UNIVERSITY OF KANSAS

Department of Mathematics

Lawrence, Kansas

A General Theory of Fibre Spaces With Structure Sheaf

by

Alexandre Grothendieck

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CONTENTS

INTRODUCTION

When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre spaces (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod (The Topology of Fibre Bundles, Princeton University Press); or the "differentiable" and "analytic" (real or complex) variants of theses notions; or the notions of algebraic fibre spaces (over an abstract field k) — one is led in a natural way to the notion of fibre space with a structure sheaf $\mathbf{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with *abelian* structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\mathbf{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $\mathrm{H}^1(X,\mathbf{G})$ the set of classes of fibre spaces on X with structure sheaf $\mathbf{G}$, $\mathbf{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\mathbf{G}$, where $\mathbf{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in $\mathbf{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the last chapter, we define the cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in the sheaf of groups $\mathbf{G}$, so that the expected classification theorem for fibre spaces with structure sheaf $\mathbf{G}$ is valid. We then proceed to a careful study of the exact cohomology sequence associated with an exact sequence of sheaves $e \longrightarrow \mathbf{F} \longrightarrow \mathbf{G} \longrightarrow \mathbf{H} \longrightarrow e$. This is the main part, and in fact the origin, of this paper. Here $\mathbf{G}$ is any sheaf of groups, $\mathbf{F}$ a subsheaf of groups, $\mathbf{H} = \mathbf{G}/\mathbf{F}$, and according to various supplementary hypotheses of $\mathbf{F}$ (such as $\mathbf{F}$ normal, or $\mathbf{F}$ normal abelian, or $\mathbf{F}$ in the center) we get an exact cohomology sequence going from $\mathrm{H}^0(X,\mathbf{F})$ (the group of section of $\mathbf{F}$) to $\mathrm{H}^1(X,\mathbf{G})$ respectively $\mathrm{H}^1(X,\mathbf{H})$ respectively $\mathrm{H}^2(X,\mathbf{G})$, with more or less additional algebraic structures involved. The formalism thus developed is quite suggestive, and as it seems useful, in particular in dealing with the problem of classification of fibre bundles with a structure group G in which we consider a sub-group F, or the problem of comparing say the topological and analytic classification for a given analytic structure group G. However, in order to keep this exposition in reasonable bounds, no examples have been given. Some complementary facts, examples, and applications for the notions developed will be given in the future. This report has been written mainly in order to serve the author for future reference; it is hoped that it may serve the same purpose, or as an introduction to the subject, to somebody else.

Of course, as this report consist in a fortunately straightforward adaptation of quite well known notions, no real difficulties had to be overcome and there is no claim for originality whatsoever. Besides, at the moment to give this report for mimeography, I hear that results analogous to those of chapter 5 were known for some years to Mr. Frenkel, who did non publish them till now. The author only hopes that this report is more pleasant to read than it was to write, and is convinced that anyhow an exposition of this sort had to be written.

Remark (added for the second edition). It has appeared that the formalism developed in this report, and specifically the results of Chapter V, are valid (and useful) also in other situations than just for sheaves on a given space X. A generalization for instance is obtained by supposing that a fixed group π is given acting on X as a group of homeomorphisms, and that we restrict our attention to the category of fibre spaces over X (and specially sheaves) on which π operates in a manner compatible with its operations on the base X. (See for instance A. Grothendieck, Sur le mémoire de Weil; Généralisations des fonctions abéliennes, Séminaire Bourbaki Décembre 1956). When X is reduced to a point, one gets (instead of sheaves) sets, groups, homogeneous spaces etc. admitting a fixed group π of operators, which leads to the (commutative and non-commutative) cohomology theory of the group π. One can also replace π by a fixed Lie group (operating on differentiable varieties, on Lie groups, and homogeneous Lie spaces). Or X, π are replaced by a fixed ground field k, and one considers algebraic spaces, algebraic groups, homogeneous spaces *defined over* k, which leads to a kind of cohomology theory of k. All this suggests that there should exist a comprehensive theory of non-commutative cohomology in suitable categories, an exposition of which is still lacking. (For the "commutative" theory of cohomology, see A. Grothendieck, Sur quelques points d'Algèbre Homologique, Tohoku Math. Journal, 1958).

