Real Numbers

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Abstract

This article contains Z Notation type declarations for the real numbers, $\mathbb{R}$, and some related objects. It has been type checked by *f*UZZ.

1 Introduction

The real numbers, $\mathbb{R}$, are foundational to many many mathematical objects such as vector spaces and manifolds, but are not built-in to Z Notation. The article provides type declarations for $\mathbb{R}$ and related objects so that they can be used and type checked in formal Z specifications.

No attempt has been made to provide axiomatic definitions of these objects since they would only be a use for proof checking. Although proof checking is highly desirable, it is beyond the scope of this article. The type declarations given here are intended to provide a basis for future axiomatization.

2 The Real Numbers

Let $\mathbb{R}$ denote the given set of real numbers.

$$[\mathbb{R}]$$

Let 0 and 1 denote the zero and unit elements of the real numbers.

$$\begin{array}{|l} 0 : \mathbb{R} \\ 1 : \mathbb{R} \end{array}$$

Define $\mathbb{R}_*$ to be the set of non-zero real numbers, also referred to as the punctured real number line.

$$\mathbb{R}_* == \mathbb{R} \setminus \{0\}$$

The usual comparison relations have the following signatures.

$$\begin{array}{|l} \_ < \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\ \_ \leq \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\ \_ > \_ : \mathbb{R} \leftrightarrow \mathbb{R} \\ \_ \geq \_ : \mathbb{R} \leftrightarrow \mathbb{R} \end{array}$$

Define $\mathbb{R}_+$ to be the set of positive real numbers.

$$\mathbb{R}_+ == \{ x : \mathbb{R} \mid x > 0 \}$$

The usual negative operator has the following signature.

$$\begin{array}{|l} - : \mathbb{R} \longrightarrow \mathbb{R} \end{array}$$

Define $\mathrm{abs}\, x$ to be $|x|$, the absolute value of the real number x.

$$\begin{array}{|l} \mathrm{abs} : \mathbb{R} \longrightarrow \mathbb{R} \\ \hline \forall x : \mathbb{R} \bullet \mathrm{abs}(x) = \mathbf{if}\ x \geq 0\ \mathbf{then}\ x\ \mathbf{else} - x \end{array}$$

The usual arithmetic operators have the following signatures.

$$\begin{array}{|l} \_ + \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ \_ - \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ \_ * \_ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ \_ / \_ : \mathbb{R} \times \mathbb{R}_* \longrightarrow \mathbb{R} \end{array}$$

Define $\mathrm{sqrt}\, x$ to be $\sqrt{x}$, the non-negative square root of the non-negative real number x.

$$\begin{array}{|l} \mathrm{sqrt} : \mathbb{R} \nrightarrow \mathbb{R} \\ \hline \mathrm{sqrt} = \{ x : \mathbb{R} \mid x \geq 0 \bullet x * x \mapsto x \} \end{array}$$

3 Open Intervals

Let a and b be real numbers. The open interval bounded by a and b is the set of all real numbers between a and b. Define $\mathrm{interval}(a, b)$ to be (a, b), the open interval bounded by a and b.

$$\begin{array}{|l} \mathrm{interval} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}\, \mathbb{R} \\ \hline \forall a, b : \mathbb{R} \bullet \\ \quad \mathrm{interval}(a, b) = \{ x : \mathbb{R} \mid a < x < b \} \end{array}$$

Clearly, $\mathrm{interval}(a, b)$ is empty if $a \geq b$.

4 Open Balls

Let x be a real number and let r be a strictly positive real number. Define $\mathrm{ball}(x, r)$ to the the open interval that contains all points within distance r of x.

$$
\begin{array}{|l}
\mathrm{ball} : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{P}\,\mathbb{R} \\
\hline
\forall\, x : \mathbb{R};\ r : \mathbb{R}_+ \bullet \\
\qquad \mathrm{ball}(x, r) = \{\, x' : \mathbb{R} \mid \mathrm{abs}(x' - x) < r \,\}
\end{array}
$$

Remark. $\mathrm{ball}(x, r) = \mathrm{interval}(x - r, x + r)$

5 Neighbourhoods

Let x be a real number. Any open ball centred at x is called a neighbourhood of it. Define $\mathrm{neigh}(x)$ to be the set of all neighbourhoods of x.

$$
\begin{array}{|l}
\mathrm{neigh} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{P}\,\mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{neigh}(x) = \{\, r : \mathbb{R}_+ \bullet \mathrm{ball}(x, r) \,\}
\end{array}
$$

Clearly, every real number has an infinity of neighbourhoods.

