Stationary surfaces for curvature functionals

Anthony Gruber

anthony.gruber@ttu.edu

Texas Tech University—Costa Rica

April 23, 2020

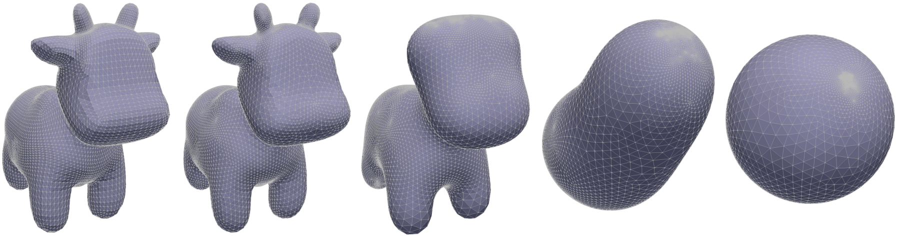
Source

Source material:

1) A. Gruber, M. Toda, H. Tran, "Stationary surfaces with boundaries" (submitted).

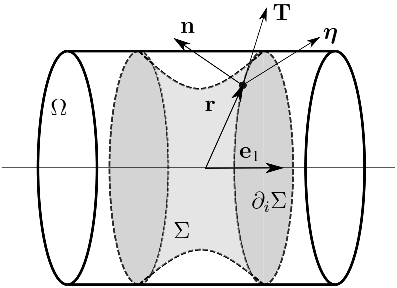
2) A. Gruber, E. Aulisa, "Computational p-Willmore Flow with Conformal Penalty" (to appear in ACM Transactions on Graphics).

3) A. Gruber, M. Toda, H. Tran, "On the variation of curvature functionals in a space form with application to a generalized Willmore energy", *Annals of Global Analysis and Geometry*. July 2019, Volume 56, Issue 1, pp 147-165, `https://doi.org/10.1007/s10455-019-09661-0`.



Outline

1. Introduction
2. Stationary surfaces and BVPs
3. Computing stationary surfaces for $\mathcal{W}^p$
4. Discretization and results



Origin: Bending energy

Let $\mathbf{r} : M \to \mathbb{R}^3$ be a smooth immersion of the surface (M, g), and let κ_1, κ_2 denote the principal curvatures of this immersion.

The study of **curvature functionals** grew primarily from a model for bending energy proposed by Sophie Germain in 1821,

$$\overline{\mathcal{B}}(\mathbf{r}) = \int_M S(\kappa_1, \kappa_2)\, dS,$$

where S is a symmetric polynomial. By Newton's theorem, this is equivalent to the functional

$$\mathcal{B}(\mathbf{r}) = \int_M F(H, K)\, d\mu_g,$$

where F is polynomial in the mean and Gauss curvatures,

$$H = \kappa_1 + \kappa_2, \qquad K = \kappa_1 \kappa_2.$$

The Willmore energy

The simplest quadratic bending energy model of this form is the (conformal) **Willmore energy**,

$$\overline{\mathcal{W}}(\mathbf{r}) = \frac{1}{4}\int_M (\kappa_1 - \kappa_2)^2 \, d\mu_g = \frac{1}{4}\int_M \left(H^2 - 4K\right) \, d\mu_g .$$

When M is closed, the Gauss-Bonnet theorem implies that this has identical extrema to the functional

$$\mathcal{W}^2(\mathbf{r}) = \frac{1}{4}\int_M H^2 \, d\mu_g ,$$

which is usually also called the Willmore energy.

(Notation: We will refer to $\overline{\mathcal{W}}, \mathcal{W}^2$ as the conformal Willmore energy and the Willmore energy, respectively.)

Critical points of $\mathcal{W}^2$ are called **Willmore surfaces** (or Willmore immersions), and arise frequently in biology and physics.

Examples of Willmore-type energies

For example, consider the following energy functionals:

Helfrich-Canham energy,

$$E_{HC}(\mathbf{r}) := \int_M k_c(2H + c_0)^2 + \overline{k}K \, d\mu_g.$$

Bulk free energy density,

$$\sigma_F(\mathbf{r}) = \int_M 2k(2H^2 - K) \, d\mu_g.$$

Hawking mass,

$$m(\mathbf{r}) = \sqrt{\frac{\mathrm{Area}\, M}{16\pi}} \left(1 - \frac{1}{16\pi} \int_M H^2 \, d\mu_g\right).$$

When M is closed, all share stationary surfaces with $\mathcal{W}^2$!

Special properties of the Willmore energy

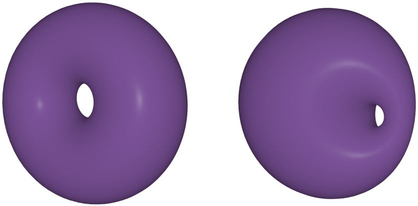
The most special property of the Willmore energy of immersed surfaces in $\mathbb{R}^3$ is its invariance under conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ (Blaschke 1929).

Sketch: Consider a conformal change of metric $g \mapsto e^{2\varphi} g$ for some function $\varphi : M \to \mathbb{R}$. A computation shows that

$$\left(H^2 - 4K\right) \mapsto e^{-2\varphi}\left(H^2 - 4K\right), \qquad d\mu_g \mapsto e^{2\varphi} d\mu_g,$$

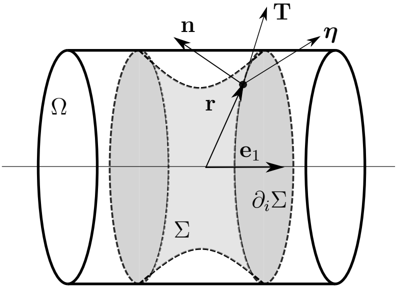
so that the Willmore energy is unchanged under a conformal transformation.

This property is beautiful, but not very physical. For example, the two pictured tori have **identical** Willmore energy!