§ I. — GENERAL FIBRE SPACES

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1. Notion of fibre space

Definition **1.1.1.** — *A* fibre space *over a space X is a triple (X, E, p) of the space X, a space E and a continuous map p of E into X.*

We do not require p to be onto, still less to be open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X, E, p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the *base space* of the fibre space, p the *projection*, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the *fibre* of x (in E).

Given two fibre spaces (X, E, p) and (X', E', p'), a *homomorphism* of the first into the second is a pair of continuous maps $f : X \longrightarrow X'$ and $g : E \longrightarrow E'$, such that $p'g = fp$, i.e. commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Then g maps fibres into fibres (but not necessarily *onto*!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X', f' a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of *isomorphism* of a fibre space (X, E, p) onto a fibre space (X', E', p') is clear: it is a homomorphism (f, g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms

Let (X, E, p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the *inverse image* of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x', y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x', y) = x'$. The map $g(x', y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a *homeomorphism* of the fibre of E' over x' *onto* the fibre of E over $f'x'$.

Suppose now, moreover, given a continuous map $f' : X'' \longrightarrow X'$ of a space X'' into X'. Then there is a canonical isomorphism of the fibre space E'' over X'', inverse image of the fibre space E by ff', and the inverse image of the fibre space E' (considered above) by f' (*transitivity* of *inverse images*). If $(x'', y) \in E''$ ($x'' \in X'', y \in E, ff'x'' = py$), it is mapped by this isomorphism into $(x'', (f'x'', y))$.

Let Y be a subspace of the base X of a fibre space E; consider the injection f of Y into X; the inverse image E' of E by f is called *fibre space induced by E* on Y, or the *restriction of E to Y*, and is denoted by $E|Y$. This is canonically homeomorphic to a subspace of E, namely the set of elements mapped by p into Y; the projection of $E|Y$ into Y is induced by p. By what has been said above, if Z is a subspace of Y, the restriction of $E|Y$ to Z is the restriction $E|Z$ of E to Z.

Again let (X, E, p) and (X', E', p') be two fibre spaces, f a continuous map $X \longrightarrow X'$. An *inverse homomorphism associated with f* is an X-homomorphism g

of the fibre space E_0 into E, where E_0 denotes the inverse image of the fibre space E' by f. That means that g is a continuous map, of the subspace E_0 of $X \times E'$ of pairs (x, y') such that $f x = p' y'$, into E, mapping for any $x \in X$ the fibre of x into E_0 (homeomorphic to the fibre of $f x$ in E'!) into the fibre $p^{-1}(x)$ of x in E. For instance, if E is itself the inverse image of E' by f, then there is a canonical inverse homomorphism of E' into E associated with f: the identity! (Though somewhat trivial, this is the most important case of inverse homomorphisms.)

1.3 Subspace, quotient, product

Let (X, E, p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E' as a fibre space with the same basis X, called a *sub-fibre-space* of E. So the sub-fibre-spaces of E are in one to one correspondence with the subsets of E; in particular, for them the notions of union, intersection etc. are defined. (Of course, in most cases we are only interested in fibre spaces the projection of which is onto; this imposes than a condition on the subspaces of E considered, which may be fulfilled for two subspaces and not for the intersection.)

Let now R be an equivalence relation in E *compatible wit the map* p, i.e. such that two elements of E congruent mod R have the same image under p. Then p defines a continuous map p' of the quotient space $E' = E/R$ into X, which turns E' into a fibre space with base X, called a *quotient fibre space of* E. So the latter are in one-to-one correspondence with the equivalence relations in E compatible with p. A quotient fibre space of a quotient fibre space of E is a quotient fibre space.

Let (X, E, p) and (X', E', p') be two fibre spaces, then (p, p') defines a continuous map of $E \times E'$ into $X \times X'$, so that $E \times E'$ appears as a fibre space over $X \times X'$, called the *product of the fibre spaces* E, E'. The fibre of (x, x') in $E \times E'$ is the product of the fibres of x in E, respectively x' in E'. Suppose now $X = X'$, and consider the inverse image of $E \times E'$ under the diagonal map $X \longrightarrow X \times X$, we get a fibre space *over* X, called the *fibre product of the fibre spaces* E, E' *over* X, denoted by $E \times_X E'$. The fibre of x in this fibre-product is the product of the fibres of x in E respectively E'. Of course, product of an arbitrary family of fibre spaces can be considered, and the usual formal properties hold.

