Real Numbers

Arthur Ryman, arthur.ryman@gmail.com

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Abstract

This article contains Z Notation type declarations for the real numbers, $\mathbb{R}$, and some related objects. It has been type checked by *f*UZZ.

1 Introduction

The real numbers, $\mathbb{R}$, are foundational to many mathematical objects such as vector spaces and manifolds, but are not built-in to Z Notation. This article provides type declarations for $\mathbb{R}$ and related objects so that they can be used and type checked in formal Z specifications.

No attempt has been made to provide complete, axiomatic definitions of all these objects since that would only be of use for proof checking. Although proof checking is highly desirable, it is beyond the scope of this article. The type declarations given here are intended to provide a basis for future axiomatization.

2 The Real Numbers

Let $\mathbb{R}$ denote the set of all real numbers.

$$[\mathbb{R}]$$

Let 0 and 1 denote the zero and unit elements of the real numbers.

$$
\begin{array}{|l}
0 : \mathbb{R} \\
1 : \mathbb{R}
\end{array}
$$

Let $\mathbb{R}_*$ denote the set of non-zero real numbers, also referred to as the *punctured* real number line.

$$\mathbb{R}_* == \mathbb{R} \setminus \{0\}$$

Let $x + y$, $x - y$, $x * y$, and $x \mathbin{/} y$ denote the usual arithmetic operations of addition, subtraction, multiplication, and division.

$$
\begin{array}{|l}
\_ + \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
\_ - \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
\_ * \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
\_ / \_ : \mathbb{R} \times \mathbb{R}_* \longrightarrow \mathbb{R}
\end{array}
$$

Let $-x$ denote the negative of x.

$$
\begin{array}{|l}
- : \mathbb{R} \longrightarrow \mathbb{R} \\
\hline
\forall x : \mathbb{R} \bullet -x = 0 - x
\end{array}
$$

Let $x < y$, $x \leq y$, $x > y$, and $x \geq y$ denote the usual comparison relations.

$$
\begin{array}{|l}
\_ < \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\
\_ \leq \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\
\_ > \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\
\_ \geq \_ : \mathbb{R} \leftrightarrow \mathbb{R}
\end{array}
$$

Let $\mathrm{abs}(x)$ denote $|x|$, the absolute value of x.

$$
\begin{array}{|l}
\mathrm{abs} : \mathbb{R} \longrightarrow \mathbb{R} \\
\hline
\forall x : \mathbb{R} \bullet \mathrm{abs}(x) = \mathbf{if}\ x \geq 0\ \mathbf{then}\ x\ \mathbf{else}\ -x
\end{array}
$$

Let $\mathbb{R}_+$ denote the set of positive real numbers.

$$\mathbb{R}_+ == \{ x : \mathbb{R} \mid x > 0 \}$$

For non-negative x, let $\mathrm{sqrt}(x)$ denote $\sqrt{x}$, the non-negative square root of x.

$$
\begin{array}{|l}
\mathrm{sqrt} : \mathbb{R} \rightarrow\!\!\!\!\!\!+\ \mathbb{R} \\
\hline
\mathrm{sqrt} = \{ x : \mathbb{R} \mid x \geq 0 \bullet x * x \mapsto x \}
\end{array}
$$

3 Open Intervals

For any real numbers a and b, let $\mathrm{interval}(a, b)$ denote the open interval bounded by a and b, typically written as (a, b).

$$
\begin{array}{|l}
\mathrm{interval} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}\,\mathbb{R} \\
\hline
\forall a, b : \mathbb{R} \bullet \\
\qquad \mathrm{interval}(a, b) = \{ x : \mathbb{R} \mid a < x < b \}
\end{array}
$$

Remark. *If* $a \geq b$ *then* $\mathrm{interval}(a, b) = \varnothing$.

4 Open Balls

For any real numbers x and r, let ball(x, r) denote the set of all real numbers within distance r of x.

$$
\begin{array}{|l}
\text{ball} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}\,\mathbb{R} \\
\hline
\forall\, x, r : \mathbb{R} \bullet \\
\qquad \text{ball}(x, r) = \{\, x' : \mathbb{R} \mid \text{abs}(x' - x) < r \,\}
\end{array}
$$

Remark. *If* $r > 0$ *then* $x \in \text{ball}(x, r)$.

Remark. *If* $r \leq 0$ *then* $\text{ball}(x, r) = \varnothing$.