Outline

1. Introduction
2. Stationary surfaces and BVPs
3. Computing stationary surfaces for $\mathcal{W}^p$
4. Discretization and results



The importance of invariance

This conformal invariance has big consequences for Willmore surfaces. In particular:

· Any composition of a minimal immersion and a conformal transformation is Willmore!

This led Blaschke to call Willmore surfaces "conformally minimal" and provides a good example of how the symmetries of a functional can be useful for constructing and studying minimizers.

With this motivation, we are interested in examining similar ideas in the general setting of curvature functionals:

Question 1: What other bending energy models are interesting, and how do their invariances (or lack thereof) influence their stationary surfaces?

Question 2: In the case of a surface with boundary, how does symmetry/behavior at the boundary of a critical surface affect its interior?

The general case

To examine these questions, we return to the general problem. First, it is advantageous to compute the first variation of $\mathcal{B}$.

To that end, let M be a surface with boundary ∂M, and let $\mathbf{r} : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a family of immersions with velocity

$$\delta_{\mathbf{X}} \mathbf{r} = \frac{d}{dt}\mathbf{r}\Big|_{t=0} = \mathbf{X}.$$

Recall that $\mathbf{r}$ carries an adapted orthonormal frame field $\{\mathbf{T}, \mathbf{n}, \boldsymbol{\eta}\}$ such that $\mathbf{T}$ is tangent to ∂M, $\mathbf{n}$ is everywhere normal to M, and $\boldsymbol{\eta} = \mathbf{T} \times \mathbf{n}$. Recall also the (scalar-valued) second fundamental form $h : TM \times TM \to \mathbb{R}$,

$$h(\mathbf{v}, \mathbf{w}) = \langle -\nabla_{\mathbf{v}} \mathbf{n}, \mathbf{w} \rangle = \langle S(\mathbf{v}), \mathbf{w} \rangle ,$$

where $S : TM \to TM$ is the self-adjoint shape operator.

The first variation of $\mathcal{B}$

Theorem: G., Toda, Tran [1]

In the notation above, the first variation of the functional $\mathcal{B}$ is given by

$$\delta_{\mathbf{X}}\mathcal{B} = \int_{\partial M} F \left\langle \mathbf{X}, \boldsymbol{\eta} \right\rangle \, ds + \int_{\partial M} \left\langle \mathbf{X}, \mathbf{n} \right\rangle \Big(h(\nabla F_K, \boldsymbol{\eta}) - \nabla_{\boldsymbol{\eta}} F_H - H \nabla_{\boldsymbol{\eta}} F_K \Big) \, ds$$
$$+ \int_{\partial M} \Big((F_H + H F_K) \nabla_{\boldsymbol{\eta}} \left\langle \mathbf{X}, \mathbf{n} \right\rangle - F_K h(\nabla \left\langle \mathbf{X}, \mathbf{n} \right\rangle, \boldsymbol{\eta}) \Big) \, ds$$
$$+ \int_M \left\langle \mathbf{X}, \mathbf{n} \right\rangle \Big(\Delta F_H + H \Delta F_K - \langle h, \mathrm{Hess}\, F_K \rangle + F_H |h|^2 + HKF_K - HF \Big) \, d\mu_g.$$

Therefore, M will be said to be a $\mathcal{B}$-**surface** provided it satisfies the Euler-Lagrange equation

$$\Delta F_H + H \Delta F_K - \langle h, \mathrm{Hess}\, F_K \rangle + F_H |h|^2 + HKF_K - HF = 0.$$

Conservation law for $\mathcal{B}$-surfaces

Additionally, the invariances of $\mathcal{B}$ under translation, rotation, and (in some cases) dilation imply the following conservation law.

Theorem: G., Toda, Tran [1]

Let

$$T = F_K\, S^2 - (F_H + HF_K)\, S + \mathbf{n} \otimes (S(\nabla F_K) - \nabla F_H - H\nabla F_K) + F\nabla \mathbf{r},$$

$$W = \Delta F_H + H\Delta F_K - \langle h, \mathrm{Hess}\, F_K \rangle + F_H |h|^2 + HKF_K - HF.$$

Then, it follows that

$$\mathrm{div}_g\, T = -W\mathbf{n}.$$

In particular, M is a $\mathcal{W}$-surface if and only if T is divergence-free.

It is easily checked that the stress tensor reduces to

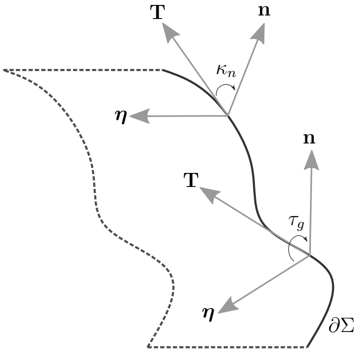
$$T = -2H\, S - 2\,\mathbf{n} \otimes \nabla H + H^2 \nabla \mathbf{r},$$

in the case of the Willmore functional $\mathcal{W}^2$, which coincides with expressions of Bernard [2] and Riviere [3].

Review: normal curvature and geodesic torsion

These variational results will lead to interesting boundary-value problems, but we first need a good description of boundary curvature. To that end, recall the **normal curvature** and **geodesic torsion** of ∂M,

$$\kappa_n = \langle \nabla_{\mathbf{T}} \mathbf{T}, \mathbf{n} \rangle = h(\mathbf{T}, \mathbf{T}),$$
$$\tau_g = \langle \nabla_{\mathbf{T}} \boldsymbol{\eta}, \mathbf{n} \rangle = h(\mathbf{T}, \boldsymbol{\eta}).$$



Note that κ_n measures how fast $\mathbf{T}$ rotates into $\mathbf{n}$ along the boundary curve, while τ_g measures how fast $\boldsymbol{\eta}$ rotates into $\mathbf{n}$. Moreover, it follows that on the boundary

$$H = \kappa_n + h(\boldsymbol{\eta}, \boldsymbol{\eta}),$$
$$K = h(\boldsymbol{\eta}, \boldsymbol{\eta})\kappa_n - \tau_g^2.$$