1.4 Trivial and locally trivial fibre spaces

Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the *trivial fibre space over X with fibre F*. All fibres are canonically homeomorphic with F. Let us determine the homomorphisms of a trivial fibre space $E = X \times F$ into another $E' = X \times F'$. More generally, we will only assume that the projection of $X \times F$ onto X is the natural one and continuous for the given topology of $X \times F$, which induces on the fibres the given topology (but the topology of $X \times F$ may not be the product topology, for instance: X and F are algebraic varieties with the Zariski topology); same hypothesis on $X \times F'$. Then a homomorphism u of E into E', inducing for each $x \in X$ a continuous map of the fibre of E over x into the fibre of E' over x, defined a function $x \longrightarrow f(x)$ of X into the set of all continuous maps of F into F', and of course the homomorphism is well determined by this map by the formula

$$u(x,y)=(x,f(x)y) \quad (x\in X, y\in F). \tag{1.4.1}$$

So the homomorphisms of E into E' can be identified with those maps f of X into the set of continuous maps of F into F' such that the map (1.4.1) is continuous. If the topologies of E and E' are the product topologies, this means that $(x,y)\longrightarrow f(x)y$ is continuous; as is well known, if moreover F is locally compact or metrizable, this means also that f is continuous when we take on the set of all continuous maps from F into F' the topology of compact convergence. If we consider a homomorphism v from E' into $E'' = X \times F''$ given by a map g of X into the set of all continuous maps of F' into F'' the homomorphism vu is given by the map $x \longrightarrow g(x)f(x)$. In order that the map (1.4.1) be injective (respectively surjective, bijective) it is necessary and sufficient that for each $x\in X$, $f(x)$ has the same property. In the bijective case, the inverse map is then defined by the function $X \longrightarrow f(x)^{-1}$. It follows that u is an isomorphism onto i and only if for each $x\in X$, f is a homeomorphism of F into F', and the map $(x,y')\longrightarrow(x,f(x)^{-1}y')$ continuous. So we get in particular (coming back to the case of trivial fibre spaces):

Proposition **1.4.1**. — *Let $E = X \times F$ and $E' = X \times F'$ be two trivial fibre spaces over X, then the isomorphisms of E onto E' can be identified with the maps f*

of X into the set of homeomorphisms of F onto F' such that $f(x)y$ and $f(x)^{-1}y'$ be continuous functions from $X \times F$ into F' respectively $X \times F'$ into F. If $E = E'$, this identification is compatible with the group structures on the set of automorphisms of E respectively the set of maps of X into the group of automorphisms of F.

Two fibre spaces E, E' over X are said to be *locally isomorphic* if each point x of X has a neighbourhood U (which can be assumed open) such that the restrictions of E and E' to U are isomorphic. This is clearly an equivalence relation. A fibre space E over X is said *locally trivial with fibre F* (F being a given space) if it is locally isomorphic to the trivial space $X \times F$.

1.5 Definition of fibre spaces by coordinate transformations

Let X be a space, (U_i) a covering of X, for each index i let E_i be a fibre space over U_i, and for any couple of indices i, j such that $U_{ij} = U_i \cap U_j \neq \varnothing$, let f_{ij} be a U_{ij}-isomorphism of $E_j|U_{ij}$ onto $E_i|U_{ij}$. On the topological sum E of the spaces E_i, let us consider the relation

(1.5.1.) $\quad y_i \in E_i|U_{ij}$ and $y_j \in E_j|U_{ij}$ are equivalent means $y_i = f_{ij}y_j$.

This is an equivalence relation, as easily checked, if and only if we have, for each triple (i, j, k) of indices such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \varnothing$, the relation

(1.5.2.) $$f_{ik} = f_{ij}f_{jk}$$

(where, in order to abbreviate notations, we wrote simple f_{ik} instead of: the isomorphism of $E_k|U_{ijk}$ onto $E_i|U_{ijk}$ induced by f_{ik}; and likewise for f_{ij} and f_{jk}). Supposing

Definition **1.5.1**. — *The fibre space over X just constructed is called the fibre space defined by the "coordinate transformatons" (f_{ij}) between the fibre spaces E_i.*

