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A computationally tractable version of the collective model

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Abstract

A computationally tractable version of the Bohr–Mottelson collective model is presented which makes it possible to diagonalize realistic collective models and obtain convergent results in relatively small appropriately chosen subspaces of the collective model Hilbert space. Special features of the proposed model are that it makes use of the beta wave functions given analytically by the softened-beta version of the Wilets–Jean model, proposed by Elliott et al., and a simple algorithm for computing $SO(5) \supset SO(3)$ spherical harmonics. The latter has much in common with the methods of Chacon, Moshinsky, and Sharp but is conceptually and computationally simpler. Results are presented for collective models ranging from the spherical vibrator to the Wilets–Jean and axially symmetric rotor–vibrator models.

1. Introduction

This paper presents a computationally tractable algebraic version of the Bohr–Mottelson (BM) collective model [1–3]. The need for such a version of the model arises because the expansion of rotational wave functions in a spherical vibrational basis is slowly convergent with the result that the diagonalization of a general coupled rotor–vibrator collective model Hamiltonian in this basis requires a large number of basis states. The proposed method constructs a basis for the collective model in which beta wave functions, centred about a non-zero equilibrium value, are obtained algebraically based on the methods of Rohoziński et al. [4], Elliott et al. [5] and Ref. [6], and new methods are developed for computing the complementary rotational and gamma wave functions. The

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motivation for this development is to be seen in the context of a sequence of steps from a phenomenological model description of nuclear collective structure to a microscopic interpretation of what it means. I follow the strategy of giving a phenomenological collective model a microscopic foundation by first formulating it in algebraic terms and subsequently constructing representations of its algebraic expression on a microscopic shell model Hilbert space.

Early advances in the shell model theory of collectivity came with the identification of seniority and symplectic symmetry with pairing [7–9] and su(3) with rotations [10]. An early description of the use of symmetry in nuclear structure was provided by Parikh [11].

Algebraic methods have subsequently had an enormous influence on collective model theory (cf. [12] for a review). Two influential developments have been: the Frankfurt version of the collective model ([13,14]) and the interacting boson model (IBM) [15]. The Frankfurt methods were developed to give solutions to the collective model in the intermediate region between the analytically solvable vibrational and rotational limits. They made major use of the algebraic structures associated with the five-dimensional harmonic oscillator. The IBM achieved a reduction of the Frankfurt program to a finite-dimensional space by compactifying the algebraic structure of the collective model to the U(6) symmetry group of the six-dimensional harmonic oscillator and restricting consideration to single (finite-dimensional) U(6) irreps. The IBM has three exactly solvable limits, corresponding to similar limits of the BM model.

Major developments towards the goal of formulating a microscopic (shell model) theory of collective states have also followed the algebraic approach (a historical survey was provided in the review article of Ref. [16]). The latter developments were based on the symplectic model [17].

The symplectic model and its several variations have not been applied widely to fit detailed nuclear data because that was not their purpose (cf. Refs. [18,19] for some recent applications). Their purpose was to obtain a fundamental explanation of nuclear collective dynamics and the way it emerges from interacting nucleons. Thus, even if one were to succeed in formulating a completely satisfactory theory of nuclear collective dynamics, it would hardly be a simple theory. Nor would it serve the purpose of every day analysis of nuclear data. For this purpose, one continues to need simple phenomenological models, albeit preferably ones with a microscopic foundation. With this concern in mind, we revisit the BM collective model with the benefit of insights acquired from the Frankfurt model, the IBM, and the symplectic model.

2. The standard solvable limits of the collective model

The classic BM collective model shares a Hilbert space with the five-dimensional harmonic oscillator. Indeed, in its harmonic vibrational limit, its spectrum and eigenstates are precisely those of the five-dimensional harmonic oscillator. Thus, it has rich algebraic, geometrical, and analytical structures all which are exploited in this paper. It has a spectrum generating algebra given by the semi-direct sum Lie algebra [hw(5)]u(5) and a corresponding dynamical group [HW(5)]U(5). The Lie algebra [hw(5)]u(5) is spanned by five ($L = 2$) pairs of d-boson (phonon) operators $\{d^\nu, d^\dagger_\nu; \nu = 0, \pm 1, \pm 2\}$ and the

infinitesimal generators $\{d^\dagger_\mu d^\nu\}$ of U(5), where the d-bosons satisfy the commutation relations $[d^\mu, d^\dagger_\nu] = \delta_{\mu\nu}$.

The collective model has three well-known algebraically solvable limits: the harmonic vibrator model, the Wilets–Jean (gamma-soft) model [20], and the axially-symmetric rigid-rotor model. These submodels are associated with dynamical subgroup chains corresponding to different paths through the set of groups

$$\begin{array}{ccccc} \big[\mathrm{HW}(5)\big]\mathrm{U}(5) & \supset & \big[\mathrm{R}^5\big]\mathrm{SO}(5) & \supset & \big[\mathrm{R}^5\big]\mathrm{SO}(3) \\ \cup & & \cup & & \cup \\ \mathrm{U}(5) & \supset & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) \end{array} \tag{1}$$

starting with [HW(5)]U(5) and ending with SO(3), where R^5 is the group with Lie algebra spanned by the quadrupole moments

$$\left\{ q_\nu = \frac{k}{\sqrt{2}}\big(d^\dagger_\nu + d_\nu\big), \quad \nu = 0, \pm 1, \pm 2 \right\} \tag{2}$$

with $d_\nu = (-1)^\nu d^{-\nu}$ and k a dimensionality constant. Thus,

$$\big[\mathrm{HW}(5)\big]\mathrm{U}(5) \supset \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3) \tag{3}$$

is a dynamical subgroup chain for the harmonic vibrator model,

$$\big[\mathrm{HW}(5)\big]\mathrm{U}(5) \supset \big[\mathrm{R}^5\big]\mathrm{SO}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3) \tag{4}$$

is a dynamical subgroup chain for the Wilets–Jean (beta-rigid, gamma-independent) model, and

$$\big[\mathrm{HW}(5)\big]\mathrm{U}(5) \supset \big[\mathrm{R}^5\big]\mathrm{SO}(5) \supset \big[\mathrm{R}^5\big]\mathrm{SO}(3) \supset \mathrm{SO}(3) \tag{5}$$

is a dynamical chain for the rigid-rotor (beta- and gamma-rigid) model. The above subgroup chains are discussed in more detail, for example, in Ref. [12].

The solution of more general collective model Hamiltonians has been tackled, for example, by Hess et al. [14] in the U(5) ⊃ SO(5) ⊃ SO(3) basis for the collective model Hilbert space of Chaćon, Moshinsky and Sharp [21]. This approach has been very influential and has applications to the IBM in which the same U(5) and SO(5) ⊃ SO(3) groups appear. Its limitation is that it is impractical for the description of collective model states of large deformation which converge slowly in a spherical U(5) basis.

3. An alternative basis for the collective model

Moving between the above solvable limits of the collective model is complicated by the 'rigidity' of two of the limits. Both the rigid-rotor model and the beta-rigid gamma-independent (Wilets–Jean) model have delta function components to their wave functions and, as a consequence, they are not realizable in the Hilbert space of the vibrator limit except as limits of sequences of normalizable wave functions. This limitation expresses the fact that rigidly-defined intrinsic quadrupole moments are unphysical and incompatible with quantum mechanics (as well as relativity theory). I therefore consider an alternative

basis for the diagonalization of collective model Hamiltonians and show that it can lead to convergent solutions for a wide range of collective model Hamiltonians.

An important feature of the collective model is that its coordinates separate into orthogonal subsets in much the same way as those of a single particle in three-dimensional space separate into radial and spherical coordinates. Associated with this observation is the fact that the Hilbert space of the collective model is a direct sum

$$\mathbb{H}^{\mathrm{CM}} = \bigoplus_{v} \mathbb{H}_v = \bigoplus_{v} \mathbb{H}_v^{\mathrm{SU}(1,1)} \otimes \mathbb{H}_v^{\mathrm{SO}(5)}, \tag{6}$$

of Hilbert spaces labelled by a seniority quantum number v, which each carry an irrep of a direct product group $\mathrm{SU}(1,1) \times \mathrm{SO}(5)$, where $\mathrm{SU}(1,1)$ is the group of scale transformations of the beta (radial) coordinate, defined below in terms of quadrupole moments, and SO(5) is the five-dimensional rotation group. As a result of this separation of variables, SO(5)-invariant collective model Hamiltonians can be diagonalized with just the Lie algebra of $\mathrm{SU}(1,1)$ as spectrum generating algebra.

