

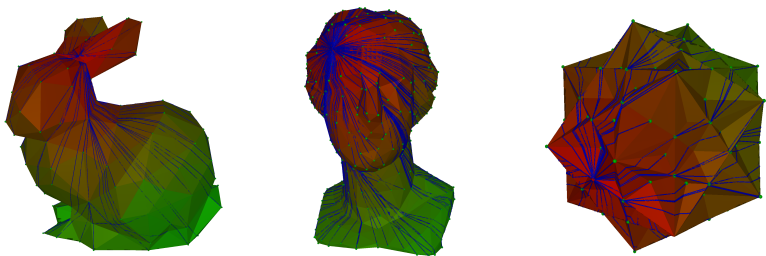
Geodesic Paths and Distances on Meshes: PDE and Computational Geometry Approaches

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Introduction



Problem Positioning

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Mesh: a set \mathbb{V} of vertices, a set \mathcal{F} of faces in \mathbb{V}^3 .

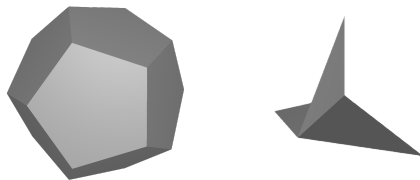


Figure: Examples of a 2-manifold (left) and non-manifold (right) mesh.

Discrete Differential Geometry

The Laplace-Beltrami operator of a function u is given by:

$$(\Delta u)_i = \frac{1}{2A_i} \sum_{e=(i,j)} (\cot \alpha_{i,j} + \cot \beta_{i,j}) (u_i - u_j). \quad (1)$$

It quantifies the symmetric deviation of the variation of the value at a point.

Physical Equations

We base ourselves on equations modelling the propagation of a phenomenon:

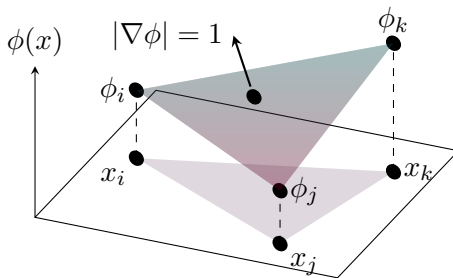
Heat Equation $\delta u_t = \frac{d}{dt} u_0;$

Poisson Equation $\Delta u = u_0$ for a fixed distribution u_0 ;

Spectral Embedding ℓ^2 distance based on eigenspaces of the laplacian Δ .

Fast Marching

We compute a variation of Dijkstra's algorithm based on the Eikonal equation $|\nabla u| = 1$ for wavefront propagation.



Improved Chen-Han (ICH) Algorithm

Idea: Propagate **windows** accross edges of the mesh, encoding shortest paths from a source.

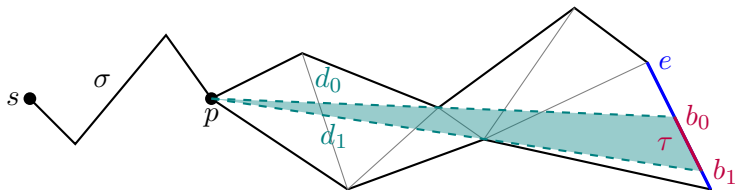


Figure: Illustration of a window $w = (s, p, e, b_0, b_1, d_0, d_1, \sigma)$.

One Angle, One Split Rule

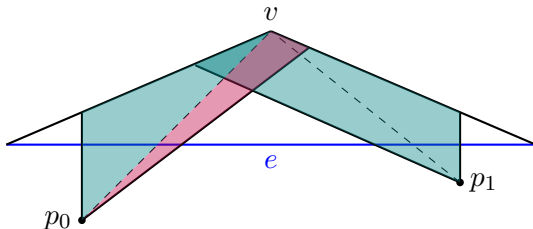
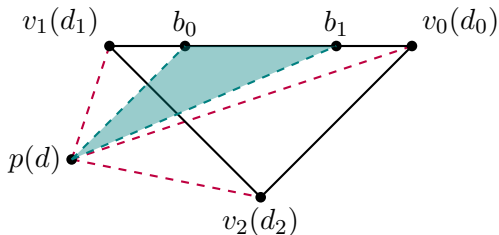


Figure: Illustration of the “one angle, one split” pruning rule: only keep three out of four windows.

Window Filtering



We can discard a window w if:

- $d + \|pb_0\| > d_0 + \|v_0b_1\|$
- $d + \|pb_1\| > d_1 + \|v_1b_1\|$
- $d + \|pb_0\| > d_2 + \|v_2b_0\|$

Compute time: Heat method

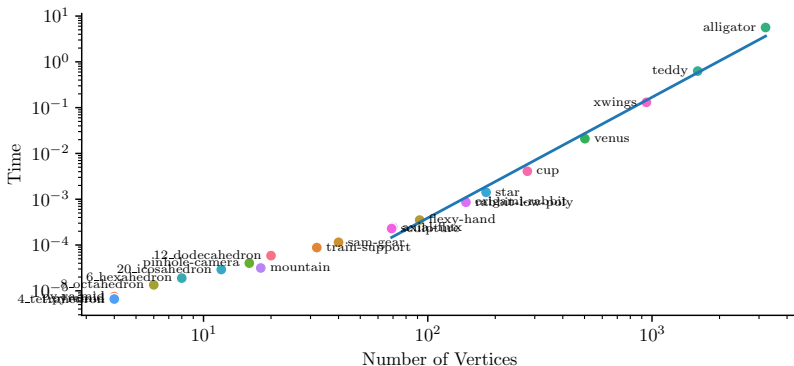


Figure: Benchmark of the *Heat method* as a function of the number of vertices. Slope: 2.63, $r^2: 0.99$.

[illegible]

Figure: Benchmark of the *Fast Marching algorithm* as a function of the number of vertices. Slope: 1.51, r^2 : 0.98.

Compute time: ICH

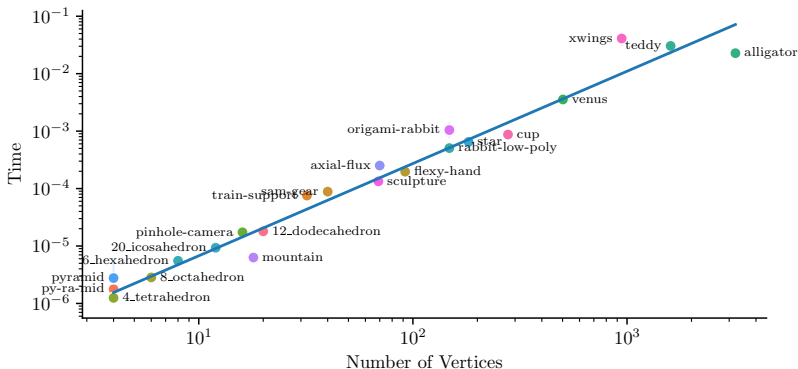


Figure: Benchmark of the *ICH algorithm* as a function of the number of vertices. Slope: 1.60, r^2 : 0.96.

Comparison of methods

Method	Theoretical Runtime	Actual Runtime	Precision	Limitations
<i>PDE-based methods</i>				
Heat	$\mathcal{O}(n_V^\omega)$	$\mathcal{O}(n_V^\omega)$ $n_V \rightarrow \infty$	Approx.	Stability, runtime, fixed sources, no paths
Poisson	$\mathcal{O}(n_V^\omega)$		Approx.	Stability, not true geodesics, runtime, fixed sources, no paths
Spectral	$\mathcal{O}(n_V^\omega)$		Approx.	Stability, not true geodesics,

Bibliography I