The identity map of E_i into E defines a map φ_i of E_i into E, which by virtue of (1.5.1.) is a one to one U_i-homomorphism of E_i onto $E|U_i$. The topology of E (by a well known transitivity property for topologies defined as the finest which…) is the finest topology on E for which the maps φ_i are continuous. Moreover, it is easy to show that in case the interiors of the U_i's already cover X, the maps φ_i are *homeomorphisms* into. Henceforth, for simplicity *we will only work with open coverings of* X, so that the preceding properties are automatically satisfied. Then φ_i *can be considered as a* U_i*-isomorphisms of* E_i *onto* $E|U_i$. Clearly

$$f_{ij} = \varphi_i^{-1}\varphi_j \tag{1.5.3.}$$

(where again, in order to abbreviate, we wrote φ_i instead of the restriction of φ_i to $e_i|U_{ij}$, φ_j instead of the restriction of φ_j to $e_j|U_{ij}$). Conversely, let E be a fibre space over X, and suppose that for each i there exists a U_i-isomorphism φ_i of E_i onto $E|U_i$, then (1.5.3.) defines, for each pair (i, j) such that $U_i \cap U_j = U_{ij} \neq \varnothing$, a U_{ij}-isomorphism of $E_j|U_{ij}$ onto $E_i|U_{ij}$, and the system (f_{ij}) satisfies obviously (1.5.2.). Therefore we can consider the fibre space E' defined by the coordinate transformations f_{ij}. Then it is obvious that the map of E into E defined by the maps φ_i is compatible with the equivalence relation in E, therefore defines a continuous map f of E' into E which is of course an X-homomorphism. Let φ_i' be the natural isomorphism of E_i onto $E'|U_i$ defined above; it is checked at once that the map of $E'|U_i$ into $E|U_i$ induced by f is $\varphi_i\varphi_i'^{-1}$, hence an isomorphism onto. It follows that f itself is an isomorphism of E' onto E, by virtue of the following easy lemma (proof left to the reader):

Lemma **1**. — *Let E, E' be two fibre spaces over X, and f an X-homomorphism of E into E', such that for any $x \in X$, exists a neighbourhood U of x such that f induces an isomorphism of $E|U$ onto (respectively, into) $E'|U$. Then f is an X-isomorphism of E onto (respectively, into) E'.*

What precedes shows the truth of:

Proposition **1.5.1.** — *The open covering (U_i) and the fibre spaces E_i over U_i being give, the fibre spaces over X which can be obtained by means of suitable coordinate transformations (f_{ij}) are exactly those, up to isomorphism, for which $E|U_i$ is isomorphic to E_i for any i.*

Consider now two systems of coordinate transformations (f_{ij}), (f'_{ij}) corresponding to the same covering (U_i), and to two systems (E_i), (E'_i) of fibre spaces over the U_i's. Let E be the fibre space defined by (f_{ij}) and E' the fibre space defined by (f'_{ij}); we will determine all homomorphisms of E into E'. If f is such a homomorphism, then for each i, $f_i = \varphi_i'^{-1} f \varphi_i$ (where f stands for the restriction of f to $E|U_i$) is a homomorphism of E_i into E'_i, and the system (f_i) satisfies clearly, for each pair (i, j) such that $U_{ij} \neq \varnothing$:

$$f_i f_{ij} = f'_{ij} f_j \tag{1.5.4.}$$

(where we write simple f_i instead of the restriction of f_i to $E_i|U_{ij}$, and likewise for f_j). The homomorphism f is moreover fully determined by the system (f_i) subject to (1.5.4) can be chosen otherwise arbitrarily, for this relation expresses exactly that the map of the topological sum E of the E_i's into the topological sum E' of the E'_i's transforms equivalent points into equivalent points, and therefore defines an X-homomorphism f of E into E'; and it is clear that the system (f_i) is nothing else but the one which is defined as above in terms of the homomorphism f. Of course, in view of lemma 1, in order that f be an isomorphism onto, (respectively, into) it is necessary and sufficient that each f_i be an isomorphism of E_i onto (respectively, into) E'_i. Thus we get:

Proposition **1.5.2.** — *Given two fibre spaces over X, E and E', defined by coordinate transformations (f_{ij}) respectively (f'_{ij}) relative to the same open covering (U_i), the X-homomorphisms f of E into E' are in one to one correspondence with systems (f_i) of U_i-homomorphisms $E_i \longrightarrow E'_i$ satisfying (1.5.4.). f is an onto-isomorphism if and only if the f_i's are, i.e. E' is isomorphic to E if and only if we can find onto-isomorphisms $f_i : E_i \longrightarrow E'_i$ such that, for any pair (i, j) of indices satisfying $U_{ij} \neq \varnothing$, we have*

$$f'_{ij} = f_i f_{ij} f_j^{-1} \tag{1.5.5.}$$

(where as usual f_i and f_j stand for restricted maps).

