$f: C \to C'$ fibred $\ni$ $R^0 f_*$ (loc. const) $\subset$ loc. const. does not seem to imply that the base change functors are heq's.

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Grothendieck's approach to the fundamental groupoid of a category C:

Consider the category of $F: C^\circ \longrightarrow$ Sets which transform all maps into isomorphisms. This category $\mathcal{L}$ is the full subcategory of locally constant objects in the topos $C^\wedge = \underline{\mathrm{Hom}}(C^\circ, \text{Sets})$. Each object X in C determines a fibre functor

$$\begin{aligned} \mathcal{L} &\longrightarrow \text{Sets} \\ F &\longmapsto F(X) \end{aligned}$$

and $\underline{\underline{\pi}}C$ can be identified with the dual of the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{L}, \text{Sets})$ consisting of the functors of the above form.

To see this we can define $\underline{\underline{\pi}}C$ by localizing C with respect to all its morphisms. Then

$$\mathcal{L} = (\underline{\underline{\pi}}C)^\wedge$$

and the inclusion of $\mathcal{L}$ in $C^\wedge$ can be viewed as the inverse image for the morphism of topoi

$$C^\wedge \longrightarrow \mathcal{L}$$

induced by the functor $C \longrightarrow \underline{\underline{\pi}}C$. Now points in $C^\wedge$ may be identified with $\mathrm{Pro}(C)$, and since $\underline{\underline{\pi}}C$ is a groupoid, $\mathrm{Pro}(\underline{\underline{\Pi}}\mathcal{C}) \cong \underline{\underline{\Pi}}\mathcal{C}$. Thus we can recover $\underline{\underline{\Pi}}\mathcal{C}$ as fibre functors on $\mathcal{L}$.

Let us now consider the functor

$$\mathcal{C} \xrightarrow{u} \underline{\underline{\Pi}}\mathcal{C}$$

and factor it in the standard way

$$\mathcal{C} \longrightarrow \tilde{\mathcal{C}} \longrightarrow \underline{\underline{\Pi}}\mathcal{C}$$

where an object of $\tilde{\mathcal{C}}$ is a triple $(x, y, u(x) \to y)$. Think of $\underline{\underline{\Pi}}\mathcal{C}$ as being the category of 1-conn. coverings of $\mathcal{C}$ and u as the functor assigning to x the pointed universal covering over x. Then we can view $\tilde{\mathcal{C}}$ as being the fibred cat. over $\mathcal{C}$ consisting of an $x \in \mathcal{C}$, a 1-conn. covering y, and a point in the fibre of y over x. Clearly $\tilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$. (In general if we have a functor $\mathcal{C} \longrightarrow \mathcal{G}$ with $\mathcal{G}$ a groupoid, then the category of $(x, y, f(x) \to y)$ is equivalent to $\mathcal{C}$.)
On the other hand, the fibre of $\tilde{\mathcal{C}}$ over y is simply the 1-conn. covering category of $\mathcal{C}$ defined by y.

Now suppose that we are given a functor

$$f : \mathcal{C} \longrightarrow \mathcal{C}'.$$

~~Suppose that f is fibred and that the base change functors $\mathcal{C}_y \longrightarrow \mathcal{C}'_{y'}$ are all heq's. We can c~~

We can consider the full subcategory $\mathcal{L}$ of $\mathcal{C}^\wedge$ consisting of those F which are locally in the image of f^*. What this means is that for every $X_0 \in \mathcal{C}$, there exists a $G : (\mathcal{C}'/fX_0)^\circ \longrightarrow \text{Sets}$ and an isomorphism

$$(*) \qquad F(X) \simeq G(fX)$$

of functors on $\mathcal{C}/X_0$. Clearly this implies that when $X \to X_0$ is $\ni$ $fX \xrightarrow{\sim} fX_0$, then $F(X) \xleftarrow{\sim} F(X_0)$. In particular F is locally constant on each fibre of f.

Conversely, suppose we are given F and X_0 and we want to construct G so that $(*)$ holds on $\mathcal{C}/X_0$. Assume f is fibred. Then given $Y \in \mathcal{C}'/fX_0$, say $u : Y \to fX_0$, put

$$G(Y) = F(u^* X_0).$$

Then because $(uv)^* = v^* u^*$ it follows that G is a well-defined functor on $\mathcal{C}'/fX_0$. If F is locally constant on each fibre of f, then given $X \in \mathcal{C}/X_0$ we have

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr] & & X_0 \\ & u^*X_0 \arrow[ur] & \end{tikzcd}$$

$$fX \xrightarrow{u = f(\alpha)} fX_0$$

and
$$F(X) \xleftarrow{\sim} F(u^* X_0) = G(fX), \quad \text{(better notation:}$$

$$F(X) \xleftarrow{\sim} F(fX \times_{fX_0} X_0) \;)$$

showing F is locally in the image of f^*.

<u>Conclude</u>: If $f: C \longrightarrow C'$ is fibred, then $F \in C^\wedge$ is locally in the image of f^* iff F is locally constant on each fibre of f.