6 Functions

Our next goal is to define continuity, limits, and differentiability. These are *local* properties of functions in the sense that they only depend on the values that the function takes in an arbitrarily small neighbourhood of a given point. We therefore restrict our attention to functions that are defined in some neighbourhood of each point in their domains. Let x be a real number. Define $\mathrm{F}(x)$ to be the set of all real-valued functions that are defined in some neighbourhood of x.

$$
\begin{array}{|l}
\mathrm{F} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{F}(x) = \{\, f : \mathbb{R} \nrightarrow \mathbb{R} \mid \exists\, U : \mathrm{neigh}(x) \bullet U \subseteq \mathrm{dom}\, f \,\}
\end{array}
$$

Let U be a subset of $\mathbb{R}$. Define $\mathrm{F}(U)$ to be the set of a real-valued functions on U that are defined in some neighbourhood of every point of U.

$$
\begin{array}{|l}
\mathrm{F} : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{F}(U) = \{\, f : U \longrightarrow \mathbb{R} \mid \forall\, x : U \bullet f \in \mathrm{F}(x) \,\}
\end{array}
$$

7 Continuity

Let f be a real-valued function and let x be a real number. The function f is said to be continuous at x if the domain of f contains some neighbourhood U of x such that for any $\epsilon > 0$ there is some $\delta > 0$ for which $f(x')$ is always within ϵ of $f(x)$ when $x' \in U$ is within δ of x.

$$\begin{aligned} &\forall \epsilon > 0 \bullet \exists \delta > 0 \bullet \forall x' \in U \bullet \\ &\qquad \left| x' - x \right| < \delta \Rightarrow \left| f(x') - f(x) \right| < \epsilon \end{aligned}$$

Continuous

$$\begin{aligned} &f : \mathbb{R} \rightarrow\!\!\!\!\!\!+ \mathbb{R} \\ &x : \mathbb{R} \\ \hline &f \in \mathrm{F}(x) \\ &\forall \epsilon : \mathbb{R}_+ \bullet \exists \delta : \mathbb{R}_+ \bullet \forall x' : \operatorname{dom} f \bullet \\ &\qquad \operatorname{abs}(x' - x) < \delta \Rightarrow \operatorname{abs}(f(x') - f(x)) < \epsilon \end{aligned}$$

Define $\mathrm{C}^0(x)$ to be the set of all functions that are continuous at x.

$$\begin{aligned} &\mathrm{C}^0 : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!\!+ \mathbb{R}) \\ \hline &\forall x : \mathbb{R} \bullet \\ &\qquad \mathrm{C}^0(x) = \{ f : \mathbb{R} \rightarrow\!\!\!\!\!\!+ \mathbb{R} \mid \mathit{Continuous} \} \end{aligned}$$

Let U be any subset of $\mathbb{R}$. Define $\mathrm{C}^0(U)$ to be the set of all functions on U that are continuous at each point in U.

$$\begin{aligned} &\mathrm{C}^0 : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \rightarrow\!\!\!\!\!\!+ \mathbb{R}) \\ \hline &\forall U : \mathbb{P}\,\mathbb{R} \bullet \\ &\qquad \mathrm{C}^0(U) = \{ f : \mathrm{F}(U) \mid \forall x : U \bullet f \in \mathrm{C}^0(x) \} \end{aligned}$$

Remark. *If $f \in \mathrm{C}^0(U)$ then U is a, possibly infinite, union of neighbourhoods.*

8 Limits

Let x and l be real numbers and let f be a real-valued function that is defined everywhere in some neighbourhood U of x, except possibly at x. The function f is said to approach the limit l at x if $f \oplus \{x \mapsto l\}$ is continuous at x.

