfies the following condition:

The strong Lefschetz "theorem". For $i \leq n = \dim X$, the map

$$L^{n-i}: H^i(X) \rightarrow H^{2n-i}(X),$$

is an isomorphism.

1.4.1. Proposition. For $j \leq n-i$, the map $L^j: H^i(X) \rightarrow H^{i+2j}(X)$ is an injection and the map $L^{n-i-j}: H^{i+2j}(X) \rightarrow H^{2n-i}(X)$ is a surjection. Consequently, if $b_i = \dim(H^i(X))$ is the i^{th} Betti number, then

$$1 = b_0 \leq b_2 \leq \dots \leq b_{2i} \quad \text{for} \quad 2i \leq n$$

$$b_1 \leq b_3 \leq \dots \leq b_{2j+1} \quad \text{for} \quad 2j+1 \leq n.$$

1.4.2. Proposition - (Primitive decomposition). For $i \leq n$, let $P^i(X)$ be the set of elements $a \in H^i(X)$, called primitive, which satisfy $L^{n-i+1}(a) = 0$. Then, for any i , any $a \in H^i(X)$ can be written uniquely in the form

$$a = \sum_{j \geq i_0} L^j a_j,$$

where $a_j \in P^{i-2j}(X)$ and $i_0 = \max(i-n, 0)$.

Indeed, these propositions are immediate consequences of the strong Lefschetz "theorem" and they permit definition of the following operators:

$$1.4.2.1. \quad \Lambda a = \sum_{j \geq i_1} L^{j-1} a_j \quad \text{where} \quad i_1 = \max(i-n, 1)$$

$$1.4.2.2. \quad {}^c\Lambda a = \sum_{j \geq i_1} j(n-i+j+1) L^{j-1} a_j$$

$$1.4.2.3. \quad {}^*a = \sum_{j \geq i_0} (-1)^{(i-2j)(i-2j+1)/2} L^{n-i+j} a_j$$

$$1.4.2.4. \quad p^j a = \delta_{ij} a_{j_0} \quad \text{for} \quad j = 0, \dots, 2n,$$

where $a \in H^i(X)$ with primitive decomposition $a = \sum_{j \geq i_0} L^j a_j$.

1.4.3. Proposition. (i) Λ and ${}^c\Lambda$ have degree -2.

(ii) For $i \leq n$, $\Lambda^{n-i}: H^{2n-i}(X) \rightarrow H^i(X)$ is inverse to L^{n-i} and ${}^c\Lambda^{n-i}: L^{n-i} P^i(X) \rightarrow P^i(X)$ is inverse to a multiple of L^{n-i} .

(iii) For all i , $*$: $H^i(X) \rightarrow H^{2n-i}(X)$, $*^2 = \text{id}$ and $\Lambda = *L*$.

(iv) Λ , ${}^c\Lambda$, $*$, π^0, \dots, π^{2n} , p^0, \dots, p^{n-1} are all given by universal, non-commutative polynomials with integer coefficients in L and p^n, \dots, p^{2n} .

Indeed, the assertions are straightforward consequences of the definitions and the formula $a_j = p^{2n-i+2j} L^{n-i+j} a$ where $a \in H^i(X)$ with primitive decomposition $a = \sum_{j \geq i_0} L^j a_j$.

1.4.4. Proposition. The operator Q -algebras generated by L and Λ , by L and ${}^c\Lambda$, by L and $*$, by L and p^n, \dots, p^{2n} are all the same and they contain p^0, \dots, p^{n-1} and π^0, \dots, π^{2n} .

Indeed, the algebra generated by L and p^n, \dots, p^{2n} contains Λ , ${}^c\Lambda$, $*$, π^0, \dots, π^{2n} , p^0, \dots, p^{n-1} by 1.4.3 (iv); that, by L and $*$ contains $\Lambda = *L*$; finally, that, by L and Λ (resp. ${}^c\Lambda$) contains p^n, \dots, p^{2n} by 1.4.3 (ii) and the following lemma.

1.4.5. Lemma. For $i \leq n$, let $\theta^i: H^*(X) \rightarrow H^*(X)$ be a map of degree $-2(n-i)$ which induces the map $L^{n-i} P^i(X) \rightarrow P^i(X)$ inverse to L^{n-i} . Then p^{2n-i} is given by a universal, non-commutative polynomial with integer coefficients in L and $\theta^0, \dots, \theta^i$.

Indeed, the assertion results by induction on i from the following, easily verified formulas:

$$\begin{aligned} \varphi_i &= \sum_{j=i}^{2n-i} \pi^j = (\text{id} - \sum_{\ell \notin [i, 2n-i]} \sum_{j \geq \ell_0} L^j p^{2n-\ell+2j} L^{n-\ell+j}) \\ p^{2n-i} &= \varphi_i \theta^i (\text{id} - \sum_{j \geq 1+n-i} L^j p^{i+2j} L^{i-n+j}) \varphi_i, \end{aligned}$$

where $\ell_0 = \max(\ell - n, 0)$.

1.4.6. Proposition. (i) ${}^c\Lambda$ is the unique operator of degree -2 which satisfies the formula

$$1.4.6.1 \quad [{}^c\Lambda, L] = \sum_{i=0}^{2n} (n-i) \pi^i.$$

(ii) Let X, Z and $X \times Z$ satisfy the strong Lefschetz "theorem" and polarize $X \times Z$ via the Segre immersion: $L_{X \times Z} = L_X \otimes \text{id} + \text{id} \otimes L_Z$. Then:

$${}^c\Lambda_{X \times Z} = {}^c\Lambda_X \otimes \text{id} + \text{id} \otimes {}^c\Lambda_Z.$$

Indeed, it results easily from the definition that ${}^c\Lambda$ satisfies 1.4.6.1. On the other hand, any operator λ which satisfies 1.4.6.1 is easily seen by induction on j to satisfy

$$1.4.6.2. \quad [\lambda, L^j] = L^{j-1} \sum_{\ell=0}^{j-1} \sum_{i=0}^{2n} (n-i) \pi^{i-2\ell}.$$

Assume λ has degree -2 and let $a \in P^i(X)$. Then $L^{n-i+2} \lambda a = \lambda L^{n-i+2} a - r L^{n-i+1} a = 0$ where r is the integer given by 1.4.6.2; hence $\lambda a = 0$. Then, for any $j \geq 1$, $\lambda L^j a = [\lambda, L^j] a + L^j \lambda a = {}^c\Lambda L^j a$. Thus, $\lambda = {}^c\Lambda$.

Finally, (ii) results formally from (i).

In the proposition below, let X satisfy the strong Lefschetz "theorem" and $f: Y \rightarrow X$ be the inclusion morphism of a smooth hyperplane section. Assume, further, that the following condition is satisfied:

The weak Lefschetz "theorem". The map $f^*: H^i(X) \rightarrow H^i(Y)$ is an isomorphism for $i \leq n-2$ and an injection for $i = n-1$, or equivalently by transposition, the map $f_*: H^i(Y) \rightarrow H^{i+2}(X)$ is an isomorphism for $i \geq n$ and a surjection for $i = n-1$.

1.4.7. Proposition. (i) $f_* f^* = L_X$ and $f^* f_* = L_Y$.

(ii) $f_* : H^1(Y) \rightarrow H^{i+2}(X)$ is injective for $i \leq n-2$ and $f^* : H^i(X) \rightarrow H^i(Y)$ is surjective for $i \geq n$.

(iii) $L_X f_* = f_* L_Y$ and $f^* L_X = L_Y f^*$.

(iv) Y satisfies the strong Lefschetz "theorem".

(v) $f_*(P^i(Y)) \subset L_X P^i(X)$ and $f^*(P^i(X)) \subset P^i(Y)$ for $i \leq n-1$, and $f^*(P^n(X)) = 0$.

(vi) f_* and f^* commute with primitive decomposition.

(vii) $\Lambda_Y = f^* \Lambda_X^2 f_*$.

Indeed, if $a \in H^*(X)$, then $f_* f^* a = f_*(f^* a \cdot 1_Y) = a \cdot f_* 1_Y = L_X a$. If $b \in H^i(Y)$ with $i \geq n-2$, then

$$f_* f^* f_* b = f_* b \cdot f_* 1_Y = f_*(b \cdot f^* f_* 1_Y) = f_* L_Y b;$$

hence, by the weak Lefschetz "theorem", $f^* f_* b = L_Y b$. Finally, it follows by transposition that, if $b \in H^i(Y)$ with $i \leq n$, then again $f^* f_* b = L_Y b$; so, the proof of (i) is complete.

Assertions (ii) and (iii) follow easily from assertion (i). Assertions (i) and (iii) imply that $L_X^{n-1} = f_* L_Y^{(n-1)-i} f^*$ for $i \leq n-1$; whence, assertion (iv).

Assertion (iii) implies that $L_X^{n-(i+2)+1} f_* = f_* L_Y^{(n-1)-i}$ for $i \leq n-1$; hence, $L_X^{n-(i+2)+1}$ is an injection on $f_* H^i(Y)$. Again, (iii) implies that $L_X^{n-(i+2)+2}$ is zero on $f_* P^i(Y)$. Thus, $f_* : P^i(Y) \rightarrow L_X P^i(X)$. Similarly, by (iii) and (i), $f_* L_Y^{n-1-i+1} f^* = L_X^{n-i+1}$ for $i \leq n$; whence, the remaining assertions of (v).

