Gr-catégories

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TABLE DE MATIÈRES

INTRODUCTION

Ce travail se compose de trois chapitres

SUMMARY

The purpose of these notes is to study the Gr-categories and give some applications of them. Below is a brief description of the organisation of the work.

Chapter I gives some definitions and results, which are used continually in the sequel, on $\otimes$-categories one can find in [2], [6], [11], [14], [15], the terminology employed in this chapter being of Neantro Saavedra Rivano [14]. A *$\otimes$-category* is a category $\mathscr{C}$ together with a *law* $\otimes$, i.e. a covariant bifunctor

$$\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

$$(X, Y) \mapsto X \otimes Y$$

An *associativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z, \quad X, Y, Z \in Ob(\mathscr{C})$$

satisfying the *pentagon axiom*, i.e. all the pentagonal diagrams

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are commutative. A $\otimes$-category together with an associativity constraint is called a *$\otimes$-associativity category*.

A *commutativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in Ob(\mathscr{C})$$

verifying the relation

$$c_{Y,X} \circ c_{X,Y} = Id_{X \otimes Y}$$

The commutativity constraint c is said to be *strict* if $c_{X,X} = Id_{X\otimes}$ for all $X \in Ob(\mathscr{C})$. A $\otimes$-category together with a commutativity constraint is a *$\otimes$-commutative category*. A $\otimes$-commutative category is *strict* if its commutativity constraint is strict.

An *unity constraint* for a $\otimes$-category $\mathscr{C}$ is a triple $(\underline{1}, g, d)$ where $\underline{1}$ is an object of $\mathscr{C}$, g and d natural isomorphisms

$$g_X : X \xrightarrow{\sim} \underline{1} \otimes X, \quad d_X : X \xrightarrow{\sim} X \otimes \underline{1}, \quad X \in Ob(\mathscr{C})$$

such that $g_{\underline{1}} = d_{\underline{1}}$. A $\otimes$-category together with an unity constraint is a *$\otimes$-unifer category*.

A $\otimes$-category $\mathscr{C}$ together with an associativity constraint a and a commutaivity constraint c is a *$\otimes$-AC category* if the *hexagonal axiom* is fulfilled, i.e. all the hexagonal diagram commutes

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A $\otimes$-category $\mathscr{C}$ together with a associativity constraint a and an unity contraint $(\underline{1}, g, d)$ is a *$\otimes$-AU category* if all the following triangles commute

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A *$\otimes$-ACU category* is a $\otimes$-AC and AU category. An object X of a $\otimes$-ACU category $\mathscr{C}$ is *invertible* if there are two objects $X', X'' \in Ob(\mathscr{C})$ such that $X' \otimes X \simeq X \otimes X'' \simeq \underline{1}$.

A *$\otimes$-functor* from a $\otimes$-category $\mathscr{C}$ to a $\otimes$-category $\mathscr{C}'$ is a pair $(F, \breve{F})$ where F is a functor $\mathscr{C} \longrightarrow cC'$ and $\breve{F}$ an isomorphism of bifunctors

$$\breve{F}_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y) \quad X, Y \in Ob(\mathscr{C})$$

A $\otimes$-functor $(F, \breve{F})$ from a $\otimes$-associative category $\mathscr{C}$ to a $\otimes$-associative category $\mathscr{C}'$ is *associative* if the following diagram commutes:

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where a is the associativity constraint of $\mathscr{C}$ and a' of $\mathscr{C}'$.

A $\otimes$-functor $(F, \breve{F})$ from a $\otimes$-commutative category $\mathscr{C}$ to a $\otimes$-commutative category $\mathscr{C}'$ is *commutative* if the following diagram commutes :

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c and c' being the commutativity constraints of $\mathscr{C}$ and $\mathscr{C}'$ respectively.

A $\otimes$-functor $(F, \breve{F})$ from a $\otimes$-category $\mathscr{C}$ with an unity constraint $(\underline{1}, g, d)$ to a $\otimes$-category $\mathscr{C}'$ with an unity constraint $(\underline{1}', g', d')$ is a*$\otimes$-unifer functor* if there exists an isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$ such that the following diagrams commute:

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It follows from the definition that the isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$, it it exists, is unique.

A $\otimes$-*AC functor* is an $\otimes$-associative and commutative functor.

A $\otimes$-*ACU functor* is a $\otimes$-associative, commutative and unifer functor.

