Vector Spaces

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Abstract

This article contains Z Notation type declarations for vector spaces and some related objects. It has been type checked by *f*UZZ.

1 Introduction

Real vector spaces are multidimensional generalizations of real numbers. They are the objects studied in linear algebra and are foundational to differential geometry.

2 Real n-tuples

2.1 $\mathbb{R}^\infty$ `\Rinf`

Let n be a natural number. A finite sequence of n real numbers is called a real n-tuple. Let $\mathbb{R}^\infty$ denote the set of all real n-tuples for any n.

$$\mathbb{R}^\infty == \operatorname{seq} \mathbb{R}$$

2.2 $\mathbb{R}$ `\Rtuples`

Let $\mathbb{R}(n)$ denote $\mathbb{R}^n$, the set of all n-tuples for some given n.

$$\begin{array}{|l}
\mathbb{R} : \mathbb{N} \longrightarrow \mathbb{P}\, \mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}(n) = \{\, v : \mathbb{R}^\infty \mid \# v = n \,\}
\end{array}$$

Remark.

$$\mathbb{R}^\infty = \bigcup \{\, n : \mathbb{N} \bullet \mathbb{R}(n) \,\}$$

2.2.1 π `\pi`

The real numbers that comprise an n-tuple are called its components. The real number $v(i)$ is the i-th component of the n-tuple v where $1 \leq i \leq n$. Let $\pi(i)$ be the projection function that maps an n-tuple v to its i-th component $v(i)$.

$$
\begin{array}{|l}
\pi : \mathbb{N}_1 \longrightarrow \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!\rightarrow \mathbb{R} \\
\hline
\forall\, i : \mathbb{N}_1 \bullet \\
\qquad \pi(i) = (\lambda\, v : \mathbb{R}^\infty \mid i \in \operatorname{dom} v \bullet v(i))
\end{array}
$$

3 Scalar Multiplication

3.1 $*$ `\smulR`

Let v be an n-tuple and let c be a real number. Scalar multiplication of v by c is the n-tuple $c * v$ defined by component-wise multiplication.

$$
\begin{array}{|l}
\_ * \_ : \mathbb{R} \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \\
\hline
\forall\, c : \mathbb{R} \bullet \\
\qquad c * \langle\rangle = \langle\rangle \\
\forall\, c : \mathbb{R};\ n : \mathbb{N}_1 \bullet \\
\qquad \forall\, v : \mathbb{R}(n);\ i : 1 \mathrel{..} n \bullet \\
\qquad\qquad (c * v)(i) = c * v(i)
\end{array}
$$

4 Vector Addition and Subtraction

4.1 $+$ `\vaddR`

Let v and w be n-tuples. Vector addition of v and w is the n-tuple $v + w$ defined by component-wise addition.

$$
\begin{array}{|l}
\_ + \_ : \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!\rightarrow \mathbb{R}^\infty \\
\hline
\operatorname{dom}(\_ + \_) = \{\, v, w : \mathbb{R}^\infty \mid \#v = \#w \,\} \\
\langle\rangle + \langle\rangle = \langle\rangle \\
\forall\, n : \mathbb{N}_1 \bullet \\
\qquad \forall\, v, w : \mathbb{R}(n);\ i : 1 \mathrel{..} n \bullet \\
\qquad\qquad (v + w)(i) = v(i) + w(i)
\end{array}
$$

4.2 $-$ \vsubR

Vector subtraction is defined similarly.

$$
\begin{array}{|l}
\_ - \_ : \mathbb{R}^\infty \times \mathbb{R}^\infty \pfun \mathbb{R}^\infty \\
\hline
\mathrm{dom}(\_ - \_) = \{\, v, w : \mathbb{R}^\infty \mid \#v = \#w \,\} \\[6pt]
\langle\rangle - \langle\rangle = \langle\rangle \\[6pt]
\forall\, n : \mathbb{N}_1 \bullet \\
\quad \forall\, v, w : \mathbb{R}(n);\ i : 1 \mathrel{..} n \bullet \\
\quad\quad (v - w)(i) = v(i) - w(i)
\end{array}
$$

Each $\mathbb{R}(n)$ is a real vector space under the operations of scalar multiplication and vector addition defined above.

5 Linear Transformations

5.1 Linear

Let n and m be natural numbers. A mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be a linear transformation if it preserves scalar multiplication and vector addition.

$$
\begin{array}{|l}
\textit{Linear} \\
\hline
n, m : \mathbb{N} \\
L : \mathbb{R}^\infty \pfun \mathbb{R}^\infty \\
\hline
L \in \mathbb{R}(n) \longrightarrow \mathbb{R}(m) \\[6pt]
\forall\, c : \mathbb{R};\ v : \mathbb{R}(n) \bullet \\
\quad L(c * v) = c * L(v) \\[6pt]
\forall\, v, w : \mathbb{R}(n) \bullet \\
\quad L(v + w) = L(v) + L(w) \\
\hline
\end{array}
$$

5.2 lin \linR

Define $\mathrm{lin}(n, m)$ to be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$
\begin{array}{|l}
\mathrm{lin} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \pfun \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\quad \mathrm{lin}(n, m) = \{\, L : \mathbb{R}^\infty \pfun \mathbb{R}^\infty \mid \textit{Linear} \,\}
\end{array}
$$