Finally, (vi) and (vii) follow easily from (v), (iii) and (i).

2. CONJECTURES OF LEFSCHETZ TYPE

Fix a Weil cohomology $X \mapsto H^*(X)$ and consider varieties X satisfying the strong Lefschetz "theorem". We shall study the following three conditions.

$A(X, L)$: For $2p \leq n = \dim X$, the injection

$$L_X^{n-2p} : A^p(X) \rightarrow A^{n-p}(X)$$

on the \mathbb{Q} -vector space $A^p(X)$ of algebraic cohomology classes is a bijection.

$B(X)$: The correspondence $\Lambda_X \in H^{2n-2}(X \times X)$, associated to L_X , is algebraic.

$C(X)$: The Künneth components $\pi^i \otimes H^{2n-i}(X) \in H^i(X)$ of the diagonal class Δ , which correspond to the projection operators

$$\pi^i : H^*(X) \rightarrow H^i(X),$$

are algebraic.

For example, if X is a curve, then X trivially satisfies all of these conditions; if X is a flag manifold (e.g., a Grassmann or a projective space), then X satisfies them by 1.2.6.

2.1. Proposition. The following conditions are equivalent:

(i) $A(X, L)$.

(ii) $A^*(X)$ is stable under the primitive projections p^n, \dots, p^{2n} .

(iii) $A^*(X)$ is stable under the operator $*$.

(iv) $A^*(X)$ is stable under the operator Λ .

(v) $A^*(X)$ is stable under the operator $c\Lambda$.

Indeed, the equivalence of (ii), (iii), (iv) and (v) results immediately from 1.4.4; the implication (iv) \rightarrow (i), from 1.4.3 (ii). Finally, if (i) holds, then $A^*(X)$ is stable under $\theta^i = \Lambda^{n-i} \pi^{2n-i}$ for $i \leq n$; whence, $A^*(X)$ is stable under p^n, \dots, p^{2n} by 1.4.5.

2.2. Corollary. $B(X)$ implies $A(X, L)$.

2.3. Proposition. The following conditions are equivalent:

- (i) $B(X)$.
- (ii) For $i \leq n$, the isomorphism $H^{2n-i}(X) \rightarrow H^i(X)$ inverse to L^{n-i} is algebraic.
- (iii) The primitive projections p^n, \dots, p^{2n} are algebraic.
- (iv) The operator $*$ is algebraic.
- (v) The operator ${}^c\Lambda$ is algebraic.

Indeed, in view of 1.3.8, the equivalence of (i), (iii), (iv) and (v) results immediately from 1.4.4; the implication (i) \rightarrow (ii), from 1.4.3 (ii). Finally, if (ii) holds, then π^n, \dots, π^{2n} are algebraic by the following lemma; so, $\theta^i = \Lambda^{n-i} \pi^{2n-i}$ is algebraic for $i \leq n$ and (iii) results from 1.4.5.

2.4. Lemma. If the map $H^{2n-j}(X) \rightarrow H^j(X)$ inverse to L^{n-j} is induced by an algebraic correspondence θ^j for $j \leq i$ where $i \leq n$, then the projections $\pi^0, \dots, \pi^i, \pi^{2n-i}, \dots, \pi^{2n}$ are algebraic. In particular, $B(X)$ implies $C(X)$.

Indeed, in view of 1.3.8, the assertion results by induction on i from the following, easily verified formulas:

$$\pi^i = \theta^i(\text{id} - \sum_{j > 2n-i} \pi^j) L^{n-i}(\text{id} - \sum_{j < i} \pi^j)$$

$$\pi^{2n-i} = t_{\pi^i}.$$

2.5. Corollary. $B(X)$ is stable under product; in other words, $B(X), B(Y)$ imply $B(X \times Y)$.

Indeed, ${}^c\Lambda_{X \times Y} = {}^c\Lambda_X \otimes \text{id} + \text{id} \otimes {}^c\Lambda_Y$ by 1.4.6 (ii). So the assertion results from 1.3.7 and 2.3.

2.6. Proposition. (i) If $A^*(X \times X)$ is stable under Künneth decomposition then X satisfies $C(X)$.

(ii) If X, Y satisfy $C(X), C(Y)$, then $A^*(X \times Y)$ is stable under Künneth decomposition.

Indeed, (i) is trivial and (ii) results from the fact that the map

$$\pi_X^i \otimes \pi_Y^j : H^{i+j}(X \times Y) \rightarrow H^i(X) \otimes H^j(Y)$$

is the projection corresponding to Künneth decomposition.

2.7. Proposition. Suppose π^{2n-i} is algebraic and let u be an endomorphism of $H^i(X)$ which is algebraic. Then the coefficients σ_j of the characteristic polynomial $P(t) = \text{Det}(1 - ut)$ of u are rational numbers. Further, if u is integrally algebraic (i.e., $u \in \gamma_{X \times X}(C^*(X \times X))$), then the σ_j are integers.

Indeed, the Newton formulas express the σ_j as polynomials with rational coefficients in the power sums

$$S_m = \alpha_1^n + \dots + \alpha_{b_i}^m$$

of the eigenvalues α_ℓ of u . And, by the trace formula (1.3.6),

$$S_m = \text{Tr}(u^m) = (-1)^i \langle u^m, \pi^{2n-i} \rangle \in \mathbb{Q}.$$

Thus, $\sigma_j \in \mathbb{Q}$.

Further, suppose that u is integrally algebraic. Then, if a is a non-zero integer such that $a\pi^{2n-1}$ is integrally algebraic, the trace formula shows that $a.S_m$ is an integer for all m . Thus, the σ_j are integers by the following lemma.

2.8. Lemma. Let A be a subring of a field K . Let $\alpha_1, \dots, \alpha_\ell$ be distinct elements of K and let p_1, \dots, p_ℓ be integers not congruent to 0 modulo the characteristic of K . Let $S_m = p_1\alpha_1^m + \dots + p_\ell\alpha_\ell^m$. Suppose there exists a non-zero element $a \in A$ such that $aS_m \in A$ for all $m \geq 1$. Then the α_i are integral over A .

Indeed, consider the matrix equation

$$\begin{bmatrix} \alpha_1 & \dots & \alpha_\ell \\ \vdots & & \vdots \\ \alpha_1^\ell & \dots & \alpha_\ell^\ell \end{bmatrix} \begin{bmatrix} p_1 \alpha_1^m \\ \vdots \\ p_\ell \alpha_\ell^m \end{bmatrix} = \begin{bmatrix} S_{m+1} \\ \vdots \\ S_{m+\ell} \end{bmatrix}.$$

Since the α_j are distinct, we may solve and find

$$p_i \alpha_i^m = \beta_{i1} S_{m+1} + \dots + \beta_{i\ell} S_{m+\ell} \quad (i = 1, \dots, \ell),$$

where the β_{ij} are independent of m . It follows that any valuation of K which is non-negative on A is non-negative on the α_i . Therefore, the α_i are integral over A .

2.9. Theorem. The following conditions are equivalent:

- (i) $B(X)$.
- (ii) $C(X)$ and $\nu(X)$: For $i \leq n-1$, there exists an isomorphism $\nu^i: H^{2n-i}(X) \simeq H^i(X)$ which is algebraic.
- (iii) $\nu(X)$ and $\rho(X)$: Let u be an endomorphism of a cohomology group $H^i(X)$. If u is algebraic, then the coefficients σ_j of the characteristic polynomial of u are rational numbers.

Indeed, by 2.3 and 2.4, (i) implies (ii); by 2.7, (ii) implies (iii).

Assume (iii), fix $i \leq n-1$ and consider the algebraic map $u = \nu^i L^{n-i}: H^i(X) \rightarrow H^i(X)$. By assumption, u is an automorphism; so, its characteristic polynomial $P(t)$ has rational coefficients. By the Cayley-Hamilton theorem, $P(u) = 0$. It follows that u^{-1} is a linear combination of $1, u, u^2, \dots$ with rational coefficients; so, u^{-1} is algebraic. Therefore, the isomorphism $H^{2n-i}(X) \rightarrow H^i(X)$ inverse to L^{n-i} , being equal to $u^{-1}\nu^i$, is algebraic. By 2.3, $B(X)$ therefore holds and the proof is complete.

2.10. Corollary. $B(X)$ is independent of the polarization L , i.e., of the embedding of X in projective space.

2.11. Corollary. Suppose X, Y satisfy $B(X)$, $B(Y)$. Let $u: H^i(X) \rightarrow H^j(Y)$ be an isomorphism which is algebraic. Then u^{-1} is algebraic. Consequently, if $a \in H^i(X)$ is such that $u(a)$ is algebraic, then a is algebraic.

Indeed, for convenience, let L^{-p} denote $\Lambda^p(p > 0)$ and set $v = L_X^{i-n} u L_Y^{m-j}$. Then, we have the commutative diagram,

$$\begin{array}{ccc} H^{2m-j}(Y) & \xrightarrow{t_u} & H^{2n-i}(X) \\ L_Y^{m-j} \uparrow & & \downarrow L_X^{i-n} \\ H^j(Y) & \xrightarrow{v} & H^i(X) \end{array}$$

Now, $w = vu$ is an automorphism of $H^1(X)$ which is algebraic; hence, by 2.8 (iii) and the Cayley-Hamilton theorem, w^{-1} is algebraic. Therefore, $u^{-1} = w^{-1}v$ is algebraic.

