June 15, 1973.

Prof Grothendieck,

The last proposition stated in the course on alg groups was

"Prop: If k is a field, G an affine alg. monoid (resp, an affine alg group).

a) $G = \varprojlim G_i$ where each G_i is an affine alg monoid of finite type over k. (resp, affine alg. groups of finite type over k).

b) If G is of finite type, ∃ a closed immersion (of monoids).

$$G \hookrightarrow GL(n)_k$$ "

I still see difficulty in accepting b). The closed immersion of monoids corresponds to a surjective mapping of coalgebras

$$A \twoheadleftarrow B$$

B an algebra representing GL(n)

A an algebra representing G,

B has an antipodal map (corresponding to the "inverse" map in the on the group object in the cat. Aff_k)

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the surjectivity $B \twoheadrightarrow A$ will induce the antipodal map on the k-algebra A and thus "force" it to be a group object (not the more general monoid object) in Aff_k.

Consider the example

$$\{0,1\} : Alg_k \longrightarrow Sets$$
$$k' \longmapsto \mathrm{Hom}_{k\text{-alg}}(k\times k, k').$$

Make $A = k\times k$ into a coalgebra over k by

$$A \xrightarrow{\pi_0} A\otimes A$$
$$(a,b) \longmapsto (a,b)\otimes(1,0) + (1,1)\otimes(0,b).$$

the two projections $p_1 : (a,b) \longmapsto a$.
$p_2 : (a,b) \longmapsto b$

represent k-valued points of $\{0,1\}$

multiplication table:

π	p_1	p_2
p_1	p_1	p_2
p_2	p_2	p_2

(This is easy to check: $(p_1,p_1) : (a,b)\otimes(c,d) \longmapsto ac$
(p_1,p_2) : " $\longmapsto ad$
(p_2,p_1) : " $\longmapsto bc$
(p_2,p_2) : " $\longmapsto bd$.

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so that
$$(p_1,p_1)\circ\pi_0 = p_1$$
$$(p_1,p_2)\circ\pi_0 = p_2$$
$$(p_2,p_1)\circ\pi_0 = p_2$$
$$(p_2,p_2)\circ\pi_0 = p_2 .$$

It is easy to check that $p_1 : A\otimes A \to k$ is a counit, that π_0 is co-associative, co-commutative. ∵ $\{0,1\}$ is a commutative monoid object in Aff_k.

But $\{0,1\}(k)$ can't be a group object in sets.
$$\{0,1\}(k) = \{p_1,p_2\}.$$

Moreover A can't be given a co-inversion or antipodal map. One way of characterizing the inverse σ for commut grps is that $\exists\,\sigma : G \to G$ such that

$$\begin{tikzcd} G \arrow[r, "I_G\times\sigma"] & G\times G \arrow[d, "\pi"] \\ & G \end{tikzcd}$$

factors through the final then initial object: i.e. that

$$\begin{tikzcd} G \arrow[r, "I_G\times\sigma"', "x\,\mapsto\,(x,x^{-1})\,\mapsto\, e"] \arrow[d] & G\times G \arrow[d, "\pi"] \\ e \arrow[r] & G \end{tikzcd}$$

commutes since the final + initial objs coincide.

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On the coalgebra level $\exists\ A \xrightarrow{\sigma_0} A$ s.t.

$$
\begin{tikzcd}
A & A \otimes A \arrow[l, "I_A \otimes \sigma_0"'] \\
k \arrow[u, "\text{structure}"] & A \arrow[l, "\text{counit}"] \arrow[u, "\pi_0"']
\end{tikzcd}
$$

Suppose σ_0 exists in this case:

$\sigma_0(c,d) = (\sigma_{01}(c,d), \sigma_{02}(c,d))$.

where

$\sigma_{0i} = P_i \circ \sigma_0$.

Now compute: the lower left route gives:

$$
\begin{tikzcd}
(a,a) \\
a \arrow[u, "\text{structure}"] & (a,b) \arrow[l, mapsto, "P_1"]
\end{tikzcd}
$$

The upper right route gives:

$$
\begin{array}{ccc}
(a,b)\cdot(\sigma_{01}(1,0), \sigma_{02}(1,0)) & \longleftarrow & (a,b)\otimes(1,0) + (1,1)\otimes(0,b) \\
+ & & \uparrow \\
(1,1)\cdot(\sigma_{01}(0,b), \sigma_{02}(0,b)) & & (a,b) \\
\shortparallel & &
\end{array}
$$

$$\big(a\sigma_{01}(1,0) + \sigma_{01}(0,b),\ b\sigma_{02}(1,0) + \sigma_{02}(0,b)\big)$$

This above is to be equal to (a,a) for all (a,b)? Perhaps I am very confused! or perhaps $GL(n)_k$ ~~could~~ should be changed to $\mathrm{End}_k(M)$.

I'm enjoying the courses very much. Many thanks for letting me attend.

F. J. Sewedis

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