Remark. $\text{ball}(x, r) = \text{interval}(x - r, x + r)$

5 Open Sets

Any subset of $\mathbb{R}$ that can be formed by taking finite intersections and arbitrary unions of open balls is said to be *open*. Let open denote the set of all open subsets of $\mathbb{R}$. The definition of open can be stated using concepts from topology but that is beyond the scope of this article. The set of all open balls is said to *generate* open. This article simply declares the type of open.

$$
\begin{array}{|l}
\mathsf{open} : \mathbb{P}(\mathbb{P}\,\mathbb{R})
\end{array}
$$

Remark. $\text{ball}(x, r) \in \mathsf{open}$

Remark. $\varnothing \in \mathsf{open}$

Remark. $\mathbb{R} \in \mathsf{open}$

6 Neighbourhoods

Let x be a real number. Any open set that contains x is called a *neighbourhood* of it. Let neigh(x) denote the set of all neighbourhoods of x.

$$
\begin{array}{|l}
\text{neigh} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{P}\,\mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \text{neigh}(x) = \{\, U : \mathsf{open} \mid x \in U \,\}
\end{array}
$$

Clearly, every real number has an infinity of neighbourhoods.

Remark. *If* $r > 0$ *then* $\text{ball}(x, r) \in \text{neigh}(x)$.

7 Functions

The following sections define continuity, limits, and differentiability. These are *local* properties of functions in the sense that they only depend on the values that the function takes in an arbitrarily small neighbourhood of any given point in their domains. It is therefore useful to first introduce the set of *locally defined* functions, namely those functions that are defined in some neighbourhood of each point of their domains.

For x a real number, let $\mathrm{F}(x)$ denote the set of all real-valued functions that are locally defined at x.

$$
\begin{array}{|l}
\mathrm{F} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!\!+\; \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{F}(x) = \{\, f : \mathbb{R} \rightarrow\!\!\!\!\!\!+\; \mathbb{R} \mid \exists\, U : \mathrm{neigh}(x) \bullet\; U \subseteq \mathrm{dom}\, f \,\}
\end{array}
$$

For U a subset of $\mathbb{R}$, let $\mathrm{F}(U)$ denote the set of all real-valued functions on U that are locally defined at each point of U.

$$
\begin{array}{|l}
\mathrm{F} : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!\!+\; \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{F}(U) = \{\, f : U \longrightarrow \mathbb{R} \mid \forall\, x : U \bullet f \in \mathrm{F}(x) \,\}
\end{array}
$$

Remark. *If $f \in \mathrm{F}(U)$ then $U \in \mathsf{open}$.*

8 Continuity

Let f be a real-valued function that is locally defined at x and let U be a neighbourhood of x contained within the domain of f. The function f is said to be *continuous* at x if for any $\epsilon > 0$ there is some $\delta > 0$ for which $f(x')$ is always within ϵ of $f(x)$ when $x' \in U$ is within δ of x.

$$
\begin{array}{l}
\forall\, \epsilon > 0 \bullet \exists\, \delta > 0 \bullet \forall\, x' \in U \bullet \\
\qquad \left|x' - x\right| < \delta \Rightarrow \left|f(x') - f(x)\right| < \epsilon
\end{array}
$$

$$
\begin{array}{|l}
\multicolumn{1}{l}{\textit{Continuous}} \\
\hline
f : \mathbb{R} \rightarrow\!\!\!\!\!\!+\; \mathbb{R} \\
x : \mathbb{R} \\
\hline
f \in \mathrm{F}(x) \\
\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x' : \mathrm{dom}\, f \bullet \\
\qquad \mathrm{abs}(x' - x) < \delta \Rightarrow \mathrm{abs}(f(x') - f(x)) < \epsilon \\
\hline
\end{array}
$$

Define $\mathrm{C}^0(x)$ to be the set of all functions that are continuous at x.

$$\begin{array}{|l}
\mathrm{C}^0 : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{C}^0(x) = \{\, f : \mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R} \mid \mathit{Continuous} \,\}
\end{array}$$

Let U be any subset of $\mathbb{R}$. Define $\mathrm{C}^0(U)$ to be the set of all functions on U that are continuous at each point in U.

$$\begin{array}{|l}
\mathrm{C}^0 : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{C}^0(U) = \{\, f : \mathrm{F}(U) \mid \forall\, x : U \bullet f \in \mathrm{C}^0(x) \,\}
\end{array}$$