So now let $\mathcal{L}$ be the full subcat of $C^\wedge$ consisting of these functors. Let $\overline{C}$ denote the category obtained by inverting all of the arrows in C which become isomorphisms in C', or equivalently inverting just the arrows in the fibres. Then we may identify

$$\mathcal{L} = \overline{C}^\wedge$$

and the inclusion of $\mathcal{L}$ in $C^\wedge$ is just the inverse image for the functor

$$C \longrightarrow \overline{C}$$

It is cleare that

$$\overline{C} \longrightarrow C'$$

is fibred and the fibre over y is

$$\overline{C}_y = \underline{\underline{\Pi}} C_y .$$

(Actually, this requires a good proof. Intuitively the base change functor

$$C_y \longrightarrow C_{y'}$$

will extend to a functor of groupoids

$$\underline{\underline{\Pi}} C_y \longrightarrow \underline{\underline{\Pi}} C_{y'}$$

and the resulting fibred category in groupoids will clearly be.

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$\bar{\mathcal{C}}$.)

We can think of objects of $\bar{\mathcal{C}}$ as the 1-connected coverings of the fibres of f. g assigns to $x \in \mathcal{C}$ the pointed 1-connected covering with basepoint over X.

~~Suppose from now on that all base change functors $\mathcal{C}_y \to \mathcal{C}_{y'}$ are heq's.~~

Suppose now that $f: \mathcal{C} \to \mathcal{C}'$ is fibred and that

(*) for any local coeff system of sets (resp. grps, resp. ab. gps) L on $\mathcal{C}$, $R^q f_*(L)$ is locally constant for $q=0$ (resp. $q \le 1$, resp. all q). Here $R^q f_*$ is computed using covariant functors so that

$$R^q f_*(L)_y = H^q(\mathcal{C}_y, L).$$

I want then to conclude that all the base change functors $\mathcal{C}_y \to \mathcal{C}_{y'}$ are heq's. I can assume $\mathcal{C}'$ is 1-connected by pulling back to any 1-conn. covering, as this doesn't change the fibres.

Now for any set S we have that $f_*(S)$ is locally constant, hence for any y

$$\begin{array}{ccccccc} H^0(\mathcal{C}', f_*(S)) & \xrightarrow{\sim} & f_*(S)_y & = & H^0(\mathcal{C}_y, S) & = & \mathrm{Hom}(\pi_0 \mathcal{C}_y, S) \\ \| & & & & & & \\ H^0(\mathcal{C}, S) & = & \mathrm{Hom}(\pi_0 \mathcal{C}, S) & & & & \end{array}$$

and so we conclude that $\pi_0 \mathcal{C}_y \xrightarrow{\sim} \pi_0 \mathcal{C}$ for all

y. ~~[illegible]~~

We can suppose C is connected since f is the sum of its restrictions to the components of C and since the $R^q f_*(L)$ decompose. Then C_y is also connected.

Let F be a covering of C, i.e. a local coefficient system on C and suppose F has a section over C_y. Since $f_* F$ is locally constant, hence constant

$$H^0(C_y, F) = (f_* F)_y \xleftarrow{\sim} H^0(C', f_* F) = H^0(C, F)$$

so the ~~[illegible]~~ section over C_y may be extended to all of C. This implies that

$$\pi_1(C_y, x) \twoheadrightarrow \pi_1(C, x)$$

for any x in C_y. (Take F to be the covering defined by the $\pi_1(C, x)$-set

$$\pi_1(C, x) / \mathrm{Im}\, \pi_1(C_y, x).)$$

Let $\tilde{C} \xrightarrow{p} C$ be the universal covering of C; it is fibred so the composite $\tilde{C} \xrightarrow{p} C \xrightarrow{f} C'$ is fibred. It is clear that $\tilde{C}_y = C_y \times_C \tilde{C}$ is the induced covering. Now if L is a local system on C, then

$$R^q(fp)_*(L) = R^q f_*(p_* L)$$

is locally constant. If we succeed in establishing that the base change $\tilde{C}_y \longrightarrow \tilde{C}_{y'}$ is an heq, it will follow that $C_y \to C_{y'}$ is. (Our problem is that we have only

$$H^*(C_{y'}, L) \xrightarrow{\sim} H^*(C_y, L)$$

for L which come from $\mathcal{C}$ and not all local coeff. systems on $\mathcal{C}_y$.) Thus we may assume $\mathcal{C}$ is 1-connected.

So we are now in the following situation. $f: \mathcal{C} \to \mathcal{C}'$ is fibred, $\mathcal{C}, \mathcal{C}'$ are 1-connected, and for all local coeff systems L on $\mathcal{C}$, $R^q f_*(L)$ is loc. const.

