Introduction
to
Functorial Algebraic Geometry

After a Summer course by

A. GROTHENDIECK

Vol. I
Affine Algebraic Geometry

SUNY at Buffalo

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INTRODUCTION

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Foreword

These notes were primarily written from tape recordings of *Grothendieck*'s lectures during his visit at SUNY in the summer of 1973. However, there recordings were supplemented by exercises, references to classical algebraic geometry, historical comments and concrete quotations of such "Bibles" as SGA, EGA, etc.[1]

Grothendieck himself does not assume any responsibility for the publication of these notes; I believe however that since no adequate "textbooks" exist today and the original publications present considerable difficulties to the beginner, a publication of this kind will help a much wider audience. This is intended as an introduction to the sources SGA, EGA,...: with concrete references to Ch., §and page number, I have completed the bibliography by referring to other introductory publications such as the *Dieudonné* articles, *Mumford*'s lecture notes, etc. Most of them contain sketchy or no proofs at all, or they are addressed to a different type of reader, cf. Macdonald-*Schemes*, addressed to classical algebraic geometers. I hope that these lecture notes, directed primarily to beginning graduate students, will bring the gap, between the previously mentioned lecture notes and the sources. To aid the newcomer, the reader will find many more details than is customary in informal publications of this type. I took advantage of some of the oral repetitions to insert *"summaries"* at the beginning of most paragraphs (mostly using the tape-recorded lectures, or my own initiative if I could not find any better source). There are many complete proofs, and others are almost complete with very few, really trivial details left to the reader.

No knowledge of "old-time" or 'classical" algebraic geometry was assumed although *Grothendieck* himself gave examples involving plane algebraic curves or surfaces, etc. In many points, especially in the introduction for future applied mathematicians and in the Summary of

[1]The names or authors and/or titles of books, papers, etc. between " " refer to the Bibliography.

the course, I tried to build some bridges with "old-time" algebraic geometry based on the study of algebraic varieties instead of *schemes*. If this might seem contrary to *Grothendieck*'s mathematical spirit, it is definitively not unfaithful to his current philosophical or sociological worries. In his prior visit to Buffalo, and in many other places as well, *Grothendieck* campaigned against *expert knowledge* and technology. How can we ignore that many people feel disappointed if they do not see the words algebraic curve or surfaces on page one in an Algebraic Geometry text? Or they complain "a priori", just by "hearsay" that there is a lot of algebra and categorical language but — where is the geometry? I try to overcome these psychological difficulties or prejudices in order to emphasize the major simplifications introduced by *Grothendieck*. The introduction for applied mathematicians is addressed to any person with a bachelor degree in Mathematics but it should be understood also by theoretical physicist and engineers…

I hope that very soon after a final revision of the whole course the second part dealing with the category of schemes will appear.

I am grateful to many colleagues and students in the audience who helped me in preparing these notes, mainly: J. Duskin, B. Fell, L. Gupta, R. Hamsher, N. Kazarinoff, M. Klun, I. Ozaki, F. C. Schanuel, G. Sicherman, J. Winthrop by correcting all kinds of mistakes, typographical, linguistic, mathematical…, and I am especially grateful first of all to *Grothendieck* who was so kind with everybody and so generous with his time. He lectured several times for periods of almost seven hours, with only a few short breaks. Who can believe that he is not interested in Mathematics anymore?

Last but not least, I am very grateful too to the typist, Mrs. Gail Berti, for her excellent job and her angelic patience, correcting and retyping the manuscript dozens of times and never once protesting.

Buffalo, June 1974
Federico Gaeta

0. Propaganda for applied mathematicians

Not more than one century ago the distinction between pure and applied mathematics was to a large extend artificial and unimportant.

1. Prerequisites

We shall assume familiarity with the basic algebraic structures: groups, rings, fields; the volumes of

2. Summary of Vol. I

In spite of all *Grothendieck*'s revolutions, algebraic geometry is still a "geometrical theory of equations". This is made clear in Chap. I starting with a very general system of polynomial equations $S = \{f_j(T_i) = 0\}$ with arbitrary index sets I, J with coefficients in a ground ring k (commutative, with unit)[2]. We shall consider solutions (a_i) $(i \in I)$ with coordinates a_i belonging *to any* k*-algebra* (cf. Ch. I, §2) in particular *we do not restrict ourselves to solutions in* k^I.

