Letter of A. Grothendieck to J. Coates[(1)]

4.1.1967

Dear Coates,

I want to add a few more comments to the talk on algebraic cycles and to what I told you on the phone.

I think the best will be to state the index conjecture right after the statement of the main results of Hodge theory, adding that this conjecture will take its whole significance only when coupled with "conjecture A" in the next paragraph. This will give more freedom in the next paragraph to express some extra relationships between various conjectures, such as A + index implies B.

In characteristic zero, state some known extra features: index theorem holds, the properties A_ℓ to D_ℓ are independent of ℓ (because of the existence of Betti cohomology, so that these properties are equivalent to the corresponding one's for rational cohomology), A and C are independent of the chosen polarisation x (for A because it is equivalent with B, for C because it can be expressed in terms of A, $C(X) = (A(X \times X) + A(Y \times Y) + ...))$

Thus the conditions without ambiguity can be called $A(X)$ to $D(X)$, without subscript ℓ and without indication of polarisation. Say too that it is known that $C(X)$ is of finite dimension over $\mathbf{Q}$, (so that A can also be expressed in terms of an equality of dimensions of C^i and C^{n-1}, which again proves it is independent of ℓ), but that this is not known in characteristic $p > 0$. Contrarily to what I hastily stated in my talk (influenced from my recollections of the characteristic 0 case) it is not clear to me if in characteristic $p > 0$ the conditions $A_\ell(X, \xi)$ and $C_\ell(X, \xi)$ are independent of the polarisation ξ; if you do not find some proof of this independence, then the possible dependence should be pointed out, as well as the fact that we do not have a proof that A to D are independent of ℓ. Of course, if the index theorem is proved for X, then $A_\ell(X, \xi) = B_\ell(X)$ is again independent of the polarisation, and analogous remark for $C_\ell(X, \xi)$.

1. This text had been transcribed by Mateo Carmona

When speaking about condition $C_\ell(X,\xi)$, emphasise at once its stability properties by products (the proof I suggested works indeed) specialisation (with possible change of characteristics), hyperplane or more generally linear sections. Give an extra proposition for the relations with the property A, via a formal proposition as follows:

Proposition. — *Conditions équivalentes sur X (variété* polarisée*) :*

(i) $C_\ell(X)$

(ii) $C_\ell(Y) et A_\ell(X\times X)$

(ii bis) $C_\ell(Y)$ *et* $A_\ell(X\times X)^\circ$*, où l'exposant* $^\circ$ *signifie qu'on se borne à exprimer la condition A pour l'homomorphisme en dimension critique* $\mathrm{H}^{2n-2}\longrightarrow \mathrm{H}^{2n+2}$.

(iii) $C_\ell(Y)$ *et* $A_\ell(X\times Y)$.

(iii bis) $C_\ell(Y)$ *et* $A_\ell(X\times X)^\circ$*, où l'exposant* $^\circ$ *signifie qu'on se borne à exprimer la condition A en dimension critique* $\mathrm{H}^{(2n-1)-1}\longrightarrow \mathrm{H}^{(2n-1)+1}$.

(iv) $C_\ell(Y)$*, et pour tout* $i\le n-1$*, l'homomorphisme naturel* $\mathrm{H}^i(Y)\longrightarrow \mathrm{H}^i(X)$ *inverse à gauche de* $\varphi^i : \mathrm{H}^i(X)\longrightarrow \mathrm{H}^i(Y)$ *(induit par* $\Lambda_X\varphi_*$*) est induit par une classe de correspondance algébrique (induisant ce qu'elle veut sur les autres* $\mathrm{H}^j(Y)$*).*

(iv bis) $C_\ell(Y)$*, et pour* $j\ge n+1$*, l'homomorphisme naturel* $\mathrm{H}^j(X)\longrightarrow \mathrm{H}^{j-2}(Y)$ *inverse à droite de* $\varphi_{j-2} : \mathrm{H}^{j-2}(Y)\longrightarrow \mathrm{H}^j(X)$ *(induit par* $\varphi^*\Lambda_X$*) est induit par une classe de correspondance algébrique (induisant ce qu'elle veut sur les autres* $\mathrm{H}^i(X)$*).*

