Topological Spaces

Arthur Ryman, arthur.ryman@gmail.com

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Abstract

This article defines topological spaces and related concepts.

1 Topological Spaces

1.1 $\mathcal{F}$ \family

Let X be a set. A *family* of subsets of X is a set of subsets of X. Let $\mathcal{F}X$ denote the set of all families of subsets of X.

$$\mathcal{F}X == \mathbb{P}(\mathbb{P}\,X)$$

1.2 *Topology*

A *topology* τ on X is a family of subsets of X, referred to as the *open* subsets of X, that satisfy the following axioms.

Topology[X]

$$\tau : \mathcal{F}X$$

$$\varnothing \in \tau$$

$$X \in \tau$$

$$\forall\, F : \mathbb{F}\,\tau \bullet \bigcap F \in \tau$$

$$\forall\, F : \mathbb{P}\,\tau \bullet \bigcup F \in \tau$$

- The empty set is open.
- The whole set is open.
- The intersection of a finite family of open sets is open.
- The union of any family of open sets is open.

1.3 *top*

Let $top[X]$ denote the set of all topologies on X.

$$top[X] == \{ \, Topology[X] \bullet \tau \, \}$$

1.4 *discrete* and *indiscrete*

The *discrete* topology on X consists of all subsets of X. The *indiscrete* topology on X consists of just X and $\varnothing$. Let $discrete[X]$ and $indiscrete[X]$ denote the discrete and indiscrete topologies on X.

$$discrete[X] == \mathbb{P}\, X$$
$$indiscrete[X] == \{\varnothing, X\}$$

Example.

$$discrete[\mathbb{N}] \in top[\mathbb{N}]$$
$$indiscrete[\mathbb{N}] \in top[\mathbb{N}]$$

1.5 *topGen*

Remark. *The intersection of a set of topologies on X is also a topology on X.*

Given a family B of subsets of X, the topology *generated by* B is the intersection of all topologies that contain B. The set B is referred to as a *basis* for the topology it generates. Let $topGen[X]\, B$ denote the topology on X generated by the basis B.

$$topGen[X] == (\lambda\, B : \mathcal{F} X \bullet \bigcap \{ \, \tau : top[X] \mid B \subseteq \tau \, \})$$

Example.

$$topGen[\mathbb{N}]\varnothing = indiscrete[\mathbb{N}]$$
$$topGen[\mathbb{N}]\{\varnothing\} = indiscrete[\mathbb{N}]$$
$$topGen[\mathbb{N}]\{\mathbb{N}\} = indiscrete[\mathbb{N}]$$

1.6 *topSpace*

Let X be a set. A *topological space* is a pair (X, τ) where τ is a topology on X. Let $topSpace[t]$ denote the set of all topological spaces (X, τ) where X is a subset of the set t.

$$topSpace[t] == \{ \, X : \mathbb{P}\, t; \tau : \mathcal{F} t \mid Topology[X] \, \}$$

Example.

$$(\mathbb{N}, \mathbb{P}\, \mathbb{N}) \in topSpace[\mathbb{N}]$$
$$(\mathbb{N}, \mathbb{P}\, \mathbb{N}) \in topSpace[\mathbb{Z}]$$