MATHEMATICS

A. GROTHENDIECK

This text has been transcribed and edited by Mateo Carmona with the collaboration of Tim Hosgood. Remarks, comments, and corrections are welcome.

https://agrothendieck.github.io/

CONTENTS

A GENERAL THEORY OF FIBRE SPACES WITH STRUCTURE SHEAF

Introduction

When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre spaces (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod ([]); or the "differeniable" and "analytic" (real or complex) variants of theses notions; or the notions of algebraic fibre spaces (over an abstract field k) - one is led in a natural way to the notion of fibre space with a structure sheaf $\mathbf{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with *abelian* structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\mathbf{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $\mathrm{H}^1(X, \mathbf{G})$ the set of classes of fibre spaces on X with structure sheaf $\mathbf{G}$, $\mathbf{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\mathbf{G}$, where $\mathbf{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in $\mathbf{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the las chapter, we define the cohomology set $\mathrm{H}^1(X,\mathbf{G})$ of X with coefficients in the sheaf of groups $\mathbf{G}$,

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I. General fibre spaces

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1 Notion of fibre space

Definition **1.1.1**. — *A* fibre space *over a space X is a triple (X,E,p) of the space X, a space E and a continuous map p of E into X.*

We do not require p to be onto, still less to b open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X,E,p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the *base space* of the fibre space, p the *projection*, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the *fibre* of x (in E).

Given two fibre spaces (X,E,p) and (X',E',p'), a *homomorphism* of the first into the second is a pair of continuous maps $f: X \longrightarrow X'$ and $g: E \longrightarrow E'$, such that $p'g = fp$, i.e. commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Then g maps fibres into fibres (but not necessarily *onto*!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X', f' a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of *isomorphism* of a fibre space (X, E, p) onto a fibre space (X', E', p') is clear: it is a homomorphism (f, g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms

Let (X, E, p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the *inverse image* of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x', y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x', y) = x'$. The map $g(x', y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a *homeomorphism* of the fibre of E' over x' *onto* the fibre of E over $f'x'$.

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1.3 Subspace, quotient, product

Let (X, E, p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E'

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1..4 Trivial and locally trivial fibre spaces

Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the *trivial fibre space over X with fibre F*.

All fibres are canonically homeomorphic with F.

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1.5 Definition of fibre spaces by coordinate transformations

Let X be a space, (U_i) a covering of X, for each

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1.6 The case of locally trivial fibre spaces

The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space over T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces

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1.7 Sections of fibre spaces

Definition **1.7.1.** — *Let* (X, E, p) *b a fibre space; a* section *of this fibre space (or, by pleonasm, a section of* E *over* X*) is a map* x *of* X *into* E *such that* ps *is the the identity map of* X*. The set of* continuous *sections of* E *is noted* $\mathrm{H}^0(X, E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x!).

The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A *section of* E *over a subset* Y of X is by definition a section of $E|Y$. If Y is open, we write $\mathrm{H}^0(Y, E)$ for the set $\mathrm{H}^0(Y, E|Y)$ of all continuous sections of E over Y.

$\mathrm{H}^0(X, E)$ as a *functor*. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. *We get thus a map, noted* f, *of* $\mathrm{H}^0(X, E)$ *into* $\mathrm{H}^0(X, E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, the so is f.

b. If f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E'' (E, E', E'' fibre spaces over X) then $(f'f) = f'f$.

Let (X, E, p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f.

II. Sheaves of sets

Throughout this exposition, we will now use the word *“section”* for *“continuous section”*.

2.1 Sheaves of sets

Definition **2.1.1**. — *Let X be a space. A* sheaf of sets *on X (or simply a* sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighbourhood U such that p induces a homeomorphism of U onto an open subset $p(U)$ of X.*

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

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2.3 Definition of a sheaf by systems of sets

2.4 Permanence properties

2.5 Subsheaf, quotient sheaf. Homeomorphism of sheaves

2.6 Some examples

a.

b.

c.

d. **Sheaf of germs of subsets**. Let X be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of U. If $V \subset U$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $\mathrm{H}^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of $P(X)$, for instance the sheaf of *germs* *of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

III. Group bundles and sheaves of groups

IV. Fibre spaces with structure sheaf

V. The classification of fibre spaces with structure sheaf

SUR LES FAISCEAUX ALGÉBRIQUES ET LES FAISCEAUX ANALYTIQUES COHÉRENTS

Le but de cet exposé [ ]

1 Généralités sur les faisceaux algébriques cohérents. (Rappels)

Soit X un espace topologique muni

ON COHERENT ALGEBRAIC AND ANALYTIC SHEAVES[1]

The aim of this talk is to generalise certain theorems of Serre. It makes fundamental use of the techniques of Serre [?, ?, ?].

1. Generalities on coherent algebraic sheaves

Let X be a topological space endowed with a sheaf of rings O. A sheaf of O-modules A (or simply an O-module) is said to be *of finite type* if, on every small-enough open subset, it is isomorphic to a quotient of O^n (for some finite integer $n \geq 0$), and *coherent* if it is of finite type and if, for every homomorphism $O^m \longrightarrow A$ on an open subset U of X, the kernel is of finite type. If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of O-modules, and if two of the modules are coherent, then so too is the third; the kernel, cokernel, image, and coimage of a homomorphism of coherent O-modules is a coherent O-module. If A and B are coherent O-modules, then so too is the sheaf ${}_O(A,B)$ of germs of homomorphisms from A to B. If O itself is coherent, then coherent O-modules are exactly the O-modules that, on small-enough open subsets, are isomorphic to the cokernel of some homomorphism $O^m \longrightarrow O^n$. For all of this, and other elementary properties, see [?, chapitre 1, paragraphe 2].

Let X be an algebraic set (over an algebraically closed field k, to illustrate the idea; but the results of this talk still hold true for schemes, and even for general arithmetic schemes...). We denote by O_X the structure sheaf of X, with its sections over an open subset $U \subset X$

[1]Translated by T. Hosgood
https://thosgood.com/translations/

being the regular functions on U. This is a sheaf of rings, and even of k-algebras.

Theorem 1

(a) O_X is a coherent sheaf of rings.

(b) If X is affine with coordinate ring $A(X)$, then, for every coherent O-module A on X, the stalks A_x are generated by the canonical image of $\Gamma(X,A)$. Furthermore, $\Gamma(X,A)$ is an $A(X)$-module of finite type, and every $A(X)$-module of finite type comes from an essentially unique coherent O-module. (Recall that $\Gamma(X,A)$ denotes the module of sections of A over X).

(c) Under the conditions of b), we have that $\mathrm{H}^i(X,A)=0$ for $i>0$.

Proof. For the proofs, which are very elementary, see [?, chapitre 2, paragraphes 2,3,4], or an talk of Cartier in the 1957 *Séminaire Grothendieck*. □

0.1 A dévissage theorem

Definition 1 Let C be an abelian category, and C' a subclass of C. We say that C' is an *exact subcategory* if, for every exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in C with two (non-zero) terms in C', the third term is also in C', and if every direct factor of any $A \in C'$ is also in C'.

Theorem 2 Let X be an algebraic set; suppose that, for every irreducible subset Y of X, we are given a coherent O_Y-module F_Y on Y that has support equal to Y. Let $K(X)$ be the abelian category of coherent algebraic sheaves on X. Then every *exact* subcategory K of $K(X)$ containing the F_Y is identical to $K(X)$.

Proof. The proof is done by induction on $n=\dim X$, with the case $n=0$ being immediate, by the second condition of Defintion 1. So suppose that $n>0$, and that the theorem is true in dimension $<n$. We can consider $K(Y)$ as a subcategory of $K(X)$ (where Y is some given closed subset of X), and then $K\cap K(Y)$ is a subcategory of $K(Y)$ satisfying the conditions of Theorem 2, and so, if $\dim Y<n$, then the induction hypothesis implies that $K(Y)=K(Y)\cap K$, i.e. $K(Y)\subset K$.

Lemma 1 Let Y be a closed subset of X, and A a coherent O_X-module such that $\operatorname{supp} A \subset Y$. Let I_Y be the sheaf of ideals of O_X defined by Y. Then there exists an integer k such that $I_Y^k A = 0$.

Proof. By "compactness" reasons, we can restrict to the case where X is affine, and then apply part *b)* of Theorem 1, noting that, if A is defined by the $A(X)$-module $M = \Gamma(X, A)$, then the ideal of $\operatorname{supp} A$ is the intersection of the minimal prime ideals associated to the annihilator of M, whence the result. □

Corollary Under the above conditions, A admits a composition series with each composition factor A_i/A_{i+1} lying in $K(Y)$.

This implies that A_i/A_{i+1} is annihilated by I_Y; we take $A_i = I_Y^i A$. In the case where $\dim Y < n$, by induction on the length of this composition series, using Definition 1 and the fact that $K(Y) \subset K$, we see that, *if* $\dim \operatorname{supp} A < n$, *then* $A \in K$.

Suppose first of all that X is irreducible. For $A \in K(X)$, let $T(A)$ be the torsion submodule of A (whose stalks are the torsion submodules of A_x).

Lemma 2 If $A \in K(X)$, then the torsion submodule $T(A)$ is also in $K(X)$, and $A = T(A)$ if and only if $\operatorname{supp} A \neq X$.

Proof. We can immediately restrict to the case where X is affine, where it is evident, by the interpretation of coherent O-modules as $A(X)$-modules of finite type. □

Using the exact sequence $0 \longrightarrow T(A) \longrightarrow A \longrightarrow A_0 \longrightarrow 0$, and that $T(A) \in K$, we see that $A \in K$ if and only if $A_0 \in K$.

Let R be the sheaf of fields over X given by the fields of fractions of the $O_{X,x}$, i.e. the sheaf of germs of rational functions, which is a constant sheaf, and we have an injective homomorphism $A_0 \longrightarrow A_0 \otimes_{O_X} R$ Representing A_0 locally as the cokernel of a homomorphism $O_X^m \longrightarrow O_X^{m'}$, we see that the tensor product $A_0 \otimes_{O_X} R$ is locally isomorphic (as sheaves of R-modules) to a sheaf of the form R^k, and thus conclude that it is *globally* isomorphic to R^k thanks to:

Lemma 3 On any irreducible algebraic set X, every locally constant sheaf is constant.