We proceed to the comparison of fibre spaces E, E' defined by coordinate transformations corresponding to different coverings, (U_i) and (U'_i), in particular to the determination of the homomorphisms of E into E' and hence of the X-isomorphisms of E and E', and therefore to the determination of whether E and E' are isomorphic. Let (V_j) be an open covering of X which is a refinement of both preceding coverings; we will show that E and E' are isomorphic to fibre spaces defined by coordinate transformations relative to this same covering (V_j), so that the problem is reduced to one already dealt with.

So let $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be two open coverings of X, the second finer that the first, that is, any V_j is contained in some U_i, i.e. there exists at least one map $\tau : J \longrightarrow I$ such that $V_j \in U_{\tau(j)}$ for any $j \in J$.

For each $i \in I$, let E_i be a fibre space over $U_{i'}$ and let $(f_{ii'})$ be a system of coordinate transforms relative to the system (E_i). For each $j \in J$, let $F_j = E_{\tau(j)}|V_j$, and let $g_{jj'}$ be the restriction of $f_{\tau(j),\tau(j')}$ to $F_j|V_{jj'}$; so $g_{jj'}$ is an isomorphism of $F_j|V_{jj'}$ onto $F_j|V_{jj'}$, and the system $(g_{jj'})$ is a system of coordinate transformations, as follows at once from the definition and (1.5.2.) applied to the system $(f_{ii'})$. Let F be the fibre space defined by the system of coordinate transformations $(g_{jj'})$; we shall define a *canonical X-isomorphism of F onto E*. For $j \in J$, let g_i be the injection map of F_j into $E_{\tau(j)}$; it is hence a map of F_j into the topological sum E of the E_i's; the system (g_j) defines a continuous map g' of the topological sum F of the f_j's into E, and as easily seen g' maps equivalent points into equivalent points. Hence g' induces a continuous map g of F into E, which clearly is an X-homomorphism. Moreover, for any j, g induces an isomorphism of $F|V_j$ onto $E|V_j$, for if we compose it with the natural isomorphism of $E|U_i$ onto E_i, we get the injection map of $E|V_j$ into E_i (we put $i = \tau(j)$). Now applying lemma 1, we see that g is an isomorphism of F onto E.

1.6 The case of locally trivial fibre spaces

The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space over T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces $E_i = T|U_i$ over U_i and coordinate transformations (f_{ij}) with respect to these. Then f_{ij} is an U_{ij}-automorphism of $T|U_{ij}$. The fibre space defined by the system (f_{ij}) of coordinate transformations will be locally isomorphic (cf. 1.4.) to T, and in virtue of proposition 1.5.1., *we obtain in this way exactly (up to isomorphism) all fibre spaces over X which are locally isomorphic to T* (by taking the open sets U_i small enough, and then a suitable system (f_{ij})).

In case T is a trivial fibre space, $T = X \times F$, we have $E_i = U_i \times F$, and

$E_i|U_{ij} = U_{ij} \times F$. Thus f_{ij} is an automorphism of the trivial fibre space $U_{ij} \times F$, and therefore, in view of proposition 1.4.1. given by a map $x \longrightarrow f_{ij}(x)$ of U_{ij} into the group of homeomorphisms of F onto itself. The equations (1.5.2.) expressing that (f_{ij}) is a system of coordinate transformations then translate into

$$f_{ik}(x) = f_{ij}(x) f_{jk}(x) \quad \text{for } x \in U_{ijk}. \tag{1.6.1.}$$

Moreover, it must not be forgotten that $x \longrightarrow f_{ij}(x)$ is submitted to the continuity condition of proposition 1.4.1. Such a system then defines in a natural way a fibre space E over X, and by what has been said it follows that this fibre bundle is locally isomorphic to $X \times F$, i.e. *locally trivial with fibre* F, and that (for suitable choice of the covering and the coordinate transformations), *we get thus, up to isomorphism, all locally trivial fibre spaces over* X *with fibre* F.