[2]The ring considered here will be commutative rings with unit. Any ring homomorphism $f : A \longrightarrow B$ preserves the unit ($f(1_A) = 1_B$). Cf. Ch. I, §2.

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0. Summary

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14. Appendix on sheaves of sets[3]

The following notes on sheaves of sets were delivered by *Grothendieck* at the beginning lectures of his course on topoi. To include this in §7 would be too digressive; thus I prefer to include it in the Appendix, which should be particularly useful for readers with a prior knowledge of FAC; at the same time it would be helpful as an introduction to the abstract approach of *Godement*'s Bible.

[3]These notes were written with the collaboration of J. Winthrop.

I am going to talk about the *theory of topoi*. I like to see it as a king of *generalization of classical general topology*. As a background we shall assume some familiarity with topological spaces, continuous maps, homeomorphisms, etc. etc. and on the other hand familiarity with the language of categories. Later we shall give some motivation for introducing something more general than topological spaces and give examples. But to understand the theory of topoi we shall also require some familiarity with the language of sheaves on a topological space. Now, I guess that this notion is not that familiar to everybody, so I will not assume anything known about it. I will review the standard theory of sheaves of sets[4] over topological spaces.

14.1 Presheaves of sets. Let X be a topological space. We consider the set $\mathscr{O} = \mathrm{Op}(X)$ of open subsets on X, i.e. the subset $\mathrm{Op}(X)$ of the power set $\mathfrak{P}(X)$[5] defining the topology on X. We recall that the axioms of a topology require that $\mathrm{Op}(X)$ contain $\varnothing$ and X itself and be stable under arbitrary unions and finite intersections. $\mathrm{Op}(X)$ is a partially ordered set (with the ordering defined by inclusion) and therefore $\mathrm{Op}(X)$ already forms a category, by abuse of language. We denote this category by $\mathscr{O}$ or $\mathrm{Op}(X)$. As in any partially ordered set C if U, V are objects $U, V \in C$ the set of "homomorphisms" $\mathrm{Hom}(U,V)$ from U to V is either empty if U is not contained in V or contains just the "inclusion map": $U \hookrightarrow V$ of U into V:

$$\mathrm{Hom}(U,V) = \begin{cases} \varnothing & U \not\subset V \\ \longrightarrow: & U \longrightarrow V \end{cases} \tag{1.1}$$

The composition of arrows $U \longrightarrow V \longrightarrow W$ is defined in the obvious way. (We have no choice.) This particular construction of a category makes sense for any partially ordered C whatever; it does not use the fact that $C = \mathrm{Op}(X)$.

In other words, the category $\mathrm{Op}(X)$ has as objects the open sets of X and as

[4] *Grothendieck* will consider mainly *sheaves of sets*, thus we shall omit this remark in the future. However later he will introduce various algebraic structures. The reader, knowing FAC, can take advantage of these lecture notes to strengthen his knowledge of sheaf theory by separating the topological properties from the algebraic ones.

[5] From the French part = subset : $\mathfrak{P}(X) = 2^X$.

arrows the graphs of the inclusion relations[6].

A *presheaf* F on X is, by definition, a contravariant functor from the category $\mathrm{Op}(X)$ to the category of sets. In other words F goes from the opposite category $\mathscr{O}^\circ$ of $\mathrm{Op}(X)$ to the category Set of sets.

Let us recall what that means:

1) To every object of the category, i.e. to every open set U of X, we associate a set $F(U)$, whose elements are called sections of F over U[7].