Proof. This is an easy consequence of the fact that every open subset of X is connected (consider a maximal open subset where the sheaf in question is constant!). □

We will thus identify $A_0 \otimes_{O_X} R$ with some R^k, which contains the sub-O_X-module O_X^k. Consider the exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_0 + O_X^k \longrightarrow Q \longrightarrow 0$$

where Q is defined as the cokernel of the injection homomorphism. We immediately see that $Q \otimes_O R = 0$, and so Q is a torsion module, and so $\operatorname{supp} Q \neq X$ (Lemma 2), whence $Q \in K$. (We implicitly make use of the fact that $A_0 + O_X^k$ is a coherent O-module, which can be easily verified). Then $A_0 \in K$ *if and only if* $A_0 + O_X^k \in K$. Similarly, the analogous exact sequence $0 \longrightarrow O_X^k \longrightarrow A_0 + O_X^k \longrightarrow Q' \longrightarrow 0$, where $\operatorname{supp} Q \neq X$, and whence $Q' \in K$, implies that $A_0 + O_X^k \in K$ *if and only if* $O_X^k \in K$. Finally, suppose that $k > 0$, i.e. that A is not a torsion O-module, i.e. that $\operatorname{supp} A = X$; then $O_X^k \in K$ *if and only if* $O_X \in K$, as follows immediately from Definition 1. So the above "if and only if"s imply that, if A is such that $\operatorname{supp} A \neq X$, then $A \in K$ *if and only if* $O_X \in K$. Taking $A = F_X$, we thus see that $O_X \in K$, whence every $A \in K(X)$ with support equal to X is in K, and since the same is true for the $A \in K(X)$ with suppose not equal to X, we indeed have that $K(X) \subseteq K$.

Now if X is not necessarily irreducible, let X_i be its irreducible components. For every coherent algebraic sheaf A on X, let A_i be the sheaf that agrees with A on X_i, and with 0 on $X \setminus X_i$; then A_i is a coherent O-module that can be identified with a quotient of A. We have a natural homomorphism $A \longrightarrow \coprod_i A_i$ from A to the direct sum of the A_i that is *injective*; let Q be its cokernel; we thus have an exact sequence $0 \longrightarrow A \longrightarrow \coprod_i A_i \longrightarrow Q \longrightarrow 0$. Since $\operatorname{supp} Q \subset \bigcup_{i \neq j} X_i \cap X_j$, we have that $Q \in K$; to prove that $A \in K$, it suffices to prove that $\coprod_i A_i \in K$, or even that each A_i is in K. But, by what we have already seen, applied to X_i, we have that $K(X_i) \in K$, whence we again conclude that $\operatorname{supp} A_i \subset X_i$ implies that $A_i \in K$, by using the Corollary of Lemma 1. This proves Theorem 2. $\square$

Remark We say that the subcategory K of $K(X)$ is *left exact* if, for every exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $K(X)$, we have that A' (respectively A) is in K provided that the two other terms are in K. The proof of Theorem 2 proves that the conclusion still holds true if we suppose that K is only left exact, *provided that* the F_Y considered as O_Y-modules are torsion free. This suffices to prove that, if X is complete, then the $\Gamma(X, A)$, for $A \in K(X)$, are vector spaces of finite dimension (since the category K of $A \in K(X)$ having this property is left exact, and contains the O_Y): this is the proof by Serre.

0.2 Complements on sheaf cohomology

Let X be a topological space, and write C^X to denote the category of abelian sheaves on X. We define, in the usual manner, injective sheaves, and we can prove the existence, for all $A \in C^X$, of a resolution $C(A)$ of A by injective sheaves, which allows us to develop the theory of right-derived functors. In particular, consider the left-exact functor $\Gamma(X,A)$ from C^X to the category C of abelian groups; its derived functors are denoted $\mathrm{H}^i(X,A)$. So

$$\mathrm{H}^i(X,A) = \mathrm{H}^i\left(\Gamma(X,C(A))\right).$$

The $\mathrm{H}^i(X,A)$ form a "cohomological functor" in A that is zero for $i < 0$, and satisfies

$$\mathrm{H}^0(X,A) = \Gamma(X,A).$$

If $f : X \longrightarrow Y$ is a continuous map from X to a space Y, then we can define, for any abelian sheaf B on Y, the abelian sheaf $f^{-1}(B)$ on X, which we call the *inverse image of* B, as well as the canonical homomorphism

$$\mathrm{H}^0(Y,B) \longrightarrow \mathrm{H}^0(X,f^{-1}(B))$$

which extends uniquely to give functorial, compatible (with the coboundary operators) homomorphisms

$$\mathrm{H}^i(Y,B) \longrightarrow \mathrm{H}^i(X,f^{-1}(B)).$$

Now let A be an abelian sheaf on X, and define its *direct image* $f_*(A)$ to be the abelian sheaf on Y whose sections over any open subset V are the sections of A over $f^{-1}(V)$. Clearly f_* is a covariant additive left-exact functor from C^X to C^Y, and, if Γ_X (resp. Γ_Y) denotes the "sections" functor on C^X (resp. C^Y), then, by definition

$$\Gamma_X = \Gamma_Y \circ f_*.$$

Furthermore, it is trivial to show that f_* sends injective sheaves to injective sheaves. From this, we easily obtain the *Leray spectral sequence of the continuous map* f, i.e. there is a cohomological spectral sequence starting with

$$ {}_2^{p,q} = \mathrm{H}^p(Y, \mathrm{R}^q f_*(A))$$

that abuts to $\mathrm{H}^\bullet(X,A)$, where the $\mathrm{R}^q f_*(A)$ are the sheaves on Y given by taking the right-derived functors of the functor $f_*\colon C^X \longrightarrow C^Y$, i.e. $\mathrm{R}^q f_*(A) = \mathrm{H}^q(f_* C(A))$. We immediately see that $\mathrm{R}^q f_*(A)$ is *the sheaf on Y associated to the presheaf* $V \mapsto \mathrm{H}^q(f^{-1}(V),A)$.

From the Leray spectral sequence, we get homomorphisms

$$\mathrm{H}^p(Y,f_*(A)) \longrightarrow \mathrm{H}^p(X,A) \tag{1}$$

whose direct definition is evident (noting that we have a natural homomorphism $f^{-1}(f_*(A)) \longrightarrow A$). Furthermore, *if* $\mathrm{R}^q f_*(A) = 0$ *for* $q > 0$, *then the homomorphisms in (1) are isomorphisms*. This follows immediately from the spectral sequence, or, even more simply, from the fact that $f_*(C(A))$ is an injective resolution of $f_*(A)$.

For the results of this section, see the 1957 *Séminaire Grothendieck*.

0.3 Supplementary results on algebraic sheaves on projective space

Let V be a finite-dimensional k-vector space, and $\mathbf{P}$ the associated projective space, the quotient of $V \setminus \{0\}$ by the algebraic group $k^\times = k \setminus \{0\}$. We see that $V \setminus \{0\}$ is a principal algebraic $k^\times$-bundle on $\mathbf{P}$, and so it defines an associated vector bundle on $\mathbf{P}$, with fibres of dimension 1; the sheaf of germs of regular sections of the *dual* bundle is denoted $O(1)$, and we denote by $O(n)$ the n-fold tensor product of $O(1)$ with itself if $n \geq 0$, and the $(-n)$-fold tensor product of the dual sheaf if $n < 0$ (in particular then, $O(0) = 0_{\mathbf{P}}$). If A is an algebraic sheaf on $\mathbf{P}$, then we let $A(n) = A \otimes_{O_{\mathbf{P}}} O_n$, and so $A(m)(n) = A(m+n)$. The definitions of $O(n)$ and of the operation $A \mapsto A(n)$ can immediately be extended to sheaves on a product $\mathbf{P} \times Y$, where Y is an arbitrary algebraic set.

Theorem 3

(a) Let Y be an affine algebraic set, and A a coherent algebraic sheaf on $\mathbf{P} \times Y$. Then, for every n large enough, $A(n)$ is generated by the module of its sections, i.e. $A(n)$ is isomorphic to some quotient of $O^k_{\mathbf{P}\times Y}$, for some integer k.

(b) For n large enough, $\mathrm{H}^i(\mathbf{P},O(n)) = 0$.

Proof. The proof is elementary; for a), see [?, page 247, théorème 1] (where the proof is given for the case where Y is a single point, but the same method works for arbitrary Y), and for b) see [?, page 259, théorème 2]. We could also give a direct proof of b) by calculating $\mathrm{H}^i(\mathbf{P}, O(n))$ using the Čech method, which can be applied here, by part (c) of Theorem 1 (see the *Séminaire Grothendieck* for more on this point), and using the well-known cover of $\mathbf{P}$ by $(r+1)$ affine open subsets, each isomorphic to k^r. □

Now suppose that $k = \mathbf{C}$ is the field of complex numbers, so that $\mathbf{P}$ is also endowed with the structure of an analytic space, which we denote by $\mathbf{P}^h$; this is itself endowed with a sheaf of analytic local rings, which we denote by O^h; finally, we can define, as above, the sheaves $O^h(n)$. With this, we have:

Corollary

(a) Let A^h be a coherent O^h-module on $\mathbf{P}^h$. Then, for all n large enough, $A^h(n)$ is isomorphic to a quotient of $(O^h)^k$, for some integer k.

(b) For n large enough, $\mathrm{H}^i(\mathbf{P}^h, O^h(n)) = 0$.

