Geodesic Paths and Distances on Meshes: PDE and Computational Geometry Approaches

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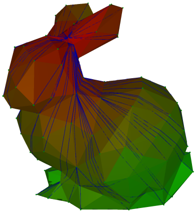
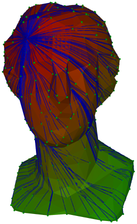
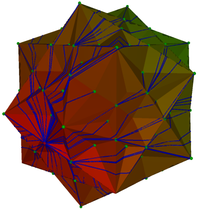
Introduction







Problem Positioning

Given a mesh of a 2-manifold embedded in $\mathbb{R}^3$ and two points on the mesh, what is the shortest path between the points?

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Mesh: a set $\mathbb{V}$ of vertices, a set $\mathcal{F}$ of faces in $\mathbb{V}^3$.

Discrete Differential Geometry

The Laplace-Beltrami operator of a function u is given by:

$$(\Delta u)_i = \frac{1}{2A_i} \sum_{e=(i,j)} \left(\cot \alpha_{i,j} + \cot \beta_{i,j}\right) \left(u_i - u_j\right). \tag{1}$$

It quantifies the symmetric deviation of the variation of the value at a point.

Physical Equations

We base ourselves on equations modelling the propagation of a phenomenon:

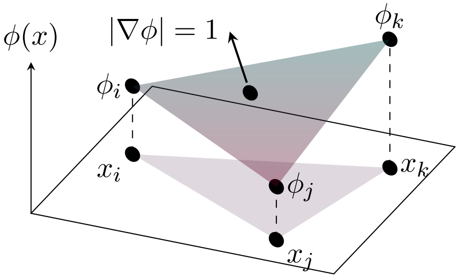
Heat Equation $\nabla u_t = \frac{\mathrm{d}}{\mathrm{d}t} u_0$;

Poisson Equation $\Delta u = u_0$ for a fixed distribution u_0;

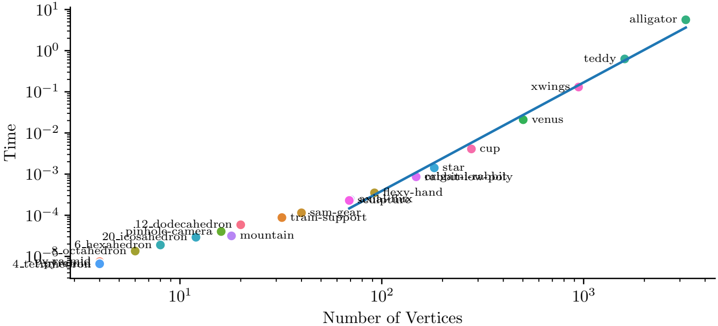
Spectral Embedding ℓ^2 distance based on eigenvectors and eigenvalues of the laplacian Δ.

Fast Marching

We compute a variation of Dijkstra's algorithm based on the Eikonal equation $|nablau| = 1$ for wavefront propagation.

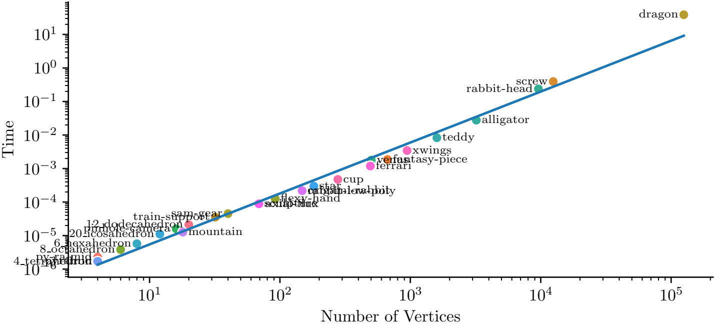


Compute Time I



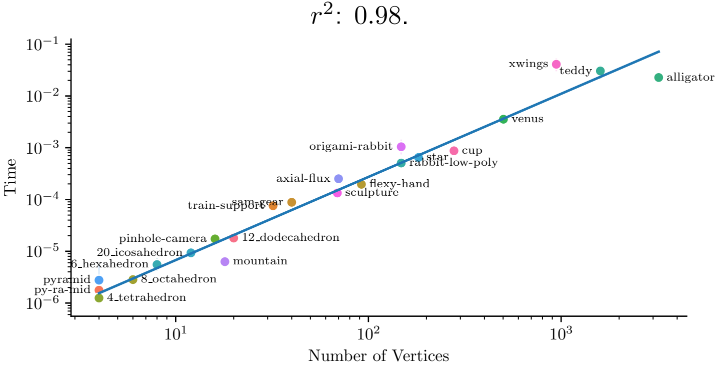
Benchmark of the time complexity of the *Heat method* on various 3D meshes, as a function of the number of vertices. Slope: 2.63, r^2: 0.99.

Compute Time II



Benchmark of the time complexity of the *Fast Marching algorithm* on various 3D meshes, as a function of the number of vertices. Slope: 1.51,

Compute Time III



Compute Time IV

Benchmark of the time complexity of the *ICH algorithm* on various 3D meshes, as a function of the number of vertices. Slope: 1.60, r^2: 0.96.

Comparison of methods

Method	Theoretical Runtime	Actual Runtime	Precision	Limitations
PDE-based methods				
Heat	$\mathcal{O}(n_{\mathbb{V}}^{\omega})$	$\underset{n_{\mathbb{V}}\to\infty}{\mathcal{O}(n_{\mathbb{V}}^{\omega})}$	Approx.	Stability, runtime, fixed sources, no paths
Poisson	$\mathcal{O}(n_{\mathbb{V}}^{\omega})$		Approx.	Stability, not true geodesics, runtime, fixed sources, no paths
Spectral	$\mathcal{O}(n_{\mathbb{V}}^{\omega})$		Approx.	Stability, not true geodesics,

Bibliography I