Let in the same way $T' = X \times F'$, and consider for the same covering (U_i) a system (f_{ij}) and a system (f'_{ij}) of coordinate transformations, the first relative to the fibre F and the second to the fibre F'. Let E and E' be the corresponding fibre spaces over X. The homomorphisms of E into E', by proposition 1.5.2., correspond to homomorphisms f_i of $E_i = U_i \times F$ into $E'_i = U_i \times F'$, satisfying conditions (1.5.4.). Now, (proposition 1.4.1.) such a homomorphism f_i is determined by a map $x \longrightarrow f_i(x)$ of U_i into the set of continuous maps of f into F' by $f_i(x,y) = (x, f_i(x)y)$, subject to the only requirement that $f_i(x)y$ is continuous with respect to the pair $(x,y) \in U_i \times F$. Then the equation (1.5.4.) translates into

$$f_i(x) f_{ij}(x) = f'_{ij}(x) f_j(x) \quad (x \in U_{ij}) \tag{1.6.2.}$$

Thus are determined the homomorphisms of E into E'. In particular, the isomorphisms of E onto E' are obtained by systems (f_i) such that $f_i(x)$ be a homeomorphism of F onto F' for any $x \in U_i$, and that $x \longrightarrow f_i^{-1}(x)$ satisfies the same continuity requirement as $x \longrightarrow f_i(x)$. The compatibility condition (1.6.2.) can then be written

$$f'_{ij}(x) = f_i(x) f_{ij}(x) f_j(x)^{-1} \quad (x \in U_{ij}) \tag{1.6.3.}$$

1.7 Sections of fibre spaces

Definition **1.7.1.** — *Let* (X, E, p) *b a fibre space; a* section *of this fibre space (or, by* *pleonasm, a section of E over X) is a map x of X into E such that ps is the the identity map of X. The set of* continuous *sections of E is noted* $\mathrm{H}^0(X,E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x!).

The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A *section of E over a subset Y* of X is by definition a section of $E|Y$. If Y is open, we write $\mathrm{H}^0(Y,E)$ for the set $\mathrm{H}^0(Y,E|Y)$ of all continuous sections of E over Y.

$\mathrm{H}^0(X,E)$ as a *functor*. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. *We get thus a map, noted f, of* $\mathrm{H}^0(X,E)$ *into* $\mathrm{H}^0(X,E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, the so is f.

b. If f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E'' (E, E', E'' fibre spaces over X) then $(f'f) = f'f$.

Let (X,E,p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f. Let s be a section of E' consider the map s' of X' into E' given by $s'x' = (x', sfx')$ (the second member belongs to E', since $fx' = psfx'$ because $px =$ identity), this is a section of E', continuous if s is continuous. Thus we get a *canonical map of* $\mathrm{H}^0(X,E)$ *into* $\mathrm{H}^0(X',E')$ (E' being the inverse image of E by f). In case $X' \subset X$ and f is the inclusion map, therefore $E' = E|X'$, then the preceding map is nothing but the *restriction map* (of $\mathrm{H}^0(X,E)$ into $\mathrm{H}^0(X',E)$ if X' open). — We leave to the reader statement and proof of an evident property of transitivity for the canonical maps just considered.

The two sorts of homomorphisms for sets of continuous sections are compatible in the following sense. Let φ be a fixed map of a space X' into X, then to any fibre space E over X correspond its inverse image E' under φ, which is a fibre space over X'; moreover, given an X-homomorphism $f : E \longrightarrow F$, it defines in a natural way an X'-homomorphism f' of E' into F'. (Er could go further and state that, for fixed φ, E' is a "functor" of E by means of the preceding definitions.)

Then the following diagram

$$\begin{array}{ccc} \mathrm{H}^0(X,E) & \xrightarrow{f_*} & \mathrm{H}^0(X,F) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X',E') & \xrightarrow{f'_*} & \mathrm{H}^0(X',F') \end{array}$$

is commutative, where the vertical arrows stand for the canonical homomorphisms defined above. The checking of course is trivial.