Thus, for example, the collective model has two other exactly solvable submodels given by simply adding a $1/\beta^2$ term to either the five-dimensional harmonic oscillator Hamiltonian [4,5] or to the five-dimensional analog of the Coulomb Hamiltonian [22]. Such models are known in three dimensions as the Davidson [23] and Kratzer models [24]. Algebraic solutions of three-dimensional central-force problems, have been studied widely in terms of the algebra of the direct product group $\mathrm{SU}(1,1) \times \mathrm{SO}(3)$ (reviewed in Ref. [25]; an historic account is given in Wybourne's book [26] and an overview of the basic methods in Ref. [27]). The natural extension of the algebraic treatment to five-dimensional space with dynamical group $\mathrm{SU}(1,1) \times \mathrm{SO}(5)$ is straightforward [6,12]. The fact that solvable central force problems remain solvable on addition of a $1/r^2$ term to the Hamiltonian follows from the observation that the commutation relations of the $\mathrm{SU}(1,1)$ spectrum generating algebras are unchanged, by the substitution $\nabla^2 \to \nabla^2 + k/r^2$. This result holds in higher-dimensional spaces.

The coordinate space for the collective model is chosen here to be the 5-dimensional space R^5 of nuclear quadrupole moments rather than the space of surface deformation parameters conventionally used. Thus, a point in R^5 is characterized by a set of quadrupole moments $\{q_M;\ M = 0, \pm 1, \pm 2\}$ or, equivalently, by a set of spherical polar coordinates $\{\beta, \gamma, \Omega\}$, where β, defined by

$$\beta^2 = |q|^2 = \sum_M |q_M|^2, \tag{7}$$

is the R^5 radius and $\{\gamma, \Omega\}$ are generalized Euler angles for a point on a four-sphere. Convenient Euler angles for the four-sphere are defined as an extension of a set Ω of Euler angles for an SO(3) rotation, by the standard Bohr expression [1] for nuclear quadrupole moments

$$q_M(\beta, \gamma, \Omega) = \beta \mathcal{Q}_M(\gamma, \Omega), \tag{8}$$

with

$$\mathcal{Q}_M(\gamma, \Omega) = \cos\gamma \mathcal{D}^2_{0,M}(\Omega) + \frac{1}{\sqrt{2}} \sin\gamma \left(\mathcal{D}^2_{2,M}(\Omega) + \mathcal{D}^2_{-2,M}(\Omega)\right),$$
$$M = 0, \pm 1, \pm 2. \tag{9}$$

Thus, $\mathcal{Q}_M$ is a quadrupole moment for a point on the unit four-sphere (the sphere of radius $\beta = 1$). As functions on the four-sphere, the quadrupole moments $\{\mathcal{Q}_M\}$ are, in fact, proportional to spherical harmonics for the fundamental 5-dimensional $v = 1$ irrep of SO(5). The algebraic structures underlying the collective model are obtained from the fundamental Heisenberg commutation relations

$$\left[\pi^M, q_N\right] = -i\hbar\delta_{M,N}, \tag{10}$$

which are realized by setting $\pi^M = -i\hbar\nabla^M \equiv -i\hbar\partial/\partial q_M$.

In its spherical harmonic vibrational limit, the collective model is assigned a Hamiltonian

$$H = -\frac{\hbar^2}{2B}\nabla^2 + \frac{1}{2}B\omega^2\beta^2. \tag{11}$$

Following the standard methods for three-dimensional central-force problems, the spectrum and wave functions for this Hamiltonian are given by

$$E_{nv} = (2n + \lambda_v)\hbar\omega, \quad n = 0, 1, 2, \ldots, \ v = 0, 1, 2, \ldots, \tag{12}$$

and

$$\Psi_{nv\sigma}(\beta, \gamma, \Omega) = \sqrt{\frac{2n!}{\Gamma(n + \lambda_v)b^5}}\left(\frac{\beta}{b}\right)^{\lambda_v - 5/2}\exp\left(-\frac{\beta^2}{2b^2}\right)L_n^{(\lambda_v - 1)}\left(\frac{\beta^2}{b^2}\right)\mathcal{Y}_{v\sigma}(\gamma, \Omega), \tag{13}$$

where $\{\mathcal{Y}_{v\sigma}\}$ is a set of SO(5) spherical harmonics for an SO(5) irrep of seniority v, $L_n^{(\lambda_v - 1)}$ is an associated Laguerre polynomial, and $\lambda_v = v + 5/2$, $b = \sqrt{\hbar/B\omega}$; note that $\mathcal{Y}_{1M} \propto \mathcal{Q}_M$.

The above results are obtained by use of the SU(1, 1) Lie algebra spanned by the operators

$$Z_1 = -\nabla^2, \qquad Z_2 = \beta^2, \qquad Z_3 = -i\left(q \cdot \nabla + \frac{5}{2}\right), \tag{14}$$

where

$$\nabla^2 = \sum_M \frac{\partial^2}{\partial q_M \partial q^M}, \qquad q \cdot \nabla = \sum_M q_M \nabla^M. \tag{15}$$

Because the commutation relations of these operators are unchanged if Z_1 is replaced by

$$Z_1' = -\nabla^2 + \frac{\beta_0^4}{\beta^2}, \tag{16}$$

it follows [6] that the harmonic vibrational limit of the collective model extends immediately to a model with beta-vibrational Hamiltonian

$$H(\beta_0) = \frac{\hbar^2}{2B}\left(-\nabla^2 + \frac{\beta_0^4}{\beta^2}\right) + \frac{1}{2}B\omega^2\beta^2. \tag{17}$$

The only change in the results of the harmonic vibrational limit for the energies $\{E_{nv}\}$ and wave functions $\{\Psi_{nv\sigma}\}$ is that the relationship $\lambda_v = v + 5/2$ is generalized to

$$\lambda_v = 1 + \sqrt{\left(v + \frac{3}{2}\right)^2 + \beta_0^4}. \tag{18}$$

This is because the replacement $Z_1 \to Z_1'$ in the SU(1, 1) Lie algebra results [6] in a modification of the value of the SU(1, 1) Casimir invariant from the value $\lambda_v(\lambda_v - 2) = v(v+3) + \frac{5}{4}$ to the value

$$\lambda_v(\lambda_v - 2) = v(v+3) + \frac{5}{4} + \beta_0^4. \tag{19}$$

The five-dimensional Kratzer model [22] is handled in a similar way as shown in the Appendix A.

With explicit wave functions, it is a simple matter to compute observable properties of model states. For example, the matrix elements of an electric quadrupole operator of the form

$$Q_M^{(E)} = Z\beta\mathcal{Q}_M \tag{20}$$

factor into products of a matrix element of β between associated Laguerre polynomials and a matrix element of a $v = 1$ SO(5) spherical harmonic. Thus, the latter matrix elements are given by elementary SO(5) Clebsch–Gordan coefficients as shown explicitly below.

The properties of the collective model Hamiltonian $H(\beta_0)$ with β_0 ranging from zero to a large value are discussed in the following section. It is seen there that the model provides a sequence of analytically solvable solutions which progress from the harmonic vibrational [HW(5)]U(5) $\supset$ U(5) limit to the asymptotic Wilets–Jean [HW(5)]U(5) $\supset$ [R^5]SO(5) limit.

Subsequent sections demonstrate that the wave functions of Eq. (13), which reduces an SU(1, 1) $\times$ SO(5) $\supset$ SO(3) dynamical subgroup chain, provide a much more rapidly convergent basis for the expansion of the eigenfunctions of an arbitrary collective model than does a basis that reduces the chain U(5) $\supset$ SO(5) $\supset$ SO(3). For a collective model Hamiltonian that is SO(5)-invariant, the eigenfunctions and spectrum are readily computed using just the Lie algebra of SU(1, 1) as a spectrum generating algebra. Convergence is then optimized by selecting a value of β_0 at the minimum of the β potential of interest.

