# COCAAG, WINTER 2025: WEEK 8 QUESTIONS

**Problem 1: Edge ideals**

(a) Write a function in Macaulay2 that, given a graph $G$ on $n$ vertices $0, 1, \dots, n-1$, creates the corresponding ideal generated by $x_i x_j$, for each edge $(i, j)$ in $G$.
(b) Take a specific example, and compute its primary decomposition, minimal primes, and Alexander dual.
(c) What is the definition of a vertex cover of a graph?
(d) For a general edge ideal $I_G$, describe its Alexander dual (i.e. what is it?).

```
needsPackage "NautyGraphs"
strs = generateGraphs(6, MinDegree => 2)
Gs = strs/stringToGraph
G = Gs_5
edges G
(edges G)/toList/sort
(edges G)/toList/sort//sort
```

**Problem 2: Algebraically independent sets** Let $I \subset R = K[x_1, \dots, x_n]$ be an ideal. We say that a subset $u \subset x = \{x_1, \dots, x_n\}$ is an *(algebraically) independent set* if $I \cap K[u] = \{0\}$.

(a) Show that the set of independent sets is a simplicial complex.
(b) Show that the independent sets of $I$ and of $\sqrt{I}$ are the same.
(c) Now let $I$ be a monomial ideal. Choose one, and find this complex.
(d) Make a conjecture based on this example (and possibly others!) as to what this complex is for a general (square-free) monomial ideal.
(e) Prove this conjecture!

**Problem 3: Tree ideals** For each positive integer $n \geq 2$, consider the following monomial ideal in $\mathbb{K}[x_1, \dots, x_n]$.

$$I_n := \left\langle \big( \prod_{x \in F} x \big)^{n-|F|+1} \mid \emptyset \neq F \subset \{x_1, \dots, x_n\} \right\rangle$$

(a) Write a Macaulay2 function to create this ideal (and ring).
(b) For $n = 2$ or $n = 3$, try computing a primary decomposition "by hand", and also the Alexander dual, and check your work with Macaulay2
(c) why are these called tree ideals? (look at standard monomials in the ideal, is there a relationship with labelled trees on $n+1$ vertices?)

```
R = QQ[a,b,c,d,e]
I = monomialIdeal(a*b, b*c, c*d^3, d*e, a*b*c^2)
irreducibleDecomposition I
primaryDecomposition I
dual I
```