Let $(F, \breve{F})$ and $(G, \breve{G})$ be $\otimes$-functors from a $\otimes$-category $\mathscr{C}$ to a $\otimes$-category $\mathscr{C}'$. A $\otimes$-*morphism* from the $\otimes$-functor $(F, \breve{F})$ to the $\otimes$-functor $(G, \breve{G})$ is a morphism of functors $\lambda : F \longrightarrow G$ such that the following diagram commutes

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Chapter II is a study of Gr-categories and Pic-categories. A Gr-*category* is a $\otimes$-AU category, the objects of which are all invertible, and the base category a groupoid (i.e. all arrows are isomorphisms). Thus a Gr-category is like a group. We obtain from this definition that if $\mathscr{P}$ is a Gr-category, the set $\pi_0(\mathscr{P})$ of the classes up to isomorphism of objects of $\mathscr{P}$, together with the operation induced by the law $\otimes$ of $\mathscr{P}$, is a group; the group $\mathrm{Aut}(\underline{1}) = \pi_1(\mathscr{P})$ is a commutative group; and for all $X \in Ob(\mathscr{P})$

$$\gamma_X : u \mapsto u \otimes Id_X = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

$$\delta_X : u \mapsto Id_X \otimes u = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

We attribute thus to a Gr-category $\mathscr{P}$ two groups $\pi_0(\mathscr{P})$ and $\pi_1(\mathscr{P})$ where $\pi_1(\mathscr{P})$ is commutative. Furthermore we can define an action of $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$ by the formula

$$su = \delta_X^{-1} \gamma_X(u)$$

for $s \in \pi_0(\mathscr{P})$ represents d by X and $u \in \pi_1(\mathscr{P})$. The commutative group $\pi_1(\mathscr{P})$ together with this action is a left $\pi_0(\mathscr{P})$-module.

Let M be a group, N a left M-module. A *preepinglage* of of type (M, N) for a Gr-category $\mathscr{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

compatible wit the action of M on N, $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$. A Gr-category *preeplingled* of type (M, N) is a Gr-category $\mathscr{P}$ together with preepinglage. Finally, an *arrow* of Gr-categories preepingled of type (M, N) $(\mathscr{P}, \varepsilon) \longrightarrow (\mathscr{P}', \varepsilon')$ is a $\otimes$-associative functor such that the following triangles commute:

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It follows from this definition that a such arrow is a $\otimes$-equivalence. Thus the set of the equivalence classes of Gr-categories preepingled of type (M,N) is equal to the set of connected components of the category of Gr-categories preepingled of type (M,N).

If we consider the cohomology group $\mathrm{H}^3(M,N)$ of the group M with coefficients N (in the sense of the group cohomology [12]) we obtain a canonical bijection between the set $\mathrm{H}^3(M,N)$ and the set of the equivalence classes of Gr-categories preepingled of type (M,N).

A Pic-*category* is a Gr-category together with a commutativity constraint which is compatible with its associativity constraint, i.e. the hexagon axiom is satisfied. Thus a Pic-category is like a commutative group. We verify immediately that a necessary condition for the existence of a Pic-category structure on a Gr-category is that $\pi_0(\mathscr{P})$ must be commutative and act trivially on $\pi_1(\mathscr{P})$. A Pic-category is *strict* if its commutativity constraint is strict.

Let M, N be abelian groups. A *preepinglage* of type (M,N) for a Pic-category $\mathscr{P}$ is a pair $\varepsilon=(\varepsilon_0,\varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

A Pic-category *preepingled* of type (M,N) is a Pic-category together with a preepinglage. We define the *arrow* of such objects in the same way as for Gr-categories.

For next propositions, let us consider two complexes of free abelian groups

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where

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so that $L_\bullet(M)$ is a truncated resolution of M. One obtains a canonical bijection between the set of the equivalence classes of Pic-categories preepingled of type (M,N) and the set $\mathrm{H}^2(Hom('L_\bullet(M),N))$. The exactitude of the complex $L(M)$ gives us e triviality of the classification of Pic-categories preepingled of type (M,N) which are strict, i.e. all Pic-categories preepingled of type (M,N) which are strict, are equivalent.

Finally chapter III gives us the construction of the solution of two universal problems: *problem of making objects "unity objects"* and *problem of reversing objects*.