2) To every inclusion $i : U \hookrightarrow V$ we associate a map:

$$p_V^U : F(V) \longrightarrow F(U) \tag{1.2}$$

between the corresponding sets (going in the opposite direction) where $p_V^U = F(i)$ is also denoted by the restriction symbol:

$$F(V) \longrightarrow F(V)|U = p_V^U(F(V)) \tag{1.3}$$

and the following "evident" axioms are satisfied

1) *Transitivity*: If another open set W contains V, i.e. $U \longrightarrow V \longrightarrow W$ are inclusions of open sets in X, then we have arrows $F(W) \longrightarrow F(V) \longrightarrow F(U)$ in the category of sets, preserving compositions. In other to the commutative diagram on the left (see below) correspond a commutative diagram on the right

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, hook] & V \arrow[d, hook] \\ & W \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} F(U) & F(V) \arrow[l] \\ & F(W) \arrow[ul] \arrow[u] \end{tikzcd} \tag{1.4}$$

In words:

[6]This is true for the category attached to an ordered set $(S,<)$: Graph $<= \{(x,y) \in S\times S | x < y\}$. In our case $S = \mathrm{Op}(X)$ and $<$ is the inclusion $\subset$.

[7]This terminology comes from an old definition of sheaves over X, in terms of an étale covering space $S \xrightarrow{p} X$ (cf. next §). If U is open in X a section over U is a map $s : U \longrightarrow S$ such that $ps = 1_U$.

Identity: F should transform identities into identities, i.e. to the identity map $U \xrightarrow{id} U$ corresponds the identity map $F(U) \longrightarrow F(U)$ from $F(U)$ to itself.

The category $\mathrm{Presh}(X) = \mathrm{Hom}(\mathcal{O}^\circ, \mathrm{Set})$ *of presheaves on* X *is defined as the category of all functors* $\mathcal{O}^\circ \longrightarrow \mathrm{Set}$, *i.e. an object of* $\mathrm{Presh}(X)$ *is a functor* $F : \mathcal{O}^\circ \longrightarrow \mathrm{Set}$.

A homomorphism $F \xrightarrow{f} G$ from a presheaf F to a presheaf G (both over X) is, by definition of *homomorphism of functors*[8], i.e. a collection of maps $F(U) \longrightarrow G(U)$ ($\forall U \in \mathrm{Ob}(\mathcal{O}(X))$) compatible with the restriction maps; i.e. for every open set U of X we have a map $F(U) \longrightarrow G(U)$, such that the following diagram commutes

$$\begin{array}{ccc} F(U) & \xrightarrow{f(U)} & G(U) \\ \uparrow & & \uparrow \\ F(V) & \xrightarrow{f(V)} & G(V) \end{array} \tag{1.6}$$

where the vertical arrows are the restriction maps corresponding to U and V. Moreover, the composition $F \xrightarrow{f} G \xrightarrow{g} H$ of morphisms of presheaves is defined by considering in an obvious way he diagram

$$\begin{array}{ccccc} F(U) & \xrightarrow{f(U)} & G(U) & \xrightarrow{g(U)} & H(U) \\ \uparrow & & \uparrow & & \uparrow \\ F(V) & \xrightarrow{f(V)} & G(V) & \xrightarrow{g(V)} & H(V) \end{array} \tag{1.7}$$

with all squares commutative.

This is the little "general nonsense" needed to construct the category $\mathrm{Presh}(X)$. So far *we have not used the properties of the category* $\mathcal{O}(X)$ of open sets, we used just the properties of morphisms of functors = "natural transformations", but we will use them now to define a particular type of presheaves, the *sheaves* over X.

14.2 Sheaves of sets. Thus, we need to introduce some axioms on presheaves *characteristic of sheaves*.

[8] *Grothendieck* prefer *"homomorphism of functors"* rather the synonymous "natural transformation"

We shall express theses axioms in terms of two properties:

Let F be a presheaf on X for every open set U of X and for every open covering $\{U_i\}_{i\in I}$ of U ($\Leftrightarrow U = \bigcup_{i\in I} U_i$)[9] we consider the restriction map of $F(U)$ into each of the $F(U_i)$ ($\forall i \in I$) and therefore a map from $F(U)$ to the product of the $F(U_i)$

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \tag{2.1}$$

Then F is *separated* iff for any choice of U and of the covering $\{U_i\}_{i\in I}$ the previous map $F(U) \longrightarrow \prod_{i\in I} F(U_i)$ is *injective*.

