①

2020-04-12

The ACM uses the subgroup chain $SO(3) \subset SO(5)$ where $SO(5)$ acts on the 5-dimensional space of quadrapole moments q_M, $M = -2, -1, 0, 1, 2$ which transform as an $L=2$ irrep of $SO(3)$. The norm on $\mathbb{R}^5$ is $|q|^2 = \sum_M |q_M|^2$.
Show that the action of $SO(3)$ on $\mathbb{R}^5$ leaves the norm invariant.

Start with $\mathbb{R}^3$ and the usual inner product

$$(u, v) = u \cdot v = \sum_{i=1}^{3} u_i \cdot v_i$$

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$

$SO(3) \subset O(3)$ $SO(3)$ is the identity component of $O(3)$

$$O(3) = \{ R : GL(3) \mid \forall u, v : \mathbb{R}^3 \cdot (Ru, Rv) = (u, v) \}$$

Now consider the action of $SO(3)$ on $\mathbb{R}^3 \otimes \mathbb{R}^3$

$$\rho : SO(3) \longrightarrow GL(\mathbb{R}^3 \otimes \mathbb{R}^3)$$

$\mathbb{R}^3 \otimes \mathbb{R}^3$ is spanned by simply bivectors of the form $u \otimes v$ for $u, v \in \mathbb{R}^3$

$$\rho R \; u \otimes v = (Ru) \otimes (Rv)$$

$\dim \mathbb{R}^3 \otimes \mathbb{R}^3 = 9$. ρ is reducible.

Consider the swap operation S

$$S \; u \otimes w = w \otimes u$$

Clearly $S^2 = I$. Therefore $\mathbb{R}^3 \otimes \mathbb{R}^3$ splits into eigenspaces of S with eigenvalues 1 & -1.

(2) Let $\Lambda^2\mathbb{R}^3 = \{b : \mathbb{R}^3 \otimes \mathbb{R}^3 \mid Sb = -b\}$ 2020-04-12

$$V^2\mathbb{R}^3 = \{b : \mathbb{R}^3 \otimes \mathbb{R}^3 \mid Sb = b\}$$

$\Lambda^3\mathbb{R}^3$ are the antisymmetric bivectors.
$V^3\mathbb{R}^3$ are the symmetric bivectors.

$\Lambda^2\mathbb{R}^3$ is spanned by $u \wedge w = u \otimes w - w \otimes u$
$\dim \Lambda^2\mathbb{R}^3 = 3$ since the following form a basis.
Let e_1, e_2, e_3 be a basis of $\mathbb{R}^3$
then $e_1 \wedge e_2$, $e_2 \wedge e_3$, $e_3 \wedge e_1$ form a basis of $\Lambda^2\mathbb{R}^3$: Prove $\Lambda^2\mathbb{R}^3 \cong \mathbb{R}^3$ as an irrep of $SO(3)$

Therefore $V^2\mathbb{R}^3$ has dimension 6.

<u>Theorem</u> $\rho \cong \rho_+ \oplus \rho_-$ $\quad \rho_+ : SO(3) \to GL(V^2\mathbb{R}^3)$
$\rho_- : SO(3) \to GL(\Lambda^2\mathbb{R}^3)$

Let $\langle e_1, e_2, e_3 \rangle$ be an orthonormal ordered basis.
define $\varepsilon = e_1 \wedge e_2 \wedge e_3$
The inner product on $\mathbb{R}3$ defines an isomorphism $u \mapsto u^T$

$$T : \mathbb{R}^3 \to \mathbb{R}^{3*} = \hom(\mathbb{R}^3, \mathbb{R})$$

$$u^T w = (u, w) = u \cdot w$$

$$\Lambda^3\mathbb{R}^3 \cong \mathbb{R} \qquad \alpha\varepsilon \mapsto \alpha$$
$$\lambda\varepsilon \mapsto \lambda.$$

Therefore we have an isomorphism

$$\alpha : \Lambda^2\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$$

$\alpha b = u$ s.t. $\quad b \wedge w = \lambda\varepsilon = (\alpha b)^T \cdot w\, \varepsilon$
$= (\alpha b, w)\,\varepsilon$

(3)

2020-04-12

Consider the action of $SO(3)$ on $\Lambda^2 \mathbb{R}^3$.

$$(\Lambda^2 \rho) R \; u \wedge w = (Ru) \wedge (Rw)$$

$$(Ru) \wedge (Rv) \wedge (Rw) = u \wedge v \wedge w \quad (\det R = 1)$$

In general $(Au) \wedge (Av) \wedge (Aw) \overset{\text{def}}{=} (\det A)\, u \wedge v \wedge w$

$T: \mathbb{R}^3 \cong \mathbb{R}^3$ if $\dim V = 3$.