2.12. Proposition. Let $f: Y \rightarrow X$ be the inclusion morphism of a smooth hyperplane section. Suppose X satisfies both Lefschetz "theorems". Then the following conditions are equivalent:

- (i) $B(X)$.
 - (ii) $B(Y)$ and, for all $i < n$, there exists an algebraic correspondence $\beta^i: H^*(Y) \rightarrow H^*(X)$ inducing the map $\Lambda_{X*} f_*: H^i(Y) \rightarrow H^i(X)$ which is the left inverse of $f^*: H^i(X) \rightarrow H^i(Y)$.
 - (iii) $B(Y)$ and, for all $i > n$, there exists an algebraic correspondence $\beta^i: H^*(X) \rightarrow H^*(Y)$ inducing the map $f^* \Lambda_X: H^i(X) \rightarrow H^{i-2}(Y)$, which is the right inverse of $f_*: H^{i-2}(Y) \rightarrow H^i(X)$.
- Indeed, assume (i). Then $\Lambda_Y = f^* \Lambda_X^2 f_*$ and $\gamma^i = \Lambda_X f_*$ are algebraic; hence, (i) implies (ii). By transposition, (ii) and (iii) are equivalent. Finally, assume (ii) and (iii). Then the composition

$$H^*(X) \xrightarrow{\beta^{2n-i}} H^*(Y) \xrightarrow{\Lambda_Y^{n-1-i}} H^*(Y) \xrightarrow{\gamma^i} H^*(X)$$

is an algebraic correspondence inducing the isomorphism $H^{2n-i}(X) \rightarrow H^i(X)$ inverse to L_X^{n-1} . Thus, by 2.3, (ii) and (iii) imply (i).

2.13. Theorem. Consider a sequence of varieties

$$X \supset Y \supset Z \supset \dots,$$

where each is a hyperplane section of the preceding. Suppose X, Y, Z, \dots all satisfy both Lefschetz "theorems", and polarize all products $X \times X, Y \times Z, \dots$ via the Segre immersion:

$$L_{X \times X} = L_X \otimes 1 + 1 \otimes L_X, \dots$$

Then the following conditions are equivalent:

- (i) $B(X)$.
- (ii) $B(Y)$ and $A(X \times X, L_{X \times X})$.
- (ii') $B(Y)$ and $A(X \times X)^0: L_{X \times X}^2: A^{n-1}(X \times X) \rightarrow A^{n+1}(X \times X)$ is bijective.
- (iii) $B(Y)$ and $A(X \times Y, L_{X \times Y})$.
- (iii') $B(Y)$ and $A(X \times Y)^0: L_{X \times Y}^2: A^{n-1}(X \times Y) \rightarrow A^{n+1}(X \times Y)$ is bijective.
- (iv) $A(X \times X)^0$ and $A(Y \times Y)^0$ and $A(Z \times Z)^0$ and \dots .
- (v) $A(X \times Y)^0$ and $A(Y \times Z)^0$ and \dots .

Indeed, by 2.12, 2.10 and 2.2, (i) implies (ii) and (iii). Clearly, (ii) implies (ii'), and (iii) implies (iii'). Since, if W is a curve, $B(W)$ holds trivially, it follows from the equivalence of (i) and (ii') that (i) and (iv) are equivalent and it follows from the equivalence of (i) and (iii') that (i) and (v) are equivalent. We proceed to prove (ii') implies (i); the proof that (iii') implies (i) is similar and so is omitted.

Let $f: Y \rightarrow X$ be the inclusion. Then we have the formulas:

$$\Lambda_X L_X = 1_X - \sum_{i=n}^{2n} L_X^{i-n} p_X^i$$

$$L_X \Lambda_X = 1_X - \sum_{i=0}^n p_X^i$$

$$f_* \Lambda_Y f^* = 1_X - \sum_{i=0}^n p_X^i - \sum_{i=n+1}^{2n} L_X^{i-n} p_X^i.$$

Hence,

$$T_X = f_* \Lambda_Y f^* + 1_X = \Lambda_X L_X + L_X \Lambda_X + p_X^n.$$

Since $p_X^n L_X = 0$ and $L_X p_X^n = 0$, it follows that

$$\mu = L_X T_X + T_X L_X = L_X^2 \Lambda_X + 2 L_X \Lambda_X L_X + \Lambda_X L_X^2.$$

Since $L_{X \times X} = L_X \otimes 1_X + 1_X \otimes L_X$ and $L_X \in H^{2n}(X \times X)$, we have by 1.3.4 that

$$L_{X \times X} w = w \circ L_X + L_X \circ w$$

for any $w \in H^*(X \times X)$; whence, it follows that

$$\mu = L_{X \times X}^2 \Lambda_X.$$

Assume (ii'); then T_X is algebraic, so μ is algebraic, so Λ_X is algebraic. The proof is now complete.

2.14. Corollary. Suppose both Lefschetz "theorems" are universally valid. Let $A(k)$, $B(k)$, $C(k)$ denote the conditions $A(X)$, $B(X)$, $C(X)$ for all varieties X over k . Then $A(k)$ and $B(k)$ are equivalent and they imply $C(k)$.

2. APPENDIX: CONJECTURES OF LEFSCHETZ TYPE AND ABELIAN VARIETIES

Fix a Weil cohomology $X \mapsto H^*(X)$. We shall prove that $B(X)$ holds if X is an abelian variety or if X is a surface such that $\dim(H^1(X)) = 2 \dim(P)$ where P is the Picard variety $((\text{Pic}_{X/k}^0)_{\text{red}})$ of X ; (e.g., $H^1(X) = H_{\text{ét}}^1(X, \mathbb{Q}_\ell) = (\varprojlim_v H_{\text{ét}}^1(X; \mathbb{Z}/\ell) \otimes \mathbb{Q}_\ell)$).

2A1. Proposition. (i) Let A be the Albanese variety of X , $\kappa: X \rightarrow A$ a canonical morphism. Then κ induces a canonical map $\alpha: H^1(A) \rightarrow H^1(X)$ which is functorial in the following sense: If $f: Y \rightarrow X$ is a morphism and $v: A_Y \rightarrow A_X$ the corresponding morphism of Albanese varieties, then the following diagram is commutative:

$$\begin{array}{ccc} H^1(A_X) & \xrightarrow{v^*} & H^1(A_Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ H^1(X) & \xrightarrow{f^*} & H^1(Y) \end{array}$$

(ii) Let P be the Picard variety of X and D a Poincaré divisor on $X \times P$. Then D induces a canonical map $\beta: H^{2n-1}(X) \rightarrow H^1(P)$, where $n = \dim(X)$, which is functorial in the following sense: If $f: Y \rightarrow X$ is a morphism and $u: P_X \rightarrow P_Y$ the corresponding morphism of Picard varieties, then the following diagram is commutative:

$$\begin{array}{ccc}
 H^{2n-1}(X) & \xrightarrow{f^*} & H^{2m-1}(Y) \\
 \beta_X \downarrow & & \downarrow \beta_Y \\
 H^1(P_X) & \xrightarrow{u^*} & H^1(P_Y)
 \end{array}$$

where $n = \dim X$, $m = \dim Y$.

Indeed, the graphs of the various canonical morphisms are all algebraically equivalent and the $(1,1)$ -Künneth components of the various Poincaré divisors are all the same; hence, α and β are canonically determined. The diagram in (i) is commutative because $v \circ \kappa_Y = \kappa_X$ or by definition of v . The diagram in (ii) is commutative because, by definition of u , the divisor $E = (f \times 1)^* D_X$ on $Y \times P_X$ is equal to $(1 \times u)^* D_Y$, where D_X and D_Y are the Poincaré divisors; hence, by 1.3.4,

$$\beta_X \circ f_* = u^* \circ \beta_Y : H^{2m-1}(Y) \rightarrow H^1(P_X).$$

2A2. Proposition. Let X and Y be varieties, E a divisor on $X \times Y$, $\phi : A_X \rightarrow P_Y$ the corresponding morphism from the Albanese variety of X to the Picard variety of Y . Then the following diagram is commutative:

$$\begin{array}{ccc}
 H^1(X) & \xleftarrow{\eta} & H^{2n-1}(Y) \\
 \alpha_X \downarrow & & \downarrow \beta_Y \\
 H^1(A_X) & \xleftarrow{\phi^*} & H^1(P_Y)
 \end{array}$$

where η is the map induced by tE .

Indeed, if D_Y is a Poincaré divisor on $X \times P_Y$ and $\kappa_X : X \rightarrow A_X$ is a canonical morphism, then ${}^tE = (1 \times \kappa_X)^* (1 \times \phi)^* D_Y$ by definition of ϕ ; hence, the diagram is commutative by 1.3.4.

2A3. Lemma. Let X be an abelian variety and $s_m : X \times \dots \times X \rightarrow X$ the m -fold sum map. Then the map $s_m^* : H^*(X) \rightarrow H^*(X \times \dots \times X)$ is given by

$$s_m^*(a) = a \otimes 1 \otimes \dots \otimes 1 + 1 \otimes a \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes a + \sum b_j \otimes c_j \otimes \dots \otimes x_j,$$

where $b_j, c_j, \dots, x_j \in \bigoplus_{i>0} H^i(X)$. Moreover, the map

$$(m\delta_X)^* : H^1(X) \rightarrow H^1(X)$$

is multiplication by m .

