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A General Theory of Fibre Spaces With Structure Sheaf

by

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https://agrothendieck.github.io/

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INTRODUCTION

When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre spaces (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod (The Topology of Fibre Bundles, Princeton University Press); or the "differentiable" and "analytic" (real or complex) variants of theses notions; or the notions of algebraic fibre spaces (over an abstract field k) - one is led in a natural way to the notion of fibre space with a structure sheaf $\mathbf{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with *abelian* structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\mathbf{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $\mathrm{H}^1(X, \mathbf{G})$ the set of classes of fibre spaces on X with structure sheaf $\mathbf{G}$, $\mathbf{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\mathbf{G}$, where $\mathbf{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $\mathrm{H}^1(X, \mathbf{G})$ of X with coefficients in $\mathbf{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the last chapter, we define the cohomology set $\mathrm{H}^1(X, \mathbf{G})$ of X with coefficients in the sheaf of groups $\mathbf{G}$, so that the expected classification theorem for fibre spaces with structure sheaf $\mathbf{G}$ is valid. We then proceed to a careful study of the exact cohomology sequence associated with an exact sequence of sheaves $e \longrightarrow \mathbf{F} \longrightarrow \mathbf{G} \longrightarrow \mathbf{H} \longrightarrow e$. This is the main part, and in fact the origin, of this paper. Here $\mathbf{G}$ is any sheaf of groups, $\mathbf{F}$ a subsheaf of groups, $\mathbf{H} = \mathbf{G}/\mathbf{F}$, and according to various supplementary hypotheses of $\mathbf{F}$ (such as $\mathbf{F}$ normal, or $\mathbf{F}$ normal abelian, or $\mathbf{F}$ in the center) we get an exact cohomology sequence going from $\mathrm{H}^0(X, \mathbf{F})$ (the group of section of $\mathbf{F}$) to $\mathrm{H}^1(X, \mathbf{G})$ respectively $\mathrm{H}^1(X, \mathbf{H})$ respectively $\mathrm{H}^2(X, \mathbf{G})$, with more or less additional algebraic structures involved. The formalism thus developed is quite suggestive, and as it seems useful, in particular in dealing with the problem of classification of fibre bundles with a structure group G in which we consider a sub-group F, or the problem of comparing say the topological and analytic classification for a given analytic structure group G. However, in order to keep this exposition in reasonable bounds, no examples have been given. Some complementary facts, examples, and applications for the notions developed will be given in the future. This report has been written mainly in order to serve the author for future reference; it is hoped that it may serve the same purpose, or as an introduction to the subject, to somebody else.

Of course, as this report consist in a fortunately straightforward adaptation of quite well known notions, no real difficulties had to be overcome and there is no claim for originality whatsoever. Besides, at the moment to give this report for mimeography, I hear that results analogous to those of chapter 5 were known for some years to Mr. Frenkel, who did non publish them till now. The author only hopes that this report is more pleasant to read than it was to write, and is convinced that anyhow an exposition of this sort had to be written.

Remark (added for the second edition). It has appeared that the formalism developed in this report, and specifically the results of Chapter V, are valid (and useful) also in other situations than just for sheaves on a given space X. A generalization for instance is obtained by supposing that a fixed group π is given acting on X as a group of homeomorphisms, and that we restrict our attention to the category of fibre spaces over X (and specially sheaves) on which π operates in a manner compatible with its operations on the base X. (See for instance A. Grothendieck, Sur le mémoire de Weil; Généralisations des fonctions abéliennes, Séminaire Bourbaki Décembre 1956). When X is reduced to a point, one gets (instead of sheaves) sets, groups, homogeneous spaces etc. admitting a fixed group π of operators, which leads to the (commutative and non-commutative) cohomology theory of the group π. One can also replace π by a fixed Lie group (operating on differentiable varieties, on Lie groups, and homogeneous Lie spaces). Or X, π are replaced by a fixed ground field k, and one considers algebraic spaces, algebraic groups, homogeneous spaces *defined over* k, which leads to a kind of cohomology theory of k. All this suggests that there should exist a comprehensive theory of non-commutative cohomology in suitable categories, an exposition of which is still lacking. (For the "commutative" theory of cohomology, see A. Grothendieck, Sur quelques points d'Algèbre Homologique, Tohoku Math. Journal, 1958).

§ I. — GENERAL FIBRE SPACES

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1. Notion of fibre space

Definition **1.1.1.** — *A* fibre space *over a space X is a triple (X, E, p) of the space X, a space E and a continuous map p of E into X.*

We do not require p to be onto, still less to be open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X, E, p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the *base space* of the fibre space, p the *projection*, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the *fibre* of x (in E).