Proof. The proof is distinctly deeper: see [?, lemme 5, page 12, and lemma 8, page 24]. It works by induction on the dimension, and makes essential use of the fact that the cohomology $\mathbf{P}^h$ with values in a coherent O^h-module is of finite dimension. □

0.4 The finiteness theorem: statement

Let $f : X \longrightarrow Y$ be a regular map of algebraic sets, and let A be an algebraic sheaf, i.e. an O_X-module, on X. Then its direct image by f, and, more generally, by the $\mathrm{R}^q f_*(A)$ (see §3), are O_Y-modules. In the case where A is coherent (or, more generally, "quasi-coherent", in the sense of Cartier, *Séminaire Grothendieck*), we can easily show that, for every *affine* open subset V of Y,

$$\Gamma(V, \mathrm{R}^q f_*(A)) = \mathrm{H}^q(f^{-1}(V), A)$$

and that the sheaves $\mathrm{R}^q f_*(A)$ are "quasi-coherent". We will given sufficient conditions for them to be coherent.

Definition 2 A morphism $f: X \longrightarrow Y$ of algebraic sets is said to be *proper* if, for every irreducible component X_i of X, the scheme X_i is complete over the scheme $f(X_i)$ (see the 1955/56 *Séminaire Cartan-Chevalley*).

A more geometric definition is the following: f is proper if, for every algebraic set Z, the corresponding map $X \times Z \longrightarrow Y \times Z$ is *closed*. Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be morphisms of algebraic sets; if f and g are proper, then gf is proper; if gf is proper, then f is proper, and g is proper if further the image of f is dense in Y. For X to be complete, it is necessary and sufficient for the morphism from X to an algebraic set consisting of a single point to be proper. If X is a locally closed subset of a complete variety X', then for $f: X \longrightarrow Y$ to be proper, it is necessary and sufficient for its graph to be closed. Combining this with Chow's lemma (§7, Lemma 4), the fact that an algebraic subset of an algebraic set over the complex numbers is closed if and only it is closed for the topology of the underlying space [?, proposition 7, page 12], and the fact that a complex projective space is compact, we easily conclude, from the above criterion that, in the "classical case", a morphism is proper if and only if the map of underlying analytic spaces is proper in the usual sense (i.e. the inverse image of a compact subset being compact); compare with [?, proposition 12, proposition 6], where a particular case is proven: X is complete if and only if it is compact.

Theorem 4 Let $f: X \longrightarrow Y$ be a proper morphism of algebraic sets. For any coherent algebraic sheaf A on X, the algebraic sheaves $\mathrm{R}^q f_*(A)$ on Y (and, in particular, the direct image $f_*(A)$) are coherent.

Proof. The proof will be given in §7. □

We state here the following corollary, obtained by taking Y to be a single point:

Corollary Let A be a coherent algebraic sheaf on a complete algebraic set. Then the $\mathrm{H}^i(X, A)$ are vector spaces of finite dimension.

0.5 An algebraic-analytic comparison theorem: statement

Let X be an algebraic set over the field of complex numbers, and denote by X^h the underlying analytic set (see [?] for proper definitions), and by O^h or O_X^h the sheaf of (analytic) local rings of X^h. The identity map $i_X : X^h \longrightarrow X$ is continuous, and we can thus consider the inverse image $i^{-1}(O_X)$, and we have a natural homomorphism of sheaves of rings $i^{-1}(O_X) \longrightarrow O_X^h$, which allows us to consider O_X^h as a sheaf of algebras over $i^{-1}(O_X)$. If now A is an O_X-module, then $i^{-1}(A)$ is an $i^{-1}(O_X)$-module, and we set

$$A^h = i^{-1}(A) \otimes_{i^{-1}(O_X)} O_X^h$$

where A^h is called the *analytic sheaf associated to A*. It is shown in [?] that the covariant functor $A \longrightarrow A^h$ is *exact*. We have a functorial homomorphism

$$i^{-1}(A) \longrightarrow A^h$$

which is injective, and gives homomorphisms (see §3)

$$\mathrm{H}^i(X, A) \longrightarrow \mathrm{H}^i(X^h, A^h). \tag{2}$$

We will see that, if X is complete, then the homomorphisms in (2) are isomorphisms. However, we will actually prove a more general result. Let

$$f : X \longrightarrow Y$$

a morphism of algebraic sets; consider the map $f^h : X^h \longrightarrow Y^h$. From the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ X^h \arrow[r, "f^h"] & Y^h \end{tikzcd}$$

we easily obtain a functorial homomorphism

$$i_Y^{-1}(f_*(A)) \longrightarrow f_*(i_X^{-1}(A))$$

for any sheaf A on X; if A is an O_X-module, then the canonical homomorphism $i_X^{-1}(A) \longrightarrow A^h$ also defines a homomorphism

$$f_*^h(i_X^{-1}(A)) \longrightarrow f_*^h(A^h).$$

The composition $i_Y^{-1}(f_*(A)) \longrightarrow f_*^h(A^h)$ of these homomorphisms is compatible with the canonical homomorphism $i_Y^{-1}(O_Y) \longrightarrow O_Y^h$ of rings of operators, whence, by tensoring with a canonical homomorphism, we obtain

$$f_*(A)^h \longrightarrow f_*^h(A^h). \tag{3}$$

This functorial homomorphism can be extended, in a unique way, to functorial homomorphisms (that commute with the coboundary operators):

$$(\mathrm{R}^q f_*(A))^h \longrightarrow \mathrm{R}^q f_*^h(A^h). \tag{4}$$

These homomorphisms have all the functorial properties that we might desire, but whose precise statements will not be given here (even though they will, of course, be essential in the proofs.)

Theorem 5 Suppose that the morphism of algebraic sets $f : X \longrightarrow Y$ is proper. Then the homomorphisms in (4) are isomorphisms.

Proof. The proof will be given in the following section. □

Taking Y to be a single point, we obtain the following:

Corollary 1 If X is a complete algebraic set, then the homomorphisms in (2) are isomorphisms.

Since $A \longrightarrow A^h$ sends coherent algebraic sheaves to coherent analytic sheaves (an immediate consequence of the exactness of the functor), the combination of Theorem 4 and Theorem 5 gives:

Corollary 2 Under the conditions of Theorem 5, the $f^h(A^h)$ are coherent analytic sheaves.

It is very plausible that, more generally, if $g : V \longrightarrow W$ is a proper holomorphic map of analytic spaces, and if F is a coherent analytic sheaf on V, then $g_*(F)$ is a coherent analytic sheaf. This is indeed true if the sets $f^{-1}(y)$ (for $y \in W$) are finite (as we can see by a classical theorem of Oka; see the 1953/54 *Séminaire Cartan*), or if W consists of a single point (by a result of Serre-Cartan, *loc. cit.*).

Corollary 3 Under the conditions of Theorem 5, suppose further that Y is an *affine* algebraic set, and let $A(Y)$ (resp. $A^h(Y)$) be the ring of regular functions on Y (resp. the ring of holomorphic functions on Y^h). Then there is a canonical isomorphism

$$\mathrm{H}^q(X^h, A^h) = \mathrm{H}^q(X, A) \otimes_{A(Y)} A^h(Y). \tag{5}$$

Proof. We have already said that $\mathrm{H}^q(X,A)$ can be identified with the module of sections of $\mathrm{R}^q f_*(A)$ over Y, and, similarly, we say that $\mathrm{H}^q(X^h,A^h)$ can be identified with the module of sections of $\mathrm{R}^q f_*^h(A^h)$ over Y. To prove this, it suffices to use the Leray spectral sequence of f^h (see §3), and to note that, identifying Y^h with a closed analytic subset of some $\mathbf{C}^n$, its cohomology with values in the coherent analytic sheaves $\mathrm{R}^q f_*^h(A^h)$ is zero in dimensions >0, by a fundamental theorem of Cartan (see the 1951/52 *Séminaire Cartan*). It thus suffices, by Theorem 5, to prove that, if B is a coherent O_Y-module on an affine variety Y, then

$$\mathrm{H}^0(Y^h,B) = \mathrm{H}^0(Y,B) \otimes_{A(Y)} A^h(Y). \tag{6}$$

But we note that both sides of this equation are exact functors in B, which means we only need to verify (6) in the case where $B = O_Y$, but then it is trivial. □

0.6 Proof of Theorems 4 and 5

The proofs follow mainly from Theorem 3, the "dévissage" of Theorem 2 (which is necessary since there is no reason for X to be isomorphic to a locally closed subset of a projective space), and the following:

Lemma 4 *(Chow's lemma.)* — Let X be a an irreducible algebraic set Then there exists an algebraic set X' that is locally closed in some projective space $\mathbf{P}$, and a proper birational morphism $g: X' \longrightarrow X$.

Recall (§5) that "proper" implies, in this case, that the graph of g is a *closed* subset of $\mathbf{P} \times X$. Here we only make use of the fact that g is *proper* and *surjective*.

Proof. We cover X by affine open subsets X_i, with each X_i locally closed in some projective space $\mathbf{P}_i$, whence we have a diagonal map $\bigcap X_i \longrightarrow \coprod \mathbf{P}_i$. We take X' to be the closure in $X \times \coprod \mathbf{P}_i$ of the graph of this diagonal map (or, really, X' is its normalisation).

Theorem 4 and Theorem 5 say that every coherent algebraic sheaf A on X satisfies a certain property. But we immediately see that, in both cases, the class K of the $A \in K(X)$ having the property in question is an *exact* subcategory (§2, Definition 1), by using the exact sequence of the $\mathrm{R}^q f_*$ (and the $\mathrm{R}^q f_*^h$) corresponding to an exact sequence of sheaves $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, and by using either the fact that, if in an exact sequence of

O_X-modules $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$, the four outer terms are coherent, then so too is C, or (in the case of Theorem 5) the classical 5 lemma. By Theorem 2, it thus suffices to find, for every irreducible closed subset Z of X, a coherent algebraic sheaf *on* Z, with support equal to Z, and belonging to K. Note that the restriction of f to Z is again proper, and so we can assume that $Z = X$, which means it suffices to find *one* coherent O_Y-module A, with support equal to X, such that $A \in K$. Consider the morphism $f' : X' \longrightarrow X$ described in Lemma 4. Since X' is embedded into $\mathbf{P}$, we can consider the sheaves $O_{X'}(n)$ on X' given by reducing the $O_{\mathbf{P}}(n)$ (see §4) modulo the sheaf of ideals defined by X' in $\mathbf{P}$. We claim that, for n large enough, the sheaf $A = f(O_{X'}(n))$ is in K (which will finish the proof, since the support of this sheaf is clearly equal to X). This will follow from:

Lemma 5 Let $g : V \longrightarrow W$ be a *proper* morphism of algebraic sets, with V a locally closed subset of a projective space $\mathbf{P}$. Let G be a coherent algebraic sheaf on V. Then, for n large enough,

$$\mathrm{R}^p f_*(G(n)) = 0$$

for $p > 0$, and $f_*(G(n))$ is coherent. Furthermore, if $k = \mathbf{C}$, then

$$\mathrm{R}^p f_*^h(G(n)^h) = 0$$

for $p > 0$, and

$$(f_*(G(n)))^h \longrightarrow f_*^h(G(n)^h)$$

is an isomorphism.

