Gr-CATEGORIES[1]
Summary of the thesis of Hoàng Xuân Sính

The purpose of these notes is to study the Gr-categories and give some applications of them. Below is a brief description of the organisation of the work.

Chapter I gives some definitions and results, which are used continually in the sequel, on $\otimes$-categories one can find in [2], [6], [11], [14], [15], the terminology employed in this chapter being of Neantro Saavedra Rivano [14]. A *$\otimes$-category* is a category $\mathscr{C}$ together with a *law* $\otimes$, i.e. a covariant bifunctor

$$\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

$$(X, Y) \mapsto X \otimes Y$$

An *associativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z, \quad X, Y, Z \in Ob(\mathscr{C})$$

satisfying the *pentagon axiom*, i.e. all the pentagonal diagrams

$$\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes T) \arrow[dr, "a_{X \otimes Y, Z, T}"] & \\
X \otimes (Y \otimes (Z \otimes T)) \arrow[ur, "a_{X,Y,Z \otimes T}"] \arrow[d, "\mathrm{id}_X \otimes a_{Y,Z,T}"'] & & ((X \otimes Y) \otimes Z) \otimes T \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[rr, "a_{X, Y \otimes Z, T}"] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[u, "a_{X,Y,Z} \otimes \mathrm{id}_T"']
\end{tikzcd}$$

[1] This text had been transcribed by Mateo Carmona
`https://agrothendieck.github.io/`

are commutative. A $\otimes$-category together with an associativity constraint is called a *$\otimes$-associativity category*.

A *commutativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in Ob(\mathscr{C})$$

verifying the relation

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$$

The commutativity constraint c is said to be *strict* if $c_{X,X} = \mathrm{id}_{X\otimes}$ for all $X \in Ob(\mathscr{C})$. A $\otimes$-category together with a commutativity constraint is a *$\otimes$-commutative category*. A $\otimes$-commutative category is *strict* if its commutativity constraint is strict.

An *unity constraint* for a $\otimes$-category $\mathscr{C}$ is a triple $(\underline{1}, g, d)$ where $\underline{1}$ is an object of $\mathscr{C}$, g and d natural isomorphisms

$$g_X : X \xrightarrow{\sim} \underline{1} \otimes X, \quad d_X : X \xrightarrow{\sim} X \otimes \underline{1}, \quad X \in Ob(\mathscr{C})$$

such that $g_{\underline{1}} = d_{\underline{1}}$. A $\otimes$-category together with an unity constraint is a *$\otimes$-unifer category*.

A $\otimes$-category $\mathscr{C}$ together with an associativity constraint a and a commutativity constraint c is a *$\otimes$-AC category* if the *hexagonal axiom* is fulfilled, i.e. all the hexagonal diagram commutes

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "a_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[r, "a_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

A $\otimes$-category $\mathscr{C}$ together with a associativity constraint a and an unity constraint $(\underline{1}, g, d)$ is a *$\otimes$-AU category* if all the following triangles commute

$$\begin{tikzcd}
X \otimes (\underline{1} \otimes Y) \arrow[rr, "a_{X,\underline{1},Y}"] & & (X \otimes \underline{1}) \otimes Y \\
 & X \otimes Y \arrow[ul, "\mathrm{id}_X \otimes g_Y"] \arrow[ur, "d_X \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

A $\otimes$-*ACU category* is a $\otimes$-AC and AU category. An object X of a $\otimes$-ACU category $\mathscr{C}$ is *invertible* if there are two objects $X', X'' \in Ob(\mathscr{C})$ such that $X' \otimes X \simeq X \otimes X'' \simeq \underline{1}$.

A $\otimes$-*functor* from a $\otimes$-category $\mathscr{C}$ to a $\otimes$-category $\mathscr{C}'$ is a pair $(F, \check{F})$ where F is a functor $\mathscr{C} \longrightarrow \mathscr{C}'$ and $\check{F}$ an isomorphism of bifunctors

$$\check{F}_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y) \quad X, Y \in Ob(\mathscr{C})$$

A $\otimes$-functor $(F, \check{F})$ from a $\otimes$-associative category $\mathscr{C}$ to a $\otimes$-associative category $\mathscr{C}'$ is *associative* if the following diagram commutes:

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "\mathrm{id}\otimes\check{F}"] \arrow[d, "a'"'] & FX \otimes F(Y \otimes Z) \arrow[r, "\check{F}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F_a"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "\check{F}\otimes\mathrm{id}"] & F(X \otimes Y) \otimes FZ \arrow[r, "\check{F}"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

where a is the associativity constraint of $\mathscr{C}$ and a' of $\mathscr{C}'$.