Let us state this property in another way. First of all the set associated with any open $U \subseteq X$ be a presheaf F is called *the set of sections of F over U*[10], and for any inclusion $U \hookrightarrow V$ the restriction map $F(V) \longrightarrow F(U)$ defines *the set of restricted sections*. Then the fact that a presheaf F is separated means that for any open covering $\{U_i\}_{i\in I}$ of U ($\in \mathrm{Ob}(\mathrm{Op}(X))$) *a section of F over U is known iff all of its restrictions to the U_i' are known* i.e. the arrow of (2.1) is an *injective arrow*, i.e. we can write instead of (2.1)

$$F(U) \hookrightarrow \prod_{i\in I} F(U_i) \tag{2.2}$$

which means, in words, that *any section of F over U can be identified with the collection of sections* of its restrictions $|U_i$ for every $i \in I$.

The second question arising, in characterizing sheaves as particular cases of presheaves, is whether any system of sections $\varphi_i = F(U_i)$, U_i open for every $i \in I$, can be obtained by restrictions from a section $F(U)$ over $U = \bigcup_{i\in I} U_i$. A necessary condition for such an $F(U)$ to exist is the "matching property":

$$\varphi_i | U_i \cap U_j = \varphi_i | U_i \cap U_j \tag{2.3}$$

for every pair $(i,j) \in I \times I$. This is clearly necessary because of the transitivity property of the restriction maps.

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[9] The union of the U_i can be defined in terms of the partial ordering in $\mathrm{Op}(X)$ by the condition that U is the Supremum of the U_i for $i \in I$.

[10] This terminology comes from an old direct definition of sheaves over X, in terms of an étale covering space $S \xrightarrow{p} X$ (cf. next §). If U is open in X a section over U is a map $S : U \longrightarrow S$ such that $ps = 1_U$.

Definition. 1.1. *We say that a presheaf F over X is a* sheaf *if for every $U \in$* Ob $\mathcal{O}(X)$ *and every open covering of U the map (2.1) (which in general is not injective) is indeed injective (i.e. F is separated) and its image consists of all elements of $\prod F(U_j)$ satisfying the "matching property" (2.3) for every pair $(i,j) \in I \times I$.* We can write Definition 1.1 in diagrammatic terms, as the condition that *the following sequence*

$$F(U) \hookrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j) \tag{2.4}$$

is exact.

To interpret (2.4) we need to apply F to the two inclusions $U_i \cap U_j \longrightarrow U_i$ and $U_i \cap U_j \longrightarrow U_j$; thus we have the two arrows $F(U_i) \longrightarrow F(U_i)|U_i \cap U_j$ and $F(U_j) \longrightarrow F(U_j)|U_i \cap U_j$. The kernel of the double arrow $F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$ means the system of $\prod_{i \in I F(U_i)}$ such that both restrictions maps agree for every pair (i,j).

The usual meaning of exactness ($\Leftrightarrow \mathrm{Im} = \mathrm{Ker}$) is then verified in (2.3).

For two sheaves F and G a *sheaf morphism $F \longrightarrow G$* is by definition the same as a presheaf morphism between F and G. Thus, we can construct a category denoted by $\mathrm{Top}(X)$ (the category of sheaves over X) which is a *full subcategory* of the category $\mathrm{Presh}(X)$ of presheaves over X:

$$\mathrm{Top}(X) \hookrightarrow \mathrm{Presh}(X) = \mathrm{Hom}(\mathcal{O}^\circ, \mathrm{Set}). \tag{2.5}$$

Now it is time to give examples to show that *this notion of sheaf is a very natural one*; i.e. that *sheaves occur very frequently*.

Examples. Let E be any set and let us define a presheaf F on X by

$$F(U) = \mathrm{Map}(U,E) = E^U \quad \forall U \in \mathrm{Ob}(\mathrm{Op}(X)) \tag{2.6}$$

i.e. $F(U)$ is the set of all maps from U to E.