$$\lim_{x' \to x} f(x') = l$$

Limit

$$f : \mathbb{R} \nrightarrow \mathbb{R}$$
$$x, l : \mathbb{R}$$

$$f \oplus \{x \mapsto l\} \in \mathrm{C}^0(x)$$

Let $\lim(x, l)$ be the set of all functions that approach the limit l at x.

$$\lim : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R})$$

$$\forall\, x, l : \mathbb{R} \bullet$$
$$\qquad \lim(x, l) = \{\, f : \mathbb{R} \nrightarrow \mathbb{R} \mid \mathit{Limit} \,\}$$

Theorem 1. *If a function f approaches some limit at x then that limit is unique.*

$$\begin{aligned}
&\forall\, x, l, l' : \mathbb{R} \bullet \\
&\qquad \forall\, f : \lim(x, l) \cap \lim(x, l') \bullet \\
&\qquad\qquad l = l'
\end{aligned}$$

Proof. Suppose we are given real numbers

$$x, l, l' \in \mathbb{R}$$

and a function

$$f \in \lim(x, l) \cap \lim(x, l')$$

Let ϵ be any positive real number

$$\epsilon > 0$$

Since f approaches limits l and l' at x there exists a real number $\delta > 0$ such that

$$\begin{aligned}
&\forall\, x' \in \mathbb{R} \mid \\
&\qquad 0 < \left| x' - x \right| < \delta \bullet \\
&\qquad\qquad \left| f(x') - l \right| < \epsilon \wedge \left| f(x') - l' \right| < \epsilon
\end{aligned}$$

For any such real number x' we have

$$\begin{aligned}
&\left| l' - l \right| \\
&\qquad = \left| (f(x') - l) - (f(x') - l') \right| && \text{[add and subtract } f(x')\text{]} \\
&\qquad \leq \left| f(x') - l \right| + \left| f(x') - l' \right| && \text{[triangle inequality]} \\
&\qquad = 2\epsilon && \text{[definition of limits]}
\end{aligned}$$

Since the above holds for any $\epsilon > 0$ we must have

$$l = l'$$

□

If f approaches the limit l at x then define $\lim(f, x) = l$. By the preceding theorem, $\lim(f, x)$ is well-defined when it exists.

$$
\begin{array}{|l}
\lim : (\mathbb{R} \rightarrowtail \mathbb{R}) \times \mathbb{R} \rightarrowtail \mathbb{R} \\
\hline
\lim = \{\, \mathit{Limit} \bullet (f, x) \mapsto l \,\}
\end{array}
$$

9 Differentiability

Let f be a real-valued function on the real numbers, let x be a real number, and let f be defined on some neighbourhood U of x.

The function f is said to be differentiable at x if the following limit holds for some number $f'(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Remark. *If f is differentiable at x then f is continuous at x.*

The geometric intuition behind the concept of differentiability is that f is differentiable at x when, very near x, the function f is approximately a straight line with slope $f'(x)$.

$$f(x+h) \approx f(x) + f'(x)h \quad \text{when} \quad |h| \approx 0$$

The slope $f'(x)$ is called the derivative of f at x.

We can read this definition as saying that the approximate slope function $m(h)$ defined for small enough, non-zero values of h by

$$m(h) = \frac{f(x+h) - f(x)}{h}$$

approaches the limit $l = f'(x)$ as $h \to 0$.

$$\lim_{h \to 0} m(h) = l = f'(x)$$

Differentiable

$$
\begin{array}{|l}
f : \mathbb{R} \rightarrowtail \mathbb{R} \\
x, l : \mathbb{R} \\
\hline
f \in \mathrm{C}^0(x) \\
\textbf{let } m == (\lambda\, h : \mathbb{R}_* \mid x + h \in \operatorname{dom} f \bullet (f(x+h) - f(x)) \,/\, h) \bullet \\
\qquad \lim(m, 0) = l
\end{array}
$$

Remark. *If f is differentiable at x then L is unique.*

Define $\mathrm{diff}(x, L)$ to be the set of all functions f that are differentiable at x with $l = f'(x)$.

$$\begin{array}{|l}
\mathrm{diff} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \\
\hline
\forall\, x, l : \mathbb{R} \bullet \\
\qquad \mathrm{diff}(x, l) = \{\, f : \mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R} \mid \mathit{Differentiable} \,\}
\end{array}$$