A $\otimes$-functor $(F, \check{F})$ from a $\otimes$-commutative category $\mathscr{C}$ to a $\otimes$-commutative category $\mathscr{C}'$ is *commutative* if the following diagram commutes:

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\check{F}"] \arrow[d, "c'"] & F(X \otimes Y) \arrow[d, "F_c"] \\
FY \otimes FX \arrow[r, "\check{F}"] & F(Y \otimes X)
\end{tikzcd}$$

c and c' being the commutativity constraints of $\mathscr{C}$ and $\mathscr{C}'$ respectively.

A $\otimes$-functor $(F, \check{F})$ from a $\otimes$-category $\mathscr{C}$ with an unity constraint $(\underline{1}, g, d)$ to a $\otimes$-category $\mathscr{C}'$ with an unity constraint $(\underline{1}', g', d')$ is a$\otimes$-*unifer functor* if there exists an isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{1}' \otimes FX \arrow[r, "\hat{F}\otimes\mathrm{id}_{FX}"] & F\underline{1} \otimes FX \arrow[d, "\check{F}"] \\
FX \arrow[u, "g'_{FX}"] \arrow[r, "Fg_X"'] & F(\underline{1} \otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes \underline{1}' \arrow[r, "\mathrm{id}_{FX}\otimes\hat{F}"] & FX \otimes F\underline{1} \\
FX \arrow[u, "d'_{FX}"] \arrow[r, "Fd_X"'] & F(X \otimes \underline{1}) \arrow[u, "\check{F}"']
\end{tikzcd}$$

It follows from the definition that the isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$, it it exists, is unique.

A $\otimes$-*AC functor* is an $\otimes$-associative and commutative functor.

A $\otimes$-*ACU functor* is a $\otimes$-associative, commutative and unifer functor.

Let $(F,\breve{F})$ and $(G,\breve{G})$ be $\otimes$-functors from a $\otimes$-category $\mathscr{C}$ to a $\otimes$-category $\mathscr{C}'$. A $\otimes$-*morphism* from the $\otimes$-functor $(F,\breve{F})$ to the $\otimes$-functor $(G,\breve{G})$ is a morphism of functors $\lambda : F \longrightarrow G$ such that the following diagram commutes

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{\breve{F}} & F(X \otimes Y) \\ {\scriptstyle \lambda_X \otimes \lambda_Y}\big\downarrow & & \big\downarrow {\scriptstyle \lambda_{X\otimes Y}} \\ GX \otimes GY & \xrightarrow{\breve{G}} & G(X \otimes Y) \end{array} \qquad X, Y \in \mathrm{ob}\,\mathscr{C}$$

Chapter II is a study of Gr-categories and Pic-categories. A Gr-*category* is a $\otimes$-AU category, the objects of which are all invertible, and the base category a groupoid (i.e. all arrows are isomorphisms). Thus a Gr-category is like a group. We obtain from this definition that if $\mathscr{P}$ is a Gr-category, the set $\pi_0(\mathscr{P})$ of the classes up to isomorphism of objects of $\mathscr{P}$, together with the operation induced by the law $\otimes$ of $\mathscr{P}$, is a group; the group $\mathrm{Aut}(\underline{1}) = \pi_1(\mathscr{P})$ is a commutative group; and for all $X \in Ob(\mathscr{P})$

$$\gamma_X : u \mapsto u \otimes \mathrm{id}_X = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

$$\delta_X : u \mapsto \mathrm{id}_X \otimes u = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

We attribute thus to a Gr-category $\mathscr{P}$ two groups $\pi_0(\mathscr{P})$ and $\pi_1(\mathscr{P})$ where $\pi_1(\mathscr{P})$ is commutative. Furthermore we can define an action of $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$ by the formula

$$su = \delta_X^{-1}\gamma_X(u)$$

for $s \in \pi_0(\mathscr{P})$ represents d by X and $u \in \pi_1(\mathscr{P})$. The commutative group $\pi_1(\mathscr{P})$ together with this action is a left $\pi_0(\mathscr{P})$-module.

