Dear Barsotti,

I would like to tell you about a result on specialization of Barsotti–Tate groups in characteristic $p$, although you have perhaps known it for a long time, as well as a corresponding conjecture (or, rather, question), whose answer you may again already know well.

First, some terminology. Let $k$ a perfect field of characteristic $p > 0$, $W$ the ring of Witt vectors over $k$, and $K$ its field of fractions. For us, an *$F$–crystal* over $k$ will mean a free module $M$ of finite type over $W$, together with a $\sigma$–linear endomorphism $F_M\colon M \to M$ (where $\sigma\colon W \to W$ is the Frobenius automorphism) such that $F_M$ is injective.[46] I am interested in considering $F$–crystals up to isogeny, an equivalence class of which I will call an *$F$–iso*–crystal. Such an $F$–iso–crystal can be interpreted as a finite dimensional $K$–vector space $E$ and a $\sigma$–linear automorphism $F_E\colon E \to E$; an $F$–isocrystal we will additionally call *effective* when there exists a lattice $M \subset E$ mapped into itself by $F_E$. The category of $F$–isocrystals is obtained from that of effective $F$–isocrystals and its natural internal tensor product by formally inverting formally the *Tate crystal*,

$$K(-1) = (K, F_{K(-1)} = p\sigma).$$

This is to say that the isocrystals $(E, F_E)$ such that $(E, p^n F_E)$ is effective (i.e., those for which the set of iterates of $(p^n F_E)$ is bounded in the natural norm structure) are precisely those of the form $E_0(n) = E_0 \otimes K(-1)^{\otimes(-n)}$, with $E_0$ an effective $F$–isocrystal.

Let us now assume $k$ to be algebraically closed. As presented in Manin's report, Dieudonné's classification theorem states that the category of $F$–isocrystals over $k$ is semi-simple and that the isomorphism classes of simple objects can be indexed by $\mathbb{Q}$—equivalently, by pairs of relatively prime integers $r, s \in \mathbb{Z}$, $r \geq 1$, $(s, r) = 1$. Over $\mathbb{F}_p$, such a pair is sent to the simple object $\mathbb{E}_{s/r} = \mathbb{E}_{r,s}$ of rank $r$ given by the formula as

$$E_{s/r} = \begin{cases} \mathbb{Q}_p[F_{s/r}]/(F_{s/r}^r - p^s) & s > 0, \\ E_{-\lambda} = (E_\lambda)^\vee & s \leq 0, \end{cases}$$

where $(-)^\vee$ denotes the linear-algebraic dual endowed with the contragredient $F$ automorphism. In Manin's report, only effective $F$–crystals are considered—and then only those such that $F_E$ is topologically nilpotent—but the observation about the Tate twist implies the result as I state it now.

[46] That is, $F(M)$ contains $p^n M$ for some $n \geq 0$.

Indexing by $\mathbb{Q}$ rather than by pairs $(s,r)$ has the advantage that we have the simple formula

$$E_\lambda \otimes E_{\lambda'} = E_{\lambda+\lambda'}.$$

More generally, if we decompose each crystal in its isotypic component corresponding to the various "slopes"[47] $\lambda \in \mathbb{Q}$, this gives a natural grading on it over the group $\mathbb{Q}$, and this grading is compatible with the tensor product structure in the following sense:

$$E(\lambda) \otimes E'(\lambda') \subset (E \otimes E')(\lambda + \lambda').$$

Let's define the sequence of slopes of a crystal $(E, F_E)$ by its isotypic decomposition, where each $\lambda$ appears rank $E(\lambda)$ many times (bearing in mind that if $\lambda = s/r$ with $(s,r) = 1$, then rank $E(\lambda)$ is a multiple of $r$). It is also convenient to give an increasing order to this sequence. This definition is still appropriate even if $k$ is not algebraically closed: by passing over to the algebraic closure of $k$, we can produce this sequence of numbers, but in fact the isotypic decomposition over $\overline{k}$ *descends* to $k$, so we even get a canonical "iso-slope"[48] decomposition over $k$:[49]