Let $u: y' \longrightarrow y$ be an arrow in $\mathcal{C}'$ and

$$u^* : \mathcal{C}_y \longrightarrow \mathcal{C}_{y'}$$

the base change functor. Let $x \in \mathcal{C}_y$ and consider

$$\pi_1(\mathcal{C}_y, x) \longrightarrow \pi_1(\mathcal{C}_{y'}, u^* x).$$

We have for any group G that $R^1 f_*(G)$ is locally constant hence

$$\begin{array}{ccc} H^1(\mathcal{C}_{y'}, G) & \xrightarrow{\sim} & H^1(\mathcal{C}_y, G) \\ \| & & \| \\ \mathrm{Hom}(\pi_1 \mathcal{C}_{y'}, G) & \xrightarrow{\sim} & \mathrm{Hom}(\pi_1 \mathcal{C}_y, G) \end{array}$$

where the Hom's are taken in the category of groups up to inner automorphisms. Since this holds for all G we can conclude that

$$\pi_1(\mathcal{C}_y, x) \xrightarrow{\sim} \pi_1(\mathcal{C}_{y'}, u^* x).$$

Suppose we now consider the factorization of f

$$\mathcal{C} \xrightarrow{g} \overline{\mathcal{C}} \xrightarrow{h} \mathcal{C}'$$

where $\overline{C}$ is obtained from C by inverting the maps in the fibres. Then $\overline{C}$ is fibred with $\overline{C}_y = \Pi\, C_y$. By what we have just shown the base changes $\overline{C}_y \to \overline{C}_{y'}$ are equivalences of groupoids.

Let x_0 be a basepoint of C and set

$$G = \pi_1(C_{fx_0}, x_0)$$

For each $y \in C'$ choose a principal G-bundle $P_y \to C_y$ which is a universal covering. This is possible as we have shown $\pi_1 C_y \simeq G$. For each map $u: y' \to y$ in C there exists at least one covering map θ_u

$$\begin{tikzcd} P_y \arrow[r, "\theta_u"] \arrow[d] & P_{y'} \arrow[d] \\ C_y \arrow[r, "u^*"] & C_{y'} \end{tikzcd}$$

compatible with the action of G. Any two choices for θ_u differ by right multiplication by an element of the center Z of G. (In effect we only have to check this for autos. Any auto. θ of P_y is determined by its effect on the fibre over the basepoint and hence is a map $P_{y,x} \to P_{y,x}$ compatible with left mult. by $\pi_1(C_y, x)$ and also right mult. by G. The only maps $G \to G$ which commute with both left & right mult. are mult. by elements of Z.) Therefore we obtain a compatible family of covering maps.

$$\begin{tikzcd} P_y/Z \arrow[r] \arrow[d] & P_{y'}/Z \arrow[d] \\ \mathcal{C}_y \arrow[r] & \mathcal{C}_{y'} \end{tikzcd}$$

whence we obtain a principal covering of $\mathcal{C}$ with group G/Z. Since $\pi_1 \mathcal{C} = 0$, it follows this covering is trivial, so restricting to the fibre $\mathcal{C}_{fx_0}$ we see that $G/Z = 0$. Thus G is abelian. So we conclude

$\pi_1 \mathcal{C}_y$ is <u>abelian</u>.

<u>Remark:</u> $\overline{\mathcal{C}} \longrightarrow \mathcal{C}'$ is a gerb for the group $G = \pi_1 \mathcal{C}_y$. It is non-trivial, otherwise we would able to find a coherent system of P_y and hence construct a non-trivial covering of $\mathcal{C}$.

Now we have reached the following problem. Consider the map $\mathcal{C} \xrightarrow{g} \overline{\mathcal{C}}$ whose fibres are essentially the universal coverings of the fibres of f. given a map

$$\begin{tikzcd} P_y \arrow[r] \arrow[d] & P_{y'} \arrow[d] \\ \mathcal{C}_y \arrow[r] & \mathcal{C}_{y'} \end{tikzcd}$$

in $\overline{\mathcal{C}}$, we know that $H^*(\mathcal{C}_{y'}, A) \xrightarrow{\sim} H^*(\mathcal{C}_y, A)$ for all abelian groups A, but we don't know this for all G-modules, $G = \pi_1 \mathcal{C}_y$. Thus for example we have

$$\begin{array}{ccccccccccc}
0 \to & H^2(G,A) & \to & H^2(\mathcal{C}_{y'},A) & \to & H^2(P_{y'},A)^G & \to & H^3(G,A) & \to & H^3(\mathcal{C}_{y'},A) \\
& \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
0 \to & H^2(G,A) & \to & H^2(\mathcal{C}_{y},A) & \to & H^2(P_{y},A)^G & \to & H^3(G,A) & \to & H^3(\mathcal{C}_{y},A)
\end{array}$$

which shows that

$$(\pi_2 P_{y'})_G \xrightarrow{\sim} (\pi_2 P_y)_G .$$

~~Thus for example it appears that it~~

I don't see how to get anything better. Thus

$$\mathcal{C} \xrightarrow{g} \bar{\mathcal{C}}$$

$Rg_*(A)$ is some complex of covariant functors on $\bar{\mathcal{C}}$ such that when pushed down to $\mathcal{C}$, it becomes locally constant.

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