Let M be a group, N a left M-module. A *preepinglage* of type (M,N) for a Gr-category $\mathscr{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

compatible wit the action of M on N, $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$. A Gr-category *preeplingled* of type (M,N) is a Gr-category $\mathscr{P}$ together with preepinglage. Finally, an *arrow* of Gr-categories preepingled of type (M,N) $(\mathscr{P},\varepsilon) \longrightarrow (\mathscr{P}',\varepsilon')$ is a $\otimes$-associative functor such that the following triangles commute:

$$\begin{tikzcd} \pi_0(\mathscr{P}) \arrow[rr] & & \pi_0(\mathscr{P}') \\ & M \arrow[ul, "\varepsilon_0"] \arrow[ur, "\varepsilon'_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_1(\mathscr{P}) \arrow[rr] & & \pi_1(\mathscr{P}') \\ & N \arrow[ul, "\varepsilon_1"] \arrow[ur, "\varepsilon'_1"'] & \end{tikzcd}$$

It follows from this definition that a such arrow is a $\otimes$-equivalence. Thus the set of the equivalence classes of Gr-categories preepingled of type (M,N) is equal to the set of connected components of the category of Gr-categories preepingled of type (M,N).

If we consider the cohomology group $\mathrm{H}^3(M,N)$ of the group M with coefficients N (in the sense of the group cohomology [12]) we obtain a canonical bijection between the set $\mathrm{H}^3(M,N)$ and the set of the equivalence classes of Gr-categories preepingled of type (M,N).

A Pic-*category* is a Gr-category together with a commutativity constraint which is compatible with its associativity constraint, i.e. the hexagon axiom is satisfied. Thus a Pic-category is like a commutative group. We verify immediately that a necessary condition for the existence of a Pic-category structure on a Gr-category is that $\pi_0(\mathscr{P})$ must be commutative and act trivially on $\pi_1(\mathscr{P})$. A Pic-category is *strict* if its commutativity constraint is strict.

Let M, N be abelian groups. A *preepinglage* of type (M,N) for a Pic-category $\mathscr{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

A Pic-category *preepingled* of type (M,N) is a Pic-category together with a preepinglage. We define the *arrow* of such objects in the same way as for Gr-categories.

For next propositions, let us consider two complexes of free abelian groups

$$L_\bullet(M) : L_3(M) \xrightarrow{d_3} L_2(M) \xrightarrow{d_2} L_1(M) \xrightarrow{d_1} L_0(M) \longrightarrow M$$

$$'L_\bullet(M) :' L_3(M) \xrightarrow{'d_3} {}' L_2(M) \xrightarrow{'d_2} {}' L_1(M) \xrightarrow{'d_1} {}' L_0(M) \longrightarrow M$$

where

$$L_0(M) = {}' L_0(M) = Z[M]$$
$$L_1(M) = {}' L_1(M) = Z[M \times M]$$
$$L_2(M) = {}' L_2(M) = Z[M \times M \times] + Z[M \times M]$$
$$L_3(M) = {}' L_3(M) + Z[M]$$
$${}' L_3(M) = Z[M \times M \times M \times M] + Z[M \times M \times M] + Z[M \times M]$$
$$d_1[x,y] = {}' d_1[x,y] = [y] - [x+y] + [x]$$
$$d_2[x,y] = {}' d_2[x,y] = [x,y] - [y-x]$$
$$d_2[x,y,z] = {}' d_2[x,y,z] = [y,z] - [x+y,z] + [x,y+z] - [x,y]$$
$$d_3[x,y,z,t] = {}' d_3[x,y,z,t] = [y,z,t] - [x+y,z,t] + [x,y+z,t] - [x,y,z+t] + [x,y,z]$$
$$d_3[x,y,z] = {}' d_3[x,y,z] = [x,y,z] - [x,z,y] + [z,x,y] - [y,z] + [x+y,z] - [x,z]$$
$$d_3[x,y] = [x,y] + [y,x] = {}' d_3[x,y]$$
$$d_3[x] = [x,x],$$

so that $L_\bullet(M)$ is a truncated resolution of M. One obtains a canonical bijection between the set of the equivalence classes of Pic-categories preepingled of type (M,N) and the set $\mathrm{H}^2(\mathrm{Hom}('L_\bullet(M),N))$. The exactitude of the complex $L(M)$ gives us e triviality of the classification of Pic-categories preepingled of type (M,N) which are strict, i.e. all Pic-categories preepingled of type (M,N) which are strict, are equivalent.