$$E = \bigoplus_{\lambda \in \mathbb{Q}} E(\lambda).$$

If we further specialize to $k = \mathbb{F}_q = \mathbb{F}_{p^a}$, and if $(E, F_E)$ is a crystal over $k$, then $F_E^a$ is a linear endomorphism of $E$ over $K$, and the slopes of the crystal are the valuations of the proper values of $F_E^a$, using the valuation of $\overline{\mathbb{Q}}_p$ normalized so that $v(q) = 1$ (i.e., $v(p) = 1/a$).[50] Thus, the sequence of slopes of the crystal defined above is just the sequence of slopes of the *Newton polygon* of the characteristic polynomial of the arithmetic Frobenius endomorphism $F_E^a$, and that data is equivalent to the data of the $p$-adic valuations of the proper values of the Frobenius!

Let us return to a generic perfect field $k$. The effective crystals are those whose slopes are positive, and those which are Dieudonné modules[51] are those whose slopes are in the closed interval $[0,1]$. Those of slope zero corresponds to ind–étale groups, and those of slope one correspond to multiplicative groups. Moreover, an arbitrary crystal decomposes canonically into a direct sum

$$E = \bigoplus_{i \in \mathbb{Z}} E_i(-i),$$

where $(-i)$ are Tate twists[52] and the $E_i$ have slopes $0 \le \lambda < 1$ (or, if we prefer $0 < \lambda \le 1$), hence correspond to isogeny classes of Barsotti–Tate groups over $k$ without multiplicative component (resp. which are connected). This remark is interesting because if $X$ is a proper and smooth scheme over $k$, then the crystalline cohomology groups $H^i(X)$ can be viewed

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[47] The terminology "slope" here, as well as the sequence of slopes occuring in any crystal, is I believe due to you, as you presented for formal groups in Pisa about three years ago. I did not appreciate then the full appropriateness of the notation and of the terminology.

[48] In French: *isopentique*.

[49] N.B. This is true only because we assumed $k$ perfect. There is a reasonable notion of $F$–crystal when $k$ is not perfect, but then we should get only a *filtration* of a crystal by increasing slopes…

[50] This is essentially the "technical lemma" in Manin's report, without his unnecessary restrictive conditions.

[51] That is: those which correspond to (not necessarily connected) Barsotti–Tate groups over $k$.

[52] This corresponds to multiplying the $F$ endomorphism by $p^i$.

as $F$-crystals, where $H^i$ has slopes between $0$ and $i$,[53] and hence this defines a whole avalanche of (isogeny classes of) Barsotti–Tate groups over $k$. Moreover, these are quite remarkable invariants whose knowledge should be thought as essentially equivalent with the knowledge of the characteristic polynomials of the "arithmetic" Frobenius, acting on (any reasonable) cohomology of $X$.[54]

Now the result about specialization of Barsotti–Tate groups. Select a Barsotti–Tate group $G$ and a second group $G'$ which is a specialization of $G$. Let $\lambda_1, \ldots, \lambda_h$ ($h =$ "height") be the slopes of $G$, and $\lambda'_1, \ldots, \lambda'_h$ the ones for $G'$. Then we have the equality

$$\sum \lambda'_i = \sum \lambda_i (= \dim G = \dim G')$$

as well as the inequalities

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \lambda'_i.$$

In other words, the "Newton polygon" of $G$ (i.e., of the polynomial $\prod_i (1 + (p^{\lambda_i})T))$ lies below the one of $G'$, and they have the same endpoints: $(0,0)$ and $(h, N)$.

