Topological Spaces

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Abstract

This article defines topological spaces and related concepts.

1 Topological Spaces

1.1 t_1, t_2, and t_3

Let t_1, t_2, and t_3 denote arbitrary sets. These will be used throughout in the statement of theorems, remarks, and examples that are parameterized by arbitrary sets.

$$[t_1, t_2, t_3]$$

1.2 $\mathcal{F}$ `\family`

Let X be a set. A *family* of subsets of X is a set of subsets of X. Let $\mathcal{F} X$ denote the set of all families of subsets of X.

$$\mathcal{F} X == \mathbb{P}(\mathbb{P}\, X)$$

1.3 *Topology*

A *topology* τ on X is a family of subsets of X, referred to as the *open* subsets of X, that satisfy the following axioms.

Topology$[X]$

$$\tau : \mathcal{F} X$$

$$\varnothing \in \tau$$

$$X \in \tau$$

$$\forall F : \mathbb{F}\, \tau \bullet \bigcap F \in \tau$$

$$\forall F : \mathbb{P}\, \tau \bullet \bigcup F \in \tau$$

- The empty set is open.
- The whole set is open.
- The intersection of a finite family of open sets is open.
- The union of any family of open sets is open.

1.4 *top*

Let $\mathit{top}[X]$ denote the set of all topologies on X.

$$\mathit{top}[X] == \{\, \mathit{Topology}[X] \bullet \tau \,\}$$

1.5 *discrete* and *indiscrete*

The *discrete* topology on X consists of all subsets of X. The *indiscrete* topology on X consists of just X and $\varnothing$. Let $\mathit{discrete}[X]$ and $\mathit{indiscrete}[X]$ denote the discrete and indiscrete topologies on X.

$$\mathit{discrete}[X] == \mathbb{P}\, X$$

$$\mathit{indiscrete}[X] == \{\varnothing, X\}$$

Example. *Let t_1 be an arbitrary set.*

$$\mathit{discrete}[t_1] \in \mathit{top}[t_1]$$

$$\mathit{indiscrete}[t_1] \in \mathit{top}[t_1]$$

1.6 *topGen*

Remark. *The intersection of a set of topologies on X is also a topology on X.*

Given a family B of subsets of X, the topology *generated by* B is the intersection of all topologies that contain B. The set B is referred to as a *basis* for the topology it generates. Let $\mathit{topGen}[X]\, B$ denote the topology on X generated by the basis B.

$$
\begin{array}{l}
[X] \\
\hline
\mathit{topGen} : \mathcal{F} X \longrightarrow \mathit{top}[X] \\
\hline
\forall\, B : \mathcal{F} X \bullet \\
\qquad \mathit{topGen}\, B = \bigcap \{\, \tau : \mathit{top}[X] \mid B \subseteq \tau \,\}
\end{array}
$$

Example. *Let t_1 be an arbitrary set.*

$$\mathit{topGen}[t_1]\varnothing = \mathit{indiscrete}[t_1]$$

$$\mathit{topGen}[t_1]\{\varnothing\} = \mathit{indiscrete}[t_1]$$

$$\mathit{topGen}[t_1]\{t_1\} = \mathit{indiscrete}[t_1]$$

1.7 *topSpace*

Let X be a set. A *topological space* is a pair (X, τ) where τ is a topology on X. Let $topSpace[X]$ denote the set of all topological spaces (X, τ).

$$topSpace[X] == \{ \tau : top[X] \bullet (X, \tau) \}$$

Example. *Let t_1 be an arbitrary set.*

$$(t_1, indiscrete[t_1]) \in topSpace[t_1]$$
$$(t_1, discrete[t_1]) \in topSpace[t_1]$$

2 Continuous Mappings

Let (X, τ) and (Y, σ) be topological spaces.

2.1 *Continuous*

A mapping $f \in X \longrightarrow Y$ is said to be *continuous* if the inverse image of every open set is open.

$$\begin{array}{|l}
\hline
\textit{Continuous}[X, Y] \\
f : X \longrightarrow Y \\
\tau : top[X] \\
\sigma : top[Y] \\
\hline
\forall U : \sigma \bullet \\
\quad f^{\sim}(\!| U |\!) \in \tau \\
\hline
\end{array}$$

2.2 C^0 `\CzeroTT`

Let $\mathrm{C}^0((X, \tau), (Y, \sigma))$ denote the set of continuous mappings from (X, τ) to (Y, σ).

$$\begin{array}{|l}
\hline\hline
[X, Y] \\
\mathrm{C}^0 : topSpace[X] \times topSpace[Y] \longrightarrow \mathbb{P}(X \longrightarrow Y) \\
\hline
\forall \tau : top[X]; \sigma : top[Y] \bullet \\
\quad \mathrm{C}^0((X, \tau), (Y, \sigma)) = \{ f : X \longrightarrow Y \mid Continuous[X, Y] \} \\
\hline
\end{array}$$

Remark. *Let t_1, t_2, and t_3 be arbitrary sets. The composition of continuous mappings is a continuous mapping.*

$$\begin{array}{l}
\forall A : topSpace[t_1]; B : topSpace[t_2]; C : topSpace[t_3] \bullet \\
\quad \forall f : \mathrm{C}^0(A, B); g : \mathrm{C}^0(B, C) \bullet \\
\quad\quad g \circ f \in \mathrm{C}^0(A, C)
\end{array}$$