Indeed, the formula for $s_m^*(a)$ follows easily when the m maps $X \rightarrow X \times \dots \times X$ defined by $x \mapsto (\dots, x, \dots)$ are considered. Moreover, since $m\delta = \Delta_m \circ s_m$ where $\Delta_m : X \rightarrow X \times \dots \times X$ is the m -fold diagonal map, $(m\delta)^* : H^1(X) \rightarrow H^1(X)$ is multiplication by m .

2A4. Lemma (Hopf's theorem). Let $H^* = \bigoplus_{i=0}^m H^i$ be a graded anticommutative algebra over a field K . Suppose that $H^m \simeq K$ and that there is a homomorphism s of H^* into the m -fold tensor product of H^* such that

$$s(a) = a \otimes a \otimes \dots \otimes 1 + 1 \otimes a \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes a + \sum b_j \otimes c_j \otimes \dots \otimes x_j,$$

where $b_j, c_j, \dots, x_j \in \bigoplus_{i>0} H^i$. Then $\dim H^1 \leq m$ and the equality holds if and only if H^* is isomorphic to the exterior algebra $\Lambda^* H^1$.

Indeed, let $a_1, \dots, a_m \in H^1$. Then

$$s(a_1 \dots a_m) = \dots + \sum a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(m)} + \dots,$$

where σ runs through the set of permutations of $1, \dots, m$. Suppose now the a_i are linearly independent. Then $s(a_1 \dots a_m) \neq 0$, so $a_1 \dots a_m \neq 0$.

Let G be the subspace of H^1 generated by a_1, \dots, a_m and consider the natural homomorphism

$$\phi: \Lambda^* G \rightarrow H^*.$$

Let $b \in \Lambda^i G$. If $b \neq 0$, then there exists $c \in \Lambda^{m-i} G$ such that $b \wedge c = a_1 \wedge \dots \wedge a_m$. So, $\phi(b) \cdot \phi(c) = a_1 \dots a_m \neq 0$, and ϕ is injective. Let $b \in H^i$, $i > 0$. Then $(a_1 \dots a_m) \cdot b = 0$; so, applying s and using the hypothesis $H^m \simeq K$, we find $0 = a_1 \dots a_m \otimes (b - c) \otimes 1 \otimes \dots \otimes 1 + \dots$ where $c \in \phi(\Lambda^i G)$. Thus, ϕ is surjective and the proof is complete.

2A5. Lemma. Let X be a curve of genus g and J the Jacobian of X . Then $\dim(H^1(J)) = \dim(H^1(X)) = 2g$ and α and β are inverse isomorphisms.

Indeed, applying 2A2 to the diagonal Δ on $X \times X$, we find $\alpha \circ \beta = \text{id}$. By the fixed-point formula 1.3.6, $2 \cdot \dim(H^1(X)) = \langle \Delta^2 \rangle$. However, the intersection class of Δ^2 , considered on Δ , is the negative canonical class; so, $\langle \Delta^2 \rangle = 2 - 2g$. Thus, $2g = \dim(H^1(X)) \leq \dim(H^1(J))$. However, by 2A3 and 2A4, $\dim(H^1(J)) \leq 2 \dim(J) = 2g$. Therefore, $\dim(H^1(X)) = \dim(H^1(J))$; whence, α and β are inverse isomorphisms.

2A6. Lemma. If $\Psi: P \rightarrow A$ is an isogeny of abelian varieties, then the map $\Psi^*: H^*(A) \rightarrow H^*(P)$ is an isomorphism.

Indeed, Ψ is surjective; so, by 1.2.4, Ψ^* is injective. However, there exist isogenies $\phi: A \rightarrow P$ and similarly $\phi^*: H^*(P) \rightarrow H^*(A)$ is injective. It follows that Ψ^* is an isomorphism.

2A7. Lemma. Let Y be a smooth one-dimensional section of X by a linear space. Let P be the Picard variety of X and J the Jacobian of Y . Then the natural homomorphism $u: P \rightarrow J$ has finite kernel and $\dim H^1_{\text{ét}}(X, \mathbb{Q}_\ell) = 2 \dim(P)$, where ℓ is an integer prime to the characteristic of the ground field k .

Indeed, consider the long exact sequence of Kummer theory,

$$H^0_{\text{ét}}(X, G_m) \xrightarrow{x \mapsto x^\ell} H^0_{\text{ét}}(X, G_m) \rightarrow H^1_{\text{ét}}(X, \mu_\ell) \rightarrow H^1_{\text{ét}}(X, G_m) \xrightarrow{x \mapsto x^\ell} H^1_{\text{ét}}(X, G_m).$$

Since X is complete, $H^0_{\text{ét}}(X, G_m) = k^*$; so, the first map is surjective. By the Hilbert theorem 90, $H^1_{\text{ét}}(X, G_m) = \text{Pic}_{X/k}(k) = P(k)$. Thus, $H^1_{\text{ét}}(X, \mu_\ell)$ is isomorphic to the points of order ℓ on P . Therefore, by Weil's theorem, $\varprojlim_{\ell} H^1_{\text{ét}}(X, \mu_\ell)$ is a free \mathbb{Z}_ℓ -module of rank $2 \cdot \dim(P)$.

To prove the kernel N of $u: P \rightarrow J$ is finite, it suffices to show that N does not contain any subgroup of order ℓ . Applying Kummer theory, we are thus reduced to showing that the map $H^1_{\text{ét}}(X, \mu_\ell) \rightarrow H^1_{\text{ét}}(Y, \mu_\ell)$ is injective. Since, by definition, these groups classify the étale coverings of X and Y with group μ_ℓ , we only have to show that any connected étale covering X' of X has a restriction $Y' = X' \times_X Y$ which is also connected. Since Y is a linear space section of X , some multiple of Y' , considered as a cycle on X' , is a linear space section of X' . Therefore, by Bertini's theorem, Y' is a specialization of an irreducible, linear space section. So, By Zariski's connectedness theorem, Y' is connected and the proof is complete.

2A8. Theorem. If X is an abelian variety of dimension n , then $\dim(H^1(X)) = 2n$ and the cup-product algebra $H^*(X)$ is isomorphic to the exterior algebra $\Lambda^* H^1(X)$.

Indeed, we may assume $n \geq 1$. Let Y be a smooth one-dimensional section of X by a linear space and J be the Jacobian of Y . By 2A7, the natural homomorphism $u: \tilde{X} = P_X \rightarrow J$ has finite kernel; so, by Poincaré's lemma of complete reducibility, J is isogenous to a product of X with another abelian variety X' . Then, $\dim(J) = \dim(X) + \dim(X')$ and by 2A6, $\dim(H^1(J)) = \dim(H^1(X)) + \dim(H^1(X'))$. However, by 2A5, $\dim(H^1(J)) = 2 \dim(J)$ and, by 2A4 and 2A3, $\dim(H^1(X)) \leq 2 \dim(X)$ and $\dim(H^1(X')) = 2 \dim(X')$. Thus, the equalities hold; so, by 2A4, $H^*(X) \simeq \Lambda^* H^1(X)$ and the proof is complete. (If it is known a priori that $\dim(H^1(X)) = 2n$, then the second assertion results immediately from 2A3 and Hopf's theorem 2A4.)

2A9. Theorem. Let $\alpha: H^1(A) \rightarrow H^1(X)$ and $\beta: H^{2n-1}(X) \rightarrow H^1(P)$ be the canonical maps (2A1). Then:

1. In general, α is an injection and β is a surjection.
2. If X is a curve, then α and β are inverse isomorphisms.
3. If X is an abelian variety, then α is the identity and β is an isomorphism.
4. If there exists an isomorphism $\nu^1: H^{2n-1}(X) \simeq H^1(X)$ which is algebraic, (e.g., if X satisfies B(X)), then α and β are isomorphisms.
5. There exists an isogeny $\Psi: P \rightarrow A$ such that the diagram

$$\begin{array}{ccc} H^1(X) & \xrightarrow{L^{n-1}} & H^{2n-1}(X) \\ \alpha \uparrow & & \downarrow \beta \\ H^1(A) & \xrightarrow{\Psi^*} & H^1(P) \end{array}$$

is commutative. Suppose α and β are isomorphisms. Then L^{n-1} is an isomorphism, and if E is a divisor on $X \times X$ defining an isogeny $\phi: A \rightarrow P$ such that $\Psi \circ \phi = m\delta_A$, then E induces the m^{th} -multiple of the map $\theta^1: H^{2n-1}(X) \rightarrow H^1(X)$ inverse to L^{n-1} .

6. If $H^1(X) = H^1_{\text{ét}}(X, Q_\ell)$, then α and β are isomorphisms.

Indeed, 2 is simply 2A5. If 5 holds, then Ψ^* is an isomorphism by 2A6; whence, 1 holds, and since $\dim(H^1(X)) = \dim(H^{2n-1}(X))$ by Poincaré duality, 3 holds. Furthermore, 4 follows from 1 and 2A2; 6 follows from 1, 2A7 and the equalities $\dim(H^1(X)) = \dim(H^{2n-1}(X))$ and $\dim(H^1(A)) = \dim(H^1(P))$.