If we have $U \hookrightarrow V$, we consider $F(V) = \mathrm{Map}(V,E)$ and the restriction map $F(V) \longrightarrow F(U)$ is defined by restriction maps from V to E to maps from U to E, i.e. if $\varphi : V \longrightarrow E$ belongs to $\mathrm{Map}(V,E)$ then $\varphi|U : U \longrightarrow E$ belongs to $\mathrm{Map}(U,E)$. Since the restriction of maps is a transitive operation, we have certainly defined a presheaf. This presheaf F defined by (2.6) is in fact a sheaf. This means that whenever an open set U of X is covered by a family U_i $(i \in I)$

then to give a map from U to E amounts to the same as to give maps from U_i to E in such a way that these maps "match up" in the $U_i \cap U_j$ for every choice of $(i, j) \in I \times I$. In fact this would be even true if the U_i would not be open in X. Therefore we have a sheaf called the *sheaf of maps from X to E*.

Now, many sheaves which occur naturally in Mathematics are subsheaves of this one, but to explain that I need to define subpresheaves and subsheaves.

Let F, G be presheaves over X. Then we say that G is a subpresheaf of F, and we write $G \hookrightarrow F$ iff the two following conditions are true:

a) For every open $U \subset X$, $G(U) \hookrightarrow F(U)$ (i.e. $G(U)$ is a subset of $F(U)$)

b) For every inclusion $U \in V$ of open sets U, V the restriction map $G(V) \longrightarrow G(U)$ is induced by $F(V) \longrightarrow F(U)$, i.e. we have an obvious commutative diagram. This is what is called in general a subfunctor of a given functor.

In other words G *is a subpresheaf of* F *iff it can be defined by a family of subsets* $G(U)$ *of the given* $F(U)$, *and the only condition that we need to impose is that the family be stable under the restriction maps*.

Now let us assume that both F and G are sheaves over X. G *is a subsheaf* of F iff G is a subpresheaf of F. So a subsheaf of F consists of subsets $G(U)$ of the $F(U)$ which are stable under the restrictions maps, according to the presheaf law, but now we need to make sure that the presheaf G is also a sheaf, thus we need an extra condition. It is evident that if G is a subpresheaf of a sheaf F, then G is separated, because if an open set U of X is covered by open U_i ($i \in I$) as before, the $G(U_i)$ determine the $F(U_i)$, and $\prod_{i \in I} F(U_i)$ determines $F(U)$ because F is a sheaf, and $F(U)$ determines $G(U)$.

We have just proved that *a subpresheaf of a separated presheaf is also separated*. Now, what would it mean that G is not only separated but also G is a sheaf? This would mean that the map $U \longrightarrow G(U)$ *is of local type*[11]. Summarizing: A subsheaf G of a sheaf F over X is defined by the collection $G(U) \subset F(U)$ ($U \in \mathrm{Ob}\, \mathscr{O}(X)$), compatible with restriction maps and in which the property of a set belonging to the $G(U)$ of local nature.

[11]We say that a property P of open sets of X is local if and only if for every open U of X, P holds in X if P holds for any open U_i of any open covering $(U_i)_{i \in I}$ of U.

Example. Let us assume that the previous set E is a topological space and consider for each U the set $\mathrm{Cont}(U,E)$ of continuous maps from U to E[12], where the composition of ps is the inclusion map of U *into* X.

Then $\mathrm{Cont}(U,E) \hookrightarrow \mathrm{Map}(U,E)$ and this property is compatible with the restriction maps (the restriction of a continuous map is continuous).

Besides we know that continuity is a property of local character: A map: $f : U \longrightarrow E$ is continuous if and only if all the restrictions $f|U_i$ to a family of opens U_i covering U are continuous! Thus we have defined a subsheaf of the sheaf of maps from X to E, viz.: the sheaf of continuous local maps $U \longrightarrow E$.