Fixed and free boundary conditions

It is now possible to formulate relevant BVP's involving bending energy models $\mathcal{B}$:

First, note that if ∂M is fixed, then $\mathbf{X} \equiv 0$ there. Further, this implies $\nabla_{\mathbf{T}} \langle \mathbf{X}, \mathbf{n} \rangle = 0$ on ∂M, so M is said to be a $\mathcal{B}$-surface with **fixed boundary** provided

$$\begin{aligned} 0 &= \Delta F_H + H \Delta F_K - \langle h, \mathrm{Hess}\, F_K \rangle + F_H |h|^2 + HKF_K - HF, \quad \text{in } M \\ 0 &= F_H + \kappa_n F_K \quad \text{on } \partial M. \end{aligned}$$

On the other hand, suppose $\mathbf{r}(\partial M) \subset \Omega$ for some smooth Ω with unit normal $\mathbf{v}$ compatible with the normal $\mathbf{n}$. Then, M is said to be a $\mathcal{B}$-surface with **free boundary** provided

$$\begin{aligned} 0 &= \Delta F_H + H \Delta F_K - \langle h, \mathrm{Hess}\, F_K \rangle + F_H |h|^2 + HKF_K - HF \quad \text{in } M, \\ 0 &= F_H + \kappa_n F_K \quad \text{on } \partial M, \\ 0 &= \left\langle \mathbf{v}, F\mathbf{n} - \Big(\nabla_{\mathbf{T}}(\tau_g F_K) + h(\nabla F_K, \boldsymbol{\eta}) - \nabla_{\boldsymbol{\eta}} F_H - H \nabla_{\boldsymbol{\eta}} F_K \Big) \boldsymbol{\eta} \right\rangle \text{ on } \partial M. \end{aligned}$$

An example of boundary control

Returning to the question of boundary influence on the interior, there is good reason to believe that boundary conditions can exert a lot of control. For example, consider the **p-Willmore energy** defined as

$$\mathcal{W}^p(\mathbf{r}) = \frac{1}{2^p} \int_M |H|^p \, d\mu_g, \qquad p \geq 1.$$

Then, the following result is known in the fixed boundary case.

Theorem: G., Toda, Tran [4]

When $p > 2$, any p-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on ∂M is minimal.

More precisely, let $p > 2$ and $\mathbf{r} : M \to \mathbb{R}^3$ be a p-Willmore immersion of the surface M with boundary ∂M. If $H = 0$ on ∂M, then $H \equiv 0$ everywhere on M.

An example of boundary control (2)

Interestingly, this is **not** true for $p = 2$, even though $\mathcal{W}^2$ is generally not conformally-invariant for surfaces with boundary. Examples of non-minimal Willmore catenoids satisfying $H = 0$ on the boundary were given in [5].

On the other hand, $\mathcal{W}^2$ is invariant under uniform dilations regardless of the surface involved. To see this, consider a dilation $\mathbf{r} \mapsto (1/t)\mathbf{r}$ for some $t > 0$. Then,

$$H \mapsto tH, \qquad d\mu_g \mapsto \frac{1}{t^2} d\mu_g,$$

and so the p-Willmore energy

$$\mathcal{W}^p(\mathbf{r}) \mapsto t^{p-2}\mathcal{W}^p(\mathbf{r}),$$

which coincides (only!) when $p = 2$.

Question 3: To what extent is scaling-invariance responsible for for the flexibility of $\mathcal{W}^2$ (or other invariant functionals) with respect to boundary conditions?

Partial answer in free-boundary case

To examine this question in our context, we say a bending energy functional $\mathcal{B}(\mathbf{r}) = \int_M F(H,K)\,d\mu_g$ is **scaling-invariant** provided that

$$F(tH, t^2K) = t^2 F(H,K),$$

for any $t > 0$. On the other hand, $\mathcal{B}$ will be called **expanding** (resp. **shrinking**) provided that

$$2F - HF_H - 2KF_K \geq 0$$
$$\left(\text{resp. } 2F - HF_H - 2KF_K \leq 0\right).$$

Theorem: G, Toda, Tran [1]

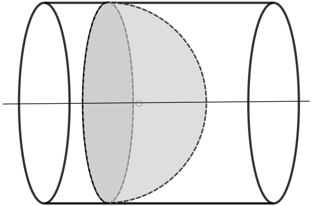
Let $\mathcal{B}$ be scaling-invariant, and $M \subset \mathbb{R}^3$ be an immersed $\mathcal{B}$-surface having free boundary with respect to $\Omega^2 \subset \mathbb{R}^3$. Suppose that M and Ω share a common axis of rotational symmetry, and Ω is strictly convex. Then, one of the following holds:

1) M is totally spherical and $F \equiv 0$ on M.
2) $F_H \equiv 0$ and F_K is constant on M.

Examples

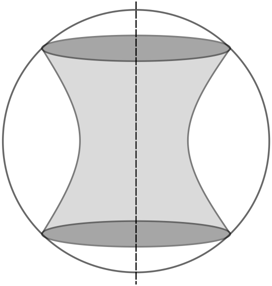
"M is totally spherical and $F \equiv 0$ on M."

Spherical cap in a cylinder, where $\mathcal{B}(\mathbf{r}) = \int_M (H^2 - K)\, d\mu_g$.



"$F_H \equiv 0$ and F_K is constant on M."

The critical catenoid in a sphere, where $\mathcal{B}(\mathbf{r}) = \int_M H^2\, d\mu_g$.



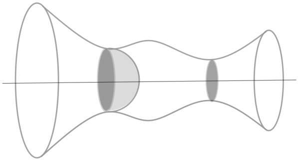
Consequences for conformal Willmore

It is known that any free-boundary conformal Willmore immersion of disk type with boundary in a plane is either a spherical cap or flat disk (B. Palmer [6]).

Our result provides complementary information in the case of rotational symmetry.

