Topological Spaces

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Abstract

This article defines topological spaces and related concepts.

1 Topological Spaces

1.1 t_1, t_2, and t_3

Let t_1, t_2, and t_3 denote arbitrary sets. These will be used throughout in the statement of theorems, remarks, and examples that are parameterized by arbitrary sets.

$$[t_1, t_2, t_3]$$

1.2 $\mathcal{F}$ \family

Let X be a set. A *family* of subsets of X is a set of subsets of X. Let $\mathcal{F}X$ denote the set of all families of subsets of X.

$$\mathcal{F}X == \mathbb{P}(\mathbb{P}\,X)$$

1.3 *Topology*

A *topology* τ on X is a family of subsets of X, referred to as the *open* subsets of X, that satisfy the following axioms.

Topology$[X]$

$\tau : \mathcal{F}X$

$\varnothing \in \tau$

$X \in \tau$

$\forall F : \mathbb{F}\,\tau \bullet \bigcap F \in \tau$

$\forall F : \mathbb{P}\,\tau \bullet \bigcup F \in \tau$

- The empty set is open.
- The whole set is open.
- The intersection of a finite family of open sets is open.
- The union of any family of open sets is open.

1.4 *top* and *tops*

Let $top[X]$ denote the set of all topologies on X.

$$
\begin{array}{l}
[X] \\
\hline
top : \mathbb{P}(\mathcal{F} X) \\
\hline
top = \{ \, Topology[X] \bullet \tau \, \}
\end{array}
$$

Let $tops[X]$ denote the set of all topologies on subsets $U \subseteq X$.

$$
\begin{array}{l}
[X] \\
\hline
tops : \mathbb{P}(\mathcal{F} X) \\
\hline
tops = \bigcup \{ \, U : \mathbb{P}\, X \bullet top[U] \, \}
\end{array}
$$

1.5 *discrete* and *indiscrete*

The *discrete* topology on X consists of all subsets of X. The *indiscrete* topology on X consists of just X and $\varnothing$. Let $discrete[X]$ and $indiscrete[X]$ denote the discrete and indiscrete topologies on X.

$$
\begin{array}{l}
[X] \\
\hline
discrete, indiscrete : \mathcal{F} X \\
\hline
discrete = \mathbb{P}\, X \\
indiscrete = \{\varnothing, X\}
\end{array}
$$

Example. *Let t_1 be an arbitrary set. Then $discrete[t_1]$ and $indiscrete[t_1]$ are topologies on t_1.*

$$discrete[t_1] \in top[t_1]$$

$$indiscrete[t_1] \in top[t_1]$$

1.6 *topGen*

Remark. *The intersection of a set of topologies on X is also a topology on X.*

Given a family B of subsets of X, the topology *generated by* B is the intersection of all topologies that contain B. The set B is referred to as a *basis* for the topology it generates. Let $topGen[X]\,B$ denote the topology on X generated by the basis B.

$$
\begin{array}{|l}
\hline
[X] \\
topGen : \mathcal{F} X \longrightarrow top[X] \\
\hline
\forall B : \mathcal{F} X \bullet \\
\qquad topGen\, B = \bigcap \{ \tau : top[X] \mid B \subseteq \tau \} \\
\hline
\end{array}
$$

Example. *Let t_1 be an arbitrary set.*

$$topGen[t_1]\varnothing = indiscrete[t_1]$$

$$topGen[t_1]\{\varnothing\} = indiscrete[t_1]$$

$$topGen[t_1]\{t_1\} = indiscrete[t_1]$$

1.7 *topSpace*

Let X be a set. A *topological space* is a pair (X, τ) where τ is a topology on X. Let $topSpace[X]$ denote the set of all topological spaces (X, τ).

$$topSpace[X] == \{ \tau : top[X] \bullet (X, \tau) \}$$

Example. *Let t_1 be an arbitrary set.*

$$(t_1, indiscrete[t_1]) \in topSpace[t_1]$$

$$(t_1, discrete[t_1]) \in topSpace[t_1]$$

1.8 *topSpaces*

Let $topSpaces[t]$ dentote the set of all topological spaces (X, τ) where X is a subset of t.

$$
\begin{array}{|l}
\hline
[t] \\
topSpaces : \mathbb{P}\, t \leftrightarrow \mathcal{F} t \\
\hline
topSpaces = \{ X : \mathbb{P}\, t; \tau : \mathcal{F} t \mid \tau \in top[X] \} \\
\hline
\end{array}
$$

Remark.

$$topSpace[t_1] \subseteq topSpaces[t_1]$$

2 Continuous Mappings

Let (X, τ) and (Y, σ) be topological spaces.

2.1 *Continuous*

A mapping $f \in X \longrightarrow Y$ is said to be *continuous* if the inverse image of every open set is open.

$$
\begin{array}{|l}
\hline
\textit{Continuous}[X, Y] \\
f : X \longrightarrow Y \\
\tau : \textit{top}[X] \\
\sigma : \textit{top}[Y] \\
\hline
\forall U : \sigma \bullet \\
\quad f^{\sim}(\!| U |\!) \in \tau \\
\hline
\end{array}
$$

2.2 C^0 `\CzeroTT`

Let A and B be topological spaces, and let $\mathrm{C}^0(A, B)$ denote the set of continuous mappings from A to B.

$$
\begin{array}{|l}
\hline
[X, Y] \\
\mathrm{C}^0 : \textit{topSpace}[X] \times \textit{topSpace}[Y] \longrightarrow \mathbb{P}(X \longrightarrow Y) \\
\hline
\forall \tau : \textit{top}[X]; \sigma : \textit{top}[Y] \bullet \\
\quad \textbf{let } A == (X, \tau); B == (Y, \sigma) \bullet \\
\quad\quad \mathrm{C}^0(A, B) = \{ f : X \longrightarrow Y \mid \textit{Continuous}[X, Y] \} \\
\hline
\end{array}
$$