$\alpha: \Lambda^2 \mathbb{R}^3 \cong \mathbb{R}^3$

$$\left(\left(\alpha(u \wedge v)\right)^T w \right) \mathcal{E} = u \wedge v \wedge w$$
$$= \left(\alpha(u \wedge v), w\right) \mathcal{E}$$

$$(\alpha(Ru \wedge Rv), w) \mathcal{E}$$
$$= Ru \wedge Rv \wedge w$$
$$= u \wedge \cancel{R} v \wedge R^{-1} w$$
$$= (\alpha(u, v), R^{-1} w) \mathcal{E}$$
$$= (R(\alpha(u, v)), w) \mathcal{E}$$
$$\Rightarrow \alpha(Ru, Rv) = R(\alpha(u, v))$$

So α is on $SO(3)$ (invariant / equivariant homomorphism)

4 Now consider $\Lambda^2\mathbb{R}^3$. 2020-04-12

$\dim \Lambda^2\mathbb{R}^3 = 6$.

The inner product g is itself an element of $\Lambda^2(\mathbb{R}^{3*})$

$$u \cdot v = (u,v) = g(u,v).$$

g is symmetric and positive definite.

$$g : \mathbb{R}^3 \cong \mathbb{R}^{3*} = \mathrm{dual}(\mathbb{R}^3)$$

define the dual basis $e^i = \lambda v \cdot g(e_i, v)$

then $g = \sum_{i,j} g_{ij} e^i \wedge e^j$

define $\beta : \mathbb{R}^{3*} \longrightarrow \mathbb{R}^3$ s.t.

$$g(\beta(u), v) = u(v)$$

use differente letter for dual vectors. e.g. f to indicate they are functions.

$$\boxed{f(v) = g(\beta f, v)}$$

Defines β
$\beta : \mathbb{R}^{3*} \longrightarrow \mathbb{R}^3$

Then extend β to tensor products.

$$\beta^{\otimes 2} g = \beta^{\otimes 2}\left(\sum \cancel{g_{ij}\, u_i \otimes v_j}\right)$$

$$g : \mathbb{R}^{3*} \otimes \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\mathbb{R}^3 \otimes \mathbb{R}^3 \overset{\cancel{\text{def}}}{=\!=} \mathbb{R}^3 \times \mathbb{R}^3 / \sim$$

⑤ 2020-04-12

Suppose $g(u,v) = \cancel{\#}$
$$g\left(\sum u_i e_i , \sum v_j e_j\right)$$

$$= \sum_i \sum_j g(e_i, e_j)\, u_i v_j$$

$$= \sum_{ij} g(e_i, e_j)\, g(e_i, u)\, g(e_j, v)$$

$$\Rightarrow \quad g = \sum_{ij} g_{ij}\, e_i^* \otimes e_j^*$$

Then $g^* = \sum_{ij} g_{ij}\, e_i \otimes e_j \in \cancel{\Lambda}\, V^2 \mathbb{R}^3$

Show that it is invariant under ~~x~~, SO(3).

$$R g^* = \sum_{ij} g_{ij} (Re_i) \otimes (Re_j)$$

$$(e_a^* \otimes e_b^*)(R g^*) = \sum_{ij} g_{ij}\, e_a^*(Re_i) \cdot e_b^*(Re_j)$$

$$= \sum_{ij} g_{ij}\, g(e_a, Re_i)\, g(e_b, Re_j)$$

$$= \sum_{ij} g_{ij}\, g(R^{-1} e_a, e_i)\, g(R^{-1} e_b, e_j)$$

$$= (R^{-1} e_a)^* \otimes (R^{-1} e_b)^* \left(\sum_{ij} g_{ij}\, e_i \otimes e_j\right)$$

$$= (R^{-1} e_a)^* \otimes (R^{-1} e_b)^* (g^*)$$

2020-04-12

$g(Ru, Rv) = g(u,v)$ by def of $SO(3)$.

$\mathbb{R}^3$ is self dual ~~for~~ under $SO(3)$.

ie. $(Ru)^* = R^* u^*$

so $g \in \mathbb{R}^{3*} \otimes \mathbb{R}^{3*}$ is invariant.

so $g^* \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is $SO(3)$-invariant.