Finally chapter III gives us the construction of the solution of two universal problems: *problem of making objects "unity objects"* and *problem of reversing objects*.

Let $\mathscr{A}$ be a $\otimes$-AC category, $\mathscr{A}'$ another $\otimes$-AC category whose base category is a groupoid, and $(T,\breve{T}) : \mathscr{A}' \longrightarrow \mathscr{A}$ a $\otimes$-AC functors. We try to make the objects TA' of $\mathscr{A}$, $A' \in Ob(\mathscr{A}')$, "unity object", i.e. we try to get:

1°) A $\otimes$-ACU category $\mathscr{P}$

2°) A $\otimes$-AC functor $(D,\breve{D}) : \mathscr{A} \longrightarrow \mathscr{P}$

3°) A $\otimes$-isomorphism

$$\lambda : (,\breve{D}) \circ (T,\breve{T}) \xrightarrow{\sim} (I_{\mathscr{P}}, \breve{I}_{\mathscr{P}})$$

where $(I_{\mathscr{P}}, \check{I}_{\mathscr{P}})$ is the $\otimes$-constant functors $\underline{1}_{\mathscr{P}}$ from $\mathscr{A}'$ to $\mathscr{P}$. The triple $(\mathscr{P}, (D, \check{D}), \lambda))$ must be universal for triples $(\mathscr{Q}, (E, \check{E}), \mu)$ satisfying 1°, 2°, 3°.

For the description of the triple $(\mathscr{P}, (D, \check{D}), \lambda)$, we introduce a quotient category of a $\otimes$-AC category as follows:

Let $\mathscr{A}$ be a $\otimes$-AC category, Y a *multiplicative subset* of $\mathscr{A}$ (that means a subset of the set of all endomorphisms of $\mathscr{A}$ such that $Id_X \in Y$ for all $X \in Ob(\mathscr{A})$ and the tensor product of two arrows of Y belongs to Y). The *$\otimes$-AC category quotient* A^Y of $\mathscr{A}$ with respect to Y is the solution of the universal problem

$$(K, \check{K}) : \mathscr{A} \longrightarrow \mathscr{B}, \quad K(u) = \text{id for all } u \in Y$$

where B is a $\otimes$-AC category and $(K, \check{K})$ a $\otimes$-AC functor.

Now let us give an idea of the construction of the triple $(\mathscr{P}, (D, \check{D}, \lambda))$ for $\mathscr{A}' \neq \varnothing$:

1° $\mathrm{Ob}(\mathscr{P}) = \mathrm{Ob}(\mathscr{A})$

2° $Hom_{\mathscr{P}}(A, B) = \varphi(A, B)_{/R_{A,B}}, A, B \in \mathrm{Ob}(\mathscr{P})$

$\varphi(A, B)$ being the set of all triples (A', B', u) where $A', B' \in \mathrm{Ob}(\mathscr{A}')$, $u \in Fl(\mathscr{A})$, $u : A \otimes TA' \longrightarrow B \otimes TB'$; $R_{A,B}$ the equivalence relation defined in $\varphi(A, B)$ as follows

$$(A'_1, B'_1, u) R_{A,B} (A'_2, B'_2, u)$$

if and only if there are objects C'_1, C'_2 and isomorphisms

$$u' : A'_1 \otimes C'_1 \xrightarrow{\sim} A'_2 \otimes C'_2, \quad v' : B'_1 \otimes C'_1 \xrightarrow{\sim} B'_2 \otimes C'_2$$

of $\mathscr{A}'$ such that the following diagram commutes in $\mathscr{A}^{\varphi}$ $\otimes$-AC quotient category of $\mathscr{A}$ with respect to the multiplicative subset of $\mathscr{A}$ generated by the endomorphisms of the form $T(c_{A',A'})$;