To prove 5, let Y be a smooth one-dimensional section of X by a linear space, $f: Y \rightarrow X$ the inclusion, J the Jacobian of Y , $u: P \rightarrow J$ the natural homomorphism and $t_u: J \rightarrow A$ the transpose of u . By 2A1, the two diagrams

$$\begin{array}{ccc} H^1(X) & \xrightarrow{f^*} & H^1(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ H^1(A) & \xrightarrow{t_u^*} & H^1(J) \end{array} \quad \begin{array}{ccc} H^1(Y) & \xrightarrow{f^*} & H^{2n-1}(X) \\ \beta_Y \downarrow & & \downarrow \beta_X \\ H^1(J) & \xrightarrow{u^*} & H^1(P) \end{array}$$

are commutative. Since $f_* f^* = L^{n-1}$ by the projection formula and $\beta_Y \circ \alpha_Y = \text{id}$ by 2A5, it follows that the diagram in 5 is commutative with $\Psi = t_u \circ u$. Further, Ψ is an isogeny because the autoduality of J is given by $t \mapsto \theta_t - \theta$ where θ is the theta divisor on J ; so, since θ is ample, the restriction of the autoduality to any abelian subvariety B of J induces an isogeny of B onto its dual. Finally, the last assertion in 5 follows from 2A6, 2A2 and 2A3.

2A10. Corollary. If X is a surface such that $\dim(H^1(X)) = 2 \dim(P)$ where P is the Picard variety of X , (e.g., $H^1(X) = H^1_{\text{ét}}(X, Q_\ell)$), then the strong Lefschetz "theorem" and condi-

tion $B(X)$ both hold. In fact, Λ_X is the class of an algebraic cycle which does not depend on the choice of cohomology theory.

Indeed, the isomorphism $\theta^i: H^{2n-i}(X) \rightarrow H^i(X)$ inverse to L^{n-i} is induced by an algebraic cycle which depends only on X ; θ^0 and θ^2 exist trivially, and θ^1 exists by virtue of part 5 of the theorem. The proof of 2.3 now shows that Λ_X is induced by an algebraic cycle depending on X .

2A11. Theorem (Lieberman). Let X be an abelian variety and Y an arbitrary variety. Then:

1. If $a \in H^*(X \times Y)$ is the class of a (given) algebraic cycle, then the Künneth components $a_{pq} \in H^p(X) \otimes H^q(Y)$ are all rationally algebraic.

2. The strong Lefschetz "theorem" for X and condition $B(X)$ both hold.

In fact, the a_{pq} and Λ_X are the classes of algebraic cycles which do not depend on the choice of cohomology theory.

Indeed, to prove 1, for each integer $m \geq 0$, consider the map

$$f_m = (m\delta_X) \times (\text{id}_Y) : X \times Y \rightarrow X \times Y.$$

Since, by 2A8, $H^*(X) \simeq \Lambda^* H^1(X)$, and by 2A3, $m\delta_X$ induces multiplication by m on $H^1(X)$, we find that, if $a \in H^r(X \times Y)$ has Künneth components a_{pq} , then

$$f_m^*(a) = \sum_{p+q=r} m^p a_{pq}.$$

Therefore, if we take $r+1$ different values for m , we can express the a_{pq} as rational linear combinations of the $f_m^*(a)$.

The strong Lefschetz theorem results easily from the observation that if $y \in H^2(X) = \Lambda^2 H^1(X)$ is the class of a hyperplane section, then, because $y^n \neq 0$, there exists a basis $e_1, \dots, e_n, t_1, \dots, t_n$ of $H^1(X)$ such that $y = \sum e_i \wedge t_i$; in fact, the basis comes from diagonalizing the non-singular, skew symmetric bilinear form on $H_1(X)$ defined by y .

By 1, the π^i are the classes of cycles depending only on X . So, by the proof of 2.9, to prove Λ is the class of a cycle depending only on X , it suffices to prove that there exists an isomorphism $\nu^i: H^{2n-i}(X) \rightarrow H^i(X)$ for $i \leq n-1$ induced by a cycle depending only on X . By 2A8, $\Lambda^* H^1(X)$ is isomorphic to $H^*(X)$ under cup product, and so by 2A3, $\Lambda^* H^{2n-1}(X)$ is isomorphic to $H^*(X)$ under Pontrjagin product ($\text{avb} = (s_2)_*(a \otimes b)$). By 2A9, an algebraic isomorphism $\nu^1: H^{2n-1}(X) \rightarrow H^1(X)$ exists, defined by a divisor E on $X \times X$ which depends only on X ; we may assume $\gamma_X(E)$ has Künneth type $(1,1)$. It now results from the following lemma that the map $\nu^i: H^{2n-i}(X) \rightarrow H^i(X)$ defined by E^i is again an isomorphism; hence, $B(X)$ holds and the proof of 2A11 is complete.

2A12. Lemma. Let X be an abelian variety, Y an arbitrary variety and $u \in H^2(X \times Y)$ an element of Künneth type $(1,1)$. Then the correspondence

$$\exp(u) = \sum u^i/i! : H^*(X) \rightarrow H^*(Y)$$

takes Pontrjagin product ($\text{avb} = (s_2)_*(a \otimes b)$) into cup product.

Indeed, consider the diagram

$$\begin{array}{ccc} H^*(X \times X) & \xrightarrow{\exp(u) \otimes \exp(u)} & H^*(Y \times Y) \\ (s_2)_* \downarrow & \searrow \exp(u) & \downarrow \Delta^* \\ H^*(X) & \xrightarrow{\exp(u)} & H^*(Y) \end{array}$$

By 1.3, it suffices to prove that the cycles $((s_2)^* \otimes 1)(\exp(u))$ and $(1 \otimes \Delta^*)(\exp(u) \otimes \exp(u)) = (1 \otimes \Delta^*)(\exp(u \otimes 1 + 1 \otimes u))$ are equal in $H^*(X \times X \times Y)$. Since $(s_2)^*$ and Δ^* are algebra homomorphisms, it suffices to prove $((s_2)^* \otimes 1)(u) = (1 \otimes \Delta^*)(u \otimes 1 + 1 \otimes u)$ with $u = a \otimes b$, $a \in H^1(X)$, $b \in H^1(Y)$. However,

$$((s_2)^* \otimes 1)(a \otimes b) = a \otimes 1 \otimes b + 1 \otimes a \otimes b = (1 \otimes \Delta^*)(a \otimes 1 \otimes b \otimes 1 + 1 \otimes a \otimes 1 \otimes b);$$

so, the proof is complete.

2A13. Remark. (Lieberman) Let X be an abelian variety, $\hat{X} = P_X$ the dual abelian variety and $u \in H^2(X \times \hat{X})$ the $(1,1)$ -Künneth component of a Poincaré divisor. It results from 2A8, 2A9 and 2A12 that the correspondence

$$\exp(u) : H^*(X) \rightarrow H^*(\hat{X})$$

is an algebraic isomorphism of the Pontrjagin algebra $H^*(X)$ with the cup-product algebra $H^*(\hat{X})$. It can be shown that if $\psi : X \rightarrow \hat{X}$ is the isogeny $x \mapsto Y_x - Y$ where Y is a smooth hyperplane section of X , then the map $*$ is $\psi^* \circ \exp(u)$ is the usual $*$ -map of the exterior algebra $\Lambda^* H^1(X)$ associated to the quadratic form on $H_1(X)$ defined by $y = \gamma_X(Y) \in \Lambda^2 H^1(X)$. Consequently, ${}^c \Lambda a = t (a \vee y^{n-1})$, where $a \in H^*(X)$ and $t \in \mathbb{Q}$ depends only on L , and if X is a Jacobian, then Poincaré's celebrated relation $(n-r)! \gamma_X(W_r) = \gamma_X(\theta^{n-r})$ holds.

3. CONJECTURES OF HODGE TYPE

In brief summary of classical Hodge theory, let X be a complex variety. Then there exists a functorial, direct sum decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X),$$

where

$$H^{p,q}(X) = H^q(X, \Omega_{X/\mathbb{C}}^p).$$

Furthermore, for $p \leq n = \dim X$, the cycle map γ_X takes $C^p(X)$ into $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Z})$; this observation led Hodge to make the following conjecture.
Hodge $P(X) : A^p(X) = H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$.

3.1. Proposition. Suppose $2p \leq n$ and Hodge $P(X)$ is true. If $q \leq p$, then Hodge $n-q(X)$ is true, and for any polarization of X , the map

$$L^{n-p-q} : A^p(X) \rightarrow A^{n-q}(X)$$

is a surjection and the map

$$L^{n-2p} : A^p(X) \rightarrow A^{n-p}(X)$$

is an isomorphism. Moreover, the map

$$L^{n-2q} : A^q(X) \rightarrow A^{n-q}(X)$$

is an isomorphism if and only if Hodge $q(X)$ holds.

Indeed, it is an immediate consequence of Lefschetz theory that the map

$$L^{n-p-q} : H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q}) \rightarrow H^{n-q, n-q}(X) \cap H^{2n-2q}(X, \mathbb{Q})$$

is a surjection; whence, the assertions.

3.2. Corollary. Suppose $A(X, L)$ holds. If Hodge $P(X)$ is true for $2p \leq n$, then Hodge $Q(X)$ is true for $q = 0, \dots, p, n-p, \dots, n$.

3.3. Corollary. If Hodge $P(X)$ is true for all p and X , then the conjectures of Lefschetz type $A(C)$, $B(C)$ and $C(C)$ (see 2.14) are all true.

