

Groups

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Abstract

This article contains Z Notation type declarations for groups and some related objects. It has been type checked by *fUZZ*.

1 Introduction

Groups are ubiquitous throughout mathematics and physics.

2 Groups

A *group* is a set G of elements x, y, \dots endowed with an associative multiplication operation $x * y$ that has an identity element $\mathbf{1}$ such that all elements have inverses x^{-1}, y^{-1}, \dots

2.1 Group

Let $Group[G]$ denote the set of all group structures defined on some given set G of elements.

$Group[G]$	_____
$_ * _ : G \times G \longrightarrow G$	
$\mathbf{1} : G$	
$_^{-1} : G \longrightarrow G$	
<hr/>	
$\forall x, y, z : G \bullet$	
$(x * y) * z = x * (y * z)$	
$\forall x : G \bullet$	
$x * \mathbf{1} = x = \mathbf{1} * x$	
$\forall x : G \bullet$	
$x * x^{-1} = \mathbf{1} = x^{-1} * x$	

- The group multiplication operation is associative.

- The element $\mathbf{1}$ is both a left and right identity element under group multiplication.
- Every group element has an inverse element that is both a left and right inverse under group multiplication.

Remark. *The group multiplication operation uniquely determines the set of elements.*

$$\begin{aligned} &\forall \text{ Group}[\mathbf{X}] \bullet \\ &\quad \mathbf{X} = \text{ran}(- * -) \end{aligned}$$

Remark. *The group multiplication operation uniquely determines the identity operation.*

$$\begin{aligned} &\forall \text{ Group}[\mathbf{X}] \bullet \\ &\quad \forall x, y : \mathbf{X} \bullet \\ &\quad \quad x * y = x \Rightarrow y = \mathbf{1} \end{aligned}$$

Remark. *The group multiplication uniquely determines the inverses.*

$$\begin{aligned} &\forall \text{ Group}[\mathbf{X}] \bullet \\ &\quad \forall x, y : \mathbf{X} \bullet \\ &\quad \quad x * y = \mathbf{1} \Rightarrow x = y^{-1} \wedge y = x^{-1} \end{aligned}$$

2.2 group

Let $\text{group}[G]$ denote the set of all group multiplication operations on G .

$\begin{aligned} &[G] \\ &\text{group} : \mathbb{P}(G \times G \rightarrow G) \\ &\text{group} = \{ \text{Group}[G] \bullet (- * -) \} \end{aligned}$

2.3 bij

Let X be a set and let $\text{bij}[X]$ denote the set of a bijections $X \rightarrowtail X$ from X to itself.

$\begin{aligned} &[X] \\ &\text{bij} : \mathbb{P}(X \rightarrow X) \\ &\text{bij} = X \rightarrowtail X \end{aligned}$
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Remark. *The composition of bijections is a bijection.*

$$\begin{aligned} &\forall f, g : \text{bij}[\mathbf{X}] \bullet \\ &\quad f \circ g \in \text{bij}[\mathbf{X}] \end{aligned}$$

Remark. *Composition is associative.*

$$\begin{aligned} \forall f, g, h : \text{bij}[X] \bullet \\ f \circ (g \circ h) = (f \circ g) \circ h \end{aligned}$$

Remark. *The identity function $\text{id } X$ acts as a left and right identity element under composition.*

$$\begin{aligned} \forall f : \text{bij}[X] \bullet \\ \text{id } X \circ f = f = f \circ \text{id } X \end{aligned}$$

Remark. *The inverse f^\sim of a bijection f is its left and right inverse under composition.*

$$\begin{aligned} \forall f : \text{bij}[X] \bullet \\ f \circ f^\sim = \text{id } X = f^\sim \circ f \end{aligned}$$

2.4 *Bij*

The preceding remarks show that set $\text{bij}[X]$ under the operation of composition has the structure of a group. Let $\text{Bij}[X]$ denote this group.

$\begin{aligned} & \text{Bij} : \text{bij}[X] \times \text{bij}[X] \longrightarrow \text{bij}[X] \\ & \text{Bij} = (\lambda f, g : \text{bij}[X] \bullet f \circ g) \end{aligned}$
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Theorem 1. *The multiplication operation $\text{Bij}[X]$ is a group.*

$$\text{Bij}[X] \in \text{group}[\text{bij}[X]]$$

3 Abelian Groups

An *abelian group* is a group in which the multiplication is commutative, i.e. $x * y = y * x$. In this case, the group multiplication is denoted as addition $x + y$, the identity element is denoted as a zero $\mathbf{0}$, and the inverse of an element is denoted as its negative $-x$.

3.1 *AbelianGroup*

Let $\text{AbelianGroup}[G]$ denote the abelian group structures on G .

$AbelianGroup[G]$ $Group[G]$ $_{-} + _{-} : G \times G \rightarrow G$ $\mathbf{0} : G$ $- : G \rightarrow G$	
$(_{-} + _{-}) = (_{-} * _{-})$ $\mathbf{0} = \mathbf{1}$ $- = (_{-}^{-1})$ $\forall x, y : G \bullet$ $x + y = y + x$	

- The group multiplication is denoted as addition.
- The group identify element is denoted as the zero element.
- The inverse of a group element is denoted as the negative of the element.
- The group addition is commutative.

3.2 *abelianGroup*

Let *abelianGroup*[*G*] denote the subset of abelian groups in *group*[*G*].

$[G]$ $abelianGroup : \mathbb{P}(G \times G \rightarrow G)$	
$abelianGroup = \{ AbelianGroup[G] \bullet (_{-} + _{-}) \}$	

Remark. The set *abelianGroup*[*G*] is a subset of *group*[*G*].

$$abelianGroup[X] \subseteq group[X]$$

Example. The integer addition is an abelian group.

$$(_{-} + _{-}) \in abelianGroup[\mathbb{Z}]$$