$$
\begin{tikzcd}
A\otimes T(A'_1\otimes C'_1) \arrow[r, "\mathrm{id}\otimes\check{T}^{-1}"] \arrow[d, "\mathrm{id}\otimes Tu'"] & A\otimes(TA'_1\otimes TC'_1) \arrow[r, "a"] & (A\otimes TA'_1)\otimes TC'_1 \arrow[r, "u_1\otimes\mathrm{id}"] & (B\otimes TB'_1)\otimes TC'_1 \arrow[d, "a^{-1}"'] \\
A\otimes T(A'_2\otimes C'_2) \arrow[d, "\mathrm{id}\otimes\check{T}^{-1}"] & & & B\otimes(TB'_1\otimes TC'_1) \arrow[d, "\mathrm{id}\otimes\check{T}"'] \\
A\otimes(TA'\otimes TC'_2) \arrow[d, "a"] & & & B\otimes T(B'_1\otimes C'_1) \arrow[d, "\mathrm{id}\otimes Tv'"'] \\
(A\otimes TA'_2)\otimes TC'_2 \arrow[r, "u_2\otimes\mathrm{id}"] & (B\otimes TB'_2)\otimes TC'_2 \arrow[r, "a^{-1}"] & B\otimes(TB'_2\otimes TC'_2) \arrow[r, "\mathrm{id}\otimes\check{T}^{-1}"] & B\otimes T(B'_2\otimes C'_2)
\end{tikzcd}
$$

We denote by $[A',B',u]$ the class which has (A',B',u) as representative

3° Composition of arrows in $\mathscr{P}$. Let $[A',B,u]:A\longrightarrow B$, $[B'',C'',v]:B\longrightarrow C$ be arrows in $\mathscr{P}$. We define

$$[B'',C'',v]\circ[A',B',u]=[A'\otimes B'',B'\otimes C',w]:A\longrightarrow C$$

where w is such that the following diagram commutes:

$$
\begin{tikzcd}
A\otimes T(A'\otimes B'') \arrow[r, "\mathrm{id}\otimes\check{T}^{-1}"] \arrow[ddddd, "\omega"] & A\otimes(TA'\otimes TB'') \arrow[r, "a"] & (A\otimes TA')\otimes TB'' \arrow[r, "u\otimes\mathrm{id}"] & (B\otimes TB')\otimes TB'' \arrow[d, "a^{-1}"] \\
& & & B\otimes(TB'\otimes TB'') \arrow[d, "\mathrm{id}\otimes c"] \\
& & & B\otimes(TB''\otimes TB') \arrow[d, "a"] \\
& & & (B\otimes TB'')\otimes TB' \arrow[dd, "v\otimes\mathrm{id}"] \\
& & & \\
C\otimes T(B'\otimes C'') \arrow[r, "\mathrm{id}\otimes\check{T}"] & C\otimes(TB'\otimes TC'') \arrow[r, "\mathrm{id}\otimes c"] & C\otimes(TC''\otimes TB') \arrow[r, "a^{-1}"] & (C\otimes TC'')\otimes TB'
\end{tikzcd}
$$

4° $\otimes$-structure on $\mathscr{P}$

$$A\otimes E\ (\text{in } \mathscr{P})=A\otimes E\ (\text{in } \mathscr{A})$$

$$[A',B',u]\otimes[E',F',v]=[A'\otimes E',B'\otimes F',w]$$

where w is defined by the commutative diagram (1)

5° ACU constraint in $\mathscr{P}$.

$([A',A',a\otimes\mathrm{id}],[A',A',c\otimes Id],(1_{\mathscr{P}}=TA'_0, g_A=[A'_0\otimes A',A',t_A],d_A=[A'_0\otimes A',A',p_A]))$

where A'_0 is a fixed object of $\mathscr{A}'$, A' an arbitrary object of $\mathscr{A}'$, g_A and d_A natural isomorphisms

$$g_A : A \longrightarrow 1_{\mathscr{P}} \otimes A, \quad d_A : A \longrightarrow A \otimes 1_{\mathscr{P}}$$

with t_A and p_A defined by the commutativity diagrams (2)

6° $(D,\check{D})$ is defined by

$$DA = A, \quad D_u = [A',A',u\otimes \mathrm{id}_{TA'}], \quad \check{D}_{A,B} = \mathrm{id}_{A\otimes B}$$