2.3 The Identity Mapping

Remark. *The identity mapping is continuous.*

$$
\begin{array}{l}
\forall \tau : \textit{top}[t_1] \bullet \\
\quad \textbf{let } A == (t_1, \tau) \bullet \\
\quad\quad \mathrm{id}\, t_1 \in \mathrm{C}^0(A, A)
\end{array}
$$

2.4 const `\const`

Let X and Y be sets and let $c \in Y$ be some given point. The mapping that sends every point of X to c is called the *constant mapping* defined by c. Let $\mathrm{const}(c)$ denote the constant mapping.

$$
\begin{array}{l}
[X, Y] \\
\hline
\mathrm{const} : Y \longrightarrow (X \longrightarrow Y) \\
\hline
\forall c : Y \bullet \\
\quad \mathrm{const}(c) = (\lambda\, x : X \bullet c)
\end{array}
$$

Remark. *The constant mapping is continuous.*

$$
\begin{array}{l}
\forall \tau : top[t_1];\ \sigma : top[t_2];\ c : t_2 \bullet \\
\quad \mathbf{let}\ A == (t_1, \tau);\ B == (t_2, \sigma) \bullet \\
\quad\quad \mathrm{const}[t_1, t_2]c \in \mathrm{C}^0(A, B)
\end{array}
$$

2.5 Composition of Continuous Mapping

Remark. *Let t_1, t_2, and t_3 be arbitrary sets. The composition of continuous mappings is a continuous mapping.*

$$
\begin{array}{l}
\forall A : topSpace[t_1];\ B : topSpace[t_2];\ C : topSpace[t_3] \bullet \\
\quad \forall f : \mathrm{C}^0(A, B);\ g : \mathrm{C}^0(B, C) \bullet \\
\quad\quad g \circ f \in \mathrm{C}^0(A, C)
\end{array}
$$

3 Induced Topology

Let $A = (X, \tau)$ be a topological space and let $U \subseteq X$ be a subset. The topology on X *induces* a topology on U. This topology is variously referred to as the *induced*, *relative*, or *subspace* topology on U.

3.1 | \inducedFam

Let ϕ be a family of subsets of X and let U be a subset of X. The family of subsets of U *induced* by ϕ is the set of intersections of the members of ϕ with U. Let $\phi \mid U$ denote the family on U induced by ϕ.

$$
\begin{array}{l}
[X] \\
\hline
\_ \mid \_ : \mathcal{F} X \times \mathbb{P}\, X \longrightarrow \mathcal{F} X \\
\hline
\forall \phi : \mathcal{F} X;\ U : \mathbb{P}\, X \bullet \\
\quad \phi \mid U = \{\, Y : \phi \bullet Y \cap U \,\}
\end{array}
$$

Remark. *If τ is a topology on X then $\tau \mid U$ is a topology on U.*

$$
\begin{array}{l}
\forall \tau : top[t_1];\ U : \mathbb{P}\, t_1 \bullet \\
\quad \tau \mid U \in top[U]
\end{array}
$$

3.2 $|_{\mathsf{top}}$ \inducedTop and $|_{\mathsf{top}}$ \inducedTopSp

Let $A = (X, \tau)$ be a topological space. Let $\tau |_{\mathsf{top}} U$ denote the topology on U induced by τ

$$
\begin{array}{l}
[X] \\
\hline
\_ |_{\mathsf{top}} \_ : \mathit{top}[X] \times \mathbb{P}\, X \longrightarrow \mathit{tops}[X] \\
\hline
\forall \tau : \mathit{top}[X];\ U : \mathbb{P}\, X \bullet \\
\quad \tau |_{\mathsf{top}} U = \tau \mid U
\end{array}
$$

Let $A |_{\mathsf{top}} U$ denote the corresponding induced topological space.

$$
\begin{array}{l}
[X] \\
\hline
\_ |_{\mathsf{top}} \_ : \mathit{topSpace}[X] \times \mathbb{P}\, X \longrightarrow \mathit{topSpaces}[X] \\
\hline
\forall \tau : \mathit{top}[X];\ U : \mathbb{P}\, X \bullet \\
\quad \mathbf{let}\ A == (X, \tau) \bullet \\
\quad\quad A |_{\mathsf{top}} U = (U, \tau |_{\mathsf{top}} U)
\end{array}
$$

Remark. *The induced topological space $A |_{\mathsf{top}} U$ is a topological space on U.*

$$
\begin{array}{l}
\forall \tau : \mathit{top}[t_1];\ U : \mathbb{P}\, t_1 \bullet \\
\quad \mathbf{let}\ A == (t_1, \tau) \bullet \\
\quad\quad A |_{\mathsf{top}} U \in \mathit{topSpace}[U]
\end{array}
$$

4 Product Topology

Let X and Y be sets and let A and B be topological spaces on them. There is a natural topology on the product set $X \times Y$ generated by the products of the open sets on X and Y.

4.1 $\times$ \prodTop

Let $A \times B$ denote the product topological space.

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \times \_ : \mathit{topSpace}[X] \times \mathit{topSpace}[Y] \longrightarrow \mathit{topSpace}[X \times Y] \\
\hline
\forall \tau : \mathit{top}[X];\ \sigma : \mathit{top}[Y] \bullet \\
\quad \mathbf{let}\ A == (X, \tau);\ B == (Y, \sigma);\ Z == X \times Y \bullet \\
\quad\quad A \times B = (Z, \mathit{topGen}[Z](\tau \times \sigma))
\end{array}
$$