Assume now, for instance that X is a differentiable variety (of any fixed differentiability class C^r $(r \geq 1), C^\infty$) and that E is also a differentiable variety (of the same C^r), then we can consider the set of local differentiable maps $\mathrm{Diff}(U,E)$ and obtain a subsheaf of the previous sheaf of continuous local maps…[13]. It is well known that the differentiability property is of purely local nature, preserved by restrictions: If U is covered by U_i then a map $f : U \longrightarrow E$ is C^r-differentiable iff the $f|U_i : U_i \longrightarrow E$ is C^r-differentiable for every $i \in I$ for any open covering $(U_i)_{i\in I}$ of U!

3) Now we can extend this ad libitum, by taking for instance instead of differentiable varieties, analytic varieties,… or algebraic varieties,…, so any kind of "variety" defines a kind of sheaves…

4) Now there is still another kind of example of subsheaves of a sheaf in terms of fibre spaces:

Let us assume now that E is a fibre space over X, i.e. E, X are topological spaces and we consider a continuous $p : E \longrightarrow X$ of E into X[14].

The triple (E,X,p) is a general continuous fibre space where X is the base space, E is the total space and p the projection.

Then every open subset U of X let us look to the set $\Gamma(U,E/X)$ of all continuous maps $s : U \longrightarrow E$ such that $ps = 1_U$. Such maps s are commonly called

[12]This case contains the previous one of $\mathrm{Map}(U,E)$ if we endow E with the discrete topology.

[13]Sheaves of germs of local (continuous C^h, differentiable $h \geq 1$, analytic), etc.

[14]p is not assumed to be onto.

section of E over U (or just local sections if we do not want to mention U). For any inclusion $U \hookrightarrow V$ we have an induced map $\Gamma(V, E/X) \longrightarrow \Gamma(U, E/X)$, i.e. the restriction maps transform V-sections in U-sections. Now it is very clear that a map $s : U \longrightarrow E$ is a continuous section of U over X if and only if the restriction maps $s|U_i \longrightarrow E$ are continuous sections over U_i for any choice of an open covering of U.

This sheaf is called the sheaf of local sections of the fibre space $E \xrightarrow{p} X$. It is because of this particular situation that the name section of F over U was introduced.

There are any other examples suggested by the audience?

Schanuel suggested that it is convenient to point out that the previous example can be obtained from this one (sheaf of local continuous sections of a fibre space) by just considering the product $E \times X \ldots$, Grothendieck agrees.

If E is endowed with the discrete topology and we introduce the product topology in $X \times E$ then the previous sheaf can be interpreted also as a sheaf of local sections of $X \times E \xrightarrow{p} X$ (where $\longrightarrow$ denotes the projection on the first factor). Then a section $s : U \longrightarrow X \times E$ be identified with a function $U \longrightarrow E$ $(x \longrightarrow (x, f(e)))$, and this property is compatible with restriction maps,...

Are there any further examples?...

...Let me point out some more examples:

Let X be a locally compact topological space, and if we associate with any open set U of X the set of *Radon* measures U, this property is compatible with restriction maps. Besides this property is local and we get a sheaf (the sheaf of Radom measures on open sets of the l.c.s. X).

If X is a differentiable variety we can consider the sheaf of distributions on X (where a distribution is a continuous linear functional over vector spaces of local functions on $X \ldots$). Distributions can be localized and we get a sheaf again.

Another example:

For every open set U of X let us assign the subsets of U closed in U. Then we have the transitivity property of restriction maps and we obtain a presheaf. This presheaf is a sheaf because $C \subset U$ is closed in U iff $C \cap U_i$ is closed in U_i for every covering of U. (i.e. the property of being closed is a local property).

In general any properties of subsets of an open set of a local nature enables to define a sheaf on X, for instance if X is an analytic variety we can define a sheaf of local analytic subsets of X.

Generally speaking one can say that sheaves are the most systematic tool to obtain global information, stating from local information, i.e. sheaves enables to "integrate" local data to global properties.

Now let $\mathscr{E} = (E, X, p)$ ($\mathscr{E}$ for short) be a fibre space over X. We associate to E a sheaf $\widetilde{E}$, called the sheaf of continuous local sections of E.