For a general collective model Hamiltonian, in which both the SU(1, 1) and SO(5) degrees of freedom are active, expressions are needed for the SO(5) spherical harmonics. They can be constructed by the methods of Chaćon et al. [21]. However, I follow a simpler construction based on the observation that the SO(5) spherical harmonics are an orthonormal basis for the Hilbert space $\mathcal{L}^2(S_4)$ of square integrable functions on the four-sphere S_4 with respect to the usual volume element [1] given, in the (γ, Ω) coordinates, by

$$\mathrm{d}v = \sin 3\gamma \, \mathrm{d}\gamma \, \mathrm{d}\Omega, \tag{21}$$

where $\mathrm{d}\Omega$ is the standard SO(3)-invariant volume element. A non-orthonormal basis for $\mathcal{L}^2(S_4)$ is simply constructed as shown in Section 5.2. This basis is then Gram–Schmidt orthogonalized to give the SO(5) spherical harmonics. This method is developed and used

to compute SO(5) Clebsch–Gordan coefficients in a following paper [28]. Thus, the facility to diagonalize any collective model Hamiltonian on the Hilbert space (6) and to compute, for example, B(E2) transition rates between its eigenstates, is increased substantially. An illustrative calculation of the spectrum and E2 transitions for an axially symmetric rotor–vibrator model is given in Section 7.

4. Results for the basic SO(5) invariant model

Fig. 1 shows the potential energy function

$$V_{\beta_0}(\beta) = \frac{\hbar\omega}{2b^2}\left(\beta^2 + \frac{\beta_0^4}{\beta^2}\right) \tag{22}$$

of the Hamiltonian (17) for $\beta_0 = 0$ and 8 together with the corresponding ground-state β wave functions. The ratio of the β width to the mean value of β, given for these wave functions by b/β_0, is proportional to $1/\beta_0\sqrt{\omega}$. Thus, as $\beta_0\sqrt{\omega}$ becomes large, the wave functions approach the Wilets–Jean rigid-beta limit.

The energy-level spectrum, given by Eqs. (12) and (18) is shown as a function of the deformation parameter β_0 in Fig. 2. This figure shows that as $\beta_0\sqrt{\omega} \to \infty$ (the Wilets–Jean limit) the energy levels asymptotically approach a harmonic oscillator sequence with the states of each asymptotic level comprising an infinite number of states. Low energy levels for specific values of β_0 are shown in more detail together with B(E2) transition rates in Figs. 3 and 4.

B(E2) transition rates are computed as follows. Let $\{|nv\sigma\rangle\}$ denote the basis states corresponding to the wave functions of Eq. (13) with λ_v related to v by Eq. (18). The B(E2) transition rate between two SO(5) levels is then defined in the standard way for the quadrupole tensor $Q^{(E)}$ by summing the squared matrix elements of the $Q^{(E)}_M$ operators over the states of the final level and averaging over initial states. Thus

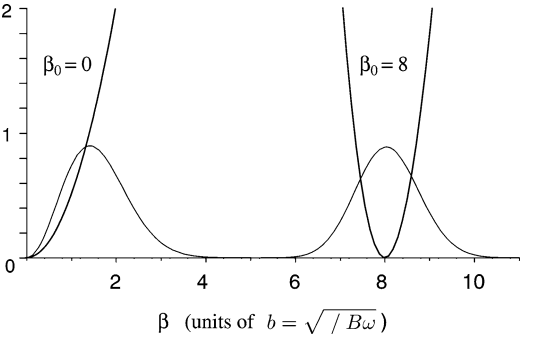

Fig. 1. The potential energy components of the Hamiltonian (17) for $\beta_0 = 0$ and 8 together with the corresponding beta wave functions multiplied by β^2. The potential is given in units of $\hbar\omega$.

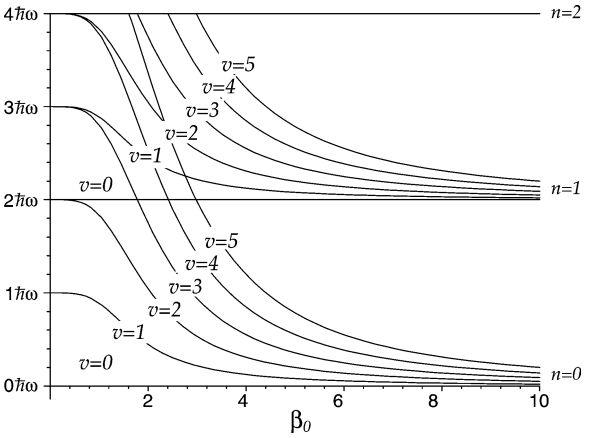

Fig. 2. Excitation energies of the five-dimensional Davidson model as a function of β_0.

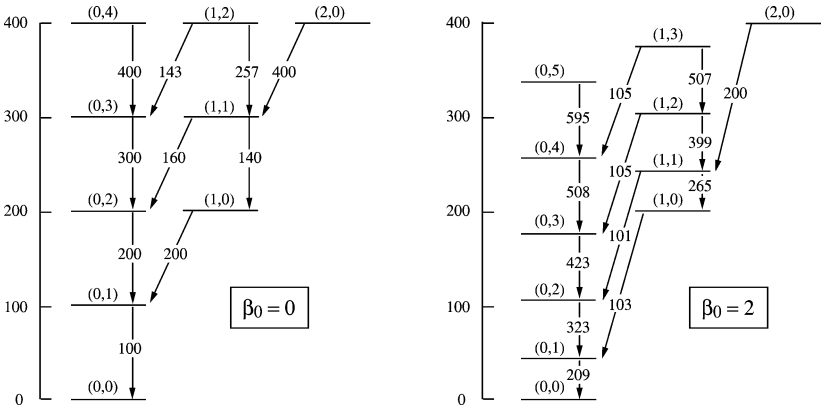

Fig. 3. Energy-levels and B(E2) transition rates for the Hamiltonian (17) with $\beta_0 = 0$ and 2. Energy levels are labelled by their SU(1, 1) × SO(5) quantum numbers (n, v). Energies and B(E2) transition rates are given in units such that the one-phonon $L = 2$ excited (0, 1) state has an excitation energy of 100 units and its B(E2) transition rate to the ground state is also 100 units in the U(5) ($\beta_0 = 0$) limit.

$$B(\mathrm{E2}; n_i v_i \to n_f v_f) = \sum_{\sigma_i \sigma_f M} \frac{1}{d(v_i)} \left| \langle n_f v_f \sigma_f | Q_M^{(E)} | n_i v_i \sigma_i \rangle \right|^2$$
$$= \frac{d(v_f)}{d(v_i)} \left| \langle n_f v_f ||| Q^{(E)} ||| n_i v_i \rangle \right|^2 = \left| \langle n_i v_i ||| Q^{(E)} ||| n_f v_f \rangle \right|^2, \tag{23}$$

where $d(v)$ is the dimension of the SO(5) irrep of seniority v and $\langle n_f v_f ||| Q^{(E)} ||| n_i v_i \rangle$ is an SO(5)-reduced matrix element of the $v = 1$ quadrupole tensor. Since the wave functions for the states $\{|n v \sigma\rangle\}$ are products of β wave functions and spherical harmonics, the matrix elements of the $Q_M^{(E)} = Z\beta \mathcal{Q}_M$ operators of Eq. (20) factor and are given by

$$\langle n_i v_i ||| Q^{(E)} ||| n_f v_f \rangle = Z \langle n_i v_i | \beta | n_f v_f \rangle \langle v_i ||| \mathcal{Q} ||| v_f \rangle, \tag{24}$$

