MATHEMATICS

A. GROTHENDIECK

This text has been transcribed and edited by Mateo Carmona.
Remarks, comments, and corrections are welcome.

https://agrothendieck.github.io/

CONTENTS

SUR LA COMPLÉTION DU DUAL D'UN ESPACE VECTORIEL LOCALEMENT CONVEXE[1]

Note[2] de M. Alexandre Grothendieck, présentée par M. Élie Cartan.

Soient E un espace vectoriel localement convexe, S un ensemble de parties bornées, convexes, symétriques et fermées de E, E'_S (resp. $\widehat{E}'_S$), l'espace des formes linéaires sur E continues (resp. dont les restrictions aux éléments de S sont continues), munis de la topologie de la convergence uniforme sur les éléments de S. Soit E_0 le sous-espace engendré par $\cup S$

[]

Proposition **1**. — *Si les éléments de S sont précompacts, pour que E'_S soit complet, il faut (et il suffit) que la restriction à E_0 de tout $u \in \widehat{E}'_S$ soit continue.*

On se ramène en effet facilement à la

Proposition **2**. — *Si les éléments de S sont précompacts, E'_S est dense dans $\widehat{E}'_S$.*

[] Comme le dual de E est le même pour la topologie donnée et sa topologie faible, et que les ensembles bornés sont faiblement précompacts, ces propositions seront d'application assez générale. Signalons la

Corollaires. —

1° Si $E_0 = E$ et si les éléments de S sont précompacts, le complété de E'_S s'identifie à $\widehat{E}'_S$.

---

[2]Séance du 6 février 1950.

C. R., 1950, i^er *Semestre*. (T. 230 N° 7.).

2° *Si E'_S est complet, il en est de même de E'_T pour $T \supset S$ (S, T ensembles de parties bornées de E, sans plus).*

En utilisant le fait que E s'identifie à un dual topologique de son dual faible, on obtient de plus :

3° *Le complété de E s'identifie à l'espace des formes linéaires sur son dual dont les restrictions aux parties équicontinues sont faiblement continues, muni de la topologie de la convergence uniforme sur ces parties. (On retrouve en particulier le complété pour la topologie faible.)*

4° *Si E est complet, toute topologie localement convexe plus fine qui a même dual, et plus généralement qui puisse se définir par une famille de seminormes* semi-continues *(soit : par un système fondamental de voisinages convexes fermés) est encore complète. [En particulier la topologie forte de Mackey associée*[3] *et la topologie induite par le bidual fort*[4] *sont encore complétes.]*

5° *Si E est complet, toute forme linéaire sur son dual dont les restrictions aux parties équicontinues sont faiblement continues est faiblement continue. On retrouve ainsi un fait connu pour les espaces de Fréchet et leurs limites inductives*[4].

---

[3] Cf. G. W. MACKEY, *Transactions of the Amer. Math. Soc.*, 57, 1945, p. 155-207 et 59, 1946, p. 530-537

[4] Cf. J. DIEUDONNÉ, et L. SCHWARTZ, *La dualité dans les espaces* (F) *et* (LF) (à paraitre aux *Annales de Grenoble*, 1950).

QUELQUES RÉSULTATS RELATIFS À LA DUALITÉ DANS LES ESPACES (F)[5]

Note[6] de M. Alexandre Grothendieck, présentée par M. Arnaud Denjoy.

Soient E un espace (F), E' son dual fort, E'' son bidual fort [?][1]. Les résultats suivants répondent partiellement à certaines questions posées dans [?].

Proposition **1**. — *Toute partie bornée dénombrable de E'' est contenue dans l'adhérence faible d'une partie bornée de E.*

[6]Séance du 30 octobre 1950

[1]J. DIEUDONNÉ et L. SCHWARTZ, La dualité dasn les espaces (F) et (LF) (à paraître aux *Annales de Grenoble*, 1950). La terminologie est celle de cet article.

CRITÈRES GÉNÉRAUX DE COMPACITÉ DANS LES ESPACES VECTORIELS LOCALEMENT CONVEXES. PATHOLOGIES DES ESPACES (LF)[2]

Note[3] de M. Alexandre Grothendieck, présentée par M. Arnaud Denjoy.

