COCAAG, WINTER 2025: WEEK 6 QUESTIONS

During "project time", work through a jupyter notebook, try exercises, or do these!

In some of the problems, you are asked to try your methods on one or some of the following varieties:

(a) the cubic surface X in $\mathbb{P}^3$: $x^3 + y^3 + z^3 + w^3 = 0$
(b) the Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$, this is what is called a K3 surface,
(c) the variety $X \subset \mathbb{P}^4$ whose ideal is given by the 3 by 3 minors of a (fairly random) 4×3 matrix of linear forms in 5 variables.
(d) Other possibly random hypersurfaces in $\mathbb{P}^n$ of degree d, e.g. $(n, d) = (4, 5), (4, 4), (4, 6)$.

Problem 1: Consider the special case $\mathbb{P}^2$. Compute from the definition in class the cohomologies of $\mathcal{O}_{\mathbb{P}^2}(d)$, for all d, verifying Serre's theorem in this case. Hint: this complex with infinitely generated modules is (over the base field) the direct sum of complexes corresponding to each monomial $x_0^a x_1^b x_2^c$, for $(a, b, c) \in \mathbb{Z}^3$. Once one sees the proof in this special case, it is pretty easy to generalize to prove the entire theorem (and, in fact, even more, that we haven't stated yet!).

Problem 2: In this exercise we prove the important theorems of Serre ((b), (c)). After proving (a), use Serre's "local duality" theorem to prove (b), (c). Here, suppose that $\widetilde{M}$ is a coherent sheaf on $\mathbb{P}^n$ where M is a finitely generated graded S-module.

(a) Show that the k-dual of M is zero in all degrees $d \gg 0$.
(b) Show that $H^i(\mathcal{O}_X(d)) = 0$, for $i > 0$ and $d \gg 0$.
(c) Show that $\mathcal{O}_X(d)$ is generated by global sections for $d \gg 0$. This means that the natural map $M_d \to H^0(\mathcal{O}_X(d))$ is an isomorphism.

Problem 3: Consider the projective variety X which is given by the zeros of $x^3 + y^3 + z^3$, in $\mathbb{P}^2$ (this is an elliptic curve).

(a) Find the dimensions of the cohomology vector spaces of $\mathcal{O}_X$.
(b) Compute "by hand" a module which sheafifies to Ω^1_X.
(c) Find the dimensions of the cohomology vector spaces of 1-forms Ω^1_X.

Problem 4: Compute the sheaf cohomology of the sheaf associated to $\mathrm{Hom}(I/I^2, S/I)$, for some choices of monomial ideals I in a polynomial ring S. (start with e.g. $I = (x^2, xy, y^2) \subset k[x, y, z]$). The H^0 of this turns out to be the tangent space to the point $[I]$ on its Hilbert scheme.

Problem 5: Consider the projective variety X which is given by the 3 by 3 minors of a (random) 3x4 matrix of linear forms in $\mathbb{P}^4$ mentioned earlier. Find the dimensions of the cohomology vector spaces of 1-forms Ω^1_X. Compute the Hodge diamond of X.

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S = ZZ/32003[a,b,c,d,e]
M = random(S^3, S^{4:-1})
I = minors(3, M)
dim I -- one more than the dimension as a projective variety
codim I,  degree I
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