Corollary: G., Toda, Tran [1]

Let $M \subset \mathbb{R}^3$ be an immersed conformal Willmore surface that has free boundary with respect to Ω. Suppose that M and Ω share a common axis of rotational symmetry, and M intersects Ω transversally. Then M must be either spherical or flat.



Note: Convexity of Ω is not necessary in this case!

How important is scaling-invariance?

On the other hand, functionals which are not dilation-invariant behave quite differently. In particular, they are (in general) much more influenced by boundary conditions.

Theorem: G., Toda, Tran [1]

Let $\mathcal{B}$ be either shrinking or expanding, and let M be a rotationally-symmetric $\mathcal{B}$-surface with (fixed) boundary. Suppose additionally that the following hold:

1) $F - h(\boldsymbol{\eta}, \boldsymbol{\eta})F_H - KF_K = 0$ on ∂M,

2) $\nabla_\eta F_H + \kappa_n \nabla_\eta F_K = 0$ on ∂M.

Then, either M is spherical or there is a constant c such that $F_H \equiv 0$, $F_K \equiv c$, and $F \equiv cK$ on M.

For example, the first case occurs when $F = (H^2 - 4K)^2$, while the second case happens if $F = H^4 + K$ and Σ is minimal.

Functionals involving only mean curvature

When the functional $\mathcal{B}$ does not depend on K (as in area, total mean curvature, p-Willmore, etc.), the following can be shown.

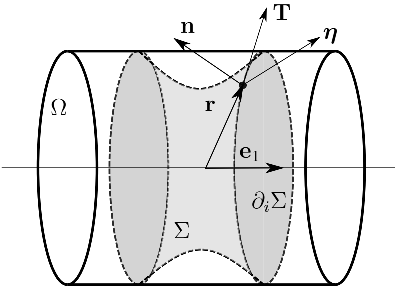
Theorem: G., Toda, Tran [1]

Let $\mathcal{B}$ be expanding or shrinking, Σ be a $\mathcal{B}$-critical surface, and $F = F(H)$ be a real analytic function of H alone. Suppose $F = F_H = \nabla_\eta F_H = 0$ on $\partial\Sigma$. Then, one of the following holds:

1) $F \equiv 0$ everywhere,
2) $F \equiv cH^2$ for some $c \in \mathbb{R}$ and $\mathcal{B}$ is scaling-invariant,
3) Σ has constant mean curvature and $F = 0$ on Σ.

Remarks: This result generalizes the p-Willmore example from before. Moreover, the real-analtyicity assumption is strictly necessary, otherwise examples can be constructed for which the theorem holds only locally.

Outline



How do we study stationary surfaces computationally?

It is also beneficial to have a computational model for stationary surfaces:

1) Helps provide intuition about the problems and allows us to "test" theoretical results.

2) Useful in producing visuals for scientific dissemination and outreach.

To that end, we now consider the problem of minimizing the p-Willmore energy of **closed surfaces** M subject to physical constraints on surface area and enclosed volume.

Note that it is necessary to constrain p-Willmore in order to have (the possibility of) closed-surface minimizers for $p > 2$, in view of our Theorem from earlier.

Model problem

More precisely, studying the constrained p-Willmore minimization means computing an immersion $u : M \to \mathbb{R}^3$ satisfying

$$\begin{aligned}
&\min_u \left(\mathcal{W}^p(u) + \lambda\mathcal{V}(u) + \nu\mathcal{A}(u)\right), \\
&s.t. \\
&V_0 = \int_M \langle u, N\rangle \, d\mu_g := \mathcal{V}(u), \\
&A_0 = \int_M d\mu_g := \mathcal{A}(u),
\end{aligned}$$

where $N : M \to \mathbb{S}^2$ is the outward-directed unit normal field.

A suitable weak formulation of this will allow for implementation using piecewise-linear finite elements.

The p-Willmore equation

It can be shown that any (unconstrained) p-Willmore surface satisfies the equation

$$0 = -\frac{p}{2}\Delta_g\left(H|H|^{p-2}\right) - pH|H|^{p-2}\left(2H^2 - K\right) + 2H|H|^p.$$

However, this is a scalar equation which is 4^{th}-order in the immersion u, and not suitable for finite-element modeling.

Instead, we must compute a weak expression which:

1) Is at most first-order in the immersion u.

2) Does not require a preferential frame in which to calculate derivatives (no moving frame).

3) Considers general variations $\varphi : M \to \mathbb{R}^3$, which may have tangential as well as normal components.

4) Avoids geometric terms that are not easily discretized, such as K and $|h|^2$.

The variational framework

To compute a usable expression for $\delta\mathcal{W}^p$, we adopt the framework of Dziuk and Elliott [7] and consider the (constrained) p-Willmore flow,

$$\dot{u} = -\delta\,\mathcal{W}^p(u) - \lambda\delta\mathcal{V}(u) - \nu\delta\mathcal{A}(u).$$

Consider a parametrization $X_0 : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ of (a portion of) the surface M, and let $u_0 : M \to \mathbb{R}^3$ be identity on M, so $u \circ X = X$.

A variation of the immersion u is a smooth function $\varphi : M \to \mathbb{R}^3$ and a 1-parameter family $u(x,t) : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $u(x,0) = u_0$ and

$$u(x,t) = u_0(x) + t\varphi(x).$$

Note that this pulls back to a variation $X : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$. Moreover, (since u is identity on $X(t)$) the time derivatives are related by

$$\dot{u} = \frac{d}{dt}u(X,t) = \nabla_{\mathbb{R}^3} u \cdot \dot{X} + u_t = \dot{X}.$$

Calculating the first variation

The trick to finding a good expression for the variation of $\mathcal{W}^p$ is to exploit the geometry of the problem.

In particular, there is the identity $\Delta_g u = HN$ for (twice) the mean curvature vector, so Dziuk noticed in [8] that the 4^{th}-order Willmore equation can be split by defining $Y = \Delta_g u$.