La première partie de cette Note

1. —

[3]Séance du 30 octobre 1950

Soit C un espace de Banach

A GENERAL THEORY OF FIBRE SPACES WITH STRUCTURE SHEAF

Introduction

When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre spaces (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod ([]); or the "differeniable" and "analytic" (real or complex) variants of theses notions; or the notions of algebraic fibre spaces (over an abstract field k) - one is led in a natural way to the notion of fibre space with a structure sheaf $\mathbf{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with *abelian* structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\mathbf{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $\mathrm{H}^1(X, \mathbf{G})$ the set of classes of fibre spaces on X with structure sheaf $\mathbf{G}$, $\mathbf{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\mathbf{G}$, where $\mathbf{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in $\mathbf{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the las chapter, we define the cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in the sheaf of groups $\mathbf{G}$,

[]

I. General fibre spaces

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1 Notion of fibre space

Definition **1.1.1.** — *A* fibre space *over a space X is a triple (X,E,p) of the space X, a space E and a continuous map p of E into X.*

We do not require p to be onto, still less to b open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X,E,p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the *base space* of the fibre space, p the *projection*, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the *fibre* of x (in E).

Given two fibre spaces (X,E,p) and (X',E',p'), a *homomorphism* of the first into the second is a pair of continuous maps $f : X \longrightarrow X'$ and $g : E \longrightarrow E'$, such that $p'g = f p$, i.e. commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Then g maps fibres into fibres (but not necessarily *onto*!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X', f' a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of *isomorphism* of a fibre space (X,E,p) onto a fibre space (X',E',p') is clear: it is a homomorphism (f,g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms

Let (X,E,p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the *inverse image* of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x',y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x',y) = x'$. The map $g(x',y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a *homeomorphism* of the fibre of E' over x' *onto* the fibre of E over $f'x'$.

[]

1.3 Subspace, quotient, product

Let (X,E,p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E'

[]

1..4 Trivial and locally trivial fibre spaces

Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the *trivial fibre space over X with fibre F*.

All fibres are canonically homeomorphic with F.

[]

1.5 Definition of fibre spaces by coordinate transformations

Let X be a space, (U_i) a covering of X, for each

[]

1.6 The case of locally trivial fibre spaces

The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space over T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces

[]

1.7 Sections of fibre spaces

Definition **1.7.1**. — *Let* (X, E, p) *b a fibre space; a* section *of this fibre space (or, by pleonasm, a section of* E *over* X*) is a map* x *of* X *into* E *such that* ps *is the the identity map of* X*. The set of* continuous *sections of* E *is noted* $\mathrm{H}^0(X, E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x!).

The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A *section of* E *over a subset* Y of X is by definition a section of $E|Y$. If Y is open, we write $\mathrm{H}^0(Y, E)$ for the set $\mathrm{H}^0(Y, E|Y)$ of all continuous sections of E over Y.

$\mathrm{H}^0(X, E)$ as a *functor*. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. *We get thus a map, noted* f*, of* $\mathrm{H}^0(X, E)$ *into* $\mathrm{H}^0(X, E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, the so is f.

b. If f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E'' (E, E', E'' fibre spaces over X) then $(f'f) = f'f$.

Let (X, E, p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f.

II. Sheaves of sets

Throughout this exposition, we will now use the word *"section"* for *"continuous section"*.

2.1 Sheaves of sets

Definition **2.1.1**. — *Let X be a space. A* sheaf of sets *on X (or simply a* sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighbourhood U such that p induces a homeomorphism of U onto an open subset $p(U)$ of X.*

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

[]

2.2

2.3 Definition of a sheaf by systems of sets

2.4 Permanence properties

2.5 Subsheaf, quotient sheaf. Homeomorphism of sheaves

2.6 Some examples

a.

b.

c.

d. **Sheaf of germs of subsets**. Let X be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of U. If $V \subset U$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $\mathrm{H}^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of $P(X)$, for instance the sheaf of *germs* *of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

III. Group bundles and sheaves of groups

IV. Fibre spaces with structure sheaf

V. The classification of fibre spaces with structure sheaf

SOME ASPECTS OF HOMOLOGICAL ALGEBRA[1]

[1]Translation by M. L. Barr and M. Barr