Define $\mathrm{diff}(x)$ to be the set of all functions that are differentiable at x.

$$\begin{array}{|l}
\mathrm{diff} : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{diff}(x) = \bigcup \{\, l : \mathbb{R} \bullet \mathrm{diff}(x, l) \,\}
\end{array}$$

Let U be any subset of $\mathbb{R}$. Define $\mathrm{diff}(U)$ to be the set of all functions on U that are differentiable at each point of U.

$$\begin{array}{|l}
\mathrm{diff} : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{diff}(U) = \{\, f : \mathrm{C}^0(U) \mid \forall\, x : U \bullet f \in \mathrm{diff}(x) \,\}
\end{array}$$

10 Derivatives

The function f' is called the derived function or the derivative of f. Define $\mathrm{deriv}(f, x)$ to be $f'(x)$.

$$\begin{array}{|l}
\mathrm{deriv} : (\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \times \mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R} \\
\hline
\mathrm{deriv} = \{\, \mathit{Differentiable} \bullet (f, x) \mapsto l \,\}
\end{array}$$

Define $\mathrm{D}(f)$ to be the derived function f'.

$$\begin{array}{|l}
\mathrm{D} : (\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \longrightarrow (\mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R}) \\
\hline
\forall\, f : \mathbb{R} \mathrel{+\!\!\!\rightarrow} \mathbb{R} \bullet \\
\qquad \mathrm{D}\, f = (\lambda\, x : \mathbb{R} \mid f \in \mathrm{diff}(x) \bullet \mathrm{deriv}(f, x))
\end{array}$$

Remark. *If f is differentiable on U then f' is not necessarily continuous on U. Counterexamples exist. If f is uniformly differentiable then f' is continuous, but I won't discuss uniform differentiability further.*

11 Higher Order Derivatives

Let n be a natural number and let x be a real number. In differential geometry we normally deal with $C^n(x)$, the set of functions that possess continuous derivatives of order $0, \ldots, n$ at x. Define $\mathrm{C}(n, x)$ to be the set of all functions that have continuous derivatives of order $0, \ldots, n$ at x.

$$\begin{array}{|l}
\mathrm{C} : \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{C}(0, x) = \mathrm{C}^0(x) \\
\forall\, n : \mathbb{N};\ x : \mathbb{R} \bullet \\
\qquad \mathrm{C}(n+1, x) = \{\, f : \mathrm{C}^0(x) \mid \mathrm{D}\, f \in \mathrm{C}(n, x) \,\}
\end{array}$$

Let n be a natural number and let U be a subset of $\mathbb{R}$. Define $\mathrm{C}(n, U)$ to be the set of all functions on U that have continuous derivatives of order $0, \ldots, n$ at every point of U.

$$\begin{array}{|l}
\mathrm{C} : \mathbb{N} \times \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, n : \mathbb{N};\ U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{C}(n, U) = \{\, f : \mathrm{F}(U) \mid \forall\, x : U \bullet f \in \mathrm{C}(n, x) \,\}
\end{array}$$

12 Smoothness

A function is said to be smooth if it possesses continuous derivatives of all orders. Let x be a real number. Define $\mathrm{C}^\infty(x)$ to be the set of all functions that are smooth at x.

$$\begin{array}{|l}
\mathrm{C}^\infty : \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, x : \mathbb{R} \bullet \\
\qquad \mathrm{C}^\infty(x) = \{\, f : \mathrm{F}(x) \mid \forall\, n : \mathbb{N} \bullet f \in \mathrm{C}(n, x) \,\}
\end{array}$$

Define $\mathrm{C}^\infty(U)$ to be the set of all functions on U that are smooth at every point of U.

$$\begin{array}{|l}
\mathrm{C}^\infty : \mathbb{P}\,\mathbb{R} \longrightarrow \mathbb{P}(\mathbb{R} \nrightarrow \mathbb{R}) \\
\hline
\forall\, U : \mathbb{P}\,\mathbb{R} \bullet \\
\qquad \mathrm{C}^\infty(U) = \{\, f : \mathrm{F}(U) \mid \forall\, x : U \bullet f \in \mathrm{C}^\infty(x) \,\}
\end{array}$$