Weakly, we have

$$0 = \int_M \langle Y, \psi \rangle \, d\mu_g + \int_M \langle du, d\psi \rangle_g \, d\mu_g,$$

for all $\psi \in H^1(M; \mathbb{R}^3)$.

With this idea, the p-Willmore functional becomes

$$\mathcal{W}^p(M) = \int_M |Y|^p \, d\mu_g,$$

which involves no explicit derivatives of u.

Calculating the first variation (3)

Problem (Closed surface p-Willmore flow with constraint)

Let $p \geq 1$, $D(\varphi) = \nabla_g \varphi + (\nabla_g \varphi)^T$, *and* $W := |Y|^{p-2} Y$. *Determine a family* $u : M \times (0, T] \to \mathbb{R}^3$ *of surface immersions with* $M(t) = u(M, t)$ *such that* $M(0)$ *has initial volume* V_0, *initial surface area* A_0, *and for all* $t \in (0, T]$ *the equations*

$$
\begin{aligned}
0 &= \int_M \langle \dot{u}, \varphi \rangle \, d\mu_g + \int_M \nu \langle du, d\varphi \rangle_g \, d\mu_g + \int_M \lambda \langle \varphi, N \rangle \, d\mu_g \\
&\quad + \int_M ((1-p)|Y|^p - p \operatorname{div}_g W) \operatorname{div}_g \varphi \, d\mu_g \\
&\quad + \int_M p \left(\langle D(\varphi) du, dW \rangle_g - \langle d\varphi, dW \rangle_g \right) d\mu_g, \qquad (1) \\
0 &= \int_M \langle Y, \psi \rangle \, d\mu_g + \int_M \langle du, d\psi \rangle_g \, d\mu_g, \\
0 &= \int_M \left\langle W - |Y|^{p-2} Y, \xi \right\rangle d\mu_g, \\
A_0 &= \int_M 1 \, d\mu_g, \qquad (2) \\
3V_0 &= \int_M \langle u, N \rangle \, d\mu_g, \qquad (3)
\end{aligned}
$$

are satisfied for some piecewise-constant λ, ν *and all* $\varphi, \psi, \xi \in H^1\left(M(t); \mathbb{R}^3\right)$.

Continuous stability

This weak formulation can be shown to monotonically decrease the p-Willmore energy.

Theorem: Aulisa, G. [9]

The closed surface p-Willmore flow is energy decreasing for $p \geq 1$, i.e.

$$0 = \int_{M(t)} |\dot{u}|^2 \, d\mu_g + \frac{d}{dt} \int_{M(t)} \left(|Y|^p + \lambda \langle u, N \rangle + \nu \right) d\mu_g$$

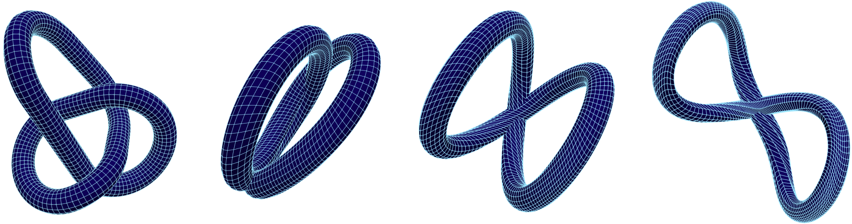
for all $t \in (0, T]$ and any piecewise-constant λ, ν.

This very nice property is what allows us to compute p-Willmore surfaces using the gradient flow, so it is important to make sure that this stability property is preserved by our chosen discretization.

We now have a beautiful way to compute any p-Willmore surface that is stationary under the constrained p-Willmore flow, right? ......

Mesh degeneration

Unfortunately, discrete computational flows typically involve some degree of **mesh sliding**, which can artificially break the simulation.



In order to prevent this, we now formulate a secondary system which will regularize the evolving surface at each time step.

This will help prevent failure which is not reflected by the continuous flow.

Conformal correction (1)

To correct mesh sliding at each time step, the goal is to enforce "Cauchy-Riemann equations" on the tangent bundle TM.

Let $u : M \to \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$ be an oriented immersion of M, and J be a complex structure (rotation operator $J^2 = -\mathrm{Id}_{TM}$) on TM. Then, if $*\alpha = \alpha \circ J$ is the usual Hodge star on forms,

Thm: Kamberov, Pedit, Pinkall [10]

The immersion u is conformal if and only if there is a Gauss map $N : M \to \mathrm{Im}\,\mathbb{H}$ such that $*du = N\,du$.

Note that,

$N \perp du(v)$ for all tangent vectors $v \in TM$.

$v, w \in \mathrm{Im}\,\mathbb{H} \longrightarrow vw = -v \cdot w + v \times w$.

Conclusion: conformality may be enforced by requiring $*du(v) = N \times du(v)$ on a basis for TM!

Conformal correction (2)

How is this accomplished on the parametrization domain V? We proceed through minimization. Define the **conformal distortion** functional,

$$\mathcal{CD}(u) = \frac{1}{2}\int_M |du\, J - N \times du|^2\, d\mu_g.$$

Then, (significant) computation establishes the variation

$$\delta\mathcal{CD}(u)\varphi = \int_M \langle Q, d\varphi\rangle_g\, d\mu_g,$$

where $Q \in T^*V \otimes TM$ is a tensor involving components of the normal N and first derivatives of the immersion u.

This can be used along with an appropriate constraint to “reparametrize” a surface after it has moved.

An appropriate constraint

Idea: If $u(x)$ (old) and $\hat{u}(x)$ (new) are close, then $(u - \hat{u})(x)$ should be orthogonal to $N(x)$ (to first order).

More formally, given $\varepsilon > 0$ and an immersion u with outer normal field N, the mesh regularization problem is to find a function $v : M \to \mathbb{R}^3$ and a Lagrange multiplier $\rho : M \to \mathbb{R}$, so that the new immersion $\hat{u} = u + v$ is the solution to

$$\min_{v} \left(\mathcal{CD}(u+v) + \varepsilon |\rho|^2 \mathcal{A}(u) + \rho \int_M \langle v, N \rangle \, d\mu_g \right).$$

This provides something close to a tangential reparametrization of M.