3.4. Remark. Hodge $P(X)$ is always true for $p = 0, 1, n-1, n$; consequently, if $n = \dim X \leq 4$, then $A(X, L)$ holds. Indeed, the cases $p = 0, n$ are trivial. By 3.1, it suffices to prove Hodge $1(X)$; let us sketch the proof of Kodaira-Spencer [5]. It suffices to show that γ_X factors as follows:

$$C^1(X) \xrightarrow{\alpha} \text{Pic}(X) \xrightarrow{\beta} H^{1,1}(X) \cap H^2(X, \mathbb{Z}),$$

where α and β are surjective. For α , we take the boundary of the exact sequence

$$0 \rightarrow O_X^* \rightarrow K_X^* \rightarrow K_X^*/O_X^* \rightarrow 0,$$

where K_X is the sheaf of meromorphic functions; for β , we take the boundary of the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow O_{X^h} \xrightarrow{\exp} O_{X^h}^* \rightarrow 0,$$

using GAGA's [10] isomorphism $\text{Pic}(X) \simeq \text{Pic}(X^h)$; a direct computation shows that the image of this boundary is precisely $H^{1,1}(X^h) \cap H^2(X^h, \mathbb{Z})$. It remains to prove that $\gamma_X = \beta \circ \alpha$. Using additivity and functoriality, we reduce to the special case $X = \mathbb{P}^N$; then, by induction, to the case $X = \mathbb{P}^1$, where the assertion is obvious.

Returning to the case of arbitrary characteristic, fix a Weil cohomology $X \mapsto H^*(X)$.

3.5. Theorem. The group $CP_{\text{num}}(X)$ of algebraic cycles of codimension p on X modulo numerical equivalence is a free group of finite rank $\leq b_{2p} = \dim(H^{2p}(X))$.

Indeed, the assertion follows easily from the existence of a Weil cohomology, e.g., étale cohomology. For, let $a_1, \dots, a_m \in C^{n-p}_{\text{hom}}(X)$ (where $n = \dim(X)$) be a base of the vector space generated over the coefficient field by the algebraic classes. Consider the homomorphism

$$\alpha : C^p_{\text{hom}}(X) \rightarrow \mathbb{Z}^m$$

defined by $\alpha(b) = (\langle b, a_1 \rangle, \dots, \langle b, a_m \rangle)$. It is easily seen that the image of α , a free group of finite rank, is precisely $CP_{\text{num}}(X)$.

3.6. Proposition. The following conditions on X are equivalent, and if the strong Lefschetz "theorem" holds for X , they imply $A(X, L)$ for any polarization L .

(i) $D(X)$: Homological equivalence of algebraic cycles on X is equal to numerical equivalence.

(ii) For all $p \leq n = \dim(X)$, the canonical pairing

$$A^p(X) \otimes Q A^{n-p}(X) \rightarrow Q$$

is non-singular.

(iii) For all $p \leq n = \dim(X)$, the cycle map γ_X induces a map, necessarily injective,

$$C_{\text{num}}^p(X) \otimes_{\mathbb{Z}} K \rightarrow H^{2p}(X),$$

where K is the coefficient field of the cohomology theory.

Indeed, the equivalence of (i), (ii) and (iii) is immediate from the definitions. If these conditions are satisfied, then $\dim_{\mathbb{Q}} A^p(X) < \infty$; hence, for $2p \leq n$, the map $L^{n-2p}: A^p(X) \rightarrow A^{n-p}(X)$, by hypothesis injective, is bijective; i.e., $A(X, L)$ holds.

For $2p \leq n = \dim(X)$, set

$$P_{\text{alg}}^p(X) = P^{2p}(X) \cap A^p(X) = \{a \in A^p(X) \mid L^{n-2p+1} a = 0\}$$

and consider the following conditions.

$IP(X, L)$: The quadratic form on $P_{\text{alg}}^p(X)$

$$a, b \mapsto (-1)^p \langle L^{n-2p} a, b \rangle$$

is positive definite.

$I(X, L)$: Condition $IP(X, L)$ holds whenever $2p \leq n$.

3.7. Proposition. Assume the weak Lefschetz theorem is universally valid (e.g., $H^*(X) = H_{\text{ét}}^*(X, \mathbb{Q}_\ell)$). Fix p and suppose that, for all varieties X of dimension $2p$, the quadratic form $a, b \mapsto (-1)^p \langle a, b \rangle$ on $P_{\text{alg}}^p(X)$ is positive definite. Then $IP(X, L)$ always holds.

Indeed, simply apply the hypotheses to a smooth p -dimensional section of X by a linear space.

3.8. Proposition. Let $2p \leq n$ and suppose that, for all $q \leq p$, the map $L^{n-2q}: A^q(X) \rightarrow A^{n-q}(X)$ is an isomorphism and that X satisfies $I^q(X, L)$. Then the quadratic form on $A^p(X)$

$$a, b \mapsto \langle a, *b \rangle,$$

where $*$ is the operator defined in 1.4.2.3, is positive definite, and consequently, the canonical pairing $A^p(X) \otimes A^{n-p}(X) \rightarrow \mathbb{Q}$ is non-singular.

Indeed, let $a = \sum L^i a_i$ and $b = \sum L^j b_j$ be the primitive decomposition of $a, b \in A^p(X)$. Then, since $L^{n-2p+i+j} a_i \cdot b_j = 0$ unless $i = j$, we have $\langle a, *b \rangle = \sum (-1)^{p-j} \langle L^{n-2(p-j)} a_j, b_j \rangle$. Thus, the quadratic form is positive definite.

3.9. Corollary. Suppose X satisfies $I(X, L)$ and the strong Lefschetz "theorem". Then $A(X, L)$ and $D(X)$ are equivalent.

3.10. Remark. Let X be a complex variety. Then X satisfies $I(X, L)$ by the Hodge index theorem. Hence, homological equivalence of algebraic 1-cycles on X is equal to numerical equivalence by 3.8, 3.1 and 3.4. Furthermore if X is an abelian variety or has dimension ≤ 4 , then by 2A11 or 3.1 and 3.4, X satisfies $A(X, L)$; hence, by 3.9, homological equivalence on X is always equal to numerical equivalence.

3.11. Theorem. Suppose X, Y satisfy the strong Lefschetz "theorem" and $B(X), B(Y)$. If $u: H^*(X) \rightarrow H^*(Y)$ is a correspondence, let u' denote its transpose with respect to the non-degenerate bilinear forms $\langle a, *x \rangle$ and $\langle c, *y \rangle$; so, $u' = {}^*X^t u {}^*Y$. Let u now be algebraic. Then:

1. u' is algebraic and $\text{Tr}(u' \circ u) \in \mathbb{Q}$.

2. If further $X \times Y$ satisfies $I(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$, then

$$\text{Tr}(u'ou) > 0,$$

when $u \neq 0$.

Indeed, 1 results immediately from 2.3, 1.3.8 and 2.7. To prove 2, note that $B(X)$ implies that the projection operators

$$q_X^{ij}: H^*(X) \rightarrow L^j P^{i-2j}(X),$$

where $j = \max(0, i-n)$ and $n = \dim(X)$, are algebraic by 2.3 and that

$$(q_Y^{rs} u q_X^{ij})' = q_X^{ij} u' q_Y^{rs}$$

by the orthogonality of primitive components. Therefore,

$$\begin{aligned} \text{Tr}(u'ou) &= \sum ((q_Y^{rs} u q_X^{ij})' o (q_Y^{rs} u q_X^{ij})) \\ &= \sum ((q_Y^{rs} u q_X^{ij})' o (q_Y^{rs} u q_X^{ij})), \end{aligned}$$

and we may assume $u = q_Y^{rs} u q_X^{ij}$.

Let $v = \Lambda_Y^s u \Lambda_X^{n-i+j}$. Then $v' = L_X^{n-i+j} u' L_Y^s$; so, $\text{Tr}(v'ov) = \text{Tr}(u'ou)$. Replacing u by v , $i-2j$ by i , and $r-2s$ by j , we may assume $u \in P^i(X) \otimes P^j(Y)$.

By 1.3.6, we now have $\text{Tr}(u'ou) = (-1)^i \langle u, {}^*X \circ u \circ {}^*Y \rangle$; by 1.3.5, ${}^*X \circ u \circ {}^*Y = ({}^*X \otimes {}^*Y)u$. Furthermore, it is easily seen that, if $L_{X \times Y} = L_X \otimes 1 + 1 \otimes L_Y$, then $u \in P^{i+j}(X \times Y)$ and

$$\binom{n-i+m-j}{n-i} [({}^*X \otimes {}^*Y)u] = (-1)^{i(i+1)/2} (-1)^{j(j+1)/2} L_{X \times Y}^{n-i+m-j} u = (-1)^{ij} (X \times Y)u.$$

Since u is algebraic, $i+j$ is even. Therefore, $I(X \times Y, L_{X \times Y})$ implies $(-1)^i \langle u, {}^*X \circ u \circ {}^*Y \rangle > 0$ when $u \neq 0$, and the proof is complete.

3.12. Corollary. Let X satisfy the strong Lefschetz "theorem" and $B(X)$, and let $X \times X$ satisfy $I(X \times X, L_X \otimes 1 + 1 \otimes L_X)$. Then the Q -algebra $\mathcal{A}^*(X)$ of algebraic correspondences is semisimple. In fact, any subalgebra of $\mathcal{A}^*(X)$ which is closed under the involution $u \mapsto u'$ is semisimple.