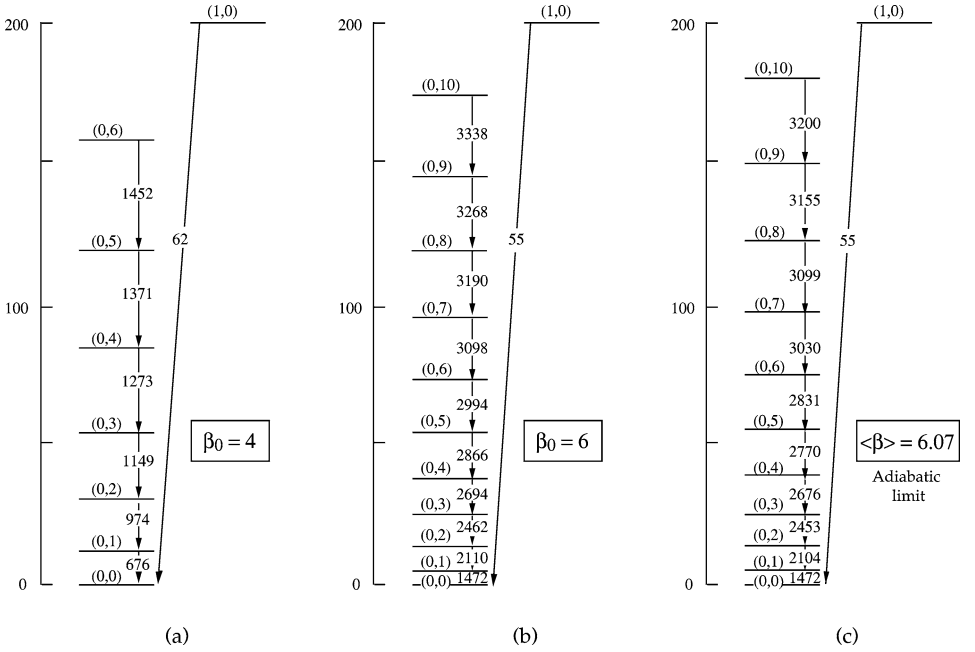

Fig. 4. Energy-levels and B(E2) transition rates for the Hamiltonian (17) with $\beta_0 = 4$ and 6 in units defined in the caption to Fig. 3. For comparison, (c) is the spectrum obtained by a simple adiabatic approximation with $\langle\beta\rangle = 6.07$ as described in the text.

with obvious notation. The β matrix element is evaluated analytically for any value of β_0 while the $\mathcal{Q}$ matrix element is independent of β_0 and can be evaluated in the U(5) ($\beta_0 = 0$) limit.

In the U(5) limit, the $Q_M^{(E)}$ operator can be expressed in terms of $L = 2$ harmonic-oscillator raising and lowering operators

$$Q_M^{(E)} = Z\frac{d_M^\dagger + d_M}{\sqrt{2}}. \tag{25}$$

It follows that $\langle 0v|||Q^{(E)}|||0v\rangle = 0$ and

$$\langle 0, v+1|||Q^{(E)}|||0v\rangle = Z\sqrt{(v+1)/2}. \tag{26}$$

Therefore, since the β integral gives $\langle 0, v+1|\beta|0v\rangle = \sqrt{(2v+5)/2}$, we infer that

$$\langle v+1|||\mathcal{Q}|||v\rangle = \sqrt{\frac{v+1}{2v+5}}, \tag{27}$$

$$\langle v|||\mathcal{Q}|||v+1\rangle = (-1)^\phi\sqrt{\frac{v+1}{2v+5}\cdot\frac{d(v+1)}{d(v)}}, \tag{28}$$

where the phase factor $(-1)^\phi$ depends on a choice of phase convention; it is set equal to 1 in Ref. [28]. Thus, for arbitrary β_0,

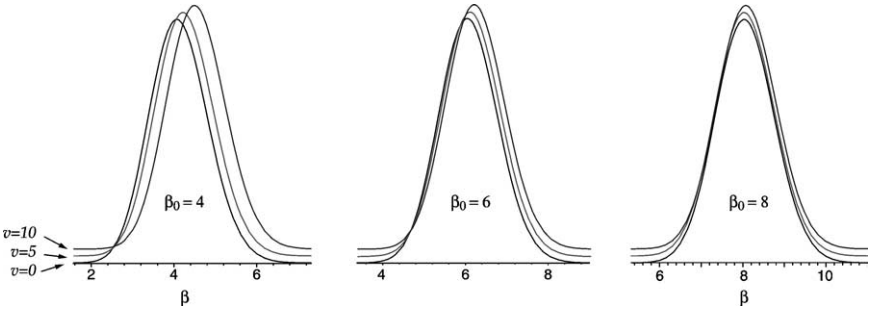

Fig. 5. Comparison of the beta components of the model wave functions for $\beta_0 = 4$, 6, 8 and for $v = 0$, 5, and 10. It is seen that the wave functions are shifted to higher values of β with increasing v due to centrifugal stretching for small values of β_0 but that as $\beta_0\sqrt{\omega} \to \infty$, the adiabatic limit is approached in which this shifting becomes more and more negligible for finite values of v. Note the different baselines used for the $v = 0$, 5, and 10 wave functions without which the three $\beta_0 = 8$ wave functions would be virtually indistinguishable.

$$B(\mathrm{E2}; n'v \to n, v-1) = \frac{Z^2 v}{2v+3}\langle n'v|\beta|n, v-1\rangle^2, \tag{29}$$

$$B(\mathrm{E2}; n'v \to n, v+1) = \frac{Z^2 (v+1)d(v+1)}{(2v+5)d(v)}\langle n'v|\beta|n, v+1\rangle^2. \tag{30}$$

It is also follows from the identity $\langle 0v|||Q^{(E)}|||0v\rangle = 0$ that all quadrupole moments are zero in this model as expected for a gamma-soft model. For the present calculations, Z^2 was given the value $Z^2 = 200$ so that $B(\mathrm{E2}; 2_1 \to 0_1) = 100$ in the U(5) ($\beta_0 = 0$) limit.

It is seen from the figures that, as the value of β_0 is increased, excited beta-vibrational bands separate and increase in energy. In the Wilets–Jean limit, in which $\beta_0\sqrt{\omega} \to \infty$, the ratio of the beta-vibrational energy to the lowest rotational excitation energy diverges. As this limit is approached, a simple adiabatic approximation gives an increasingly good approximate solution to the results of the Hamiltonian (17) as illustrated by comparison of Fig. 4(b) and (c).

The adiabatic approximation follows by expanding the expression for the energy (12) with λ_v given by Eq. (18), for large values of β_0,

$$E_{nv} = \left[2n + \frac{1}{2\beta_0^2}v(v+3)\right]\hbar\omega + 0\left(\frac{1}{\beta_0^4}\right) + \text{const}, \tag{31}$$

and neglecting terms of order $1/\beta_0^4$ and higher. This approximation corresponds to neglecting the centrifugal coupling between the rotational and beta-vibrational degrees of freedom. Perturbation theory shows that this approximation gives accurate results so long as the rotational energy $v(v+3)/2\beta_0^2$, for the gamma-soft rotor, is small in comparison with the beta-vibrational energy $\hbar\omega$.

The effective decoupling of the beta-vibrational and rotational degrees of freedom in the adiabatic limit, is evidenced by the independence of the beta wave function on the seniority quantum number v in this limit. A comparison is made between the $v = 0$, 5, and 10 beta wave functions in Fig. 5 for $\beta_0 = 4$, 6, and 8. It is seen that centrifugal stretching is substantial for $\beta_0 = 4$ but negligible for $\beta_0 \approx 8$ and $v \lesssim 10$.

5. More general collective model Hamiltonians

An extension of the above model to admit gamma as well as deformed-beta equilibrium shapes, is given, for example, by adding a gamma-dependent potential to the Hamiltonian (17),

$$H(\beta_0,\chi)=\frac{\hbar^2}{2B}\left(-\nabla^2+\frac{\beta_0^4}{\beta^2}\right)+\frac{1}{2}B\omega^2\beta^2+\chi V(\gamma). \tag{32}$$

To illustrate, I consider the potential

$$V(\gamma)=-\cos 3\gamma, \tag{33}$$

which has an axially symmetric minimum. This potential produces a very tractable model because, to within a constant, $\cos 3\gamma$ is the SO(5) $v=3$, $L=0$ spherical harmonic. An even simpler model is obtained by assuming a value of $\beta_0\sqrt{\omega}$ for which the β and SO(5) degrees of freedom are adiabatically decoupled. The latter choice is not necessary but simplifies the calculations considerably.