Remark. *If $f \in \mathrm{C}^0(U)$ then U is a, possibly infinite, union of neighbourhoods.*

9 Limits

Let x and l be real numbers and let f be a real-valued function that is defined everywhere in some neighbourhood U of x, except possibly at x. The function f is said to approach the limit l at x if $f \oplus \{x \mapsto l\}$ is continuous at x.

$$\lim_{x' \to x} f(x') = l$$

$$\begin{array}{|l|}
\hline
\mathit{Limit} \\
f : \mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R} \\
x, l : \mathbb{R} \\
\hline
f \oplus \{x \mapsto l\} \in \mathrm{C}^0(x) \\
\hline
\end{array}$$

Let $\lim(x, l)$ be the set of all functions that approach the limit l at x.

$$\begin{array}{|l}
\lim : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R}) \\
\hline
\forall\, x, l : \mathbb{R} \bullet \\
\qquad \lim(x, l) = \{\, f : \mathbb{R} \rightarrow\!\!\!\!\!+ \mathbb{R} \mid \mathit{Limit} \,\}
\end{array}$$

Theorem 1. *If a function f approaches some limit at x then that limit is unique.*

$$\begin{array}{l}
\forall\, x, l, l' : \mathbb{R} \bullet \\
\qquad \forall\, f : \lim(x, l) \cap \lim(x, l') \bullet \\
\qquad\qquad l = l'
\end{array}$$

Proof. Suppose we are given real numbers

$$x, l, l' \in \mathbb{R}$$

and a function

$$f \in \lim(x, l) \cap \lim(x, l')$$

Let ϵ be any positive real number

$$\epsilon > 0$$

Since f approaches limits l and l' at x there exists a real number $\delta > 0$ such that

$$\begin{array}{l} \forall\, x' \in \mathbb{R} \mid \\ \quad 0 < \left| x' - x \right| < \delta \bullet \\ \qquad \left| f(x') - l \right| < \epsilon \wedge \left| f(x') - l' \right| < \epsilon \end{array}$$

For any such real number x' we have

$$\begin{array}{ll} \left| l' - l \right| & \\ \quad = \left| (f(x') - l) - (f(x') - l') \right| & \text{[add and subtract } f(x')\text{]} \\ \quad \leq \left| f(x') - l \right| + \left| f(x') - l' \right| & \text{[triangle inequality]} \\ \quad = 2\epsilon & \text{[definition of limits]} \end{array}$$

Since the above holds for any $\epsilon > 0$ we must have

$$l = l'$$

□

If f approaches the limit l at x then define $\lim(f, x) = l$. By the preceding theorem, $\lim(f, x)$ is well-defined when it exists.

$$\begin{array}{|l} \lim : (\mathbb{R} \nrightarrow \mathbb{R}) \times \mathbb{R} \nrightarrow \mathbb{R} \\ \hline \lim = \{\, \mathit{Limit} \bullet (f, x) \mapsto l \,\} \end{array}$$

10 Differentiability

Let f be a real-valued function on the real numbers, let x be a real number, and let f be defined on some neighbourhood U of x.

The function f is said to be differentiable at x if the following limit holds for some number $f'(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Remark. *If f is differentiable at x then f is continuous at x.*

The geometric intuition behind the concept of differentiability is that f is differentiable at x when, very near x, the function f is approximately a straight line with slope $f'(x)$.

$$f(x+h) \approx f(x) + f'(x)h \quad \text{when} \quad |h| \approx 0$$

The slope $f'(x)$ is called the derivative of f at x.

We can read this definition as saying that the approximate slope function $m(h)$ defined for small enough, non-zero values of h by

$$m(h) = \frac{f(x+h) - f(x)}{h}$$

approaches the limit $l = f'(x)$ as $h \to 0$.

$$\lim_{h \to 0} m(h) = l = f'(x)$$

Differentiable

$f : \mathbb{R} \rightarrowtail\!\!\!\!\!\!+ \ \mathbb{R}$
$x, l : \mathbb{R}$

$f \in \mathrm{C}^0(x)$

let $m == (\lambda\, h : \mathbb{R}_* \mid x + h \in \operatorname{dom} f \bullet (f(x+h) - f(x)) \,/\, h) \bullet$
$\quad \lim(m, 0) = l$

Remark. *If f is differentiable at x then L is unique.*

Define $\mathrm{diff}(x, L)$ to be the set of all functions f that are differentiable at x with $l = f'(x)$.