Corrollaire. — *Ces conditions équivalent aussi à*

(v) $A_\ell(X\times X)+A_\ell(Y\times Y)^\circ+A_\ell(Z\times Z)+\ldots$*, où* $X\supset Y\supset Z$ *est une suite décroissante de sections hyperplanes.*

(vi) $A_\ell(X\times Y)^\circ+A_\ell(Y\times Z)^\circ+\ldots$*, avec les mêmes notations.*

Of course, the products and hyperplane sections are endowed with the polarisations stemming from the polarisation on X. The conditions (v) and (iv) have the slight interest that they allow to express the conjecture $A(k)=C(k)$ in terms of $A(T)^\circ$ for every T of even (resp. odd dimension), where the upper $^\circ$ means that it is sufficient to look at what happens in critical dimensions.

For thenproof of the proposition, I told you already the equivalence of (i) and (ii), (ii bis). The equivalence of (iv) and (iv bis) is trivial by transposition, they imply (i) because $\mathrm{H}^{2n-i}(X)\longrightarrow \mathrm{H}^i(X)$ is the composition $\mathrm{H}^{2n-i}(X)\longrightarrow \mathrm{H}^{2n-i-2}(Y)\longrightarrow \mathrm{H}^i(Y)\longrightarrow \mathrm{H}^i(X)$ where the extreme arrows are the ones of (iv bis) and (iv) and the middle one is induced by $\Lambda_Y^{(n-1)-i}$, and they are implied by (iii bis) because of the formula

$$(\Lambda_X\varphi_*)L_Y+L_X(\Lambda_X\varphi_*)=(\varphi_*\Lambda_Y\varphi^*+id_X)\varphi_*.$$

On the other hand (iii) $\Rightarrow$ (iii bis) is trivial, and so is (i) $\Rightarrow$ (iii) because of the stabilities. N.B. $(\varphi^*\Lambda_X)L_X + L_Y(\varphi^*\Lambda_X) = \varphi^*(\varphi_*\Lambda_Y\varphi^* + id_X)$.

For the list of the known facts, you can state that:

1) In arbitrary characteristic, $C(X)$ is known if dim $X \leq 2$, because more generally, it is known that in arbitrary dimension n, $\mathrm{H}^{2n-1}(X) \longrightarrow \mathrm{H}^1(X)$ is induced by an algebraic correspondence class; also, in arbitrary dimension, it is known that π_0, π_{2n}, π_1, π_{2n-1} are algebraic (trivial for the first two, not quite trivial for the two next one's). If dim $X = 3$, it is not known however, even in characteristic 0, if $C(X)$ or only $D(X)$ hold, nor $A(X)$ and $B(X)$ in characteristic $p > 0$, also if []

By the way, the fact that the π_1 for a surface are algebraic was pointed out (Tate tells me) by Hodge in Algebraic correspondences between surfaces, Proc, London Math. Soc. Series 2, Vol XLIV, 1938, p. 226. It is rather striking that this statement should not have struck the algebraic geometers more, and has fallen into oblivion for nearly thirty years!

2) In characteristic 0, $A(X)$ is known for dim $X \leq 4$. But $A(X)^\circ$ is not known if dim $X = 5$; the first interesting case would be for a variety $X \times X$, X of dimension 3 and Y a hyperplane section, as this would prove $C(X)$, see above.

Thus the main problems arise already for 1-cycles on threefolds, and partially even in characteristic 0. Urged by Kleiman's question, I will look again at my old scribbles on that subject (when I pretend to reduce the "strong" form of Lefschetz to the "week" one). As for the suggestion I made on the phone, to try to get any X as birationally equivalent to a non singular X', which is a specialisation of a non singular X'', itself birationally equivalent to a non singular hypesurface - this cannot work as Serre pointed out, because such an X would have to be simply connected ! Thus if one wants to reduce somehow to the case of hypersurfaces, one will have to work also with singular ones, and see how to reformulate for singular varieties the standard conjectures...

Sincerely yours