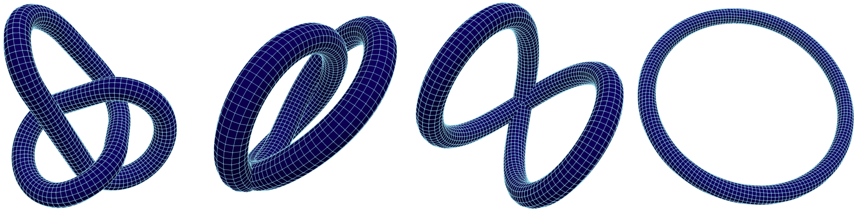
Note the penalty term in the minimization, which is beneficial for saddle point problems involving a mixture of linear and constant finite-elements.

Conformal correction (5)

Formulated weakly, the goal of this procedure is find a new immersion $\hat{u} = u + v$ and a multiplier ρ which satisfy the system

$$0 = \int_M \rho \langle \varphi, N \rangle \, d\mu_g + \int_M \langle Q, d\varphi \rangle_g \, d\mu_g,$$
$$0 = \int_M \psi \langle v, N \rangle \, d\mu_g + \varepsilon \int_M \psi \rho \, d\mu_g,$$

for all $\varphi, \psi \in H^1(M; \mathbb{R}^3)$.



As can be seen, this works quite well in keeping meshes well-behaved as they evolve.

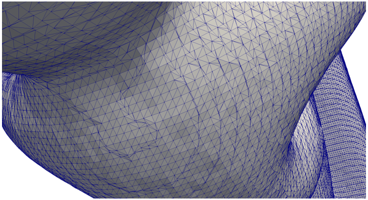
Conformal to what?

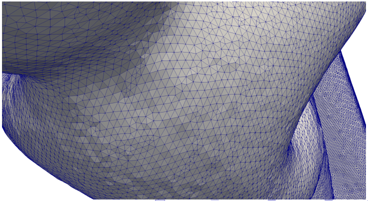
To actually implement the conformal penalty regularization procedure, it is important to specify a reference triangulation.

In practice this is done by letting the largest angle of each element adjust the others.

Algorithm 1 Generation of target angles

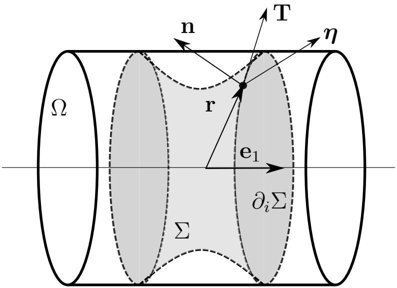
Require: Reference triangulation $\mathcal{T}$ of the closed surface M.
  for $T \in \mathcal{T}$ **do**
    for vertex $1 \leq i \leq 3$ **do**
      Compute $m_i = \#$ of adjacent elements
      $\alpha_i \leftarrow \alpha_i / m_i$
    end for
    Determine maximum vertex angle α_i.
    if $\alpha_i > \alpha_j$ for all $j \neq i$ **then**
      for vertices $j \neq i$ **do**
        $\alpha_j \leftarrow \alpha_j (\pi - \alpha_i) / \left(\sum_{k \neq i} \alpha_k\right)$
      end for
    else
      for vertices $1 \leq j \leq 3$ **do**
        $\alpha_j \leftarrow \alpha_j \pi / \left(\sum_{k=1}^{3} \alpha_k\right)$.
      end for
    end if
  end for





Outline

1. Introduction
2. Stationary surfaces and BVPs
3. Computing stationary surfaces for $\mathcal{W}^p$
4. Discretization and results



The spatial discretization

We assume the smooth surface M is polygonally approximated by nondegenerate simplices T_h, so that

$$M_h = \bigcup_{T_h \in \mathcal{T}_h} T_h.$$

Denoting the nodes of this triangulation by $\{a_j\}_{j=1}^N$, the standard nodal basis $\{\phi_i\}$ on $M_h(t)$ satisfies $\phi_i(a_j, t) = \delta_{ij}$.

The space of piecewise-linear finite elements on $M_h(t)$ is then

$$S_h(t) = \mathrm{Span}\{\phi_i\} = \{\phi \in C^0(M_h(t)) \,:\, \phi|_{T_h} \in \mathbb{P}_1(T_h), T_h \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(T_h)$ denotes the space of linear polynomials on T_h.

Note that in practice we allow not only triangulations, but also quadrangulations of the continuous surface M.

What about the temporal discretization?

A good discretization of the continuous systems discussed should:

1) Preserve (at least empirically) the energy-decrease of the p-Willmore flow.
2) Be robust to noise and other numerical artifacts.
3) Be relatively fast to implement.

The simplest thing to do is to linearize the problem at each time step, effectively pushing the nonlinearities into the time domain. This is the standard strategy of Dziuk and Elliott in [7].

Instead, we choose our discretization "as centrally as possible". More precisely, let $\tau > 0$ be a fixed temporal stepsize, and denote $u_h^k = u_h(\cdot, k\tau)$.