Indeed, the corollary follows from the theorem and the following lemma.

3.13. Lemma. Let A be a finite dimensional Q -algebra with an involution $u \mapsto u'$, i.e., a Q -linear map $A \rightarrow A$ such that $(uv)' = v'u'$ and $(u')' = u$, and with a trace, i.e., a Q -linear functional $\sigma: A \rightarrow Q$ such that $\sigma(uv) = \sigma(vu)$ and $\sigma(u'u) \neq 0$ when $u \neq 0$. Then A is semisimple.

Indeed, if u were a non-zero element of the radical of A , then $u'u$ would be nilpotent, but $u'u \neq 0$ as $\sigma(u'u) \neq 0$. Say $(u'u)^{2m} = 0$, but $v = (u'u)^{2m-1} \neq 0$. Then $\sigma(v'v) = \sigma(0) = 0$, a contradiction.

3.14. Corollary. Let X, Y be varieties satisfying the strong Lefschetz "theorem" and $B(X), B(Y)$. Let

$$u: H^i(X) \rightarrow H^j(Y)$$

be an algebraic correspondence.

1. If $X \times Y$ satisfies $I(Y \times Y, L_X \otimes 1 + 1 \otimes L_Y)$ and u is an injection, then u has a left

inverse $v: H^j(Y) \rightarrow H^i(X)$ which is algebraic. Consequently, if $a \in H^i(X)$ is such that $u(a)$ is algebraic, then a is algebraic.

2. If $X \times X$ satisfies $I(X \times X, L_X \otimes 1 + 1 \otimes L_X)$ and u is a surjection, then u has a right inverse $v: H^j(Y) \rightarrow H^i(X)$ which is algebraic. Consequently, if $b \in H^j(Y)$ is algebraic, then there exists an algebraic $a \in H^i(X)$ such that $b = u(a)$.

Indeed, to prove 1, let $y = u'ou$ and $x = uou'$. Then $x' = x$; hence, x is semisimple by 3.12 and so, $\text{Ker}(x) = \text{Ker}(x^2) = \text{Ker}(uyu')$. Since u is injective, u' is surjective; it follows that $y: H^i(X) \rightarrow H^i(X)$ is injective, and so is an automorphism. Hence, by 2.7 (iii) and the Cayley-Hamilton theorem, y^{-1} is algebraic. Therefore, $v = y^{-1}u'$ is a left inverse of u , which is algebraic. The proof of 2 is similar.

3.15. Lemma. Let $E^* = \bigoplus_{\nu=-n}^n E^\nu$ be a graded, non-commutative ring with 1 and $\pi^0, \dots, \pi^{2n} \in E^*$ elements which satisfy the following five conditions:

- (i) $(\pi^i)^2 = \pi^i$ for $i = 0, \dots, 2n$.
- (ii) $\pi^i \pi^j = 0$ if $i \neq j$ for $i, j = 0, \dots, 2n$.
- (iii) $\sum_{i=0}^{2n} \pi^i = 1$.
- (iv) $E^\nu = \bigoplus_i \pi^{i+\nu} E^* \pi^i$.
- (v) For $i = 0, \dots, n$, there exist elements $v^i \in E^{2n-2i}$ and $w^i \in E^{-(2n-2i)}$ such that $(w^i v^i - 1) \pi^i = 0$ and $(v^i w^i - 1) \pi^{2n-i} = 0$.

Then the π^i are uniquely determined by these conditions.

Indeed, by (iii) and (iv), $u \in E^0$ if and only if $u = \sum \pi^i u \pi^i$; whence, by (i), (ii) and (iii), if and only if $\pi^i u = \pi^i u \pi^i = u \pi^i$ for all i . In particular, the π^i are in the center $Z(E^0)$ of E^0 .

Proceeding by induction on $i \leq n$, suppose $\pi^0, \dots, \pi^{i-1}, \pi^{2n-i+1}, \dots, \pi^{2n}$ are uniquely determined. Let $\varphi^i = 1 - \sum_{\alpha \notin [i, 2n-i]} \pi^\alpha$. Then, by (iii), $\varphi^i = \sum_{\alpha=i}^{2n-i} \pi^\alpha$; so, by (i),

(ii) and (iv), $\varphi^i E^{2n-2i} \varphi^i = E^{2n-2i} \pi^i$. If $E^{2n-2i} \pi^i u = 0$, then $w^i v^i \pi^i = 0$; hence, by (v), $\pi^i u = 0$. Therefore, the right annihilator of π^i is uniquely determined. However, $\pi^i \in Z(E^0)$ and an idempotent of a commutative ring is completely determined by its annihilator. Similarly, π^{2n-i} is uniquely determined.

3.16. Theorem. Suppose both Lefschetz "theorems" are universally valid. Then the following two conditions are equivalent:

- (i) The standard conjectures hold; i.e., $B(X)$ (or, equivalently, $A(X)$) and $I(X, L)$ are satisfied by all varieties X over k .
- (ii) For all varieties X over k and all integers p such that $2p \leq n = \dim(X)$, $D(X)$ holds and the quadratic form $a, b \mapsto (-1)^p \langle L^{n-2p} ab \rangle$ is positive definite on the set of elements $a \in C^p_{\text{num}}(X)$ such that $L^{n-2p+1} a = 0$.

Moreover, if these conditions hold for several cohomology theories, then:

1. The operators $\Lambda, {}^c\Lambda, *, p^0, \dots, p^{2n}, \pi^0, \dots, \pi^{2n}$ are the classes of algebraic cycles which do not depend on the theory. In fact, given L , these cycles are determined modulo numerical equivalence by certain intrinsic properties.

2. The Betti numbers $b_i = \dim(H^i(X))$ do not depend on the theory.

3. The characteristic polynomial of an endomorphism induced by a rationally (resp. integrally) algebraic cycle has rational (resp. integer) coefficients which do not depend on the theory.

4. If an algebraic cycle induces a map $H^i(X) \rightarrow H^j(Y)$ which is bijective (resp. injective, resp. surjective), then, in any other theory $X \mapsto H^*(X)$, the cycle induces a map $H^i(X) \rightarrow H^j(Y)$ which is bijective (resp. injective, resp. surjective). In fact, the inverse (resp. one left inverse, resp. one right inverse) may be induced by an algebraic cycle which does not depend on the theory.

Indeed, the equivalence of (i) and (ii) results immediately from 3.9 (and 2.14). If these

conditions hold, then π^0, \dots, π^{2n} are the classes of algebraic cycles by 2.4. By 3.15 applied to the ring of algebraic correspondences, these cycles are uniquely determined modulo homological or, what is the same, numerical equivalence by 3.15 (i)-(v). By 2.3, $c\Lambda$ is the class of an algebraic cycle, which, therefore, is uniquely determined modulo numerical equivalence by 1.4.6.1. Finally, p^n, \dots, p^{2n} (resp. $\Lambda, *, p^0, \dots, p^{n-1}$) are given by universal (non-commutative) polynomials with rational coefficients in L and $c\Lambda$ by 1.4.3 (ii) and 1.4.5 (resp. in L and p^n, \dots, p^{2n} by 1.4.3 (iv)). Thus, 1 holds.

By 1, the π^i are intrinsically determined. Therefore, 2 results from the formula $b_i = (-1)^i \langle \Delta, \pi^{2n-i} \rangle$ of 1.3 (i), and 3 results from the proof of 2.7. Further, a correspondence $u: H^*(X) \rightarrow H^*(Y)$ induces a map $u': H^i(X) \rightarrow H^i(Y)$ if and only if $\pi_Y^\ell u \pi_X^i = 0$ for $\ell \neq j$, and u' is injective (resp. surjective, resp. bijective) if and only if there exists a correspondence $v: H^*(Y) \rightarrow H^*(X)$ such that $v u \pi_X^i = \pi_X^i$ (resp. ...); hence, 4 results from 3.14.

4. FORMALISM OF THE WEIL CONJECTURES

Let the ground field k have characteristic $p > 0$ and let X be a k -variety of dimension n which is defined over the finite field with q -elements F_q ; i.e.,

$$X = X' \otimes_{F_q} k,$$

where X' is an F_q -scheme. The zeta function $Z(t)$ of X is defined by the formula

$$\log Z(t) = \sum_{s=1}^{\infty} N_s t^s / s,$$

where N_s is the number of points of X rational over F_{q^s} . We propose to study $Z(t)$ formally, as suggested by Weil, by interpreting $Z(t)$ in terms of the representation of the Frobenius endomorphism F of X on the groups of a Weil cohomology $X \mapsto H^*(X)$.

The Frobenius endomorphism F of X is defined [4] as the base extension $F' \otimes_{F_q} k$ of the endomorphism F' of X' which is the identity map on the topological space and the q^{th} -power homomorphism on the structure sheaf. So, if $x \in X(k)$, x is rational over F_{q^s} if and only if $F^s(x) = x$. Therefore, if we let f^s be the graph of F^s , then, since f^s meets the diagonal Δ transversally, we have

$$N_s = \langle f^s, \Delta \rangle.$$

Let $X \mapsto H^*(X)$ be a Weil cohomology. It follows from the definition that F is a finite morphism; so, by 1.2.4, F^s induces an automorphism

$$f_1^s: H^i(X) \rightarrow H^i(X).$$

By the Lefschetz fixed-point formula 1.3.6., we have

$$\begin{aligned} \log Z(t) &= \sum_{i=0}^{2n} (-1)^i \operatorname{Tr} \left(\sum_{s=1}^{\infty} f_1^s t^s / s \right) \\ &= \sum_{i=0}^{2n} (-1)^i \operatorname{Tr} \log (1 / (1 - f_1 t)) \end{aligned}$$

$$= \sum_{i=0}^{2n} (-1)^{i+1} \log \text{Det} (1 - f_i t).$$

Taking antilogarithms, we obtain the first part of the following theorem; the second part follows from 2.6.