Therefore g^* spans a 1-dimensional $SO(3)$-invariant subspace of $\vee^2 \mathbb{R}^3$.

$SO(3)$ is a compact Lie group so the space $\vee^2 \mathbb{R}^3$ splits

$$\vee^2 \mathbb{R}^3 \cong D_0 \oplus D_2$$

$\dim D_0 = 1$

$\dim D_2 = 5$ $\quad D_2$ is irreducible.

in an orthornormal basis

$$g^* = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

what is the basis of D_2?

$e_1 \otimes e_2 + e_2 \otimes e_1$, $e_1 \otimes e_3 + e_3 \otimes e_1$,

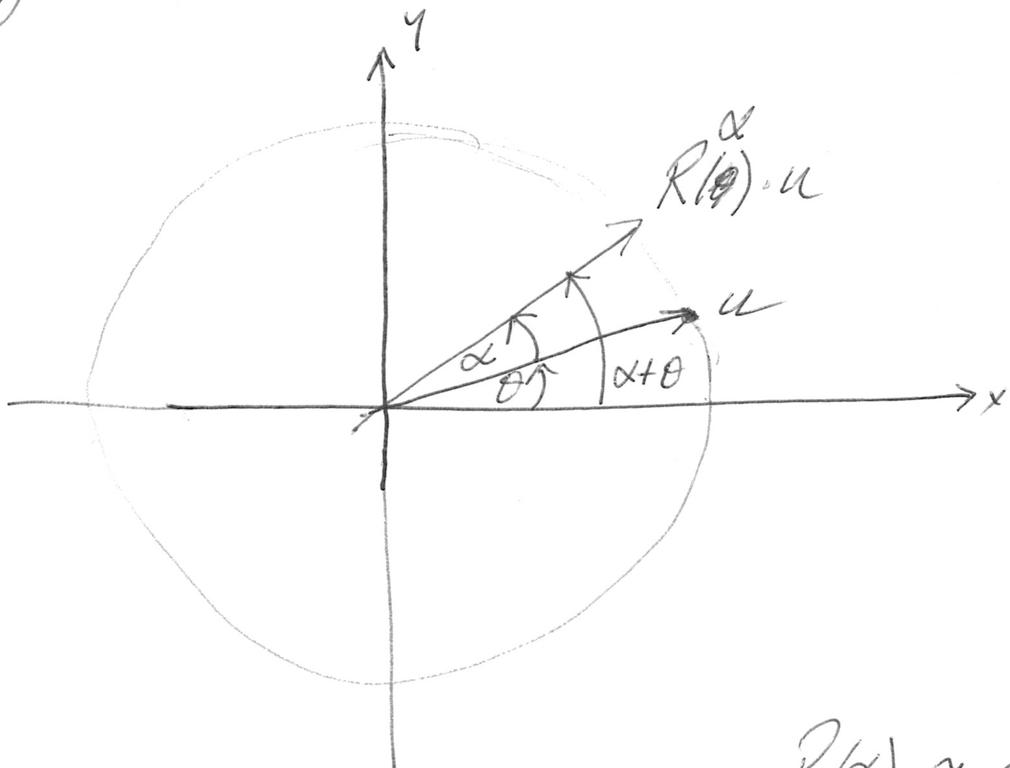
$e_2 \otimes e_3 + e_3 \otimes e_2$, $e_1 \otimes e_1$, $e_2 \otimes e_2$

Look at the action of $SO(2)$ & the eigen values of rotation about e_3.

$R_z(\theta)\, e_1 = \cos\theta \cdot e_1 \pm \sin\theta \cdot e_2$

$\qquad e_2 = \pm \sin\theta\, e_1 + \cos\theta \cdot e_2$

2020-04-12



$u = x + iy$ $\qquad R(\alpha) \sim e^{i\alpha}$

$R(\alpha)u = e^{i\alpha}(x+iy)$

$= (\cos\alpha + i\sin\alpha)(x+iy)$

$= x\cos\alpha + ix\sin\alpha$

$+ iy\cos\alpha - y\sin\alpha$

$= (x\cos\alpha - y\sin\alpha) + i(x\sin\alpha + y\cos\alpha)$

$R(\alpha)\cdot(x,y) = (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$

$= \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$

$SO(2) = \{\alpha : [0, 2\pi]\} \downarrow \circ \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\}$