5.1. The adiabatic map

For each value of the β quantum number n, the SO(5) quantum number v runs over the complete set of non-negative integers $v=0,1,2,\ldots$. Moreover, just as the set of SO(3) spherical harmonics $\{Y_{LM}\}$ spans the Hilbert space $\mathcal{L}^2(S_2)$ of square integrable functions on the two-sphere, so the set of SO(5) spherical harmonics $\{\mathcal{Y}_{v\sigma}\}$ spans the Hilbert space $\mathcal{L}^2(S_4)$ of square-integrable functions on the four-sphere (isomorphic to the factor space SO(5)/SO(4)). It follows that

$$\mathcal{L}^2(S_4)=\bigoplus_{v=0}^{\infty}\mathbb{H}_v^{\mathrm{SO}(5)}. \tag{34}$$

Observe also that the SU(1,1) Hilbert spaces $\mathbb{H}_v^{\mathrm{SU}(1,1)}$ featuring in Eq. (6) are all isomorphic to the common Hilbert space $\mathbb{H}^{\mathrm{SU}(1,1)}=\mathcal{L}^2(\mathrm{R}^+)$ of functions of $\beta\in\mathrm{R}^+$ (the positive radial line) that are square integrable with respect to the measure $\beta^4\,\mathrm{d}\beta$. Thus, we can define an *adiabatic map* from $\mathbb{H}_v^{\mathrm{SU}(1,1)}$ to the isomorphic Hilbert space $\mathbb{H}_0^{\mathrm{SU}(1,1)}$ in which the basis wave functions map

$$\phi_{nv}\mapsto\phi_n\equiv\phi_{n0}. \tag{35}$$

Under this map

$$\mathbb{H}^{\mathrm{CM}}\to\mathbb{H}_0^{\mathrm{SU}(1,1)}\otimes\mathcal{L}^2(S_4). \tag{36}$$

The corresponding map of the Hamiltonian is given by the observation that, for β_0^4 large compared to $(v+3/2)^2$,

$$\lambda_v=1+\sqrt{\left(v+\frac{3}{2}\right)^2+\beta_0^4}\to\lambda_0+\frac{1}{2\beta_0^2}v(v+3). \tag{37}$$

Since $v(v+3)$ is an eigenvalue of the SO(5) Casimir operator $\mathcal{C}^{\mathrm{SO}}(5)$, it follows that the adiabatic map sends the Hamiltonian $H(\beta_0)$ to

$$H(\beta_0) \to H_0(\beta_0) + \frac{\hbar\omega}{2\beta_0^2}\mathcal{C}^{\mathrm{SO}(5)}, \tag{38}$$

where $H_0(\beta_0)$ is the SU(1,1) Hamiltonian $H(\beta_0)$ restricted to the scalar $v=0$ representation on $\mathbb{H}_0^{\mathrm{SU}(1,1)}$. Thus, the Hamiltonian $H_0(\beta_0)$ gives a harmonic sequence of β-vibrational states with energies $\{n\hbar\omega,\ n=0,\ldots,\infty\}$. The Hamiltonian $H(\beta,\chi)$ of Eq. (32) likewise maps, in the adiabatic limit, to

$$H(\beta_0,\chi) \to H_{\mathrm{ad}}(\beta_0,\chi) = H_0(\beta_0) + \frac{\hbar\omega}{2\beta_0^2}\mathcal{C}^{\mathrm{SO}(5)} + \chi V(\gamma). \tag{39}$$

I consider the Hamiltonian

$$\mathcal{H}(\chi) = \mathcal{C}^{\mathrm{SO}(5)} - \chi\cos 3\gamma, \tag{40}$$

as a component of the adiabatic Hamiltonian

$$H_{\mathrm{ad}}(\beta_0,\chi) = H_0(\beta_0) + \frac{\hbar\omega}{2\beta_0^2}\big[\mathcal{C}^{\mathrm{SO}(5)} - \chi\cos 3\gamma\big]. \tag{41}$$

Then, if the energy levels of $\mathcal{H}(\chi)$ are given by $\{\mathcal{E}_{\alpha L}\}$, the energy levels of $H_{\mathrm{ad}}(\beta_0,\chi)$ are given by

$$E_{n\alpha L} = n\hbar\omega + \frac{\hbar\omega}{2\beta_0^2}\mathcal{E}_{\alpha L}. \tag{42}$$

5.2. Basis wave functions and matrix elements

The following construction builds on the methods of Chaćon, Moshinsky, and Sharp [21] used by Hess et al. [14] but is conceptually and computationally simpler.

The primary observation is that the Hilbert space of square integrable functions on the four-sphere is spanned by polynomials of the elementary $v=1$, $L=2$ quadrupole moment functions $\{\mathcal{Q}_M\}$ of Eq. (9). A basis is then constructed by first forming a minimal set of angular-momentum-coupled wave functions with angular momentum projection $M=L$ from which a complete set of $M=L$ wave functions can be generated by taking products of the wave functions in this minimal set. A huge advantage is gained by proceeding in this way because all the wave functions generated have good angular momentum and form a complete set of highest-weight wave functions for the SO(3) irreps in $\mathcal{L}^2(S_4)$.

A suitable minimal set of wave functions is found by examination of the angular momentum content of the $\mathcal{L}^2(S_4) = \bigoplus_v \mathbb{H}_v^{(\mathrm{SO}(5))}$ space. Knowing the SO(5) $\to$ SO(3) branching rules for the irreps appearing in the collective model [29], it is possible to arrange the angular-momentum states of $\mathcal{L}^2(S_4)$ into K bands, each having the same sequence of angular-momentum states as those of rigid-beta axially-symmetric rotor bands of the same K, and with the sequence of K bands being in one-to-one correspondence with those

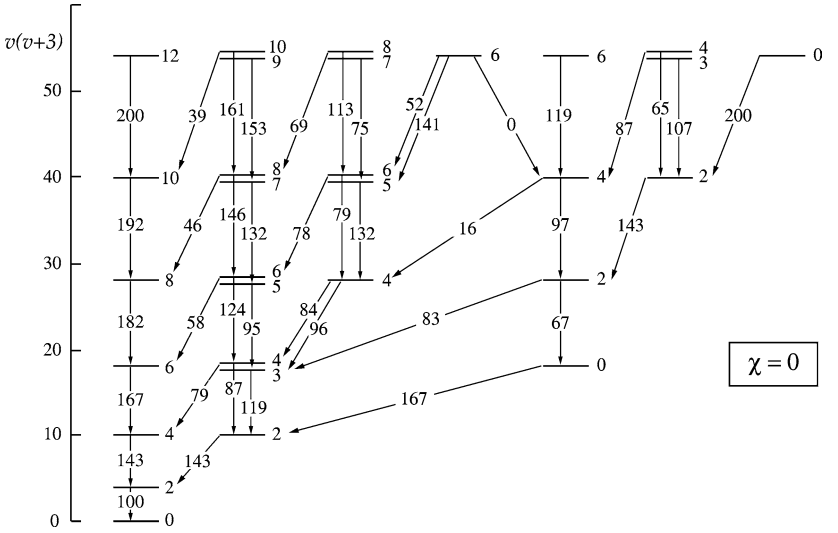

Fig. 6. The low-energy spectrum of the Hamiltonian (40) in the SO(5) limit in which seniority v is a good quantum number. Some pairs of degenerate energy levels are separated slightly for clarity. The B(E2) transition rates shown are in units such that $B(\text{E2: } 12_1 \to 00_1) = 100$. A more complete set of transition rates is given (precisely) in Table 1. Quadrupole moments vanish for states of good seniority.

of a sequence of gamma-vibrational bands. More precisely, the band-heads of the K bands appear with increasing seniority v in the sequences

$$\begin{array}{lccccccccccc} v = 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\ K = 0 & & 2 & & 4 & & 6 & & 8 & & 10 & \ldots \\ & & & 0 & & 2 & & 4 & & 6 & & \ldots \\ & & & & & & 0 & & 2 & & 4 & \ldots \end{array} \tag{43}$$