By associating with every fibre space E over X the sheaf $\widetilde{E}$ we obtain a functorial correspondence $E \longrightarrow \widetilde{E}$. In other words if we have a morphism $E \longrightarrow F$ of fibre spaces over X (that means a continuous map between total spaces making commutative the triangle below):

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

this allows us to define a morphism $\widetilde{E} \longrightarrow \widetilde{F}$. In fact, whenever we have a section $s : U \longrightarrow E$ on an open set U of X, we obtain a section fs of F over the same U. This maps $s \longrightarrow fs$ are compatible with restriction maps so we are going to obtain a homomorphism of sheaves $\widetilde{f} : \widetilde{E} \longrightarrow \widetilde{F}$. The map $f \longrightarrow \widetilde{f}$ for a variable f is compatible with composition of maps between fibre spaces over X and thus *we obtain a functor*:

$$\mathrm{Fib}(X) \longrightarrow \mathrm{Top}(X) \tag{1.3}$$

form the category of fibre spaces over X (denoted by $\mathrm{Fib}(X)$) *to the category* $\mathrm{Top}(X)$ of sheaves over X.

The first question that might come to our minds is, can we reconstruct an object in $\mathrm{Fib}(X)$ just by knowing its image in $\mathrm{Top}(X)$?, i.e. can we reconstruct a fibre space $\mathscr{E}$ over X in terms of $\widetilde{E}$; (the corresponding sheaf). In formal terms we *would like to know* whether or not the functor (1.3) *is fully faithful*?

We shall see that is not so, as we can convince ourselves quite readily. In order to see it let us consider the particular case when X is reduced to a single point e,

$X = \{e\}$. Then the category of fibre spaces over a one-point space is just equivalent to the category of topological spaces, because for any such spaces E there is just one map $E \longrightarrow \{e\}$.

Therefore $\mathrm{Fib}(e) \approx \mathfrak{T}$ (category of all topological spaces). On the other hand, what is the category of sheaves $\mathrm{Top}\{e\}$ over a one-point space $\{e\}$? We consider again the maps associating with $\{e\}$ the sections $\{e\} \longrightarrow E$. Up to a canonical equivalence a sheaf over a one-point space is known iff we know the corresponding E. Thus the functor (1.3) reduces, when $X = \{e\}$ to a functor: $\mathfrak{T} \longrightarrow \mathrm{Set}$ from the category of all topological spaces to the category of sets, which associates with the topological space E the underlying abstract set, i.e. we we obtain a forgetful functor, in which we just "forget" the topology of E! Now it is obvious *that this functor* is not fully faithful, i.e. we can't recover the topology of E from its underlying set. Therefore to give a fibre space $\mathscr{E}$ over X is something much more precise that to give the sheaf $\widetilde{E}$ of local continuous sections!

14.3 The category $\acute{E}t(X)$ of étale coverings of X. We can wonder now whether or not we can define some full subcategory $\acute{E}t(X)$ of the category $\mathrm{Fib}(X)$ of fibre spaces over X, such that the restriction to $\acute{E}t(X)$ of the functor (1.3), becomes fully faithful. For instance in the case of a one-point space $\{e\}$, which are the topological spaces whose topology is known if we know the corresponding underlying set ? There are several choices. One of them would be to introduce the discrete topology: for a given set S there is just one discrete topology over S. Therefore if we take the restriction of the functor (1.3) to the category of discrete topological spaces over $\{e\}$ we obtain an equivalence of categories. Now we want to generalize this categorical equivalence to the general case of a general basis X. Thus we want to define a full subcategory of $\mathrm{Fib}(X)$ which in the $\{e\}$ case reduces to the category of discrete topological spaces over $\{e\}$. The property which looks "nice" is thus of a topological space E which is *étale over X*. We shall define it!

A continuous map $\pi : E \longrightarrow X$ between topological spaces is called *étale* if it is a local isomorphism, in the following sense:

For every point $x \in E$ there exists an open neighbourhood $V \subset E$ of x such that $\pi(V)$ is open in X such that π induces a map from V into $\pi(V)$ which is a homeomorphism, which means that π looks like a collection of local homeomorphisms between some open sets of the space E upstairs and their projections $\pi(V)$ downstairs.