Then, $M_h^{k+\frac{1}{2}}$ is the image of the immersion $u_h^{k+\frac{1}{2}} = (1/2)\left(u_h^k + u_h^{k+1}\right)$, and for any field quantity F,

$$F_h^{k+\frac{1}{2}} = \frac{1}{2}\left(F_h^k + F_h^{k+1}\right).$$

Discrete p-Willmore flow

Problem

Let u, Y, W, λ, ν be as in Problem 1. Given the discrete data u_h^k, Y_h^k, W_h^k at time $t = k\tau$, the p-Willmore flow problem is to find functions $u_h^{k+1}, Y_h^{k+1}, W_h^{k+1}, \lambda_h, \nu_h$ which satisfy the system of equations

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle Y_h^{k+\frac{1}{2}}, \psi_h \right\rangle d\mu_{g_h} + \int_{M_h^{k+\frac{1}{2}}} \left\langle du_h^{k+1}, d\psi_h \right\rangle_{g_h} d\mu_{g_h},$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle \left(W_h^{k+\frac{1}{2}} - \left| Y_h^{k+\frac{1}{2}} \right|^{p-2} Y_h^{k+\frac{1}{2}} \right), \xi_h \right\rangle d\mu_{g_h}, \tag{4}$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle du_h^{k+\frac{1}{2}}, \left(du_h^{k+1} - du_h^k \right) \right\rangle_{g_h} d\mu_{g_h}, \tag{5}$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle \left(u_h^{k+1} - u_h^k \right), N_h^{k+\frac{1}{2}} \right\rangle d\mu_{g_h}, \tag{6}$$

$$\begin{aligned} 0 &= \int_{M_h^{k+\frac{1}{2}}} \frac{\left\langle \left(u_h^{k+1} - u_h^k \right), \varphi_h \right\rangle}{\tau} d\mu_{g_h} + \int_{M_h^{k+\frac{1}{2}}} \lambda_h \left\langle \varphi_h, N_h^{k+\frac{1}{2}} \right\rangle d\mu_{g_h} \\ &+ \int_{M_h^{k+\frac{1}{2}}} \nu_h \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} + (1-p) \int_{M_h^{k+\frac{1}{2}}} \left| Y_h^{k+\frac{1}{2}} \right|^p \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} \\ &- p \int_{M_h^{k+\frac{1}{2}}} \left(\mathrm{div}_{g_h} W_h^{k+\frac{1}{2}} \right) \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} - p \int_{M_h^{k+\frac{1}{2}}} \left\langle dW_h^{k+1}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} \\ &+ p \int_{M_h^{k+\frac{1}{2}}} \left\langle D(\varphi_h) du_h^k, dW_h^k \right\rangle_{g_h} d\mu_{g_h}, \end{aligned} \tag{7}$$

for all $\varphi_h, \psi_h, \xi_h \in S_h$.

Discrete conformal penalty regularization

Problem (Discrete conformal penalty regularization)

Let $\varepsilon > 0$ be fixed, let $\hat{u}, u, N, \rho$ be as before, and let $\tilde{N} = (1/2)\left(N + \hat{N}\right)$. Given u_h^{k+1}, N_h^{k+1}, solving the discrete conformal penalty regularization problem means finding functions $\hat{u}_h^{k+1}, \rho_h$ which satisfy the system

$$0 = \int_{M_h^{k+1}} \rho_h \left\langle \varphi_h, \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h} + \int_{M_h^{k+1}} \left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h},$$

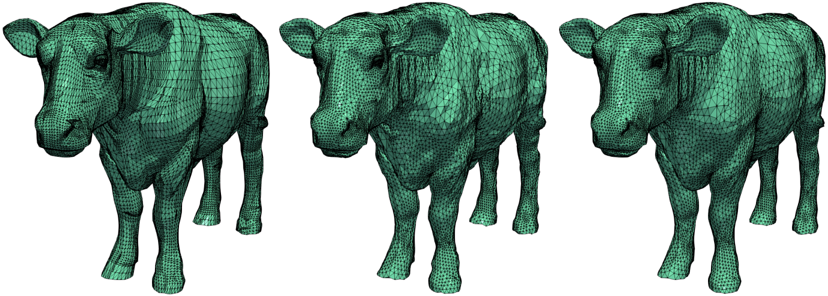
$$0 = \int_{M_h^{k+1}} \psi_h \left\langle \left(\hat{u}_h^{k+1} - u_h^{k+1} \right), \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h} + \varepsilon \int_{M_h^{k+1}} \psi_h \, \rho_h \, d\mu_{g_h},$$

for all $\varphi_h, \psi_h \in S_h$ and where $\left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h}$ refers to the discretization on the known surface M_h^{k+1} of the analogous continuous quantity, which involves components of the known normal N_h^{k+1} and derivatives of the unknown immersion $\hat{u}_h^{k+1}$, computed with respect to M_h^{k+1}.

Linear vs. Nonlinear regularization

Since the only nonlinearity in the discrete conformal penalty regularization comes from $\tilde{N}$, it is easy to adapt our method to require only a linear solve by instead using the known N.

Below is a performance comparison on a cow with 34.5k triangles. Original mesh (left), linear algorithm (middle), nonlinear algorithm (right).



Main algorithm

Fist, note the following (linear) systems used to generate the initial curvature data.

$$0 = \int_{M_h^k} \left\langle Y_h^k, \psi_h \right\rangle d\mu_{g_h} + \int_{M_h^k} \left\langle du_h^k, d\psi_h \right\rangle_{g_h} d\mu_{g_h}, \tag{8}$$

$$0 = \int_{M_h^k} \left\langle \left(W_h^k - \left| Y_h^k \right|^{p-2} Y_h^k \right), \xi_h \right\rangle d\mu_{g_h}. \tag{9}$$

Algorithm 2 p-Willmore flow with conformal penalty

Require: Closed, oriented surface immersion $u_h^0 : M_h^0 \to \mathbb{R}^3$; real numbers $\varepsilon, \tau > 0$, integer $k_{\max} \geq 1$.

while $0 \leq k \leq k_{\max}$ **do**

Solve (8) for Y_h^k

Solve (9) for W_h^k

Solve the p-Willmore flow problem for $u_h^{k+1}, Y_h^{k+1}, W_h^{k+1}, \lambda_h, \nu_h$