This band structure, albeit with different energies, is precisely that of the axially-symmetric rotor–gamma–vibrator model (i.e., no beta-vibrational bands) as can be seen in the spectrum of the Hamiltonian $\mathcal{H}(\chi)$ for $\chi = 0$ of Fig. 6. It is now seen that a complete set of $M = L$ coupled polynomials in $\{\mathcal{Q}_M\}$ is generated by taking multiple products of the four generating functions

$$\Phi_{002} \propto \mathcal{Q}_2, \tag{44}$$

$$\Phi_{022} \propto [\mathcal{Q} \otimes \mathcal{Q}]_{22}, \tag{45}$$

$$\Phi_{023} \propto [\mathcal{Q} \otimes \mathcal{Q} \otimes \mathcal{Q}]_{33}, \tag{46}$$

$$\Phi_{100} \propto [\mathcal{Q} \otimes \mathcal{Q} \otimes \mathcal{Q}]_{0}. \tag{47}$$

These functions generate a linearly-independent basis of $M = L$ polynomials

$$\Phi_{tKL} = [\Phi_{002}]^{n_1}[\Phi_{022}]^{n_2}[\Phi_{023}]^{n_3}[\Phi_{100}]^{t}, \tag{48}$$

with

$$K = 2n_2 + 2n_3, \qquad L = 2n_1 + 2n_2 + 3n_3, \quad n_3 = 0 \text{ or } 1. \tag{49}$$

If a state belonging to a K band is labelled by the index $t = i - 1$ when it appears in the ith occurrence of K (e.g., $t = 0$ for states of the lowest $K = 0$ band and $t = 1$ for states of the first excited $K = 0$ band), then the polynomials of Eq. (48) are in one-to-one correspondence with the states labelled by the K-band system given above; t is then the number of Φ_{100} zero-coupled triplets as in the basis construction of [21].

The polynomials $\{\Phi_{tKL}\}$ do not form an orthonormal basis. However, their overlaps are readily evaluated using the inner product for $\mathcal{L}^2(S_4)$ defined by the volume element (21). Moreover, since a polynomial Φ_{tKL} of degree N in $\mathcal{Q}$ does not contain admixtures of wave functions of seniority greater than N, it is a simple matter to sequentially Gram–Schmidt orthogonalize these polynomial functions and obtain an orthonormal basis of wave functions $\{\Psi_{vKLM}\}$ of good SO(5) seniority v. (Note that the label K of the orthonormal basis is not a good quantum number; like the Vergados K label of the SU(3) model it is just a convenient label which makes a useful correspondence with the rotor model.) The procedure is described in detail in Ref. [28].

In manipulating $\mathcal{L}^2(S_4)$ wave functions, it is convenient to expand them in the form

$$\Psi(\gamma,\Omega) = \sum_{\kappa\geqslant 0}^{\text{even}} F_\kappa(\gamma)\sqrt{\frac{2L+1}{16\pi^2(1+\delta_{\kappa 0})}}\big[\mathcal{D}^L_{\kappa M}(\Omega) + (-1)^L\mathcal{D}^L_{-\kappa M}(\Omega)\big]. \tag{50}$$

Thus, a wave function $\Psi \in \mathcal{L}^2(S_4)$ is defined by a set of functions $\{F_0(\gamma), F_2(\gamma), \ldots\}$. In particular, the above generating functions are given by

$$\Phi_{002} \sim \{\cos\gamma, \sin\gamma\}, \tag{51}$$

$$\Phi_{022} \sim \{\cos 2\gamma, \sin 2\gamma\}, \tag{52}$$

$$\Phi_{023} \sim \{0, \sin 3\gamma\}, \tag{53}$$

$$\Phi_{100} \sim \{\cos 3\gamma\}. \tag{54}$$

The overlaps of wave functions are given in this representation by

$$\langle\Psi_{LM}|\Psi'_{L'M'}\rangle = \delta_{LL'}\delta_{MM'}\sum_{\kappa\geqslant 0}^{\text{even}}\int F^*_{L\kappa}(\gamma)F'_{L\kappa}(\gamma)\sin 3\gamma\,\mathrm{d}\gamma \tag{55}$$

and matrix elements of the interaction $\cos 3\gamma$ by

$$\langle\Psi_{LM}|\cos 3\gamma|\Psi'_{L'M'}\rangle = \delta_{LL'}\delta_{MM'}\sum_{\kappa\geqslant 0}^{\text{even}}\int F^*_{L\kappa}(\gamma)F'_{L\kappa}(\gamma)\cos 3\gamma\sin 3\gamma\,\mathrm{d}\gamma. \tag{56}$$

In the present calculation, the needed matrix elements and overlap integrals were evaluated analytically using the interactive mathematics computer program 'MAPLE'.

6. E2 transition rates and quadrupole moments

When the adiabatic approximation is valid and the excited beta bands lie at high energies, the matrix elements of the quadrupole operators $Q^{(E)}_M = Z\beta\mathcal{Q}_M$ between lowlying states factor

$$\langle\Psi_L||Q^{(E)}||\Psi_{L'}\rangle = Z\langle\beta\rangle\langle\Psi_L||\mathcal{Q}||\Psi_{L'}\rangle. \tag{57}$$

Moreover the expectation value $\langle\beta\rangle$, which varies continuously with β_0 can itself be treated as a free parameter. Thus, it only remains to compute the reduced matrix elements of $\mathcal{Q}$. This is straightforward using the wave functions constructed according to the methods of the previous section. For example, in the SO(5) ($\chi = 0$) limit, the $v = 0$, $L = 0$ and $v = 1$, $L = 2$ states have wave functions

$$|00\rangle \sim \sqrt{\frac{3}{16\pi^2}}, \tag{58}$$

$$|12M\rangle \sim \sqrt{\frac{15}{16\pi^2}}\left[\cos\gamma\mathcal{D}^2_{0M}(\Omega) + \frac{1}{\sqrt{2}}\sin\gamma\left(\mathcal{D}^2_{2M}(\Omega) + \mathcal{D}^2_{-2,M}(\Omega)\right)\right]. \tag{59}$$

Thus, with $\mathcal{Q}_M$ given by Eq. (9),

$$\langle 12||\mathcal{Q}||00\rangle = 1 \tag{60}$$

and from the definition

$$B(\text{E2: } L_i \to L_f) = Z^2\langle\beta\rangle^2\frac{|\langle L_f||\mathcal{Q}||L_i\rangle|^2}{2L_i+1}, \tag{61}$$

it follows that

$$B(\text{E2: } 12 \to 00) = \frac{1}{5}Z^2\langle\beta\rangle^2. \tag{62}$$

The calculated results can be used with arbitrary values of the factor $Z\langle\beta\rangle$. The transition rates shown in the figures are in units such that, in the SO(5) limit (in which $\chi = 0$ and v is a good quantum number; cf. Fig. 6)

$$B(\text{E2: } 12 \to 00) = 100. \tag{63}$$

Thus, B(E2) values are given explicitly in these units by the expressions

$$B(\text{E2: } L_i \to L_f) \equiv 100\frac{B(\text{E2: } L_i \to L_f)}{B(\text{E2: } 12 \to 00)} = 500\frac{|\langle L_i||\mathcal{Q}||L_f\rangle|^2}{2L_i+1}. \tag{64}$$

Quadrupole moments are given in the corresponding units by

$$Q^{(E)}(L) = \sqrt{500}(LL, 20|LL)\frac{\langle L||\mathcal{Q}||L\rangle}{\sqrt{2L+1}}, \tag{65}$$

where $(LL, 20|LL)$ is a standard SO(3) Clebsch–Gordan coefficient.