Given two fibre spaces (X, E, p) and (X', E', p'), a *homomorphism* of the first into the second is a pair of continuous maps $f : X \longrightarrow X'$ and $g : E \longrightarrow E'$, such that $p'g = fp$, i.e. commutativity holds in the diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

Then g maps fibres into fibres (but not necessarily *onto*!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X', f' a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of *isomorphism* of a fibre space (X,E,p) onto a fibre space (X',E',p') is clear: it is a homomorphism (f,g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms

Let (X,E,p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the *inverse image* of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x',y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x',y) = x'$. The map $g(x',y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a *homeomorphism* of the fibre of E' over x' *onto* the fibre of E over $f'x'$.

Suppose now, moreover, given a continuous map $f' : X'' \longrightarrow X'$ of a space X'' into X'. Then there is a canonical isomorphism of the fibre space E'' over X'', inverse image of the fibre space E by ff', and the inverse image of the fibre space E' (considered above) by f' (*transitivity* of *inverse images*). If $(x'',y) \in E''$ ($x'' \in X'', y \in E, ff'x'' = py$), it is mapped by this isomorphism into $(x'',(f'x'',y))$.

Let Y be a subspace of the base X of a fibre space E; consider the injection f of Y into X; the inverse image E' of E by f is called *fibre space induced by* E on Y, or the *restriction of* E *to* Y, and is denoted by $E|Y$. This is canonically homeomorphic to a subspace of E, namely the set of elements mapped by p into Y; the projection of $E|Y$ into Y is induced by p. By what has been said above, if Z is a subspace of Y, the restriction of $E|Y$ to Z is the restriction $E|Z$ of E to Z.

Again let (X,E,p) and (X',E',p') be two fibre spaces, f a continuous map $X \longrightarrow X'$. An *inverse homomorphism associated with* f is an X-homomorphism g

of the fibre space E_0 into E, where E_0 denotes the inverse image of the fibre space E' by f. That means that g is a continuous map, of the subspace E_0 of $X \times E'$ of pairs (x, y') such that $fx = p'y'$, into E, mapping for any $x \in X$ the fibre of x into E_0 (homeomorphic to the fibre of fx in E'!) into the fibre $p^{-1}(x)$ of x in E. For instance, if E is itself the inverse image of E' by f, then there is a canonical inverse homomorphism of E' into E associated with f: the identity! (Though somewhat trivial, this is the most important case of inverse homomorphisms.)

1.3 Subspace, quotient, product

Let (X, E, p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E'

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1.4 Trivial and locally trivial fibre spaces

Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the *trivial fibre space over X with fibre F*.

All fibres are canonically homeomorphic with F.

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1.5 Definition of fibre spaces by coordinate transformations

Let X be a space, (U_i) a covering of X, for each

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1.6 The case of locally trivial fibre spaces

The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space over T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces

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1.7 Sections of fibre spaces

Definition **1.7.1**. — *Let* (X,E,p) *b a fibre space; a* section *of this fibre space (or, by pleonasm, a section of* E *over* X*) is a map* x *of* X *into* E *such that* ps *is the the identity map of* X*. The set of* continuous *sections of* E *is noted* $\mathrm{H}^0(X,E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x!).

The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A *section of* E *over a subset* Y of X is by definition a section of $E|Y$. If Y is open, we write $\mathrm{H}^0(Y,E)$ for the set $\mathrm{H}^0(Y,E|Y)$ of all continuous sections of E over Y.

$\mathrm{H}^0(X,E)$ as a *functor*. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. *We get thus a map, noted* f, *of* $\mathrm{H}^0(X,E)$ *into* $\mathrm{H}^0(X,E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, the so is f.

b. If f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E'' (E, E', E'' fibre spaces over X) then $(f'f) = f'f$.

Let (X,E,p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f.

§ II. — SHEAVES OF SETS

Throughout this exposition, we will now use the word *"section"* for *"continuous section"*.

2.1. Sheaves of sets

Definition **2.1.1**. — *Let X be a space. A* sheaf of sets *on X (or simply a* sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighbourhood U such that p induces a homeomorphism of U onto an open subset $p(U)$ of X.*

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

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2.2

2.3 Definition of a sheaf by systems of sets

2.4 Permanence properties

2.5 Subsheaf, quotient sheaf. Homeomorphism of sheaves

2.6 Some examples

a.

b.

c.

d. **Sheaf of germs of subsets**. Let X be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of U. If $V \subset U$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $\mathrm{H}^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of $P(X)$, for instance the sheaf of *germs of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

§ III. — GROUP BUNDLES AND SHEAVES OF GROUPS

3.1. Sheaves of sets

§ IV. — FIBRE SPACES WITH STRUCTURE SHEAF

4.1. The definition

§ V. — THE CLASSIFICATION OF FIBRE SPACES WITH STRUCTURE SHEAF

5.1. The functor $\mathrm{H}^1(X, \mathbf{G})$ and its interpretation