When this property holds we say also that π is an *étale morphism* between the topological spaces E, X. This is in fact a pretty old one in the theory of functions of one complex variable, where certain maps appear which are étale over an open subset of the complex plane.

Let us give some examples of *"étaleness"*.

The most evident case is the inclusion map $U \longrightarrow X$ of an open set U into X. This is the standard model!

Another example: Let us take a discrete topological space I, i.e. an abstract set endowed I endowed with its discrete topology and let us consider the product space $E = I \times X$. This means that we take the disjoint some of the copies $a \times X$ ($a \in I$), which are open in E. Then the projection map $I \times X \longrightarrow X$ reduces to the "identity map" $(a, x) \mapsto x$ on these open copies of X.

Now we shall construct the category $\acute{E}t(X)$ of étale covering spaces of X, which is a subcategory of Fib(X). Let us look at the restriction functor (1.3) (going from Fib(X) to the category Top(X) of sheaves over X) to the subcategory $\acute{E}t(X)$. Then we get the generalisation of the case of discrete spaces over a one point space $\{e\}$ that we were looking for! We obtain an *equivalence of categories*:

$$\acute{E}t(X) \xrightarrow{\approx} \mathrm{Top}(X) \tag{3.1}$$

on their own (étale coverings of $X \Leftrightarrow$ objects of $\acute{E}t(X)$) and we can jiggle back and forth between both languages. It turns out that for certain operations that we can perform on sheaves, the language of sections is by far the most convenient and in other the language of étale spaces is better.

Examples. Now maybe I should give some examples. Is there a suggestion?

Question: Is there no translation in English for the French adjective *étale*?

Answer: No, this is a question that was raised about fifteen years ago. In French we say: *un espace étalé dans un autre…*, which mans a space "spread out over another", but in terms o a morphism, to say that it is "spread out" doesn't look good, so why not introduce another word into English…? Duskin asks why not say just a local homeomorphism? Grothendieck's answer is that when we deal in analogous contexts with differentiable, analytic or algebraic spaces we would like to use the same word, since the formal properties are the same (instead of introducing local diffeomorphisms, local biholomorphic maps, etc). It is better to have a word which applies to all these particular cases…[15].

Question: Is it true that the oldest examples of étale maps come from the construction of Riemann surfaces with several copies of the $\mathbf{C}$-plane?

Answer: Yes, provided you drop the branch points! All right, I will give this example:

1) Let us take the map $f : \mathbf{C} \longrightarrow \mathbf{C}$ of the affine complex line, in itself given by $x \longrightarrow x^n$ $(n \geq 2)$. Then $f(0) = 0$. The *the restriction* $f|\mathbf{C}^*$ *onto itself* $(f(0) = 0)$, $\mathbf{C}^* = \mathbf{C} - \{0\}$ *is étale*. In fact the fibre over an $x \neq 0$ on the second plane is the set $x_1, x_2, \ldots, x_n$ of the n n^{th}-roots of x in such a way that in choosing one of them, the others are obtained by multiplication with the n^{th}-roots of unity $\exp(2\pi i k n^{-1})$ $(h = 0, 1, \ldots, n-1)$. Then for any open neighbourhood U of x not containing 0 obtain n copies of U covering U homeomorphically.

   Thus we see a difference in behavior of the map according to whether or not $x = 0$. The restriction to a neighbourhood of zero is not étale (0 is a ramification point).

2) The previous example can be extended to any dominant morphism $X \xrightarrow{f} Y$ ($\Leftrightarrow f(X)$ is dense in Y) between two complex irreducible non singular curves. Throwing out finitely many points of Y (ramification points) the restriction of the map f to $X - \bigcup_R f^{-1}(R)$ (R-ramification points) is étale.

3) A third example of covering *étale everywhere* is the covering map $\hat{X} \longrightarrow X$ of a topological connected manifold by its universal covering space.

[15] Later on in private conversation *Grothendieck* told me that also in his native German the word *étale* is used,…instead of looking for a translation.