(1)

$$\begin{tikzcd}
(A\otimes TA')\otimes(E\otimes TE') \arrow[r, "u\otimes v"] \arrow[d, "a"'] & (B\otimes TB')\otimes(F\otimes TF') \arrow[d, "a"] \\
((A\otimes TA')\otimes E) \arrow[d, "a^{-1}\otimes\mathrm{id}"'] & ((B\otimes TB')\otimes F)\otimes TF' \arrow[d, "a^{-1}\otimes\mathrm{id}"] \\
(A\otimes(TA'\otimes E))\otimes TE' \arrow[d, "(\mathrm{id}\otimes c)\otimes\mathrm{id}"'] & (B\otimes(TB'\otimes F))\otimes TF' \arrow[d, "(\mathrm{id}\otimes c)\otimes\mathrm{id}"] \\
(A\otimes(E\otimes TA'))\otimes TE' \arrow[d, "a\otimes\mathrm{id}"'] & (B\otimes(F\otimes TB'))\otimes TF' \arrow[d, "a\otimes\mathrm{id}"] \\
((A\otimes E)\otimes TA')\otimes TE' \arrow[d, "a^{-1}"'] & ((B\otimes F)\otimes TB')\otimes TF' \arrow[d, "a^{-1}"] \\
(A\otimes E)\otimes(TA'\otimes TE') \arrow[d, "\mathrm{id}\otimes\check{T}"'] & (B\otimes F)\otimes(TB'\otimes TF') \arrow[d, "\mathrm{id}\otimes\check{T}"] \\
(A\otimes E)\otimes T(A'\otimes E') \arrow[r, "w"] & (B\otimes F)\otimes T(B'\otimes F')
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
A\otimes(TA'_0\otimes TA') \arrow[r, "\mathrm{id}\otimes\check{T}"] \arrow[d, "a"'] & A\otimes T(A'_0\otimes A') \arrow[d, "t_A"] \\
(A\otimes TA'_0) \arrow[r, "c\otimes\mathrm{id}"] & (TA'_0\otimes A)\otimes TA'
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(TA'_0\otimes TA') \arrow[r, "\mathrm{id}\otimes\check{T}"] \arrow[d, "a"'] & A\otimes T(A'_0\otimes A') \arrow[d, "p_a"] \\
(A\otimes TA'_0)\otimes TA' \arrow[r, equal] & (A\otimes TA'_0)\otimes TA
\end{tikzcd}$$

7° The ⊗-isomorphism

$$\lambda : (D,\breve{D})\circ(T,\breve{T}) \xrightarrow{\sim} (T_{\mathscr{P}},\breve{I}_{\mathscr{P}})$$

is defined by natural isomorphisms

$$DTA' = TA' \xrightarrow{\lambda_{A'}=[A'_0,A',c_{TA',TA'_0}]} I_{\mathscr{P}}A' = TA'_0 \quad A' \in \mathrm{ob}\,\mathscr{A}'$$

$\mathscr{P}$ is called the *⊗-ACU category* of the ⊗-AC category $\mathscr{A}$ with respect to $(A',(T,\breve{T}))$.

For the problem of reversing objects, let us consider a ⊗-category $\mathscr{C}$ with a ACU constraint $(a,c,(1,g,d))$ a ⊗-category $\mathscr{C}'$ with a ACU constraint $(a',c',(1',g',d'))$, the base category of which is a groupoid, and a ⊗-ACU functor $(F,\breve{F}) : \mathscr{C}' \longrightarrow \mathscr{C}$. We try to find a ⊗-ACU category $\mathscr{P}$ and a ⊗-ACU functor $(D,\breve{D}) : \mathscr{C} \longrightarrow \mathscr{P}$ having the following properties

1° DFX' is invertible in $\mathscr{P}$ for all $X' \in \mathrm{Ob}(\mathscr{C}')$

2° For all ⊗-ACU functor $(E,\breve{E})$ from $\mathscr{C}$ to a ⊗-ACU category $\mathscr{Q}$ such that EFX' is invertible in $\mathscr{Q}$ for all $X' \in \mathrm{Ob}(\mathscr{C}')$, there exists a ⊗-ACU functor $(E',\breve{E}')$, unique up to ⊗-isomorphism, from $\mathscr{P}$ to $\mathscr{Q}$ such that $(E,\breve{E}) \simeq (E',\breve{E}'\circ(D,\breve{D}))$.