First we will show how this lemma will imply the previous one. Applying the lemma to $f' : X' \longrightarrow X$, we immediately see, from the definitions, and from the fact that $\mathrm{R}^p f_*(O(n)) = 0$ for $p > 0$, that

$$\mathrm{R}^p(ff')_*(O(n)) = \mathrm{R}^p f_*(f'(O(n))) = \mathrm{R}^p f_*(A).$$

But the first object is zero for $p > 0$ and large enough n, by Lemma 5 applied to $ff' : X' \longrightarrow Y$, and so $\mathrm{R}^p f_*(A) = 0$ for $p > 0$, and, a fortiori, $\mathrm{R}^p f_*(A)$ is coherent for $p > 0$; similarly, $f_*(A)$ is coherent, since $f_*(A) = (ff')_*(O(n))$, and so it suffices to apply Lemma 5 to ff'. This thus proves that $A \in K$ in the setting of Theorem 4. In the setting of Theorem 5, the same argument proves that, if n is large enough, $\mathrm{R}^p f_*^h(A^h) = 0$ for $p > 0$, and, a fortiori, the homomorphisms in (4) are isomorphisms for $q > 0$; similarly, the homomorphism

$(f_*(A))^h \longrightarrow f_*^h(A^h)$ is an isomorphism, since both the domain and codomain can be identified (respectively) with $((ff')_*(O(n)))^h$ and $f^h(A^h) = f'^h_*(O(n)^h)$ (since $A^h = (f'(O(n)))^h = f'^h_*(O(n)^h)$, by Lemma 5 applied to $ff'\colon X' \longrightarrow Y$).

It thus remains only to prove Lemma 5. Since the graph V' of g is a closed subset of $\mathbf{P} \times W$, isomorphic to V, we can, by identifying sheaves on V with sheaves on V' (and thus on $\mathbf{P} \times W$), suppose that $V = \mathbf{P} \times W$, and that g is the projection homomorphism. Furthermore, we can suppose that W is affine, and even that $W = k^m$.

We first prove Lemma 5 in the case where $F = O_k$. For an arbitrary field k, this thus implies that $\mathrm{H}^p(\mathbf{P} \times W, O(n)) = 0$ for $p > 0$ and n large enough, and that $\mathrm{H}^0(\mathbf{P} \times W, O(n))$ is a module of finite type over the coordinate ring $A(W)$ of W. Since $O_{\mathbf{P}\times W}(n)$ is the "tensor product" (in the sense of algebraic sheaves) of the sheaves $O_{\mathbf{P}}(n)$ on $\mathbf{P}$ and O_W on W, the Künneth formula (whose proof, in this setting, is elementary) applies, and we thus obtain the stated result, taking into account the fact that $\mathrm{H}^i(W, O) = 0$ for $i > 0$ (part (c) of Theorem 1) and part (b) of Theorem 3, since then

$$\mathrm{H}^i(\mathbf{P} \times W, O_{\mathbf{P}\times W}(n)) = \mathrm{H}^i(\mathbf{P}, O_{\mathbf{P}}(n)) \otimes_F A(W)$$

is zero for $i > 0$ and n large enough, and is of finite type over $A(W)$ when $i = 0$, since $\mathrm{H}^0(\mathbf{P}, O_{\mathbf{P}}(n))$ is clearly of finite dimension. When $k = \mathbf{C}$, we must prove that, for n large enough,

$$\mathrm{H}^i(\mathbf{P}^h \times W', O(n)^h) = 0$$

for $i > 0$ and W' any Stein open subset of W^h, and also that $f_*(O(n)^h)$ can be identified with $(f_*(O(n)))^h$, i.e. with $\mathrm{H}^0(\mathbf{P}, O(n)) \otimes O_W^h$; or, in other words, that

$$\mathrm{H}^i(\mathbf{P}^h \times W', O(n)) = \mathrm{H}^0(\mathbf{P}, O(n)) \otimes \mathrm{H}^0(X', O_W^h)$$

for every Stein open subset W' of W. But $H^\bullet(\mathbf{P}^h \times W', O(n)^h)$ can be calculated by a *vectorial-topological variant of the Künneth theorem* (using the fact that the space $\mathrm{H}^0(W', O_{W'}^h)$ is *nuclear*; see the 1953/54 *Séminaire Schwartz*); taking into account the fact that $\mathrm{H}^i(W', O_W) = 0$ for $i > 0$, we see that it is equal to $\mathrm{H}^i(\mathbf{P}^h, O(n)^h) \otimes \mathrm{H}^0(W', O_w)$, by a fundamental theorem of Cartan concerning Stein varieties (which, for our purposes here, it suffices to know for a polycylinder and the structure sheaf. where it is an easy consequence of the aforementioned vectorial-topological Künneth theorem). The above claims then follow from corollary (b) of Theorem 3, taking into account the fact that $\mathrm{H}^0(\mathbf{P}^h, O(n)^h) = \mathrm{H}^0(\mathbf{P}, O(n))$ (which is proven in the proof of that corollary).

To prove Lemma 5 in the general case, we proceed by induction on p, since the lemma is trivial for p large enough, for dimension reasons. By part (a) of Theorem 3, A is isomorphic to a quotient of some $O(m)^k = L$, i.e. we have an exact sequence $0 \longrightarrow A' \longrightarrow L \longrightarrow A \longrightarrow 0$, whence, for all n, an exact sequence

$$0 \longrightarrow A'(n) \longrightarrow L(n) \longrightarrow A(n) \longrightarrow 0$$

which gives an exact sequence

$$\mathrm{R}^p f_*(A'(n)) \longrightarrow \mathrm{R}^p f_*(L(n)) \longrightarrow \mathrm{R}^p f_*(A(n)) \longrightarrow \mathrm{R}^{p+1} f_*(A'(n)).$$

By the induction hypothesis, the last term in this sequence is zero for n large enough, and so too is $\mathrm{R}^p f_*(L(n))$ when $p > 0$, by what we have already proven, whence $\mathrm{R}^p f_*(A(n)) = 0$ for n large enough and $p > 0$. If $p = 0$, then the same exact sequence proves that, for n large enough, $f_*(A(n))$is coherent, since $f_*(L(n))$ is coherent, and $f_*(A'(n))$ is anyway quasi-coherent. In the case where $k = \mathbf{C}$, we can prove, in the same way, that $\mathrm{R}^p f_*^h(A(n)^h) = 0$ for n large enough and $p > 0$. It remains only to show that, for n large enough, $(f_*(A(n)))^h \longrightarrow f^h(A^h)$ is bijective. For this, we write A as the cokernel of a homomorphism $L' \longrightarrow L$, where L and L' are isomorphic to direct sums of sheaves of the form $O(m)$ for various m (which is possible by part (a) of Theorem 3). By the above, for n large enough $f_*(A(n))$ and $f_*^h(A(n)^h)$ can be identified (respectively) with the cokernel of $f_*(L'(n)) \longrightarrow f(L(n))$ and the cokernel of $f^h(L'(n)^h) \longrightarrow f^h(L(n)^h)$; taking into account the fact that the functor $B \longrightarrow B^h$ is exact, we thus obtain a homomorphism of exact sequences

$$\begin{array}{ccccccc}
(f_*(L'(n)))^h & \longrightarrow & (f_*(L(n)))^h & \longrightarrow & (f_*(A(n)))^h & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
f_*^h * (L'(n)^h) & \longrightarrow & f_*^h * (L(n)^h) & \longrightarrow & f_*^h * (A(n)^h) & \longrightarrow & 0
\end{array}$$

Since, for n large enough, the first two vertical arrows are isomorphisms, so too is the third, by the five lemma, which finishes the proof. $\square$

Remark The last paragraph of this proof can be simplified if we use the fact that A admits a finite resolution by sheaves that are direct sums of sheaves of the form $O(m)$ for various m;

but this fact is less elementary than part (a) of Theorem 3, and so we wanted to avoid using it.

0.7 Algebraic and analytic sheaves on a compact algebraic variety

We are going to complete Corollary 1 of Theorem 5:

Theorem 6 Let X be a complete algebraic set over $\mathbf{C}$. Then every coherent analytic sheaf F on X^h is isomorphic to a sheaf A^h, where A is an essentially unique coherent algebraic sheaf on X.

The uniqueness of A follows from:

Corollary 1 With X as above, let A and B be coherent algebraic sheaves on X. Then the natural homomorphism

$$ {}_{O_X}(A,B) \longrightarrow {}_{O_X^h}(A^h,B^h) \tag{7}$$

is bijective.