4.1. Theorem. The zeta function $Z(t)$ of X is a rational function of t ; more precisely, $Z(t)$ has the form

$$Z(t) = \frac{P_1(t) \dots P_{2n-1}(t)}{P_0(t) P_2(t) \dots P_{2n}(t)}$$

where $P_i(t)$ is the characteristic polynomial $\text{Det}(1 - f_i t)$ of the automorphism f_i of $H^i(X)$ induced by the Frobenius endomorphism of X . If, further, X satisfies the conjecture $C(X)$ of Lefschetz type, then the $P_i(t)$ have integer coefficients.

To derive the functional equation of the zeta function, we first prove two lemmas.

4.2. Lemma. Let $H^* = \bigoplus_{i=0}^{2n} H^i$ be a graded algebra over a field, which satisfies Poincaré

duality. Let $g = \bigoplus g_i$ be a linear endomorphism of H^* of degree 0 and suppose:

(i) g is multiplicative; i.e., $g(a.b) = g(a).g(b)$ for all $a, b \in H^*$.

(ii) $g_{2n} = \text{id}$.

Then g is an automorphism and $g_i^{-1} = {}^t g_{2n-i}$ for all i .

Indeed, let $a \in H^i$. If $a \neq 0$, then, by duality, there exists $b \in H^{2n-i}$ such that $a.b \neq 0$. Thus, by (i) and (ii), we have $g(a).g(b) = g(a.b) \neq 0$; hence, $g(a) \neq 0$. Therefore, g is injective and, since H^* is finite dimensional, g is an automorphism.

Let $a \in H^i$ and $b \in H^{2n-i}$. Then, by (ii) and (i), we have

$$\langle g_i^{-1}(a).b \rangle = \langle g_{2n}(g_i^{-1}(a).b) \rangle = \langle a.g_{2n-i}(b) \rangle.$$

Therefore, $g_i^{-1} = {}^t g_{2n-i}$ as asserted.

4.3. Lemma. Let k be a perfect field, L a separably generated extension of k of transcendence degree n . Then

$$[L : L^q] = q^n.$$

Indeed, let x_1, \dots, x_n be a separating transcendence base for L over k . Let $L_0 = k(x_1, \dots, x_n)$ and consider the diagram

$$\begin{array}{ccc} & L & \\ L^q & \swarrow \quad \searrow & L_0 \\ & L_0^q & \end{array}$$

The extension L^q/L_0^q is isomorphic to L/L_0 and so is separable. It follows that $[L : L^q] = [L_0 : L_0^q]$. However, it is clear that $[L_0 : L_0^q] = q^n$; so, the proof is complete.

To apply 4.2, define a linear endomorphism $g = \bigoplus g_i$ of degree 0 of $H^*(X)$ by

$$g_i = f_i / (\sqrt{q})^i,$$

where $f = \oplus f_i$ is the algebra endomorphism of $H^*(X)$ induced by the Frobenius endomorphism of X . Here and from now on, we need to assume that the coefficient field of the cohomology theory contains \sqrt{q} ; so, base extend the theory if necessary. It is clear that g is multiplicative and, by 4.3, $g_{2n} = \text{id}$. By 4.2, therefore, $g_i^{-1} = {}^t g_{2n-i}$.

Let α_{ij} be the eigenvalues of f_i . Since the eigenvalues of ${}^t g_{2n-i}$ are the same as those of g_{2n-i} , it follows by equating the eigenvalues of g_i^{-1} and ${}^t g_{2n-i}$ that the sets

$$\left\{ \frac{(\sqrt{q})^i}{\alpha_{ij}} \right\} \quad \text{and} \quad \left\{ \frac{\alpha_{2n-i,j}}{(\sqrt{q})^{2n-i}} \right\}$$

differ only by a change of indexing. Since $Z(t) = \prod_{i,j} (1 - \alpha_{ij} t)^{(-1)^i}$ by 4.1, an explicit computation now yields the following theorem.

4.4. Theorem. The zeta function $Z(t)$ of X satisfies the functional equation

$$Z(1/q^n t) = (-1)^{\chi} q^{n\chi/2} t^{\chi} Z(t),$$

where

$$\chi = \sum_{i=0}^{2n} (-1)^i \dim H^i(X)$$

is the Euler number.

4.5. Remark. By the Lefschetz fixed-point formula,

$$\chi = \langle \Delta^2 \rangle,$$

where Δ is the diagonal class on $X \times X$. Hence, the functional equation, derived with the aid of a Weil cohomology, is independent of cohomology. Furthermore, this derivation does not depend on any unproved conjectures.

4.6. Lemma. Let g be an endomorphism of $H^*(X)$.

1. Suppose: (i) g is of degree 0.

(ii) g is multiplicative.

(iii) $g(y) = y$ where y is the class of a hyperplane section of X .

Then g is an automorphism and $g^{-1} = {}^t g = g'$.

2. Let X satisfy the strong Lefschetz "theorem" and $B(X)$, and let $X \times X$ satisfy $I(X \times X, L_X \otimes 1 + 1 \otimes L_X)$. Suppose that $g \in A^*(X \times X) \otimes Q(\alpha)$ for some $\alpha \in R$ and that $g'og = \text{id}$. Then g is semisimple and its eigenvalues have absolute value 1.

Indeed in 1, $g(y^n) = y^n \neq 0$, so $g H^{2n}(X) = \text{id}$. Hence, by 4.2, g is an automorphism and $g^{-1} = {}^t g$. Consequently, ${}^t g$ satisfies (i), (ii) and (iii). It follows that ${}^t g$ induces an automorphism of $P^i(X)$ and that ${}^t g$ commutes with $*$. Therefore, $g' = {}^* {}^t g^* = {}^t g$.

In 2, by 3.11, the hypotheses imply that the pairing

$$h, h_1 \mapsto \text{Tr}(h' o h_1)$$

is an inner product on the $Q(\alpha)$ -algebra generated by g . Since $g'og = \text{id}$, left translation by g preserves this inner product. It follows that g is semisimple and its eigenvalues have absolute value 1.

To apply 4.6 to the normalized Frobenius $g = \oplus f_i / (\sqrt{q})^i$, we have only to check (iii). Let then Y be a section of X by a hyperplane which is defined over Fq . Then the Frobenius endomorphism of X clearly permutes the irreducible components of Y ; so, since y is

the class of Y , 4.3 applied to these components yields that $g(y) = y$. Therefore, 4.6 applies and, together with 4.1 and 2.4, gives the following theorem.

4.7. Theorem ("Riemann hypothesis"). Let X satisfy the strong Lefschetz "theorem" and the conjecture $B(X)$ of Lefschetz type, and let $X \times X$ satisfy the Hodge index conjecture $I(X \times X, L_X \otimes 1 + 1 \otimes L_X)$. Then the zeta function $Z(t)$ of X has the form

$$Z(t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots P_{2n}(t)}$$

where, for $i = 0, \dots, 2n = 2 \dim(X)$, $P_i(t)$ is a polynomial with integer coefficients and with roots of absolute value $(\sqrt{q})^i$. Furthermore, $P_i(t)$ is the characteristic polynomial of the automorphism f_i of $H^i(X)$ induced by the Frobenius endomorphism of X , and f_i is semisimple.

REFERENCES

- [1] M. Artin, A. Grothendieck, J. L. Verdier, Cohomologie étale des schémas (cité SGA4), to appear North-Holland Pub. Cie.
- [2] C. Chevalley, Les classes d'équivalence rationnelle, I et II, Séminaire C. Chevalley, E. N. S., Paris (1958).
- [3] A. Grothendieck, Sur quelques propriétés fondamentales en theory des intersections, Séminaire C. Chevalley, E. N. S., Paris (1958).
- [4] Houzel, Morphisme de Frobenius et rationalité des fonctions L , in Cohomologie ℓ -adique et fonctions L (cité SGA5 XIII), to appear North-Holland Pub. Cie.
- [5] Kodaira-Spencer, Divisor class groups on algebraic varieties, Proc. N. A. Sci, USA, vol. 39 (1953).
- [6] S. Lefschetz, L'analysis situs et la géométrie algébrique, Gauthiers-Villiers, Paris (1950).
- [7] D. Lieberman, Numerical and homological equivalence of algebraic cycles on Hodge manifolds, to appear.
- [8] D. Lieberman, Higher Picard varieties, to appear.
- [9] P. Samuel, Relations d'équivalence en géométrie algébrique, Proc. Internat. Congress of Math., Cambridge, England (1958).
- [10] J. -P. Serre, Géométrie algébrique et géométrie analytique, Annales de l'Institut Fourier, t. VI (1956).
- [11] J. -P. Serre, Analogues Kähleriens de certaines conjectures de Weil, Annals of Math. vol. 71 no. 2 (1960).
- [12] A. Weil, Variétés Kähleriennes, Alt. Scient. et Ind. 1267, Hermann, Paris.