under $\mathbb{R}^2 \subset \mathbb{R}^3$

$(x,y) \longmapsto (x,y,0)$

$SO(2) \subset SO(3)$

$SO(2) = \{R : SO(3) \mid Re_3 = e_3\}$

$e_1 = (1,0,0) \quad e_2 = (0,1,0) \quad e_3 = (0,0,1)$

$\cancel{R_\alpha e_1 =}$ $\begin{pmatrix}x \\ y\end{pmatrix} = xe_1 + ye_2$

$= (x\cos\alpha - y\sin\alpha)e_1 + (x\sin\alpha + y\cos\alpha)e_2$

$R_\alpha\begin{pmatrix}x \\ y\end{pmatrix} = (x\cos\alpha - y\sin\alpha)e_1 + (x\sin\alpha + y\cos\alpha)e_2$

$R_\alpha(xe_1 + ye_2) = (x\cos\alpha - y\sin\alpha)e_1 + (x\sin\alpha + y\cos\alpha)e_2$

$R_\alpha e_1 = \cos\alpha\, e_1 + \sin\alpha\, e_2 \qquad R_\alpha e_2 = -\sin\alpha\, e_1 + \cos\alpha\, e_2$

(8) 2020-04-12

$$R_\alpha(e_1\otimes e_1 + e_2\otimes e_2) = (\cos\alpha\, e_1 + \sin\alpha\, e_2)\otimes(\cos\alpha\, e_1 + \sin\alpha\, e_2)$$
$$+ (-\sin\alpha\, e_1 + \cos\alpha\, e_2)\otimes(-\sin\alpha\, e_1 + \cos\alpha\, e_2)$$

$$= \cos^2\alpha\, e_1\otimes e_1 + \cancel{\cos\alpha\cdot\sin\alpha\cdot e_1\otimes e_2}$$
$$+ \cancel{\sin\alpha\cos\alpha\, e_2\otimes e_1} + \sin^2\alpha\, e_2\otimes e_2$$
$$+ \sin^2\alpha\, e_1\otimes e_1 - \cancel{\sin\alpha\cos\alpha\, e_1\otimes e_2}$$
$$- \cancel{\sin\alpha\cos\alpha\, e_2\otimes e_1} + \cos^2\alpha\, e_2\otimes e_2$$

$$= e_1\otimes e_1 + e_2\otimes e_2. \quad \checkmark \qquad \Rightarrow \text{invariant.}$$

$$R_\alpha(e_1\otimes e_2 + e_2\otimes e_1)$$
$$= (R_\alpha e_1)\otimes(R_\alpha e_2) + (R_\alpha e_2)\otimes R_\alpha(e_1)$$
$$= (R_\alpha e_1)\vee(R_\alpha e_2)$$
$$= (\cos\alpha\, e_1 + \sin\alpha\, e_2)\vee(-\sin\alpha\, e_1 + \cos\alpha\, e_2)$$
$$= \cancel{(+\cos\alpha - \sin\alpha)\, e_1\vee e_1}$$
$$- \cos\alpha\cdot\sin\alpha\, e_1\vee e_1$$
$$+ \cos^2\alpha\, e_1\vee e_2$$
$$- \sin^2\alpha\, e_1\vee e_2$$
$$+ \sin\alpha\cos\alpha\, e_2\vee e_2$$

$$(e^{i\alpha})^2 = e^{i2\alpha} = \cos 2\alpha + i\sin 2\alpha$$
$$= (\cos\alpha + i\sin\alpha)^2$$
$$= \cos^2\alpha - \sin^2\alpha + 2i\cos\alpha\sin\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\sin 2\alpha = 2\cos\alpha\sin\alpha$$
$$R_\alpha(e_1\vee e_2) = \sin\alpha\cos\alpha\,(e_2\vee e_2 - e_1\vee e_1)$$
$$+ (\cos^2\alpha - \sin^2\alpha)\, e_1\vee e_2$$
$$= \frac{1}{2}\sin 2\alpha\,(e_2\vee e_2 - e_1\vee e_1)$$
$$+ \cos 2\alpha\, e_1\vee e_2$$

But $\{e_1\vee e_1, e_1\vee e_2, e_2\vee e_2\}$ probably transform like 2α.
$e_1\vee e_3$, $e_2\vee e_3$ like α
$e_3\vee e_3$ ~~like~~ $0 = \alpha$