Proof. This homomorphism comes from, by taking sections, the monomorphism of sheaves

$$ i_X^{-1}({}_{O_X}(A,B)) \longrightarrow {}_{O_X^h}(A,B) $$

(where denotes the sheaf of germs of homomorphisms), but we already know that

$$ ({}_{O_X}(A,B))^h = {}_{O_X^h}(A^h,B^h) \tag{8}$$

(an almost immediate consequence of the fact that $C \longrightarrow C^h$ is exact), and so, by applying Corollary 1 of Theorem 5 to the sheaf ${}_{O_X}(A,B)$ with $i=0$, the result follows. □

From Corollary 1 and the exactness of the functor $C \longrightarrow C^h$ also follows the fact that, if F and G are coherent analytic sheaves on X that come from algebraic sheaves, and if u is a homomorphism from F to G, then the kernel, cokernel, image, and coimage of u all also come from algebraic sheaves. In particular, if X is embedded into a projective space $\mathbf{P}$, then every coherent analytic sheaf on X is isomorphic to the cokernel of a homomorphism

$L^h \longrightarrow L'^h$, where L and L' are direct sums of finitely many sheaves of the form $O(k)$ (part (a) of the Corollary of Theorem 3); it thus follows that Theorem 6 is also true if X is *projective* (Serre]Theorem 6.

Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence of coherent analytic sheaves on X^h, and suppose that F' and F'' come from coherent algebraic sheaves; we then claim that so too does F. Suppose that $F' = A'^h$ and $F'' = A''$, where A' and A'' are coherent algebraic sheaves; it suffices to show that the set $\mathrm{Ext}^1_{O_X}(X;A'',A')$ of classes of O-module extensions of A'' by A' can be identified with the analogous set $\mathrm{Ext}^1_{O^h_X}(X^h;A''^h,A'^h)$. But, more generally, we have canonical homomorphisms

$$\mathrm{Ext}^i_{O_X}(X;A'',A') \longrightarrow \mathrm{Ext}^i_{O^h_X}(X;A''^h,A'^h) \tag{9}$$

(defined without any restrictions on X, A', or A''), which are here isomorphisms, as follows from the spectral sequence of Ext of sheaves of modules (see the 1957 *Séminaire Grothendieck*), from the elementary local relations

$$({}^i_{O_X}(A,B))^h = {}^i_{O^h_X}(A^h,B^h) \tag{10}$$

that generalise (8) (with denoting the *sheaf* Exts), and from Corollary 1 of Theorem 5; this implies that the initial pages of the spectral sequences of both the domain and codomain of the morphism in (9) are identical.

Proof of Theorem 6. We can now prove Theorem 6, by induction on $n = \dim X$, with the theore]Theorem 6 being trivial when $n = 0$. So suppose that $n > 0$, and that the theorem is true in dimensions $< n$. Proceeding as in the end of the proof of Theorem 2, we can restrict to the case where X is irreducible. So consider the map $f: X' \longrightarrow X$ considered in Chow's lemma (Lemma 4), with X' a *projective* variety, and f a *birational* morphism. For every analytic sheaf F on X, let

$$F' = f^{-1}(F) \otimes_{f^{-1}(O^h_X)} O^h_X$$

(where the tensor product makes sense, since O^h_X is a module over $f^{-1}(O^h_X)$, which can be identified with a subsheaf (of rings) of O^h_X). It is easy to prove that, if F is coherent, then so too is F'. Furthermore, there is a natural homomorphism

$$F \longrightarrow f^h_*(F')$$

and, in the current setting, this homomorphism is bijective outside of an algebraic set Y of dimension $< n$ (where Y is the set of points of X over which f is not biregular). We thus have an exact sequence

$$0 \longrightarrow T \longrightarrow F \longrightarrow f_*^h(F') \longrightarrow T' \longrightarrow 0$$

where T and T' have support contained inside Y. Using the analogue of Lemma 1 of §2 (thanks to the compactness of X), we find that T (and even T') admits a composition series with composition factors that are coherent analytic sheaves *on* Y. These quotients are in fact "algebraic", by the induction hypothesis; thus so too are their extensions T and T'. Furthermore, since X' is projective, F' is also "algebraic", by what we have already said, and thus so too is $f_*^h(F')$, by Theorem 5 applied to $f: X' \longrightarrow X$ and $F' = B^h$. Thus the kernel of $f_*^h(F') \longrightarrow T'$ is also algebraic, and thus so too is F, which is an extension of this kernel by T. Thus we have proved Theorem 6. □

Sur une note de Mattuck-Tate

1. Dans un travail récent [?], Mattuck et Tate déduisent l'inégalité fondamentale de A. Weil qui établit l'hypothèse de Riemann pour les corps de fonctions [?] comme conséquence facile du théorème de Riemann-Roch pour les surfaces. En essayant de comprendre la portée exacte du leur méthode, je suis tombé sur l'énoncé suivant, connu en fait depuis 1937 [?] [?] [?] (comme me l'a signalé J. P. Serre), mais apparemment peu connu et utilisé:

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2. Nous allons déduire sur X, nous désignerons par $l(D)$ la dimension de l'espace vectoriel des fonctions f sur X telles que $(f) \geq -D$ donc $l(D)$ ne dépend que de la classe de D. Rappelons *l'inégalité de Riemann-Roch*

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3. Ce qui précède n'utilisait pas à proprement parler la méthode de Mattuck-Tate (si ce n'est en utilisant l'inegalité de Riemann-Roch sur les surfaces). Nous allons indiquer maintenant comment la méthode de ces auteurs, convenablement généralisée, donne d'autres inégalités que celle de A. Weil. Nous nous appuierons sur le

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Remarques. Le corollaire 1 devient faux si on ne fait pas l'hypothèse que $K/2$ est encore une classe de diviseurs. En effet, toutes les hypothèses sauf cette dernière sont vérifiées si X est un surface non singulière *rationnelle*. Or, à partir d'une telle surface, on construit facilement une surface birationnellement équivalente par éclatements successifs, dont l'index τ soit < 0 (contrairement à (3.7 ter)). En effet, on vérifie aisément que lorsqu'on fait éclater un point dans une surface non singulière projective, l'index diminue d'une unité. (Cette remarque, ainsi que l'interprétation de l'inégalité (3.7) a l'aide de l'index, m'a été signalée par J. P Serre).

La disparité des énoncés qu'on déduit du théorème (3.2) est due au fait qu'il n'est pas relatif à un élément arbitraire de l'espace vectoriel E de Néron-Séveri introduit plus haut, mais à un élément du "lattice" provenant des diviseurs sur X. On notera d'ailleurs que dans le cas particulier où X est le produit des deux courbes C et C', le théorème 3.2 ne contient rien de plus que l'inégalité de A. Weil.

The cohomology theory of abstract algebraic varieties

It is less than four years since cohomological methods (i.e. methods of Homological Algebra) were introduced into Algebraic geometry in Serre's fundamental paper [?], and it seems already certain that they are to overflow this part of mathematics in the coming years, from the foundations up to the most advanced parts. All we can do here is to sketch briefly some of the ideas and results. None of these have been published in their final form, but most of them originated in or were suggested by Serre's paper.

Let us first give an outline of the main topics of cohomological investigation in Algebraic geometry, as they appear at present. The need of a theory of cohomology for 'abstract' algebraic varieties was first emphasized by Weil, in order to be able to give a precise meaning to his celebrated conjectures in Diophantine geometry [?]. Therefore the initial aim was to find the *'Weil cohomology' of an algebraic variety*, which should have as coefficients something 'at least as good' as a field of *characteristic 0*, and have such formal properties (e.g. duality, Künneth formula) as to yield the analogue of Lefschetz's 'fixed-point formula'. Serre's general idea has been that the usual 'Zariski topology' of a variety (in which the closed sets are the algebraic subset) is a suitable one for applying methods of Algebraic Topology. His first approach was hoped to yield at least the right Betti numbers of a variety, it being evident from the start that it could not be considered as the Weil cohomology itself, as the coefficient field for cohomology was the ground field of a variety, and therefore not in general of characteristic 0. In fact, even the hope of getting the 'true' *Betti numbers* has failed, and so have other attempts of Serre's [?] to get Weil's cohomology by taking the cohomology of the variety with values, not in the sheaf of local rings themselves, but in the sheaves of Witt-vectors constructed on the latter. He gets in this way modules over the ring $W(k)$ of infinite Witt vectors on the ground field k, and $W(k)$ is a ring of characteristic 0 even if k is of characteristic $p \neq 0$. Unfortunately, modules thus obtained over $W(k)$ may be infinitely generated, even when the variety V is an abelian variety [?]. Although interesting relations must certainly exist between these cohomology groups and the 'true ones', it seems certain now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the *connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other* (as explained quite unsystematically in Serre's tentative Mexico paper [?]), and by Serre's idea that a 'reasonable' algebraic principal fiber space with structure group G, defined on a variety V, if it is not locally trivial, should become locally trivial on some covering of V *unramified* over a given point of V. This has been the starting point of a definition of the Weil cohomology (involving both 'spatial' and Galois cohomology), which seems to be the right one, and which gives clear suggestions how Weil's conjectures may be attacked by the machinery of Homological algebra. As I have not begun theses investigations seriously as yet,and as moreover this theory has a quite distinct flavor from the one of the theory of algebraic coherent sheaves which we shall now be concerned with, we shall not dwell any longer on Weil's cohomology. Let us merely remark that the definition alluded to has already been the starting-point of a theory of cohomological dimension of fields, developed recently by Tate [?].

The second main topic for cohomological methods is the *cohomology theory of algebraic coherent sheaves*, as initiated by Serre. Although inadequate for Weil's purposes, it is at present yielding a wealth of new methods and new notions, and gives the key even for results which were not commonly thought to be concerned with sheaves, still less with cohomology, such as Zariski's theorem on 'holomorphic functions' and his 'main theorem' - which can be stated now in a more satisfactory way, as we shall see, and proved by the same uniform elementary methods. The main parts of the theory, at present, can be listed as follows:

(a) General finiteness and asymptotic behaviour theorems.

(b) Duality theorems, including (respectively identical with) a cohomological theory of residues.

(c) Riemann-Roch theorem, including the theory of Chern classes for algebraic coherent sheaves.

(d) Some special results, concerning mainly abelian varieties.