In the SO(5) limit, when $\chi = 0$ and v is a good quantum number,

$$\frac{\langle v+1, L_i||\mathcal{Q}||v, L_f\rangle}{\sqrt{2L_i+1}} = (vL_f, 12||v+1, L_i)\langle v+1|||\mathcal{Q}|||v\rangle, \tag{66}$$

Table 1
Values of the $B(\text{E2: } vL_i \to v-1, L_f)/B(\text{E2: } 12_1 \to 00_1)$ ratio computed exactly as rational numbers

v	L_i	$L_f = L-2$	$L-1$	L	$L+1$	$L+2$
1	2_1	1				
2	4_1	10 / 7				
	2_2			10 / 7		
3	6_1	5 / 3				
	4_2	55 / 63		50 / 63		
	3_1		25/21		10 / 21	
	0_2					5 / 3
4	8_1	20 / 11				
	6_2	150 / 121		70 / 121		
	5_1	21 / 22	105 / 242		52 / 121	
	4_3		26 / 26	910 / 1089		64 / 3267
	2_3	2 / 3			5 / 6	7 / 22
5	10_1	25 / 13				
	8_2	19 / 13		6 / 13		
	7_1	120 / 91	3 / 13		34 / 91	
	6_3	16065 / 20449	340 / 1001	1224 / 1573		256 / 13013
	5_2		189 / 143	6 / 13	20 / 143	
	4_4	75 / 77		1728 / 11011	7 / 11	245 / 1573
	2_4			10 / 7		45 / 91
6	12_1	2				
	10_2	92 / 57		22 / 57		
	9_1	55 / 36	11 / 76	56 / 171		
	8_3	1254 / 1105	19 / 117	847 / 1235		640 / 37791
	7_2	68 / 91	22 / 35	121 / 273	19 / 105	
	6_4	0	32300 / 22841	4598 / 8785	1292 / 22841	52 / 8785
	6_5	1004 / 845	3136 / 212095	544 / 3263	91125 / 169676	15827 / 169676
	4_5	17 / 26		34 / 39	17 / 78	10 / 39
	3_2		15 / 14		22 / 35	3 / 10
	0_3	2				

where $(vL_f, 12||v+1, L_i)$ is a reduced SO(5) Clebsch–Gordan coefficient and $\langle v+1|||\mathcal{Q}|||v\rangle$ is the SO(5)-reduced matrix element given by Eq. (28). Thus, in the SO(5) limit,

$$\frac{B(\text{E2: } v+1, L_i \to vL_f)}{B(\text{E2: } 12 \to 00)} = \frac{5(v+1)}{(2v+5)}(vL_f, 12||v+1, L_i)^2. \tag{67}$$

Low-lying values of these ratios are given in Table 1. Quadrupole moments vanish in the SO(5) limit because $\mathcal{Q}$ is a $v=1$ tensor operator and, hence, its matrix elements satisfy the well-known $\Delta v = \pm 1$ selection rule.

7. Results

The low-energy spectrum of the Hamiltonian (40) is shown for $\chi = 0$, 25 and 50 in Figs. 6, 7, and 8, respectively. The results were obtained by diagonalization of the

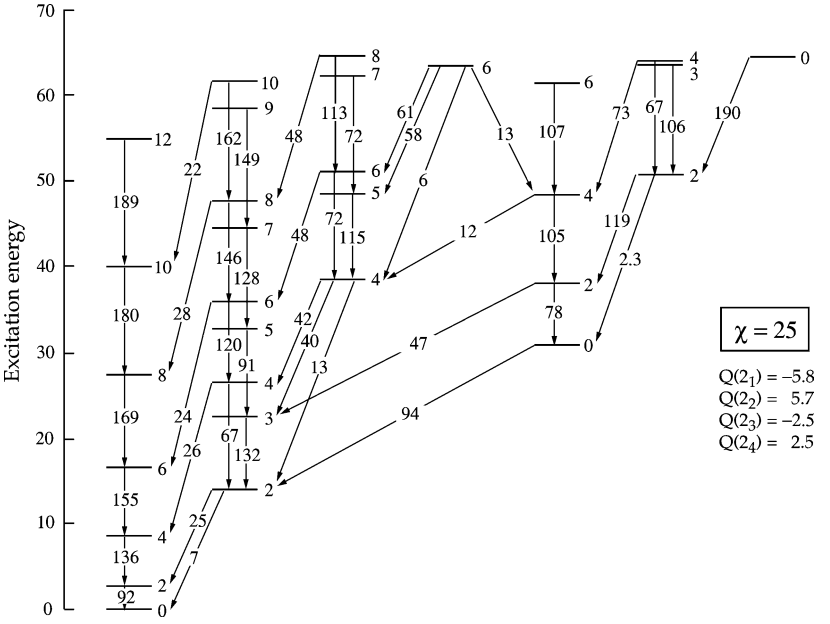

Fig. 7. The low-energy spectrum and B(E2) transition rates for the Hamiltonian (40) with $\chi = 25$. The units are as defined in the caption to Fig. 6. For χ non-zero, seniority is no longer a good quantum number and quadrupole moments are no longer zero.

Hamiltonian in a basis of 12 states for each angular momentum; it was ascertained that this number is sufficient to obtain results of the desired accuracy for $\chi \leqslant 50$. It can be seen from these figures that the spectrum and E2 transition rates progress with increasing χ from those of the Wilets–Jean gamma-independent model (at $\chi = 0$) to those of the adiabatic axially-symmetric rotor–vibrator model. In particular, the K bands acquire the characteristics of an axially-symmetric rotor model and sequences of excited gamma-vibrational bands emerge. This is most evident in the sequence of $K = 2$, 4 and 6 bands, which appear as one-, two-, and three-phonon gamma-vibrational bands, and the move of the first excited $K = 0$ band-head towards the energy of the two-phonon $K = 4$ gamma-vibrational energy.

It is instructive to compare the results of Fig. 8 with the sequence of rotational bands expected in the standard axially symmetric rotor model shown in Fig. 9.

The comparison shows that the rotational dynamics are adiabatic relative to the gamma-vibrations for the lowest energy states but that significant departures become evident at higher energies. Increasing the value of χ would surely reduce the rotational–vibrational coupling and give results closer to those of the adiabatic rotor model. However, by the same token, the gamma-vibrational bands would rise to higher energies. It is of interest to see if some of the effects of rotation–vibration coupling, e.g., the non-rotational behaviour of some of the E2 transitions in the two-phonon $K = 4$ gamma band, are realized in rotational nuclei having low-energy gamma bands. It would also be interesting to see if the results of Fig. 8 can be reproduced more accurately in the adiabatic rotor model by taking the rotation–vibration coupling into account in perturbation theory.

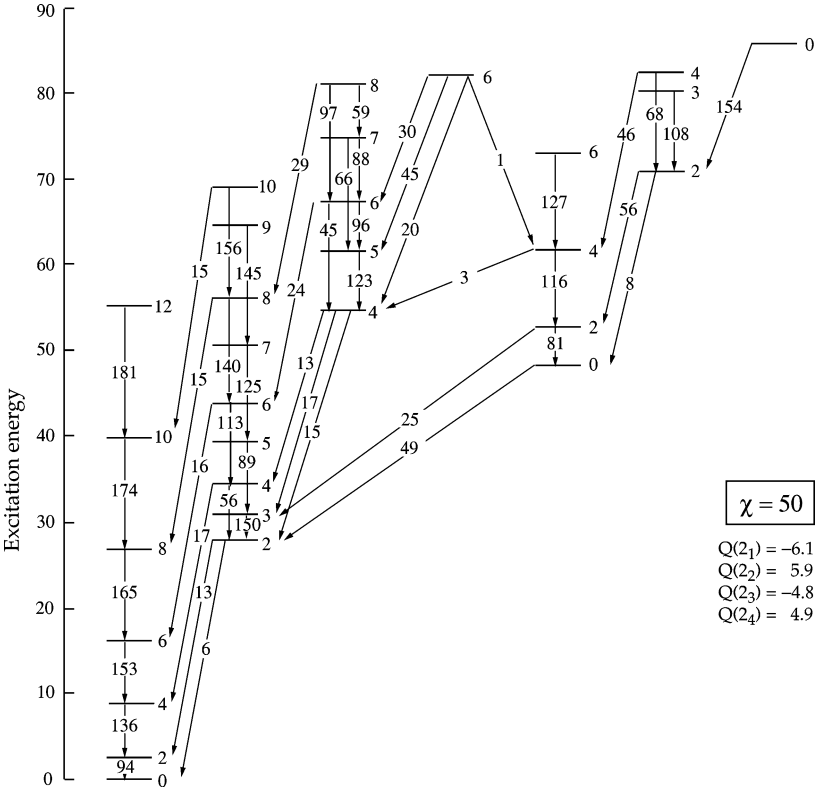

Fig. 8. The low-energy spectrum and B(E2) transition rates for the Hamiltonian (40) with $\chi = 50$. The units are as defined in the caption to Fig. 6. For χ non-zero, seniority is no longer a good quantum number and quadrupole moments are no longer zero.