This problem is reduced by the first by putting $\mathscr{A}' = \mathscr{C}'$, $\mathscr{A} = \mathscr{C}\times\mathscr{C}'$, $TX' = (FX',X')$ and by remarking that if $\mathscr{C}$, $\mathscr{C}'$, $\mathscr{Q}$ are ⊗-ACU categories, $\mathrm{Hom}^{\otimes,ACU}(\mathscr{C},\mathscr{Q})$ the category of all ⊗-ACU functors from $\mathscr{C}$ to $\mathscr{Q}$, then there is a canonical equivalence of categories

$$\mathrm{Hom}^{\otimes,ACU}(\mathscr{C}\times\mathscr{C}',\mathscr{Q}) \longrightarrow \mathrm{Hom}^{\otimes,ACU}(\mathscr{C},\mathscr{Q})\times\mathrm{Hom}^{\otimes,ACU}(\mathscr{C}',\mathscr{Q})$$

The ⊗-ACU category $\mathscr{P}$ thus defined is called the *⊗-category of fractions* of the category $\mathscr{C}$ with respect to $(\mathscr{C}',(F,\breve{F}))$. The ⊗-category of fractions of $\mathscr{C}^{is}$ with respect to $(\mathscr{C}^{is},(\mathrm{id}_{\mathscr{C}^{is}},\mathrm{id}))$ is a Pic-category which is called the Pic-*envelope* of the category $\mathscr{C}$, and denoted by $\mathrm{Pic}(\mathscr{C})$.

For an application of the Pic-envelope, we take $\mathscr{C} = P(R)$, category of all finitely generated R-modules (R a ring) and $\mathscr{P} = \mathrm{Pic}(P(R))$, then one obtain

$$\pi_0(\mathscr{P}) \simeq K^0(R)$$

$$\pi_1(\mathscr{P}) \simeq K^1(R)$$

where $K^0(R)$is the Grothendieck group and $K^1(R)$ the whitehead group [1].

The use of the $\otimes$-category of fractions of a $\otimes$-ACU category gives us the following result:

Let $\mathscr{C}$ be a $\otimes$-ACU category, Z an arbitrary object of $\mathscr{C}$ different from the unity object $\underline{1}$, S the functor from $\mathscr{C}$ to $\mathscr{C}$ defined by

$$X \mapsto X \otimes Z.$$

The *suspension category* of the $\otimes$-ACU category $\mathscr{C}$ defined by the object Z is the triple $(\mathscr{P}, i, p)$ which solves the universal problem for triples $(\mathscr{Q}, j, q)$ where $\mathscr{Q}$ is a category, j a functor from $\mathscr{C}$ to $\mathscr{Q}$, and q an equivalence of categories from $\mathscr{Q}$ to $\mathscr{Q}$, so that the following diagram commutes

$$\begin{tikzcd} \mathscr{C} \arrow[r, "S"] \arrow[d, "j"'] & \mathscr{C} \arrow[d, "j"] \\ \mathscr{Q} \arrow[r, "q"] & \mathscr{Q} \end{tikzcd}$$

up to natural isomorphism. In the case where $\mathscr{C}$ is the homotopy category of pointed topological spaces $_*$ together with the smash [] (the smash [] of two spaces X and Y, with the base points x_0 and y_0, is obtained from the product $X \times Y$ by [] the subset [] to a single point which is taken as the base point of []), and the usual ACU constraint; and Z is the 1-sphere S^1 hence S^1 is the suspension functor, we get the well-known definition of the suspension category.

Let $\mathscr{C}'$ be the $\otimes$-stable subcategory of $\mathscr{C}$ generated by Z and $\mathscr{P}$ the $\otimes$-category of fractions of $\mathscr{C}$ with respect to $(\mathscr{C}', (F, \mathrm{id}))$ where $F : \mathscr{C}' \longrightarrow \mathscr{C}$ is the inclusion functor. One obtains a functor $G : \mathscr{P} \longrightarrow \mathscr{P}$ from the suspension category to the $\otimes$-category of fractions of $\mathscr{P}$. If G is not faithful, that is the case of the homotopy category of pointed topological spaces $_*$ together with the smash $\wedge$ and the 1-sphere S^1; then it is impossible to construct in $\mathscr{P}$ a law $\otimes$ such that $\mathscr{P}$ together with this law is a $\otimes$-ACU category, iZ invertible in $\mathscr{P}$, and i embedded in a pair $(i, \check{i})$ which is a $\otimes$-ACU functor from $\mathscr{C}$ to $\mathscr{P}$.

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