The third main topic consists in the *application of the cohomological methods to local algebra*. Initiated by Koszul and Cartan-Eilenberg in connection with Hilbert's 'theorem of syzygies', the systematic use of these methods is mainly due again to Serre. The results are the *characterization* of regular local rings as those whose global cohomological dimension is finite, the clarification of *Cohen-Macaulay's equidimensionality theorem* by means of the notion of *cohomological codimension* [?], and specially the possibility of giving (for the first time as it seems) a *theory of intersections*, really satisfactory by its algebraic simplicity and its generality. Serre's result just quoted, that regular local rings are the the only ones of finite global cohomological dimension, accounts for the fact that only for such local rings does a satisfactory theory of intersections exist. I cannot give any details here on these subjects, nor on various results I have obtained by means of a *local duality theory*, which seems to be the tool which is to replace differential forms in the case of unequal characteristics, and gives, in the general context of commutative algebra, a clarification of the notion of residue, which as yet was not at all well understood. The motivation of this latter work has been the attempt to get a global theory of duality in cohomology for algebraic varieties admitting arbitrary singularities, in order to be able to develop intersection formulae for cycles with arbitrary singularities, in a non-singular algebraic variety, formulas which contain also a 'Lefschetz formula mod p' [?]. In fact, once a proper local formalism is obtained, the global statements become almost trivial. As a general fact, it appears that, to a great extent, the 'local' results already contain a global one; more precisely, global results on varieties of dimension n can frequently be deduced from corresponding local ones for rings of Krull dimension $n+1$.

We will therefore

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Tapis de Quillen

Relation entre catégories et ensembles semi-simpliciauxA toute catégorie C, on associe un ensemble semi-simplicial $S(C)$, trouvant ainsi un foncteur pleinement fidèle

$$S :\longrightarrow .$$

Les systèmes locaux d'ensemble sur SC correspondent aux foncteurs sur C qui transforment toute flèche en isomorphisme (i.e. qui se factorisent par le groupoïde associé à C). Les H^i sur SC d'un tel système local (H^0 pour ensembles, H^1 pour groupes, H^i quelconques pour groupes abéliens) s'interprètent en termes des foncteurs $\varprojlim^{(i)}$ dérivés de $\varprojlim$, ou si on préfère, des H^i (du *topos* C). On voit ainsi à quelle condition un foncteur $C \longrightarrow C'$ induit un homotopisme $SC \longrightarrow SC'$: en vertu du critère cohomologique de Artin-Mazur, il f et s que pour tout système de coefficients F' sur C', l'homomorphisme naturel $\varprojlim_{C'}^{(i)} F' \longrightarrow \varprojlim_{C}^{(i)} F$ soit un isomorphisme (pour les i pour lesquels cela a un sens).

A C on peut associer le topos $\widetilde{C}$, qui varie de façon *covariante* avec C. (NB le foncteur $C \mapsto \widetilde{C}$ n'a plus rien de pleinement fidèle, semble-t-il ??).

Les systèmes de coefficients ensemblistes sur C (les foncteurs $C^\circ \longrightarrow$ Ens transformant isomorphismes en isomorphismes) correspondent aux faisceaux localement constants i.e. les objets localement constants de $\widetilde{C}$, définis intrinsèquement en termes de $\widetilde{C}$. Ainsi, le fait pour un foncteur $F : C \longrightarrow C'$ d'induire une homotopisme $S(C) \longrightarrow S(C')$ ne dépend que du morphisme de topos $\widetilde{F} : \widetilde{C} \longrightarrow \widetilde{C'}$ induit, et signifie que pour tout faisceau localement constant F' sur C' i.e. sur $\widetilde{C'}$, les applications induites $\mathrm{H}^i(\widetilde{C'}, F') \longrightarrow \mathrm{H}^i(\widetilde{C}, \widetilde{F}^*(F'))$ sont des isomorphismes (pour les i pour lesquels cela a un sens).

On a aussi un foncteur évident

$$T :\longrightarrow ,$$

en associant à tout ensemble semi-simplicial X la catégorie $T(X) = \Delta_{/X}$ des simplexes sur X, dont l'ensemble des objets est la réunion disjointe des X_n... (c'est une catégorie fibrée sur la catégorie Δ des simplexes types, à fibres les catégories discrètes définies par les X_n). Ceci posé, Quillen prouve que pour tout X, $ST(X)$ est isomorphe canoniquement à X dans la catégorie homotopique construite avec , et que pour toute C, la catégorie $TS(C)$ est canoniquement “homotopiquement équivalente à C” i.e. canoniquement isomorphe a C dans la catégorie quotient de obtenue en inversant les foncteurs qui sont des homotopismes. Ces isomorphismes sont fonctoriels en X. Il en résulte formellement qu'un morphisme $f : X \longrightarrow Y$ dans est un homotopisme si et seulement si en est ainsi de $T(f) : T(X) \longrightarrow T(Y)$, d'où des foncteurs $S' :' \longrightarrow '$ et $T' :' \longrightarrow '$ entre les catégories “homotopiques”, construites avec resp , qui sont quasi-inverses l'un de l'autre.

De plus, Quillen contruit un isomorphisme canonique et fonctoriel dans $'$ entre C et la catégorie opposée C°, ou ce qui revient au même, un isomorphisme canonique et fonctoriel dans $'$ entre $S(C)$ et $S(C^\circ)$. La définition est telle que le foncteur induit sur les systèmes locaux sur C transforme le foncteur contravariant F sur C, transformant toute flèche en flèche inversible, en le foncteur covariant (i.e. contravariant sur C°) ayant mêmes valeurs sur les objets, et obtenu sur les flèches en remplaçant $F(u)$ par $F(u)^{-1}$; en d'autres termes, l'effet de l'homotopisme de Quillen sur les groupoïdes fondamentaux est l'isomorphisme évident entre les groupoïdes fondamentaux de C et de C°, compte tenu que le deuxième est l'opposé du premier. Comme application, Quillen obtient une interprétation faisceautique de la cohomologie d'un ensemble semi-simplicial à coefficients dans un système local covariant F (défini classiquement par le complexe cosimplicial des $C^n(F) = \amalg_{x \in X_n} F(x)$): on considère le système local contravariant défini par F, on l'interprète comme un faisceau sur $T(X)$ i.e. objet de $_{/X}$, et on prend sa cohomologie. - Cependant, quand F est un système de coefficients covariant pas nécessairement local, on n'a toujours pas d'interprétation de ses groupes de cohomologie classiques en termes faisceautiques; ni, lorsque F est contravariant, de son homologie, ou inversement de sa cohomologie faisceautique en termes classiques.

A propos de la notion de foncteur qui est un homotopisme. Quillen montre qu'un tel foncteur $F : C \longrightarrow C'$ induit une équivalence entre la sous-catégorie triangulée $\mathbb{D}^b_{lc}(C')$ de la catégorie dérivée bornée de celle des faisceaux abéliens sur C', dont les faisceaux de cohomologie sont des systèmes locaux, et la catégorie analogue pour C; et réciproquement. On peut dans cet énoncé introduire aussi n'importe quel anneau de base (à condition de le supposer $\neq 0$ dans le cas de la réciproque); la partie dire vaut aussi avec un anneau de coefficients par nécessairement constant, mains constant tordu. Je pense que ce résultat (facile) doit pouvoir se généraliser ainsi.

Soit $f : X \longrightarrow X'$ un morphisme de topos qui soit tel que pour tout faisceau localement constant sur X', f induise un isomorphisme sur les cohomologies (avec cas non commutatif inclus). Supposons que X et X' soit *localement homotopiquement trivial*, i.e. que pour tout entier $n \geq 1$, tout objet U ait un recouvrement par des $U_i \longrightarrow U$, tels que a) tout système local sur U devient constant sur U_i, et toute section sur U devient constant sur U_i et b) pour tout groupe abélien G, les $\mathrm{H}^j(U,G) \longrightarrow \mathrm{H}^j(U_i,G)$ sont nuls pour $1 \leq j \leq n$ [2]. Alors le foncteur $\mathbb{D}^b_{lc}(X') \longrightarrow \mathbb{D}^b_{lc}(X)$ induit par f est une équivalence. Même énoncé si on met dans le coup un système local d'anneaux sur X'. Enfin, f induit une équivalence entre la catégorie des coefficients locaux sur C et celle des coefficients locaux sur C'.

n-catégories, catégories n-uples, et Gr-catégories

Point de vue "motivique" en théorie du cobordisme

[2]Attention, cette condition n'est typiquement pas satisfaite par les schémas sur []

Standard Conjectures on Algebraic Cycles

0.8 Introduction

We state two conjectures on algebraic cycles, which arose fro an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties. These are not really new, and they were worked out about three years ago independently by Bombieri and myself.

The first is an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures), and is inspired by and formally analogous to Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field.

The second is a statement of positivity, generalising Weil's well-known positivity theorem in the theory of abelian varieties. It is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero.

WHAT REMAINS TO BE PROVED OF WEIL'S CONJECTURES? Before stating our conjectures, let us recall what remains to be proved in respect of the Weil conjectures, when approached through ℓ-adic cohomology.

Let $X/\mathbf{F}_q$ be a smooth irreducible projective variety of dimension n over the finite field $\overline{\mathbf{F}}_q$ with q elements, and ℓ a prime different from the characteristic. It has then been proved by M. Artin and myself that the Z-function of X can be expressed as

$$
\begin{aligned}
Z(t) &= \frac{L'(t)}{L(t)}, \\
L(t) &= \frac{L_0(t)L_2(t)\dots L_{2n}(t)}{L_1(t)L_3(t)\dots L_{2n-1}(t)}, \\
L_i(t) &= \frac{1}{P_i(t)},
\end{aligned}
$$

where $P_i(t) = t^{\dim \mathrm{H}^i(\overline{X})} Q_i(t^{-1})$, Q_i being the characteristic polynomial of the action of the Frobenius endomorphism of X on $\mathrm{H}^i(\overline{X})$ (here H^i stands for the i^{th} ℓ-adic cohomology group and $\overline{X}$ is deduced from X by base extension to the algebraic closure of $\mathbf{F}_q$). But it has not been proved so far that

(a) the $P_i(t)$ have integral coefficients, independent of $\ell (\neq \mathrm{char}\, \mathbf{F})$;

(b) the eigenvalues of the Forbenius endomorhisms on $\mathrm{H}^i(\overline{X})$, i.e., the reciprocals of the roots of $P_i(t)$, are of absolute value $q^{i/2}$.

