11.9.1967

Dear Anantharaman,

Matsumura proved that if X is proper over a field k, then $\underline{\mathrm{Aut}}_{X/k}$ is representable by a group scheme loally of finite type over k. I think I can systematize the key step of his argument in the following way. Consider a scheme S, and a morphism

$$\phi : Z \longrightarrow X$$

of S-schémes which are proper, flat and of finite presentation. Let Y be locally of finite présentation and separated over S, then ϕ induces a homomorphisme of functors $u \rightsquigarrow u\phi$:

$$\phi' : \underline{\mathrm{Hom}}_S(X,Y) \longrightarrow \underline{\mathrm{Hom}}_S(Z,Y) .$$

Then one can define a subfunctor of $\underline{\mathrm{Hom}}_S(X,Y)$ where ϕ' is "unramified" in a rather obvious sense, and this turns out to be an "open subfunctor", say $\underline{\mathrm{Hom}}_S(X,Y;\phi)$. Now look at the induced homomorphism

$$\underline{\mathrm{Hom}}_S(X,Y;\phi) \longrightarrow \underline{\mathrm{Hom}}_S(Z,Y) .$$

Using the main result of Murre's talk, one can prove that the latter morphism is representable by unramified separated morphisms locally of finite presentation; as a consequence, if $\underline{\mathrm{Hom}}_S(Z,Y)$ is representable, so is $\underline{\mathrm{Hom}}_S(X,Y;\phi)$.

To get, given X and Y, a representability theorem for $\underline{\mathrm{Hom}}_S(X,Y)$, one tries to find morphisms $\phi_i : Z_i \longrightarrow X$ as above, such that the open subfunctors $\underline{\mathrm{Hom}}_S(X,Y;\phi_i)$ cover $\underline{\mathrm{Hom}}_S(X,Y)$ (as a fpqc sheaf), and such that the functors $\underline{\mathrm{Hom}}_S(Z_i,Y)$ are all representable. If for instance S is the spectrum of a field k, and if X has "enough" points radicial over k (which is always true if k is alg. closed) then we can take for Z_i all finite subschemes of X whose points are radicial over k, and we get

30

that $\underline{Hom}_S(X,Y)$ is representable (any Y locally of finite presentation (and separated) over k); if we do not make any assumption on X except properness over k, the previous assumption becomes true after finite ground-field extension k'/k, so that we get that for every Y as above, $\underline{Hom}_S(X,Y)\times_S \mathrm{Spec}(k')$ is representable. From this Matsumuras theorem stated at the beginning follows in a standart way by descebt arguments. The result holds too for $\underline{Isom}_k(X,Y)$ instead of $\underline{Aut}_k(X)$, but as you probably know, $\underline{Hom}_S(X,Y)$ is not always representable, even if X is a quadratic extension of S=Spec k, Y being proper non projective.

Over an arbitrary base S, one can give a fairly general statement of a representability theorem, the points radicial over k used above being replaced by suitable flat subschemes of X. As ~~applications~~ particular cases, we get for instance that if X has integral geometric fibers and a section along which X is smooth, then $\underline{Hom}_S(X,Y)$ is representable; and if X has reduced geometric fibers, then $\underline{Hom}_S(X,Y)$ is represéntable locally for the étale topology over S. Also, if Y is quasi-projective over S=Spec k, then $\underline{Hom}_S(X,Y)$ is representable.

To fix the ideas, I gave the statements for $\underline{Hom}_S(X,Y)$, but one has quite analoguous resultst of course for the $\underline{\mathrm{P}}_{X/S}$ P/X fonctors, which I guess will imply rather formally the other ones.

If you are interested, I can send you a photocopy of the statement of the general theorem of representability I alluded to above, and a couple of corollaries (I already listed here the most striking ones).

Sincerely your's

31