6 The Dot Product

6.1 $\cdot$ `\dotR`

The *inner* or *dot* product of n-tuples v and w is the real number $v \cdot w$ defined by the sum of the component-wise products.

$$
\begin{array}{|l}
\_ \cdot \_ : \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrowtail\!\!\!\!\!\!+ \mathbb{R} \\
\hline
\mathrm{dom}(\_ \cdot \_) = \{\, v, w : \mathbb{R}^\infty \mid \# v = \# w \,\} \\
\langle\rangle \cdot \langle\rangle = 0 \\
\forall\, x, y : \mathbb{R};\ v, w : \mathbb{R}^\infty \mid \# v = \# w \bullet \\
\qquad (\langle x \rangle \frown v) \cdot (\langle y \rangle \frown w) = x * y + v \cdot w
\end{array}
$$

Each $\mathbb{R}(n)$ is a real inner product space under the operation of dot product defined above.

7 The Norm

7.1 norm `\normR`

The norm $\|v\|$ of the n-tuple v is the positive square root of its dot product with itself.

$$\|v\| = \sqrt{v \cdot v}$$

Define $\mathrm{norm}(v)$ to be $\|v\|$.

$$
\begin{array}{|l}
\mathrm{norm} : \mathbb{R}^\infty \longrightarrow \mathbb{R} \\
\hline
\forall\, v : \mathbb{R}^\infty \bullet \\
\qquad \mathrm{norm}(v) = \mathrm{sqrt}(v \cdot v)
\end{array}
$$

The concepts of continuity, limits, and differentiability extend to functions between normed vector spaces such as $\mathbb{R}^n$.

7.2 ball `\ballRn`

Let $\mathrm{ball}(n, v, r)$ denote the open ball in $\mathbb{R}(n)$ of radius $r \in \mathbb{R}$ centred at $v \in \mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathrm{ball} : \mathbb{N} \times \mathbb{R}^\infty \times \mathbb{R} \rightarrowtail\!\!\!\!\!\!+ \mathbb{P}\, \mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N};\ v : \mathbb{R}^\infty;\ r : \mathbb{R} \mid v \in \mathbb{R}(n) \bullet \\
\qquad \mathrm{ball}(n, v, r) = \{\, w : \mathbb{R}(n) \mid \mathrm{norm}(v - w) < r \,\}
\end{array}
$$

7.3 balls `\ballsRn`

Let balls(n) denote the family of all open balls in $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\text{balls} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \text{balls}(n) = \{\, v : \mathbb{R}(n);\ r : \mathbb{R} \bullet \text{ball}(n, v, r) \,\}
\end{array}
$$

7.4 $\tau_\mathbb{R}$ `\tauRn`

The usual topology on $\mathbb{R}(n)$ is the topology generated by the open balls in $\mathbb{R}(n)$. Let $\tau_\mathbb{R}(n)$ denote the usual topology on $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\tau_\mathbb{R} : \mathbb{N} \longrightarrow \mathcal{F}\,\mathbb{R}^\infty \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \tau_\mathbb{R}(n) = \mathit{topGen}[\mathbb{R}(n)](\text{balls}(n))
\end{array}
$$

Remark. *If* $n \in \mathbb{N}$ *then* $\tau_\mathbb{R}(n)$ *is a topology on* $\mathbb{R}(n)$.

$$\forall\, n : \mathbb{N} \bullet \tau_\mathbb{R}(n) \in \mathit{top}[\mathbb{R}(n)]$$

7.5 $\mathbb{R}_\tau$ `\Rtaun`

Let $\mathbb{R}_\tau(n)$ denote the topological space defined by the usual topology on $\mathbb{R}(n)$.

$$
\begin{array}{|l}
\mathbb{R}_\tau : \mathbb{N} \longrightarrow \mathit{topSpaces}[\mathbb{R}^\infty] \\
\hline
\forall\, n : \mathbb{N} \bullet \\
\qquad \mathbb{R}_\tau(n) = (\mathbb{R}(n), \tau_\mathbb{R}(n))
\end{array}
$$

8 Continuity

A mapping f from $\mathbb{R}(n)$ to $\mathbb{R}(m)$ is said to be continuous if it is continuous with respect to the usual topologies. Let $\mathrm{C}^0(n, m)$ denote the set of these continuous mappings.

$$
\begin{array}{|l}
\mathrm{C}^0 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{R}^\infty \nrightarrow \mathbb{R}^\infty) \\
\hline
\forall\, n, m : \mathbb{N} \bullet \\
\qquad \mathrm{C}^0(n, m) = \mathrm{C}^0(\mathbb{R}_\tau(n), \mathbb{R}_\tau(m))
\end{array}
$$

9 Differentiability

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let $x \in \mathbb{R}^n$. Then f is differentiable at x if there exists a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that f is approximately linear very near x.

$$f(x+h) \approx f(x) + L(h) \quad \text{when} \quad \|h\| \approx 0$$

Let $\mathrm{C}^\infty(x, n, m)$ denote the set of all functions $f \in \mathbb{R}(n) \twoheadrightarrow \mathbb{R}(m)$ that are smooth at $x \in \mathbb{R}(n)$.