Our first conjecture meets question (a). The first and second together would, by an idea essentially due to Serre [?], imply (b).

0.9 A weak form of conjecture 1

From now on, we work with varieties over a ground field k which is algebraically closed and of arbitrary characteristic. Then (a) leads to the following question: If f is an endomorphism of a variety X/k and $\ell \neq \mathrm{char}\, k$, f induces

$$f^i : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^i(x),$$

and each of these f^i has a characteristic polynomial. *Are the coefficients of these polynomials rational integers, and are they independent of ℓ?* When X is smooth and proper of dimension n, the same question is meaningful when f is replaced by any cycle of dimension n in $X \times X$, considered as an algebraic correspondence.

In characteristic zero, one sees that this is so by using integral cohomology. If $\mathrm{char}\, k > 0$, one feels certain that this is so, but this has not been proved so far.

Let us fix for simplicity an isomorphism

$$\ell^{\infty k^* \simeq \mathbf{Q}_\ell / \mathbf{Z}_\ell} \qquad \text{(a heresy!)}.$$

We than have a map

$$: F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathrm{H}^{2i}_\ell(X)$$

which associates to an algebraic cycle its cohomology class. We denote by $C^i_\ell(X)$, and refer to its elements as *algebraic cohomology classes*.

A known result, due to Dwork-Faton, shows that for the integrality question (not to speak of the independence of the characteristic polynomial of ℓ), it suffices to prove that

$$f_i^N \in \frac{1}{m}\mathbf{Z} \quad \text{for every } N \geq 0,$$

where m is a fixed positive integer[3]. Now, the graph Γ_{f^N} in $X \times X$ of f^N defines a cohomology class on $X \times X$, and if the cohomology class Δ of the diagonal in $X \times X$ is written as

$$\Delta = \sum_0^n \pi_i$$

where π_i are the projections of Δ onto $\mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$ for the canonical decomposition $\mathrm{H}^n(X \times X) \simeq \sum_{i=0}^n \mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$, a known calculation shows that

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\Gamma_{f^N})\pi_i \in \mathrm{H}^{4n}(X \times X) \approx \mathbf{Q}_\ell.$$

Assume that the π_i are algebraic. Then $\pi_i = \frac{1}{m}(\prod_i)$, where $\prod_i$ is an algebraic cycle, hence

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\prod_i \Gamma_{f^N}) \in \frac{1}{m}\mathbf{Z}$$

and we are through.

WEAK FORM OF CONJECTURE 1. $(C(X))$: The elements π_i^ℓ are algebraic, (and come from an element of $F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is independent of ℓ).

N.B.

1. The statement in parenthesis is needed to establish the independence of P_i on ℓ.
2. If $C(X)$ and $C(Y)$ hold, $C(X \times Y)$ holds, and more generally, the Künneth components of any algebraic cohomology class on $X \times Y$ are algebraic.

[3]This was pointed out to me by S. Kleimann.

0.10 The conjecture 1 (of Lefschetz type)

Let X be smooth and projective, and $\xi \in \mathrm{H}^2(X)$ the class of a hyperplane section. Then we have a homomorphism

$$(*) \qquad \cup\xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X) \quad (i \le n).$$

It is expected (and has been established by Lefschetz [?], [?] over the complex field by transcendental methods) that this is an isomorphism for all characteristics. For $i = 2j$, we have the commutative square

[]

Our conjecture is then: $(A(X))$:

(a) *(*) is always an isomorphism (the mild form);*

(b) *if $i = 2j$. (*) induces an isomorphism (or equivalently, an epimorphism) $C^j(X) \longrightarrow C^{n-j}(X)$.*

N.B. If $C^j(X)$ is assumed to be finite dimensional, (b) is equivalent to the assertion that $\dim C^{n-j}(X) \le \dim C^j(X)$ (which in particular implies the equality of these dimensions in view of (a)).

An equivalent formulation of the above conjecture (for all varieties X as above) is the following.

$(B(X))$: *The Λ-operation (c.f. [?]) of Hodge theory is algebraic.*

By this, we mean that there is an algebraic cohomology class λ in $\mathrm{H}^*(X \times X)$ such that the map $\Lambda : \mathrm{H}^*(X) \longrightarrow \mathrm{H}^*(X)$ is got by lifting a class from X to $X \times X$ by the first projection, cupping with λ and taking the image in $\mathrm{H}^*(X)$ by the Gysin homomorphism associated to the second projection

Note that $B(X) \Rightarrow A(X)$, since the algebricity of λ implies that of λ^{n-i}, and λ^{n-i} provides an inverse to $\cup\xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X)$. On the other hand, it is easy to show that $A(X \times X) \Rightarrow B(X)$ and this proves the equivalence of conjectures A and B.

The conjecture seems to be most amenable in the form of B. Note that $B(X)$ is stable for products, hyperplane sections and specialisations. In particular, since it holds for projective spaces, it is also true or smooth varieties which are complete intersections in some projective space. (As a consequence, we deduce for such varieties the wished-for integrality theorem for the Z-function!). It is also verified for Grassmannians, and for abelian varieties (Liebermann [?]).

I have an idea of a possible approach to Conjecture B, which relies in turn on certain unsolved geometric questions, and which should be settled in any case.

Finally, we have the implication $B(X) \Rightarrow C(X)$ (first part), since the π_i can be expressed as polynomials with coefficients in $\mathbf{Q}$ of λ and $L = \cup\xi$. To get the whole of $C(X)$, one should naturally assume further that there is an element of $F(X \times X) \otimes_{\mathbf{Z}} \mathbf{Q}$ which gives λ for every ℓ.

0.11 Conjecture 2 (of Hodge type)

For any $i \leq n$, let $P^i(X)$ be the 'primitive part' of $\mathrm{H}^i(X)$, that is, the kernel of $\cup\xi^{n-i+1}$: $\mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i+2}(X)$, and put $C^j_{Pr}(X) = P^{2j} \cap C^j(X)$. On $C\widehat{\Theta}_{Pr}(X)$, we have a $\mathbf{Q}$-valued symmetric bilinear form given by

$$(x, y) \longrightarrow (-1)^j K(xy\xi^{n-2j})$$

where K stands for the isomorphism $\mathrm{H}^{2n}(X) \simeq \mathbf{Q}_\ell$. Our conjecture is then that

$((X))$: *The above form is positive definite.*

One is easily reduced to the case when $\dim X = 2m$ is even, and $j = m$.

REMARKS.

(1) In characteristic zero, this follows readily from Hodge theory [?].

(2) $B(X)$ and $Hdg(X \times X)$ imply, by certain arguments of Weil and Serre, the following: if f is an endomorphism of X such that $f^*(\xi) = q\xi$ for some $q \in \mathbf{Q}$ (which is necessarily > 0), then the eigenvalues of $f_{\mathrm{H}^i(X)}$ are algebraic integers of absolute value $q^{i/2}$. Thus, this implies all of Weil's conjectures.

(3) The conjecture $Hdg(X)$ together with $A(X)(a)$ (the Lefschetz conjecture in cohomology) implies that numerical equivalence of cycles is the same as cohomological equivalence for any ℓ-adic cohomology if and only if $A(X)$ holds.

(4) In view of (3), $B(X)$ and $Hdg(X)$ imply that numerical equivalence of cycles coincides wit $\mathbf{Q}_\ell$-equivalence for any ℓ. Further the natural map

$$\mathrm{Z}^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \longrightarrow \mathrm{H}^i_\ell(X)$$

is a monomorphism, and in particular, we have

$$\dim_{\mathbf{Q}} C^i(X) \leq \dim_{\mathbf{Q}_\ell} \mathrm{H}^i_\ell(X).$$

Note that for the deduction of this, we do not make use of the positivity of the form considered in (X),but only the fact that it is non-degenerate.

Another consequence of $Hdg(X)$ and $B(X)$ is that the stronger version of $B(X)$, viz. that λ comes from an algebraic cycle with rational coefficients *independent of* ℓ, holds.

0.12 Conclusions

The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties.

Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry.

CURRICULUM VITAE DE ALEXANDRE GROTHENDIECK

Né le 28 mars 1928 à Berlin, de mère allemande et de père apatride, émigré de Russie en 1921, mes parents émigrent d'Allemagne en 1933, participent à la révolution espagnole; je les rejoins en mai 1939. Mes parents sont internés, d'abord mon père en 1939, puis ma mère en 1940 avec moi. Mon père est déporté du camp de Vernet en août 1942 pour Auschwitz et est resté disparu; ma mère meurt en 1957 des suites d'une tuberculose contractée au camp de concentration. Je reste près de deux ans dans des camps de concentration français, puis suis recueilli par une maison d'enfants du "Secours suisse" au Chambon-sur-Lignon, où je termine mes études de lycée en 1945. Études de licence (mathématiques) à Montpellier 1945-48, auditeur libre à l'École Normale Supérieure à Paris en 1948-49, où je suis le premier séminaire Cartan sur la théorie des faisceaux, et un cours de Leray du Collage de France sur la théorie de Schauder du degré topologique dans les espaces localement convexes. De 1949 à 1953 je poursuis des recherches à Nancy sur les espaces vectoriels topologiques, comme élève de J. Dieudonné et de L. Schwartz, aboutissement à ma thèse de doctorat en 1953, sur la théorie des produits tensoriels topologiques et des espaces nucléaires, publiée dans les "Memoirs of the American Mathematical Society". Je passe alors deux ans à l'Université de Sao Paulo (Brésil), où je continue et mène à leur aboutissement naturel certaines recherches liées aux produits tensoriels topologiques [?], [?], mais en même temps, sous l'influence de J. P. Serre, commence à me familiariser avec des questions de topologie algébrique et d'algèbre homologique. Ces dernières continueront à m'occuper jusqu'à aujourd'hui, et son encore très loin d'être menés à leur terme. Ce sont elles qui m'occuperont surtout pendant l'année 1955 passée à l'Université du Kansas (USA) ; j'y développe une théorie commune pour la théorie de Cartan-Eilenberg des foncteurs dérivés des foncteurs de modules et la théorie de Leray-Cartan de la cohomologie des faisceaux [?], et développe des notions de "cohomologie non commutative" dans le contexte des faisceaux et des espaces fibrés à faisceau structural, qui trouveront leur cadre naturel quelques années plus tard avec la théorie des topos (aboutissement naturel du point de vue faisceautique en topologie générale) [?], [?].