Let $\mathscr{A}$ be a $\otimes$-AC category, $\mathscr{A}'$ another $\otimes$-AC category whose base category is a groupoid, and $(T,\breve{T}) : \mathscr{A}' \longrightarrow \mathscr{A}$ a $\otimes$-AC functors. We try to make the objects TA' of

$\mathscr{A}, A' \in Ob(\mathscr{A}')$, "unity object", i.e. we try to get:

1°) A ⊗-ACU category $\mathscr{P}$

2°) A ⊗-AC functor $(D, \check{D}) : \mathscr{A} \longrightarrow \mathscr{P}$

3°) A ⊗-isomorphism

$$\lambda : (,\check{D}) \circ (T, \check{T}) \xrightarrow{\sim} (I_{\mathscr{P}}, \check{I}_{\mathscr{P}})$$

where $(I_{\mathscr{P}}, \check{I}_{\mathscr{P}})$ is the ⊗-constant functors $\underline{1}_{\mathscr{P}}$ from $\mathscr{A}'$ to $\mathscr{P}$. The triple $(\mathscr{P}, (D, \check{D}), \lambda))$ must be universal for triples $(\mathscr{Q}, (E, \check{E}), \mu)$ satisfying 1°, 2°, 3°.

For the description of the triple $(\mathscr{P}, (D, \check{D}), \lambda)$, we introduce a quotient category of a ⊗-AC category as follows:

Let $\mathscr{A}$ be a ⊗-AC category, Y a *multiplicative subset* of $\mathscr{A}$ (that means a subset of the set of all endomorphisms of $\mathscr{A}$ such that $Id_X \in Y$ for all $X \in Ob(\mathscr{A})$ and the tensor product of two arrows of Y belongs to Y). The *⊗-AC category quotient* A^Y of $\mathscr{A}$ with respect to Y is the solution of the universal problem

$$(K, \check{K}) : \mathscr{A} \longrightarrow \mathscr{B}, \quad K(u) = Id \text{ for all } u \in Y$$

where B is a ⊗-AC category and $(K, \check{K})$ a ⊗-AC functor.

Now let us give an idea of the construction of the triple

Let $\mathscr{C}$ be a ⊗-ACU category, Z an arbitrary object of $\mathscr{C}$ different from the unity object $\underline{1}$, S the functor from $\mathscr{C}$ to $\mathscr{C}$ defined by

$$X \mapsto X \otimes Z.$$

The *suspension category* of the ⊗-ACU category $\mathscr{C}$ defined by the object Z is the triple $(\mathscr{P}, i, p)$ which solves the universal problem for triples $(\mathscr{Q}, j, q)$ where $\mathscr{Q}$ is a category, j a functor from $\mathscr{C}$ to $\mathscr{Q}$, and q an equivalence of categories from $\mathscr{Q}$ to $\mathscr{Q}$, so that the following diagram commutes

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up to natural isomorphism. In the case where $\mathscr{C}$ is the homotopy category of pointed topological spaces $\underline{\mathrm{Htp}}_*$ together with the smash []

Let $\mathscr{C}'$ be the ⊗-stable subcategory of $\mathscr{C}$ generated by Z and $\mathscr{P}$ the ⊗-category of fractions of $\mathscr{C}$ with respect to $(\mathscr{C}', (F, Id))$ where $F : \mathscr{C}' \longrightarrow \mathscr{C}$ is the inclusion functor. One obtains a functor $G : \mathscr{P} \longrightarrow \mathscr{P}$ from the suspension category to the $\otimes$-category of fractions of $\mathscr{P}$. If G is not faithful, that is the case of the homotopy category of pointed topological spaces $\underline{\mathrm{Htp}}_*$ together with the smash $\wedge$ and the 1-sphere S^1; then it is impossible to construct in $\mathscr{P}$ a law $\otimes$ such that $\mathscr{P}$ together with this law is a $\otimes$-ACU category, iZ invertible in $\mathscr{P}$, and i embedded in a pair $(i, \breve{i})$ which is a $\otimes$-ACU functor from $\mathscr{C}$ to $\mathscr{P}$.

§ I. — $\otimes$-CATÉGORIES ET $\otimes$-FONCTEURS

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§ II. — Gr-CATÉGORIES ET Pic-CATÉGORIES

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§ III. — Pic-ENVELOPPE D'UNE $\otimes$-CATÉGORIE ACU

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