Massy 22. 10. 1969

Dear Kostant,

I read again through your papers in the Amer. Journ. on "The principal three-dimensional subgroup ..." and "Lie group representations on polynomial rings", prompted by some nice work of Brieskorn on Klein singularities. I very much appreciate this work of yours, and would appreciate getting reprints of any later work you may have available. As I already felt when your work appeared, it cries for careful reconsideration in the framework of Alg. Geometry, and suggests number of interesting problems. I wonder if you know the answer to some of them. For instance, I feel it would be of interest to classify (for a given complex simple adjoint Lie group, say) quadruples (T,B,T',B'), where T and T' are maximal tori "in apposition" (in your terminology), and B and B' Borel subgroups of G containing T resp. T'. The number of conjugacy classes of such animals is equal to $\mathrm{card}(W)^2/hz\phi(h)$, where W is the Weyl group, h the Coxeter number, z the order of the center of G, and $\phi(h) = \mathrm{card}\,(\underline{Z}/h\underline{Z})^*$ is the Euler indicatrix; if you want to classify such data with moreover a generating element (principal regular in your terminology) given for the finite group of order h $T\cap N(T')$, then we get $\mathrm{card}(W)^2/hz$ choices. One of the reasons I think this question is interesting is that the stablity subgroup in G of such a quadruple is $T\cap T' = e$,* hence such a structure makes G entirely "rigid" up to exterior automorphisms. Among the first questions one may wish to ask in this direction are the following: is it true that for some, or for all, such quadruples, B and B' are "in general position" i.e. $B\cap B'$ is a torus (necessarily maximal), so that B and B' are "opposite" to each other with respect to the latter ? Is there in any sense a <u>distinguished</u> conjugacy class of such quadruples (which should then be, of course, invariant by action of exterior automorphisms) ? What are the groups of automorphisms of such quadruples (they

*NB: $N(T)\cap N(T')$ is an extension of $(\mathbb{Z}/h\mathbb{Z})^*$ by $\mathbb{Z}/h\mathbb{Z} \times \underset{\text{center of } G}{Z}$, hence of order $hz\varphi(h)$

are isomorphic to subgroups of the group of exterior automorphisms of G) ?

In a different direction, it would be interesting to know more about the canonical morphism $\underline{g} \longrightarrow \underline{h}/W$, where $\underline{g}$ is the Lie algebra of G and $\underline{h}$ a Cartan subalgebra, and of the orbits of G on $\underline{g}$, and the analoguous morphism of Steinberg $G \longrightarrow T/W$ and the orbits of G acting on itself. Do you know for instance exactly how varies the rank (= rang of the tangent map) of this ~~applixxxx~~ map ? There are some reasons to believe that ~~thisx~~ for points of unipotent orbits of dimension n-r-2 (just ~~thexnextxlowerxafter~~ the next lower after the maximal dimension n-r), the rank is r-1 in case all roots have same length (that is the corresponding diagrams $A_{n}, D_{n}, E_6, E_7, E_8$ are those corresponding to Klein singularities), and $<$ r-1 in all other cases; in the first case, I would expect that there is just one orbit as stated, by the way, and maybe this property will be caracteristic of the "homogeneous" case when all roots have same length. Also, what can be said of the dimension of the set of all Borel subgroups containing a given element $g \in G$, resp. whose Lie algebra contains a given element $x \in \underline{g}$? If 2N = n-r is the number of roots and if the dimension of the orbit of the given element is 2N-2i, I would expect the dimension of the previous variety to be equal to i; the inequality $\leqslant$ i would imply, by generalizing a depth argument of Brieskorn, that the singularities of the fibers of your map $\underline{g} \rightarrow \underline{h}/W$ (resp. Steinberg's $G \rightarrow T/W$ in case G is simply connected instead of adjoint) are "rational", i.e. if F is such a fiber and $F' \xrightarrow{f} F$ a resolution of singularities of F, than $R^i f_*(\underline{O}_{F'}) = 0$ for $i > 0$. Maybe the answer to most of the questions I am asking are well known and I am just ignorant; for some of them, it would indeed be scandalous that the experts do not know the answer ! In any case, I would be grateful for any comment you would have on any of my questions.

Sincerely yours

A Grothendieck

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