A partir de 1956 je suis resté en France, à l'exception de séjours de quelques semaines ou mois dans des universités étrangères. De 1950 à 1958 j'ai été chercheur au CNRS, avec le grade de directeur de recherches en 1958. De 1959 à 1970 j'ai été professeur à l'Institut des Hautes Études Scientifiques. Ayant découvert à la fin de 1959 que l'IHES était subventionné depuis trois ans par le Ministère des Armées, et après des essais infructueux pour inciter mes collègues à une action commune sans équivoque contre la présence de telles subventions, je quitte l'IHES en septembre 1970.

Depuis 1959 je suis marié à une française, et je suis père de quatre enfants. Je suis apatride depuis 1940, et ai déposé une demande de naturalisation française au printemps 1970.

Depuis 1956 jusqu'à une date récente, mon intérêt principal s'est porté sur la géométrie algébrique. Mon intérêt pour la topologie, la géométrie analytique, l'algèbre homologique ou le langage catégorique a été constamment subordonné aux multiples besoins d'un vaste programme de construction de la géométrie algébrique, dont une première vision d'ensemble remonte à 1958. Ce programme est poursuivi systématiquement dans [?], [?], d'abord dans un isolement relatif, mais progressivement avec l'assistance d'un nombre croissante de chercheurs de valeur. Il est loin d'être achevé à l'heure actuelle. L'extraordinaire crise écologique que nous aurons à affronter dans les décades qui viennent, rend peu probable qu'il le sera jamais. Elle nous imposera d'ailleurs une perspective et des critères de valeur entièrement nouveaux, qui réduiront à l'insignifiance ("irrelevance") beaucoup des plus brillants progrès scientifiques de notre siècle, dans la mesure où ceux-ci restent étrangers au grand impératif évolutionniste de notre temps : celui de la survie. Cette optique s'est imposée à moi avec une force croissante au cours de discussions avec de nombreux collègues sur la responsabilité sociale des scientifiques, occasionnées par ma situation à l'IHES depuis la fin de 1969. Elle m'a conduit en juillet 1970 à m'associer à la fondation d'un mouvement international et interprofessionnel "Survivre", et à consacrer aux questions liées à la survie une part importante de mon énergie. Dans cette optique, la seule valeur de mon apport comme mathématicien est de me permettre aujourd'hui, grâce à l'estime professionnelle et personnelle acquise parmi mes collègues, de donner plus de force à mon témoignage et à mon action en faveur d'une stricte subordination de toutes nos activités, y compris nos activités de scientifiques, aux impératifs de la survie, et à la promotion d'un ordre stable et humain sur notre planète, sans lequel la survie de notre espèce ne serait ni possible, ni désirable.

A Grothendieck

Principales publication

Espaces Vectoriels Topologiques

[?] Critères de compacité dans les espaces fonctionnels généraux, Amer. J. 74 (1952), p. 168-186

Topologie et algèbre homologique

Géométrie analytique

Géométrie algébrique

ESQUISSE THÉMATIQUE DES PRINCIPAUX TRAVAUX MATHÉMATIQUES

Les numéros entre crochets renvoient, soit à la bibliographie sommaire jointe à mon Curriculum Vitae (numéros de [?] à [?]), soit au complément à cette bibliographie placée à la fin du présent rapport (numéros entre [?] et [?]). Enfin, nous avons joint en dernière page une liste par ordre alphabétique des auteurs de certains des travaux cités dans les présent rapport qui ont été directement suscités ou influencés par les travaux de A. Grothendieck; le renvoi à cette dernière bibliographie se fait par le sigle [*] derrière le nom de l'auteur cité, comme pour I. M. Gelfand [*].

1. Analyse Fonctionnelle ([?] à [?], [?])

Mes travaux d'Analyse Fonctionnelle (de 1949 à 1953) ont porté surtout sur la théorie des espaces vectoriels topologiques. Parmi les nombreuses notions introduites et étudiées (produits tensoriels topologiques [5,6], applications nucléaires et applications de Fredholm [5,6,7], applications intégrales et ses variantes diverses [5,6], applications de puissance p-ième sommable [5], espaces nucléaires [5], espaces (DF) [4], etc.), c'est la notion d'*espace nucléaire* qui a connu la meilleure fortune : elle a fait jusqu'à aujourd'hui l'objet de nombreux séminaires et publications. En particulier, un volume du traité de I. Gelfand [*] sur les "Fonctions Généralisées" lui est consacré. Une des raisons de cette fortune provient sans doute de la théorie des probabilités, car il s'avère que parmi tous les EVT, c'est dans les espaces nucléaires que la théorie de la mesure prend la forme la plus simple (théorème de Minlos).

Les résultats de [6], plus profonds, semblent avoir été moins bien assimilés par les développements ultérieurs, mais ils apparaissent comme source d'inspiration dans un certain nombre de travaux délicats assez récents sur des inégalités diverses liées à la théorie des espaces de Banach, notamment ceux de Pelczynski. Signalons également les résultats assez fins de [6] et de [8 bis] sur les propriétés de décroissance de la suite des valeurs propres de certains opérateurs dans les espaces de Hilbert et dans les espaces de Banach généraux.

Références : L. Schwartz, J. Dieudonné, I. Gelfand, P. Cartier, J. L. Lions.

2. Algèbre Homologique ([?], [?], [?], [?])

Depuis 1955, me plaçant au point de vue de "l'usager" et non celui de spécialiste, j'ai été amené continuellement à élargir et à assouplir le langage de l'algèbre homologique, notamment sous la poussée des besoins de la géométrie algébrique (théories de dualité, théories du type Riemann-Roch, cohomologies ℓ-adiques, cohomologies du type de De Rham, cohomologies cristallines…). Deux directions principales à ces réflexions : développement d'une algèbre homologique non commutative (amorcée dans [10 bis] et systématisée dans la thèse de J. Giraud [*]); théorie des catégories dérivées (dévelopée systématiquement par J. L. Verdier, exposée dans Hartshorne [*], Illusie [*] et [[?] SGA 4 Exp. XVIII]). Ces deux courants de réflexion sont d'ailleurs loin d'être épuisés, et sont sans doute appelés à se rejoindre, soit au sein d'une "algèbre homotopique" dont une esquisse préliminaire a été faite par Quillen [*], soit dans l'esprit de la théorie des n-catégories, particulièrement bien adaptée à l'interprétation géométrique des invariants cohomologiques (cf. le livre cité de J. Giraud et le travail de Mme. M. Raynaud [*]).

Références : J.L. Verdier, P. Deligne, D. Quillen, P. Gabriel.

3. Topologie ([?], SGA 4, [?])

Jusqu'à présent, c'est surtout le K-invariant des espaces topologiques que j'avais introduit à l'occasion de mes recherches sur le théorème de Riemann-Roch en géométrie algébrique, qui a connu la fortune la plus brillante, étant le point de départ de très nombreuses recherches en topologie homotopique et topologie différentielle. De nombreuses constructions que j'avais introduits pour les besoins de la démonstration algèbrqiue du théorème de Riemann-Roch (telles les opérations λ_i et leurs liens avec les opérations du groupe symétrique) sont devenues pratique courante non seulement en géométrie algébrique et en algèbre, mais également en topologie et en théorie des nombres, notamment dans les travaux de mathématiciens commme Atiyah, Hirzebruch, Adams, Quillen, Bass, Tate, Milnor, Karoubi, Shih, etc...

Plus fondamental me semble néanmoins l'élargissement de la topologie générale, dans l'esprit de la théorie des faisceaux (développée initialement par J. Leray), contenu dans le point de vue des topos ([[?], SGA 4]). J'ai introduit ces topos à partir de 1958 en partant du besoin de définir une cohomologie ℓ-adique des variétés algébriques (plus généralement, des schémas), qui convienne à l'interprétation cohomologique des célèbres conjectures de Weil. En effet, la notion traditionnelle d'espace topologique ne suffit pas à traiter le cas des variétés algébriques sur un corps autre que le corps des complexes, la topologie proposée précédemment par Zariski ne donnant pas lieu à des invariants cohomologiques "discrets" raisonnables. A l'heure actuelle, le point de vue des topos, et la notion de "localisation" correspondante, font partie de la pratique quotidienne du géomètre algébriste, et il commence à se répandre également en théorie des catégories et en *logique mathématique* (avec la démonstration par B. Lawvere [*] du théorème de Cohen d'indépendance de l'axiome du continu, utilisant une adaptation convenable de la notion de topos). Il n'en est pas encore de même en topologie et en géométrie différentielle et analytique, malgré certains premiers essais dans ce sens (comme la tentative de démonstration par Sullivan d'une conjecture d'Adams en K-théorie, par réduction à une propriété de l'opération de Frobenius sur les variétés algébriques en car. $p > 0$).

Références : M. Atiyah, F. Hirzebruch, H. Bass, J. Leray, M. Artin, D. Quillen, M. Karoubi...

4. Algèbre ([?], [?], [?])

Comme l'algèbre homologique, l'algèbre a été pour moi un outil à développer, et non un but en soi. J'ai parlé au par. 2 de mes contributions à l'algèbre homologique, et au par. 3 de mes contributions à la K-théorie; celle-ci comprend une partie purement algébrique (qui, une fois étendue en une théorie des K^i supérieurs, finira par devenir une partie de l'algèbre homologique ou homotopique). Ainsi, un certain nombre de mes résultats en géométrie algébrique se spécialisent en des résultats en algèbre pure, comme la relation $K(A[t]) \simeq K(A)$, où A est un anneau. Mises à part ces retombées, on peut signaler les contributions ci-dessous.