I arrived at this result through a generalization of Dieudonné theory for Barsotti–Tate groups over an arbitrary base $S$ of characteristic $p$, which allows me to manufacture an $F$-crystal over $S$, heuristically thought of as an $S$-*family* of $F$-crystals in the sense outlined above. Using this, the result just stated is but a particular case of the analogous statement about specialization of arbitrary crystals. Now this latter statement is not hard to prove at all: passing to $\bigwedge^h E$ and $\bigwedge^h E'$, the equality is reduced to the case of a family of rank one crystals, and even further to the statement that such a family is just a twist of some fixed power of the (constant) Tate crystal. The general inequality (2) is reduced, passing to $\bigwedge^j E$ and $\bigwedge^j E'$, to just the first inequality $\lambda_1 \leq \lambda'_1$. Raising both $E$ and $E'$ to an $r^{\text{th}}$ tensor power such that $r\lambda_1$ is an integer, we may assume that $\lambda_1$ is an integer, and a Tate twist allows us to assume that $\lambda_1 = 0$, so the statement boils down to the following: if the general member of the family is an *effective* crystal, so are all others. This is readily checked in terms of the explicit definition of a crystal over $S$.

The conjecture I have in mind is as follows: the equality and inequality family above are necessary conditions for $G'$ to be a specialization of $G$, and I would like them to also be sufficient. More explicitly, start with a Barsotti–Tate group $G_0 = G'$, and take its formal modular deformation in characteristic $p$ (over a modular formal variety $S$ of dimension $dd^*$, $d = \dim G_0$, $d^* = \dim G_0^*$). For the Barsotti–Tate group $G$ over $S$ so-obtained, we want to know if every sequence of rational numbers $\lambda_i$, lying between $0$ and $1$ and satisfying the equality and inequality family, occurs as the sequence of slopes of a fiber of $G$ at some point of $S$. This does not seem too unreasonable, as the set of all $(\lambda_i)$ satisfying these conditions is finite, as is the set of slope-types of all possible fibers of $G$ over $S$.

I should mention that the inequality family was suggested to me by the following beautiful conjecture of Katz: if $X$ is smooth and proper over a finite field $k$, with Hodge numbers in dimension $i$ given by $h^0 = h^{0,i}$, $h^1 = h^{1,i-1}$, …, $h^i = h^{i,0}$, and if we consider the characteristic polynomial of the arithmetic Frobenius $F^a$ operating on some reasonable

[53] This is not proved now in complete generality, but is proved in $X$ lifts formally to characteristic zero, and I certainly believe it to be true generally.

[54] However, the arithmetic Frobenius is not really defined, unless $k$ is finite!

cohomology group of $X$ (say, $\ell$–adic for $\ell \neq p$, or crystalline), then the Newton polygon of this polynomial should be *above* the one of the polynomial $\prod_i (1+p^i T)^h i$. In a very heuristic and also very suggestive way, this could now be interpreted (without needing to assume $k$ finite) as stating that $H^i_{\text{cris}}(X)$ is a specialization of a crystal whose sequence of slopes is 0 $h^0$ times, 1 $h^1$ times, ..., $i$ $h^i$ times. If $X$ lifts formally to characteristic 0, then we can introduce also the Hodge numbers of the lifted variety, which satisfy

$$(h')^0 \leq h^0, \ldots, (h')^i \leq h^i,$$

and one should expect a strengthening of Katz's conjecture to hold, with the $(h')^j$ replaced by the $h^j$. Thus the transcendental analog of an $F$–crystal in characteristic $p$ seems to be something like a Hodge structure of a Hodge filtration, and the sequence of slopes of such a structure should be defined as the sequence in which $j$ enters with multiplicity $(h')^j = \operatorname{rank} \operatorname{gr}^j$.[55] I have some idea how Katz's conjecture with the $h^i$s (not the $(h')^i$s, at least for the time being) may be attacked by the machinery of crystalline cohomology, at least at the level of the first inequality among the family. At the same time, the formal argument involving exterior powers, outlined afterwards, gives the feeling that it is really the first inequality $\lambda_1 \leq \lambda'_1$ that is essential, and the others should follow once we have a good general framework.

I would very much appreciate your comments on this general nonsense—again, I imagine that most of it is quite familiar to you, under a different terminology.

Very sincerely yours,

A. Grothendieck
Bures May 11, 1970

[55]N.B.: Katz made his conjecture only for global complete intersections. However, I would not be as cautious as he!