Solve the mesh regularization problem for $\hat{u}_h^{k+1}, \rho_h$

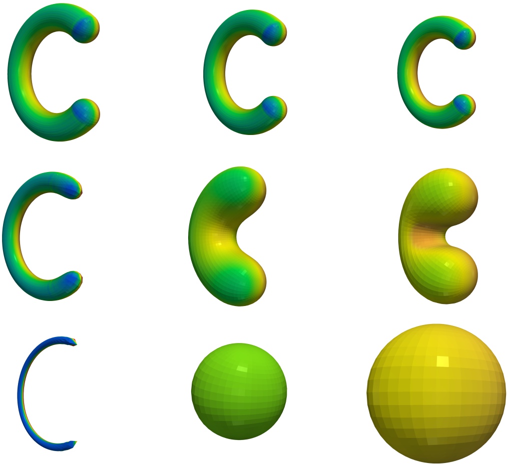
$u_h^{k+1} = \hat{u}_h^{k+1}$

$k = k + 1$

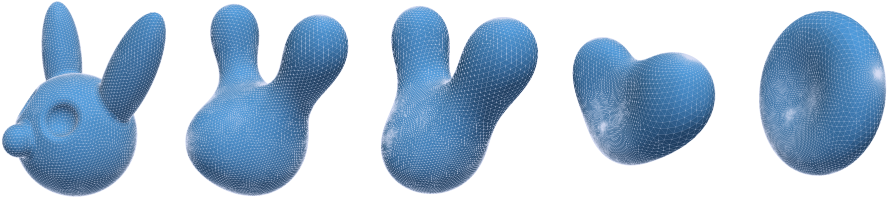
end while

Unconstrained flow comparison on a letter C

MCF (0-Willmore) | Willmore flow (2-Willmore) | 4-Willmore flow



Constrained 2-Willmore flow

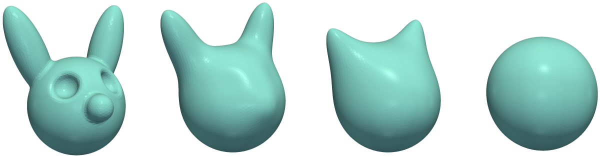


Area and volume constrained 2-Willmore flow of a rabbit-dog.

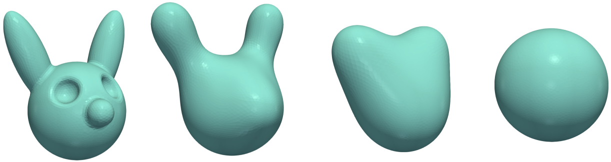
Volume constrained 2-Willmore flow of a genus 4 statue mesh.

Volume-constrained flow comparison on a rabbit-dog

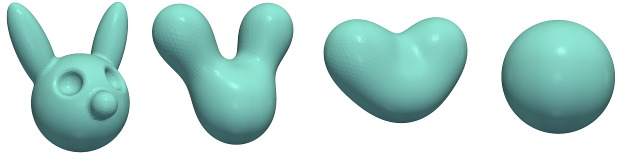
MCF (0-Willmore)



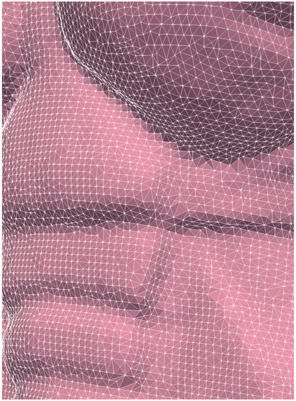
Willmore flow (2-Willmore)

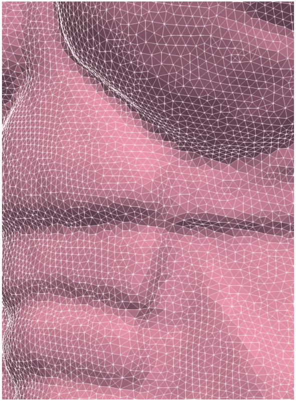


4-Willmore flow



Mesh edit of a cartoon armadillo





Area preserving 2-Willmore flow of a cow

Area and volume preserving 2-Willmore flow of a cow

Almost isometric 2-Willmore flow of a torus knot

Almost-isometric 2-Willmore flow of a torus knot (again)

Future work

Theoretical:

1) Investigate the stability of $\mathcal{B}$-stationary surfaces.

2) Use the established conservation law to (hopefully) lower solution regularity requirements.

3) Try to say something about the $\mathcal{B}$-surface flow.

Computational:

1) Find a reasonable way to conformally-correct on the moving surface itself, not just its tangent space (higher-order approximation).

2) Develop rigorous guarantees on the consistency and stability of the discretization.

3) Extend these ideas to surfaces with boundary, as well as other curvature flows of interest (Ricci flow, Yamabe flow, etc.)

Thanks

Thank you!

References I

Anthony Gruber, Magdalena Toda, and Hung Tran.
Stationary surfaces with boundaries, 2019.

Yann Bernard.
Noether's theorem and the willmore functional.
Advances in Calculus of Variations, 9(3):217–234, 7 2016.

Tristan Rivière.
Analysis aspects of willmore surfaces.
Inventiones mathematicae, 174(1):1–45, Oct 2008.

A. Gruber, M. Toda, and H. Tran.
On the variation of curvature functionals in a space form with application to a generalized willmore energy.
Annals of Global Analysis and Geometry, May 2019.

Anna Dall'Acqua, Klaus Deckelnick, and Glen Wheeler.
Unstable willmore surfaces of revolution subject to natural boundary conditions.
Calculus of Variations and Partial Differential Equations, 48, 11 2013.

References II

Bennett Palmer.
Uniqueness theorems for willmore surfaces with fixed and free boundaries.
Indiana University Mathematics Journal, 49(4):1581–1601, 2000.

G. Dziuk and C.M. Elliott.
Finite element methods for surface pdes.
Acta Numerica, 22:289–396, 2013.

G. Dziuk.
Computational parametric Willmore flow.
Numerische Mathematik, 111(1):55, 2008.

Anthony Gruber and Eugenio Aulisa.
Computational p-willmore flow with conformal penalty, 2019.

G. Kamberov, F. Pedit, and U. Pinkall.
Bonnet pairs and isothermic surfaces.
Duke Math. J., 92(3):637–644, 04 1998.