6.1.1966

Dear Coates,

Here a few more comments to my talk on the conjectures. The following proposition shows that the conjecture $C_\chi(X)$ is independant of the choosen polarization, and has also some extra interest, in showing the part playd by the fact that $H^i(X)$ should be "motive-theoritetically" isomorphic to its natural dual $H^{2n-i}(X)$ (as usual, I drop the twist for simplicity).

<u>Proposition</u> The condition $C_\chi(X)$ is equivalent also to each of the follwing conditions:

a) $D_\chi(X)$ holds, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \to H^i(X)$ which is algebraic (i.e. induced by an algebraic correspondance class; we do not make any assertion on what it induces in degrees different from 2n-i).

b) For every endomorphism $H^i(X) \to H^i(X)$ which is algebraic, the coefficients of the characteristic polynomial are rational, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \to H^i(X)$ which is algebraic.

Proof. I sketched already how $D_\chi(X)$ implies the fact that for an algebraic endomorphism of $H^i(X)$, the coefficients of the characteristic polynomial are rational numbers, Therefore we know that a) implies b), and of course $C_\chi(X)$ implies a). It remains to prove that b) implies $C_\chi(X)$. Let $u: H^{2n-i}(X) \to H^i(X)$ be the given isomorphism which is algebraic, and $v: H^i(X) \to H^{2n-i}(X)$ ~~an~~ the algebraic isomorphism in the opposite direction, induced by L_X^{n-i}. Then $uv = w$ is an automorphism of $H^i(X)$ which is algebraic, and the Hamilton-Cayley formula $w^b - \sigma_1(w)w^{b-1} + \ldots + (-1)^b \sigma_b(w) = 0$ (where the $\sigma_i(w)$ are the coefficients of the car. pol. of w) whos that w^{-1} is a linear combination of the w^i, with coefficients of the type $\pm \sigma_i(w)/\sigma_b(w)$ (NB $b = \operatorname{rank} H^i$). The assumption implies that these coefficients are rational, which implies that w^{-1} is algebraic, and so is $w^{-1}u = v^{-1}$, which was to be proved.

NB In char. 0, ~~itxis~~ the statement simplifies to: $C(X)$ equivalent to the existence of algebraic isomorphisms $H^{2n-i}(X) \to H^i(X)$, (as the preliminary condition in b) is then automatically satisfied). Maybe with some extra care this can be proved too in arbitrary characteristics.

<u>Corollary</u> Assume X and X' satisfy condition C_i , and let $u: H^i(X) \to H^{i+2j}(X')$ $(j \in \mathbb{Z})$ be an isomorphism which is algebraic. Then u^{-1} is algebraic.

Indeed, the two spaces can be identified "algebraically" (both directions !) to their dual, so taht the transpose of u can be viewed as an isomorphism $u' : H^{i+2j}(X') \to H^i(X)$. Thus $u'u$ is an algebraic automorphism w of $H^i(X)$, and by the previous argument we see that w^{-1} is algebraic, hence so is $u^{-1} = w^{-1}u'$.

As a consequence, we see that if $x \in H^i(X)$ is such that $u(x)$ is algebraic (i being now assumed to be even), then so is x. The same result should hold in fact if u is a monomorphism, the reason being that in this case there should exist a left-inverse which is algebraic; this exists indeed in a case like $H^{n-1}(X) \to H^{n-1}(Y)$ (where we take the left inverse $\wedge_X \phi_*$). But to get it in general, it seems we need moreover the Hodge index relation. (The complete yoga then being that we have the category of motives which is semi-simple !). Without speaking of motives, and staying down on earth, it would be nice to explain in the notes that $C(X)$ together with the index relation $I(X \times X)$ implies that the ring of correspondance classes for X is semi-simple, and how one deduces from this the existence of left and right inverses as looked for above.

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This could be given in an extra paragraph (which I did not really touch upon in the talk), containing also the deduction of the Weil conjectures from the conjectures C and A.

A last and rather trivial remark is the following. Let's introduce variants $A'_\chi(X)$ and $A''_\chi(X)$ as follows:

$A'_\chi(X)$: if $2i \leqslant n-1$, any element x of $H^i(X)$ whose image in $H^i(Y)$ is algebraic, is algebraic.

$A''_\chi(X)$: if $2i \geq n-1$, any algebraic element of $H^{i+2}(X)$ is the image of an algebraic element of $H^i(Y)$.

Let us consider also the specifications $A'_\chi(X)^\circ$ and $A''_\chi(X)^\circ$, where we restrict to the critical dimensions $2i = n-1$ if n odd, $2i = n-2$ if n even. All these conditions are in the nature of "week" Lefschets relations, and they are trivially implied by $A_\chi(X)$ resp. $C_\chi(Y)$ (in the first case, applying ϕ we see that $L_X x$ is algebraic; in the second, we take $y = \Lambda_Y \phi^+(x))$. The remark then is that these pretended "week" variants in fact imply the full Lefschets relations for algebraic cycles, namely:

<u>Proposition</u> $A_\chi(X)$ is equivalent to the conjunction $C_\chi(Y) + A'_\chi(X\times X)^\circ + A''_\chi(X\times X)^\circ$, hence (by induction) also to the conjunction of the conditions $A'^{\,\circ}_\chi$ and $A''^{\,\circ}_\chi$ for all of the varieties $X\times X, Y\times Y, Z\times Z$ .... . Analoguous statement with $X\times Y, Y\times Z$ etc instead of $X\times X, Y\times Y$ etc.

This comes from the remark that ~~if X is even-dimensional (say a square~~ $A_\chi(X)^\circ$ follows from the conjuction of $A'_\chi(X)^\circ$ and $A''_\chi(X)^\circ$, as one sees by decomposing $L_X^{\,2}: H^{2m-2}(X) \to H^{2m+2}(X)$ into $H^{2m-2}(X) \xrightarrow{\varphi^*} H^{2m-2}(Y) \xrightarrow{\varphi_*} H^{2m}(X) \xrightarrow{L_X} H^{2m+2}(X)$ if $\dim X = 2m$ is even, and $H^{2m+1-1}(X) \longrightarrow H^{2m+1+1}(X)$ into $H^{2m}(X) \xrightarrow{\varphi^*} H^{2m}(Y) \xrightarrow{\varphi_*} H^{2m+2}(X)$ if $\dim X = 2m+1$ is odd.

Sincerely your's

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