Particular case: replacing X by an open subset U of X, and taking for X' an open subset V of U and φ the inclusion map $V \longrightarrow U$, we get that for any fibre spaces E, F over X and X-homomorphism $f : E \longrightarrow F$, the following diagram is commutative:

(1.7.2.)
$$\begin{array}{ccc} \mathrm{H}^0(U,E) & \longrightarrow & \mathrm{H}^0(U,F) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(V,E) & \longrightarrow & \mathrm{H}^0(V,F) \end{array}$$

where the vertical arrows are the restriction maps, and the horizontal arrows are the maps defined by f (or, strictly speaking, by the restrictions of f to $E|U$ respectively $E|V$). In words: the "homomorphisms" between spaces of sections over open sets defined by X-homomorphisms of fibre spaces commute with the restriction operators.

Determination of sections. Let us come back to the conditions of the definition 1.5.1.; we keep the notations of that section. Let s be a section of the fibre space E, and for any i let $s_i = \varphi_i^{-1}s$; then s_i is a section of E_i over U_i, and from $s = \varphi_i s_i = \varphi_j s_j$ over U_{ij} we get $s_i = \varphi_i^{-1}\varphi_j s_j = f_{ij}s_j$:

(1.7.3.)
$$s_i = f_{ij}s_j$$

where again we write s_i, s_j instead of: restriction of s_i, s_j to U_{ij}.

Of course, s is entirely determined by the system (s_i), for s is give over U_i by $s = \varphi_i s_i$. On the other hand, the system (s_i) subject to (1.7.3.) can be otherwise arbitrary, for these conditions express precisely that for $x \in X$, the element $\varphi_i s_i(x)$ of E obtained by taking a U_i containing x does not depend on i, and may therefore be denoted by $s(x)$: Then the $\varphi_i^{-1}s$ determined by the above definition are of course nothing else than the s_i's we started with. Let us note also that in order that the section s be continuous, it is necessary and sufficient that each s_i be continuous. We thus obtain:

Proposition **1.7.1.** — *Let E be the fibre space defined by coordinate transformations (f_{ij}) relative to an open covering (U_i) of X and fibre spaces E_i over U_i. Then there is a canonical one to one correspondence between sections of E and systems (s_i) of sections of E_i over U_i, satisfying (1.7.3.).* Continuous *sections correspond to systems of* continuous *sections.*

Let again, as in section 1.5, be given *two* systems (E_i) and (E'_i) of fibre spaces over the U_i's and two corresponding systems of coordinate transformations (f_{ij}) and (f'_{ij}), let E and E' be the corresponding fibre spaces, and f an X-homomorphism of E into E', defined by virtue of proposition 1.5.2., by a system (f_i) of U_i-homomorphisms of E_i into E'_i satisfying (1.5.4.). Let s be a section of E, given by a system (s_i) of sections of E_i over U_i. *Then the systems $(f_i s_i)$ of sections of E'_i over U_i defined the section $f\,s$* (trivial).

The reader may check, as an exercise, how the canonical maps of spaces of sections considered above in this section, can be made explicit for fibre spaces given by means of coordinate transformations.

§ II. — SHEAVES OF SETS

Throughout this exposition, we will now use the word *"section"* for *"continuous section"*.

2.1. Sheaves of sets

Definition **2.1.1**. — *Let X be a space. A* sheaf of sets *on X (or simply a* sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighbourhood U such that p induces a homeomorphism of U onto an open subset $p(U)$ of X.*

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

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2.2. $\mathrm{H}^\circ(A, E)$ for arbitrary $A \subset X$

2.3 Definition of a sheaf by systems of sets

2.4 Permanence properties

2.5 Subsheaf, quotient sheaf. Homomorphisms of sheaves

2.6. Some examples

a.

b.

c.

d. **Sheaf of germs of subsets**. Let X be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of U. If $V \subset U$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $\mathrm{H}^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of $P(X)$, for instance the sheaf of *germs of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

§ III. — GROUP BUNDLES AND SHEAVES OF GROUPS

3.1. Fibre spaces with composition law

3.2. Group bundles and sheaves of groups

3.3. Sub-group-bundles and quotient-bundles. Subsheaves and quotient sheaves

3.4. Fibre spaces with group bundle of operators

3.5. The sheaf of germs of automorphisms

3.6. Particular cases

§ IV. — FIBRE SPACES WITH STRUCTURE SHEAF

§ V. — THE CLASSIFICATION OF FIBRE SPACES WITH STRUCTURE SHEAF