$\mathrm{diff} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R})$

$\forall\, x, l : \mathbb{R} \bullet$
$\quad \mathrm{diff}(x, l) = \{\, f : \mathbb{R} \nrightarrow \mathbb{R} \mid \mathit{Differentiable} \,\}$

Define $\mathrm{diff}(x)$ to be the set of all functions that are differentiable at x.

$\mathrm{diff} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R})$

$\forall\, x : \mathbb{R} \bullet$
$\quad \mathrm{diff}(x) = \bigcup \{\, l : \mathbb{R} \bullet \mathrm{diff}(x, l) \,\}$

Let U be any subset of $\mathbb{R}$. Define $\mathrm{diff}(U)$ to be the set of all functions on U that are differentiable at each point of U.

$\mathrm{diff} : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R})$

$\forall\, U : \mathbb{P}\,\mathbb{R} \bullet$
$\quad \mathrm{diff}(U) = \{\, f : \mathrm{C}^0(U) \mid \forall\, x : U \bullet f \in \mathrm{diff}(x) \,\}$

11 Derivatives

The function f' is called the *derived* function or the *derivative* of f. Define $\operatorname{deriv}(f, x)$ to be $f'(x)$.

$$
\begin{array}{|l}
\operatorname{deriv} : (\mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R}) \times \mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R} \\
\hline
\operatorname{deriv} = \{\, \mathit{Differentiable} \bullet (f, x) \mapsto l \,\}
\end{array}
$$

Define $\mathrm{D}(f)$ to be the derived function f'.

$$
\begin{array}{|l}
\mathrm{D} : (\mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R}) \longrightarrow (\mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R}) \\
\hline
\forall f : \mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R} \bullet \\
\qquad \mathrm{D}\, f = (\lambda\, x : \mathbb{R} \mid f \in \operatorname{diff}(x) \bullet \operatorname{deriv}(f, x))
\end{array}
$$

Remark. *If f is differentiable on U then f' is not necessarily continuous on U. Counterexamples exist. If f is uniformly differentiable then f' is continuous, but I won't discuss uniform differentiability further.*

12 Higher Order Derivatives

Let n be a natural number and let x be a real number. In differential geometry we normally deal with $C^n(x)$, the set of functions that possess continuous derivatives of order $0, \ldots, n$ at x. Define $\mathrm{C}(n, x)$ to be the set of all functions that have continuous derivatives of order $0, \ldots, n$ at x.

$$
\begin{array}{|l}
\mathrm{C} : \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R}) \\
\hline
\forall x : \mathbb{R} \bullet \\
\qquad \mathrm{C}(0, x) = \mathrm{C}^0(x) \\
\\
\forall n : \mathbb{N};\ x : \mathbb{R} \bullet \\
\qquad \mathrm{C}(n + 1, x) = \{\, f : \mathrm{C}^0(x) \mid \mathrm{D}\, f \in \mathrm{C}(n, x) \,\}
\end{array}
$$

Let n be a natural number and let U be a subset of $\mathbb{R}$. Define $\mathrm{C}(n, U)$ to be the set of all functions on U that have continuous derivatives of order $0, \ldots, n$ at every point of U.

$$
\begin{array}{|l}
\mathrm{C} : \mathbb{N} \times \mathbb{P}\, \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrowtail\!\!\!\!+ \mathbb{R}) \\
\hline
\forall n : \mathbb{N};\ U : \mathbb{P}\, \mathbb{R} \bullet \\
\qquad \mathrm{C}(n, U) = \{\, f : \mathrm{F}(U) \mid \forall x : U \bullet f \in \mathrm{C}(n, x) \,\}
\end{array}
$$

13 Smoothness

A function is said to be smooth if it possesses continuous derivatives of all orders. Let x be a real number. Define $\mathrm{C}^\infty(x)$ to be the set of all functions that are smooth at x.

$$
\begin{array}{|l}
\mathrm{C}^\infty : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{C}^\infty(x) = \{\, f : \mathrm{F}(x) \mid \forall\, n : \mathbb{N} \bullet f \in \mathrm{C}(n, x) \,\}
\end{array}
$$

Define $\mathrm{C}^\infty(U)$ to be the set of all functions on U that are smooth at every point of U.

$$
\begin{array}{|l}
\mathrm{C}^\infty : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{C}^\infty(U) = \{\, f : \mathrm{F}(U) \mid \forall\, x : U \bullet f \in \mathrm{C}^\infty(x) \,\}
\end{array}
$$