8. Summary

The Wilets–Jean model is an algebraic model with an $[R^5]SO(5) \supset SO(5) \supset SO(3)$ dynamical subgroup chain. However, as observed above, the states that reduce this dynamical chain are strictly zero-width beta-rigid states. Complete rigidity is unphysical. Nevertheless, it is possible for the WJ model to work well even when the beta fluctuations of physical states are relatively large. This is because, it is hard to measure the beta-width of physical states; indeed it is invariably unclear as to the nature of excited $K = 0$ excitations. Many intrinsic excitations of a nucleus can give rise to excited $K = 0$ bands and even the collective model has two-phonon $K = 0$ gamma bands.

To avoid the problems associated with the beta-rigidity of the WJ model, we considered in this paper a more physical collective model which, in its gamma-soft limit, has an $SU(1,1) \times SO(5) \supset SO(5) \supset SO(3)$ dynamical subgroup chain. For large values of the parameter $\beta_0\sqrt{\omega}$, the low-energy states of this model acquire properties that are essentially indistinguishable from those of the less physical WJ model. In the WJ limit of this model, the beta vibrational frequency is infinite for a finite value of β_0 and, hence, beta vibrational

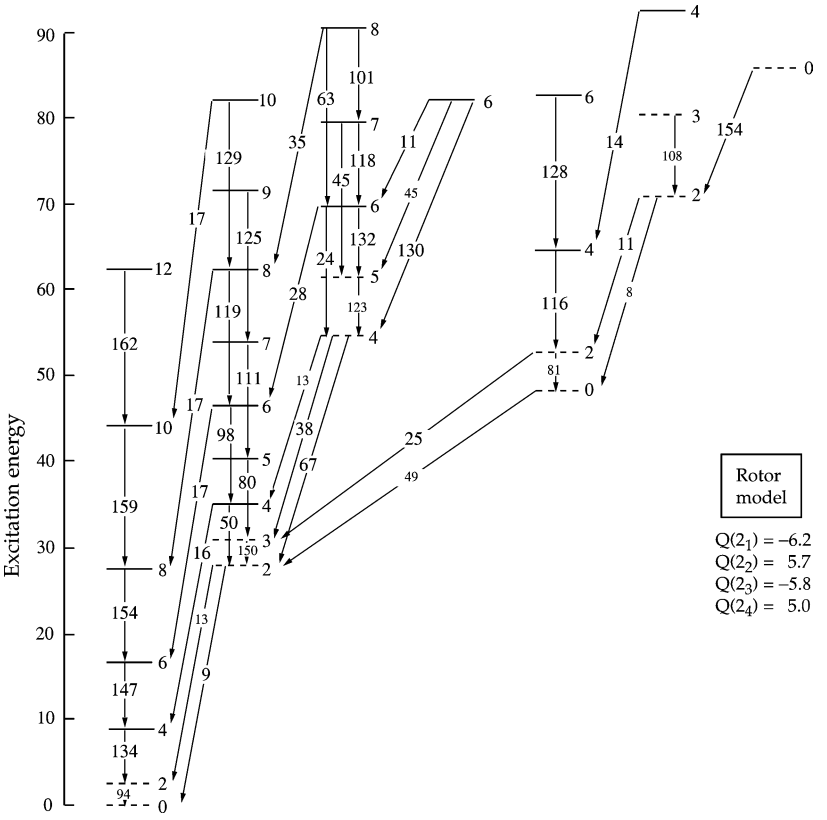

Fig. 9. The low-energy spectrum and B(E2) transition rates given by the axially symmetric rotor model with band-head energies and moments of inertia adjusted so that the dashed energy levels are fitted to those of Fig. 8. The intrinsic quadrupole moments of the model were also adjusted to fit the E2 transition rates shown in smaller type.

excitations are not observable. However, neither an infinite beta-vibrational frequency nor a zero beta-width for the collective wave functions is necessary to obtain the results of the WJ model. All that is needed is an adiabatic decoupling of the rotational and beta-vibrational degrees of freedom. The signature of such decoupling is that the beta wave functions in the $SU(1,1) \times SO(5)$ model become essentially independent of the excitation energy for the range of energies of interest. That such is the case when the rotational energies are small compared to the one-phonon beta-vibrational energy is illustrated in Fig. 5.

Relaxing the rigidity of the WJ model and replacing its $[R^5]SO(5)$ dynamical group with $SU(1,1) \times SO(5)$, results in a more physical collective model. More importantly, it makes it possible to construct a basis in which arbitrary collective model Hamiltonians can be expanded in a rapidly convergent manner. For example, the solutions of the Hamiltonian (41) given in Figs. 6–8 were obtained by diagonalization of 12×12 matrices. For values of $\chi > 50$, the dimensions would have to be increased to ensure accurate results. But clearly, much larger dimensions can be accommodated with available computers.

Acknowledgements

The author wishes to thank J.L. Wood and the referee for constructive suggestions.

Appendix A. The five-dimensional Kratzer model

The five-dimensional Kratzer model, proposed by Fortunato and Vitturi [22], has a similar algebraic solution to that of the Elliott–Evans–Park model [5]. A straightforward extension of the methods used for the hydrogen atom to the Hamiltonian

$$H = -\frac{\hbar^2}{2B}\nabla^2 - \frac{k}{\beta}, \tag{A.1}$$

of a five-dimensional Coulomb problem gives the energy levels

$$E_{n\tau} = -\frac{k^2 B}{2\hbar^2 \nu_{n\tau}^2}, \quad \nu_{n\tau} = n + \lambda_\tau/2, \tag{A.2}$$

and wave functions

$$\Psi_{n\tau}(\beta) = \sqrt{\frac{n!}{\Gamma(n+\lambda_\tau)2\nu_{n\tau}^6 b^5}}\left(\frac{\beta}{b\nu_{n\tau}}\right)^{(\lambda_\tau-4)/2} \times \exp\left(-\frac{\beta}{2b\nu_{n\tau}}\right) L_n^{(\lambda_\tau-1)}\left(\frac{\beta}{b\nu_{n\tau}}\right)\mathcal{Y}_{\tau\sigma}(\omega), \tag{A.3}$$

where now $\lambda_\tau = 2\tau + 4$ and $b = \hbar^2/(2kB)$.

These results are obtained by use of the SU(1, 1) Lie algebra spanned by the operators

$$Z_1 = -2\beta\nabla^2, \qquad Z_2 = 2\beta, \qquad Z_3 = -2i\left(q\cdot\nabla + \frac{5}{2}\right). \tag{A.4}$$

Again it is seen that, the commutation relations of this Lie algebra are unchanged if Z_1 is replaced by

$$Z_1' = 2\beta\left(-\nabla^2 + \frac{\beta_0^4}{\beta^2}\right). \tag{A.5}$$

It follows that the above results for the Kepler Hamiltonian, extend immediately to the five-dimensional Kratzer model with Hamiltonian

$$H(\varepsilon) = \frac{\hbar^2}{2B}\left(-\nabla^2 + \frac{\beta_0^4}{\beta^2}\right) - \frac{k}{\beta}. \tag{A.6}$$

However, now the relationship $\lambda_\tau = 2\tau + 4$ is replaced by

$$\lambda_\tau = 1 + 2\sqrt{\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4}. \tag{A.7}$$

This follows because the replacement $Z_1 \to Z_1'$ in the SU(1, 1) Lie algebra modifies the value of the SU(1, 1) Casimir invariant from the value $\lambda_\tau(\lambda_\tau - 2) = 4\tau(\tau + 3) + 8$ to the value

$$\lambda_\tau(\lambda_\tau - 2) = 4\tau(\tau + 3) + 8 + 4\beta_0^4. \tag{A.8}$$

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