Topological Spaces

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Abstract

This article defines topological spaces and related concepts.

1 Topological Spaces

1.1 t_1, t_2, and t_3

Let t_1, t_2, and t_3 denote arbitrary sets. These will be used throughout in the statement of theorems, remarks, and examples that are parameterized by arbitrary sets.

$$[t_1, t_2, t_3]$$

1.2 $\mathcal{F}$ `\family`

Let X be a set. A *family* of subsets of X is a set of subsets of X. Let $\mathcal{F} X$ denote the set of all families of subsets of X.

$$\mathcal{F} X == \mathbb{P}(\mathbb{P}\, X)$$

1.3 *Topology*

A *topology* τ on X is a family of subsets of X, referred to as the *open* subsets of X, that satisfy the following axioms.

Topology$[X]$

$$\tau : \mathcal{F} X$$

$$\varnothing \in \tau$$

$$X \in \tau$$

$$\forall F : \mathbb{F}\, \tau \bullet \bigcap F \in \tau$$

$$\forall F : \mathbb{P}\, \tau \bullet \bigcup F \in \tau$$

- The empty set is open.
- The whole set is open.
- The intersection of a finite family of open sets is open.
- The union of any family of open sets is open.

1.4 *top*

Let $top[X]$ denote the set of all topologies on X.

$$top[X] == \{ Topology[X] \bullet \tau \}$$

1.5 *discrete* and *indiscrete*

The *discrete* topology on X consists of all subsets of X. The *indiscrete* topology on X consists of just X and $\varnothing$. Let $discrete[X]$ and $indiscrete[X]$ denote the discrete and indiscrete topologies on X.

$$discrete[X] == \mathbb{P}\, X$$
$$indiscrete[X] == \{\varnothing, X\}$$

Example. *Let t_1 be an arbitrary set.*

$$discrete[t_1] \in top[t_1]$$
$$indiscrete[t_1] \in top[t_1]$$

1.6 *topGen*

Remark. *The intersection of a set of topologies on X is also a topology on X.*

Given a family B of subsets of X, the topology *generated by* B is the intersection of all topologies that contain B. The set B is referred to as a *basis* for the topology it generates. Let $topGen[X]\, B$ denote the topology on X generated by the basis B.

$$\begin{array}{|l}
\hline [X] \\
topGen : \mathcal{F} X \longrightarrow top[X] \\
\hline \forall\, B : \mathcal{F} X \bullet \\
\qquad topGen\, B = \bigcap \{ \tau : top[X] \mid B \subseteq \tau \} \\
\hline
\end{array}$$

Example. *Let t_1 be an arbitrary set.*

$$topGen[t_1]\varnothing = indiscrete[t_1]$$
$$topGen[t_1]\{\varnothing\} = indiscrete[t_1]$$
$$topGen[t_1]\{t_1\} = indiscrete[t_1]$$

1.7 *topSpace*

Let X be a set. A *topological space* is a pair (X, τ) where τ is a topology on X. Let *topSpace*$[X]$ denote the set of all topological spaces (X, τ).

$$topSpace[X] == \{ \tau : top[X] \bullet (X, \tau) \}$$

Example. *Let t_1 be an arbitrary set.*

$$(t_1, indiscrete[t_1]) \in topSpace[t_1]$$
$$(t_1, discrete[t_1]) \in topSpace[t_1]$$

2 Continuous Mappings

Let (X, τ) and (Y, σ) be topological spaces.

2.1 *Continuous*

A mapping $f \in X \longrightarrow Y$ is said to be *continuous* if the inverse image of every open set is open.

Continuous$[X, Y]$

$$f : X \longrightarrow Y$$
$$\tau : top[X]$$
$$\sigma : top[Y]$$

$$\forall U : \sigma \bullet$$
$$\quad f^{\sim}(\!| U |\!) \in \tau$$

2.2 C^0 `\CzeroTT`

Let $\mathrm{C}^0((X, \tau), (Y, \sigma))$ denote the set of continuous mappings from (X, τ) to (Y, σ).

$[X, Y]$

$$\mathrm{C}^0 : topSpace[X] \times topSpace[Y] \longrightarrow \mathbb{P}(X \longrightarrow Y)$$

$$\forall \tau : top[X]; \sigma : top[Y] \bullet$$
$$\quad \mathrm{C}^0((X, \tau), (Y, \sigma)) = \{ f : X \longrightarrow Y \mid Continuous[X, Y] \}$$

Remark. *Let t_1, t_2, and t_3 be arbitrary sets. The composition of continuous mappings is a continuous mapping.*

$$\forall A : topSpace[t_1]; B : topSpace[t_2]; C : topSpace[t_3] \bullet$$
$$\quad \forall f : \mathrm{C}^0(A, B); g : \mathrm{C}^0(B, C) \bullet$$
$$\quad\quad g \circ f \in \mathrm{C}^0(A, C)$$