

Manifolds

Arthur Ryman, `arthur.ryman@gmail.com`

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Abstract

This article contains Z Notation type declarations for manifolds and some related objects. It has been type checked by *fUZZ*.

1 Introduction

Manifolds can be defined in several ways. The way I prefer to think about them is that, first of all, they are based on topological spaces. A manifold is therefore a topological space with some additional structure. This additional structure allows one to regard a manifold as, locally, being like an open subset of \mathbb{R}^n for some natural number n referred to as the dimension of the manifold. In the following, let M be a topological space of dimension n .

2 Charts

A chart ϕ on M is a continuous injection of some open subset $U \subseteq M$ into \mathbb{R}^n . A chart gives every point $p \in U$ in its domain of definition a tuple of n real number coordinates.

$$\phi : U \rightarrow \mathbb{R}^n \tag{1}$$

2.1 Transition Functions

Let U, V, W be open subsets of M with $W = U \cap V$. Let $\phi : U \rightarrow \mathbb{R}^n$ and $\psi : V \rightarrow \mathbb{R}^n$ be charts. Every point $p \in W$ is therefore given two, typically distinct, tuples of coordinates. The mapping from one coordinate tuple to the other is called the transition function defined by the pair of charts. Let $t_{\phi, \psi}$ denote that transition function that maps the ϕ coordinates to the ψ coordinates.

$$\forall x \in \phi(W) \bullet t_{\phi, \psi}(x) = \psi(\phi^{-1}(x)) \tag{2}$$

2.2 Compatible Charts

Let \mathcal{F} be some family of partial injections from \mathbb{R}^n to \mathbb{R}^n , e.g. continuous, differentiable, smooth, defined on the open subsets.

$$\mathcal{F} \subseteq \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \quad (3)$$

A pair of charts are said to be compatible with respect to \mathcal{F} when their transition functions belong to \mathcal{F} .

3 Atlases

A set of pairwise compatible charts that cover M is called an atlas for M . An atlas gives M a manifold structure. If the charts are only required to be continuous then M is called a topological manifold. If the charts are required to be differentiable then the atlas is called a differential or differentiable structure and M is called a differentiable manifold. Infinitely differentiable charts are called smooth charts. We are only concerned with smooth charts and manifolds.

In general, we normally consider an atlas to be a maximal set of charts. A given set of mutually compatible charts belongs to a unique maximal atlas. The given set is said to generate the maximal atlas.

4 Smooth Mappings

Mappings from one smooth manifold to another are called smooth when they are smooth when expressed in their coordinate charts. A smooth mapping that has a smooth inverse is called a diffeomorphism.

5 Tangent Vectors

A tangent vector X at the point $p \in M$ is a mapping from the set of smooth functions at p to \mathbb{R} that satisfies the following for all $c \in \mathbb{R}$ and $f, g \in C^\infty(M, p)$

$$X(cf) = cX(f) \quad (4)$$

$$X(f + g) = X(f) + X(g) \quad (5)$$

$$X(fg) = g(p)X(f) + f(p)X(g) \quad (6)$$

A smooth curve $\gamma : \mathbb{R} \rightarrow M$ defines a tangent vector X at $p = \gamma(0)$ by

$$X(f) = \left. \frac{df(\gamma(t))}{dt} \right|_{t=0} \quad (7)$$

6 Tangent Bundles

The set of all tangent vectors at p is denoted M_p or $T_p(M)$. It is an n -dimensional vector space and is called the tangent space at p . The set of all tangent spaces is called the tangent bundle and is denoted $T(M)$

$$T(M) = \{ (p, X) \mid p \in M, X \in M_p \} \quad (8)$$

The tangent bundle $T(M)$ is a smooth vector bundle over M under the natural projection $\pi : T(M) \rightarrow M, \pi(p, X) = p$.